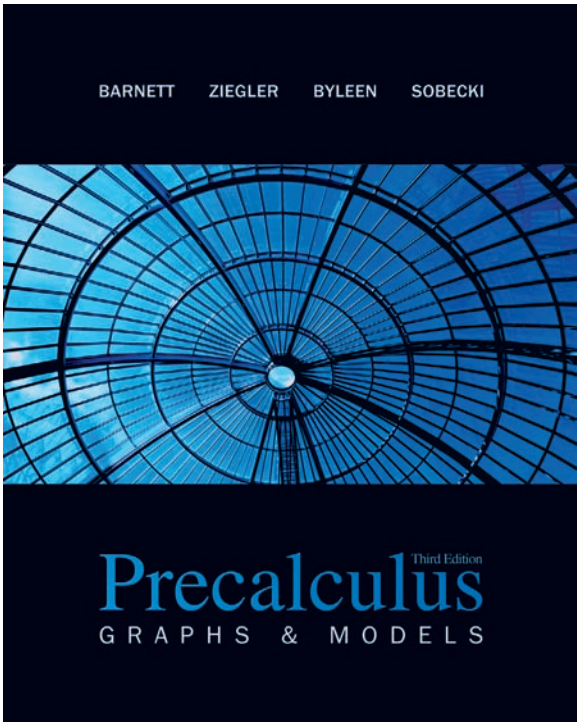
BARNETT ZIEGLER BYLEEN SOBECKI

Precalculus

Third Edition

GRAPHS & MODELS



THIRD EDITION

Precalculus

GRAPHS AND MODELS

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College Algebra with Trigonometry: Graphs and Models

This book is the same as *Precalculus: Graphs and Models* except that the trigonometric functions are introduced by right triangle trigonometry. This text assumes the use of a graphing calculator.
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Preface

Enhancing a Tradition of Success

We take great satisfaction from the fact that more than 100,000 students have learned college algebra or precalculus from a Barnett Series textbook. Ray Barnett is one of the masters of college textbook writing. His central approach is proven and remains effective for today's students.

The third edition of *Precalculus: Graphs and Models* has benefited greatly from the numerous contributions of new coauthor Dave Sobecki of Miami University Hamilton. Dave brings a fresh approach to the material and many good suggestions for improving **student accessibility.** Every aspect of the revision focuses on making the text more relevant to students, while retaining the precise presentation of the mathematics for which the Barnett name is renowned.

Specifically we concentrated on the areas of writing, worked examples, exercises, technology, and design. Based on numerous reviews, advice from expert consultants, and direct correspondence with many users of previous editions, we feel that this edition is more relevant than ever before. We hope you will agree.

Writing Without sacrificing breadth or depth or coverage, we have rewritten explanations to make them clearer and more direct. As in previous editions, the text emphasizes computational skills, real-world data analysis and modeling, and problem solving rather than theory.

Examples In the new edition, even more solved examples in the book provide graphical solutions side-by-side with algebraic solutions. By seeing the same answer result from their symbol manipulations *and* from graphical approaches, students gain insight into the power of algebra and make important conceptual and visual connections.

Likewise, we added expanded color annotations to many examples, explaining the solution steps in words. Each example is then followed by a similar matched problem for the student to solve. Answers to the matched problems are located at the end of each section for easy reference. This active involvement in learning while reading helps students develop a more thorough understanding of concepts and processes.

Exercises With an eye to improving student performance *and* to make the book more useful for instructors, we have extensively revised the exercise sets. We added hundreds of new writing questions as well as exercises at the easy to moderate level and expanded the variety of problem types to ensure a gradual increase in difficulty level throughout each exercise set.

Technology Although technology is employed throughout, we strive to balance algebraic skill development with the use of technology as an aid to learning and problem solving. We assume that students using the book will have access to one of the various graphing calculators or computer programs that are available to perform the following operations:

- Simultaneously display multiple graphs in a user-selected viewing window
- Explore graphs using trace and zoom

- Approximate roots and intersection points
- Approximate maxima and minima
- Plot data sets and find associated regression equations
- Perform basic matrix operations, including row reduction and inversion

Most popular graphing calculators perform all of these operations. The majority of the graphing calculator images in this book are "screen shots" from a Texas Instruments TI-84 Plus graphing calculator. Students not using that TI calculator should be able to produce similar results on any calculator or software meeting the requirements listed. The proper use of such calculators is covered in Section 1-1.

A Central Theme

In the Barnett series, the function concept serves as a unifying theme. A brief look at the table of contents reveals this emphasis. A major objective of this book is the development of a library of elementary functions, including their important properties and uses. Employing this library as a basic working tool, students will be able to proceed through this book with greater confidence and understanding.

Design: A New Book with a New Look

The third edition of *Precalculus: Graphs and Models* presents the subject in the precise and straightforward way that users have come to rely on in the Barnett textbooks, now updated for students in the twenty-first century. The changes to the text are manifested visually in a new design. We think the pages of this edition offer a more contemporary and inviting visual backdrop for the mathematics.

Features

New to the Third Edition

An extensive **reworking of the narrative** throughout the chapters has made the language less formal and more engaging for students.

A **new full-color design** gives the book a more contemporary feel and will appeal to students who are accustomed to high production values in books, magazines, and nonprint media.

Even more examples now feature **side-by-side solutions** integrating algebraic and graphical solution methods. This format encourages students to investigate mathematical principles and processes graphically and numerically, as well as algebraically.

The increased use of **annotated algebraic steps,** in small colored type, to walk students through each critical step in the problem-solving process helps students follow the authors' reasoning and improve their own problem-solving strategies.

An **Annotated Instructor's Edition** is now available and contains answers to exercises in the text, including answers to section, chapter review, and cumulative review exercises. These answers are printed in a second color, adjacent to corresponding exercises, for ease of use by the instructor.

More **balanced exercise sets** give instructors maximum flexibility in assigning homework. We added exercises at the easy to moderate level and expanded the variety of problem types to ensure a gradual increase in difficulty level throughout each exercise set. The division of exercise sets into A (routine, easy mechanics), B (more difficult mechanics), and C (difficult mechanics and some theory) is no longer explicit

in the student edition of the text: the letter designations appear only in the Annotated Instructor's Edition. This change was made in order to avoid fueling students' anxiety about challenging exercises. As in previous editions, students at all levels can be challenged by the exercises in this text.

Hundreds of new **writing questions** encourage students to think about the important concepts of the section *before* solving computational problems. Problem numbers that appear in blue indicate problems that require students to apply their reasoning and writing skills to the solution of the problem.

Features Retained

Examples and matched problems introduce concepts and demonstrate problem-solving techniques using side-by-side algebraic and graphical solution methods. Each carefully solved example is followed by a similar Matched Problem for the student to work through while reading the material. Answers to the matched problems are located at the end of each section, for easy reference. This active involvement in the learning process helps students develop a more thorough understanding of algebraic and graphical concepts and processes.

Graphing calculator technology is integrated throughout the text for visualization, investigation, and verification. Graphing calculator screens displayed in the text are actual output. Although technology is employed throughout, the authors strive to balance algebraic skill development with the use of technology as an aid to learning and problem solving.

Annotated steps of examples and developments are found throughout the text to help students through the critical stages of problem solving. Think Boxes (color dashed boxes) are used to enclose steps that, with some experience, many students will be able to perform mentally.

Applications throughout the third edition give the student substantial experience in modeling and solving real-world problems, fulfilling a primary objective of the text. Over 650 application exercises help convince even the most skeptical student that mathematics is relevant to everyday life. Chapter Openers are written to highlight interesting applications and an Applications Index is included to help locate applications from particular fields.

Explore-discuss boxes are interspersed throughout each section. They foster conceptual understanding by asking students to think about a relationship or process before a result is stated. Verbalization of mathematical concepts, processes, and results is strongly encouraged in these investigations and activities.

Group activities at the end of each chapter involve multiple concepts discussed in the chapter. These activities strongly encourage the verbalization of mathematical concepts, results, and processes. All of these special activities are highlighted to emphasize their importance.

Foundations for calculus icons are used to mark concepts that are especially pertinent to a student's future study of calculus.

Interpretation of graphs icons are used to mark exercises that ask students to make determinations about equations or functions based on graphs.

Key Content Changes

Chapter 1, Functions, Graphs, and Models Functions are now introduced in terms of relations, providing more flexibility in discussing correspondence between sets of objects.

Chapter 2, Modeling with Linear and Quadratic Functions Quadratic inequalities are now covered using a more general test point method, so that all nonlinear inequalities are solved using the same method.

Chapter 3, Polynomial and Rational Functions Section 3-1 (Polynomial Functions and Models) has been split into two sections, so Section 3-2 is now entirely devoted to division of polynomials, including remainder and factor theorems crucial to the study of zeros of polynomials later in the chapter. The chapter also features a new section (3-6) on direct, inverse, joint, and combined variation. This new section supports the book's emphasis on mathematical modeling.

Chapter 8, Modeling with Systems of Equations and Inequalities A new Section 8-2 on Systems of Linear Equations in Three Variables was added, while Section 8-1 on Systems of Linear Equations in Two Variables was reorganized.

Chapter 9, Matrices and Determinants The material on matrix solutions to linear systems is now found in Section 9-1.

Supplements

MathZone McGraw-Hill's MathZone is a complete online tutorial and homework management system for mathematics and statistics, designed for greater ease of use than any other system available. Instructors have the flexibility to create and share courses and assignments with colleagues, adjunct faculty, and teaching assistants with only a few clicks of the mouse. All algorithmic exercises, online tutoring, and a variety of video and animations are directly tied to text-specific materials.

MathZone is completely customizable to suit individual instructor and student needs. Exercises can be easily edited, multimedia is assignable, importing additional content is easy, and instructors can even control the level of help available to students while doing their homework. Students have the added benefit of full access to the study tools to individually improve their success without having to be part of a MathZone course.

MathZone has automatic grading and reporting of easy-to-assign algorithmically generated problem types for homework, quizzes, and tests. Grades are readily accessible through a fully integrated grade book that can be exported in one click to Microsoft Excel, WebCT, or BlackBoard.

MathZone offers

- Practice exercises, based on the text's end-of-section material, generated in an unlimited number of variations, for as much practice as needed to master a particular topic.
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ALEKS ALEKS (**A**ssessment and **LE**arning in **K**nowledge **S**paces) is a dynamic online learning system for mathematics education, available over the Web 24/7. ALEKS assesses students, accurately determines their knowledge, and then guides them to the material that they are most ready to learn. With a variety of reports, Textbook Integration Plus, quizzes, and homework assignment capabilities, ALEKS offers flexibility and ease of use for instructors.

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- ALEKS with AI-2 gives instructors increased control over the scope and sequence of student learning. Students using ALEKS demonstrate a steadily increasing mastery of the content of the course.
- ALEKS offers a dynamic classroom management system that enables instructors to monitor and direct student progress toward mastery of course objectives.
- See www.aleks.com.

Student's Solutions Manual Prepared by Dave Sobecki, the *Student's Solutions Manual* provides comprehensive, worked-out solutions to all of the odd-numbered exercises from the text. The steps shown in the solutions match the style of solved examples in the textbook.

Video Lectures on Digital Video Disk (DVD) In the videos, J. D. Herdlick of St. Louis Community College at Meramec introduces essential definitions, theorems, formulas, and problem-solving procedures and then works through selected exercises from the textbook, following the solution methodology employed in the text. In addition, new instructional videos on graphing calculator operations help students master the most essential calculator skills used in the precalculus course. The video series is available on DVD or online as an assignable element of MathZone. The DVDs are closed-captioned for the hearing impaired, subtitled in Spanish, and meet the Americans with Disabilities Act Standards for Accessible Design. Instructors may use them as resources in a learning center, for online courses, and/or to provide extra help to students who require extra practice.

NetTutor Available through MathZone, NetTutor is a revolutionary system that enables students to interact with a live tutor over the World Wide Web. NetTutor's Web-based, graphical chat capabilities enable students and tutors to use mathematical

notation and even to draw graphs as they work through a problem together. Students can also submit questions and receive answers, browse previously answered questions, and view previous live-chat sessions. Tutors are familiar with the textbook's objectives and problem-solving styles.

CTB (Computerized Test Bank) Online Available through MathZone, this computerized test bank, utilizing algorithm-based testing software, enables users to create customized exams quickly. This user-friendly program enables instructors to search for questions by topic, format, or difficulty level; to edit existing questions or to add new ones; and to scramble questions and answer keys for multiple versions of the same test. Hundreds of text-specific open-ended and multiple-choice questions are included in the question bank. Sample chapter tests and final exams in Microsoft Word® and PDF formats are also provided.

Instructor's Solutions Manual Prepared by Dave Sobecki, and available on MathZone, the *Instructor's Solutions Manual* provides comprehensive, worked-out solutions to all exercises in the text. The methods used to solve the problems in the manual are the same as those used to solve the examples in the textbook.

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Producing this new edition with the help of all these extremely competent people has been a most satisfying experience.

EXAMPLE 4 Using Exponential Function Properties

Find all solutions to $4^{x-3} = 8$.

SOLUTIONS

Algebraic Solution

Notice that the two bases, 4 and 8, can both be written as a power of 2. This will enable us to use Property 2 to equate exponents.

$$4^{x-3} = 8 \quad \text{Express 4 and 8 as powers of 2.}$$
$$(2^2)^{x-3} = 2^3 \quad (a^y)^y = a^{xy}$$
$$2^{2x-6} = 2^3 \quad \text{Property 2}$$
$$2x - 6 = 3 \quad \text{Add 6 to both sides.}$$
$$2x = 9 \quad \text{Divide both sides by 2.}$$
$$x = \tfrac{9}{2}$$

CHECK

$$4^{(9/2)-3} = 4^{3/2} = (\sqrt{4})^3 = 2^3 \stackrel{\checkmark}{=} 8$$

Graphical Solution

Graph $y_1 = 4^{x-3}$ and $y_2 = 8$. Use the INTERSECT command to obtain $x = 4.5$ (Fig. 10).

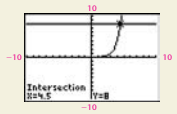


Figure 10

Side-by-Side Solutions: Algebraic and Graphical

Many solved examples in the book provide graphical solutions side-by-side with algebraic solutions. By seeing the same answer result from their symbol manipulations *and* from graphical approaches, students gain insight into the power of algebra and make important conceptual and visual connections.

Examples and Matched Problems

Integrated throughout the text, completely worked examples and practice problems are used to introduce concepts and demonstrate problem-solving techniques—algebraic, graphical, and numerical. Each **Example** is followed by a similar **Matched Problem** for the student to work through while reading the material. Answers to the matched problems are located at the end of each section, for easy reference. This active involvement in the learning process helps students develop a thorough understanding of algebraic concepts and processes.

EXAMPLE 5 Home Ownership Rates

Table 3 Home Ownership Rates

Year	Home Ownership Rate (%)
1940	43.6
1950	55.0
1960	61.9
1970	62.9
1980	64.4
1990	64.2
2000	67.4

The U.S. Census Bureau published the data in Table 3 on home ownership rates.

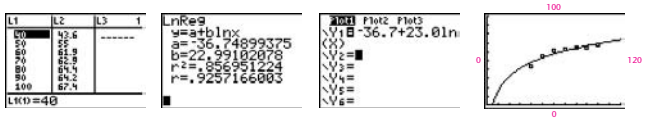
(A) Let x represent time in years with $x = 0$ representing 1900, and let y represent the corresponding home ownership rate. Use regression analysis on a graphing calculator to find a logarithmic function of the form $y = a + b \ln x$ that models the data. (Round the constants a and b to three significant digits.)

(B) Use your logarithmic function to predict the home ownership rate in 2010.

SOLUTIONS

(A) Figure 1 shows the details of constructing the model on a graphing calculator.

(B) The year 2010 corresponds to $x = 110$. Evaluating $y_1 = -36.7 + 23.0 \ln x$ at $x = 110$ predicts a home ownership rate of 71.4% in 2010.



(a) Data (b) Regression equation (c) Regression equation entered in equation editor (d) Graph of data and regression equation

Figure 1

MATCHED PROBLEM 5

Refer to Example 5. The home ownership rate in 1995 was 64.7%.

(A) Find a logarithmic regression equation for the expanded data set.

(B) Predict the home ownership rate in 2010.

ANSWERS TO MATCHED PROBLEMS

1. 95.05 decibels **2.** 7.80 **3.** 2.67 **4.** 1 kilometer per second less

5. (A) $-31.5 + 21.7 \ln x$ (B) 70.5%

2-6 Exercises

In Problems 1–8, determine the validity of each statement. If a statement is false, explain why.

1. If $x^2 = 5$, then $x = \pm\sqrt{5}$ **2.** $\sqrt{25} = \pm 5$

3. $(x + 5)^2 = x^2 + 25$ **4.** $(2x - 1)^2 = 4x^2 - 1$

5. $(\sqrt{x - 1} + 1)^2 = x$ **6.** $(\sqrt{x - 1})^2 + 1 = x$

7. If $x^3 = 2$, then $x = 8$ **8.** If $x^{1/3} = 2$, then $x = 8$

In Problems 9–14, transform each equation of quadratic type into a quadratic equation in u and state the substitution used in the transformation. If the equation is not an equation of quadratic type, say so.

9. $2x^{-6} - 4x^{-3} = 0$ **10.** $\frac{4}{7} - \frac{3}{x} + \frac{6}{x^2} = 0$

11. $3x^3 - 4x + 9 = 0$ **12.** $7x^{-1} + 3x^{-1/2} + 2 = 0$

13. $\frac{10}{9} + \frac{4}{x^2} - \frac{7}{x^4} = 0$ **14.** $3x^{3/2} - 5x^{1/2} + 12 = 0$

15. Explain why squaring both sides of an equation sometimes introduces extraneous solutions.

16. Would raising both sides of an equation to the third power ever introduce extraneous solutions? Why or why not?

17. Write an example of a false statement that becomes true when you square both sides. What would every possible example have in common?

18. How can you recognize when an equation is of quadratic type?

In Problems 19–32, solve algebraically and confirm graphically, if possible.

19. $\sqrt{4x - 7} = 5$ **20.** $\sqrt{4 - x} = 4$

21. $\sqrt{5x + 6} + 6 = 0$ **22.** $\sqrt{10x + 1} + 8 = 0$

23. $\sqrt[3]{x + 5} = 3$ **24.** $\sqrt[3]{x - 3} = 2$

25. $\sqrt{x + 5} + 7 = 0$ **26.** $3 + \sqrt{2x - 1} = 0$

27. $y^4 - 2y^2 - 8 = 0$ **28.** $x^4 - 7x^2 - 18 = 0$

29. $3x = \sqrt{x^2 - 2}$ **30.** $x = \sqrt{5x^2 + 9}$

31. $\sqrt{x^2 - 5x} = \sqrt{x - 8}$ **32.** $\sqrt{2x + 3} = \sqrt{x^2 - 12}$

In Problems 33–56, solve algebraically and confirm graphically, if possible.

33. $\sqrt{5n + 9} = n - 1$ **34.** $m - 13 = \sqrt{m + 7}$

35. $\sqrt{3x + 4} = 2 + \sqrt{x}$ **36.** $\sqrt{3w - 2} - \sqrt{w} = 2$

37. $2x^{2/3} + 3x^{1/3} - 2 = 0$ **38.** $x^{2/3} - 3x^{1/3} - 10 = 0$

39. $(m^2 - m)^2 - 4(m^2 - m) = 12$

40. $(x^2 + 2x)^2 - (x^2 + 2x) = 6$

41. $\sqrt{u - 2} = 2 + \sqrt{2u + 3}$ **42.** $\sqrt{3t + 4} + \sqrt{t} = -3$

43. $\sqrt{3y - 2} = 3 - \sqrt{3y + 1}$

44. $\sqrt{2x - 1} - \sqrt{x - 4} = 2$

45. $\sqrt{7x - 2} - \sqrt{x + 1} = \sqrt{3}$

46. $\sqrt{3x + 6} - \sqrt{x + 4} = \sqrt{2}$

47. $\sqrt{4x^2 + 12x + 1} - 6x = 9$

48. $6x - \sqrt{4x^2 - 20x + 17} = 15$

49. $3n^{-2} - 11n^{-1} - 20 = 0$ **50.** $6x^{-2} - 5x^{-1} - 6 = 0$

51. $9y^{-4} - 10y^{-2} + 1 = 0$ **52.** $4x^{-4} - 17x^{-2} + 4 = 0$

53. $y^{1/2} - 3y^{1/4} + 2 = 0$ **54.** $4x^{-1} - 9x^{-1/2} + 2 = 0$

55. $(m - 5)^4 + 36 = 13(m - 5)^2$

56. $(x - 3)^4 + 3(x - 3)^2 = 4$

57. Explain why the following "solution" is incorrect:

$$\sqrt{x + 3} + 5 = 12$$
$$x + 3 + 25 = 144$$
$$x = 116$$

58. Explain why the following "solution" is incorrect.

$$\sqrt{x^2 - 16} = 2x + 3$$
$$x - 4 = 2x + 3$$
$$-7 = x$$

In Problems 59–62, solve algebraically and confirm graphically, if possible.

59. $\sqrt{5 - 2x} - \sqrt{x + 6} = \sqrt{x + 3}$

60. $\sqrt{2x + 3} - \sqrt{x - 2} = \sqrt{x + 1}$

61. $2 + 3y^{-4} = 6y^{-2}$ **62.** $4m^{-2} = 2 + m^{-4}$

In Problems 63–66, solve two ways: by isolating the radical and squaring, and by substitution. Confirm graphically, if possible.

63. $m - 7\sqrt{m} + 12 = 0$ **64.** $y - 6 + \sqrt{y} = 0$

65. $t - 11\sqrt{t} + 18 = 0$ **66.** $x = 15 - 2\sqrt{x}$

Balanced Exercise Sets

Precalculus: Graphs and Models, third edition, contains more than 5,500 problems. Each exercise set is designed so that an average or below-average student will experience success and a very capable student will be challenged. Exercise sets are found at the end of each section in the text. The Annotated Instructor's Edition features A (routine, easy mechanics), B (more difficult mechanics), and C (difficult mechanics and some theory) designations to denote these levels and help instructors in the assignment building process. Problem numbers that appear in blue indicate problems that require students to apply their reasoning and writing skills to the solution of the problem.

Applications

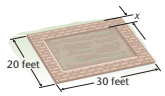
One of the primary objectives of this book is to give the student substantial experience in modeling and showing real-world problems. More than 650 application exercises help convince even the most skeptical student that mathematics is relevant to everyday life. The most difficult application problems are marked with two stars (* *), the moderately difficult application problems with one star (*), and easier application problems are not marked. An Application Index is included immediately preceding Chapter 1 to locate particular applications.

APPLICATIONS

99. AIR SEARCH A search plane takes off from an airport at 6:00 A.M. and travels due north at 200 miles per hour. A second plane takes off at 6:30 A.M. and travels due east at 170 miles per hour. The planes carry radios with a maximum range of 500 miles. When (to the nearest minute) will these planes no longer be able to communicate with each other?

100. NAVIGATION A speedboat takes 1 hour longer to go 24 miles up a river than to return. If the boat cruises at 10 miles per hour in still water, what is the rate of the current?

101. CONSTRUCTION A gardener has a 30 foot by 20 foot rectangular plot of ground. She wants to build a brick walkway of uniform width on the border of the plot (see the figure). If the gardener wants to have 400 square feet of ground left for planting, how wide (to two decimal places) should she build the walkway?

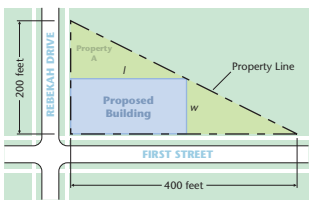


102. CONSTRUCTION Refer to Problem 101. The gardener buys enough bricks to build 160 square feet of walkway. Is this sufficient to build the walkway determined in Problem 101? If not, how wide (to two decimal places) can she build the walkway with these bricks?

103. CONSTRUCTION A 1,200 square foot rectangular garden is enclosed with 150 feet of fencing. Find the dimensions of the garden to the nearest tenth of a foot.

104. CONSTRUCTION The intramural fields at a small college will cover a total area of 140,000 square feet, and the administration has budgeted for 1,600 feet of fence to enclose the rectangular field. Find the dimensions of the field.

105. ARCHITECTURE A developer wants to erect a rectangular building on a triangular-shaped piece of property that is 200 feet wide and 400 feet long (see the figure).

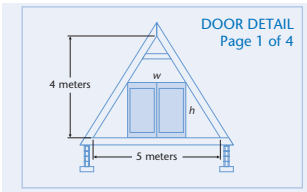


(A) Express the area $A(w)$ of the footprint of the building as a function of the width w and state the domain of this function. [*Hint:* Use Euclid's theorem* to find a relationship between the length l and width w.]

(B) Building codes require that this building have a footprint of at least 15,000 square feet. What are the widths of the building that will satisfy the building codes?

(C) Can the developer construct a building with a footprint of 25,000 square feet? What is the maximum area of the footprint of a building constructed in this manner?

106. ARCHITECTURE An architect is designing a small A-frame cottage for a resort area. A cross-section of the cottage is an isosceles triangle with a base of 5 meters and an altitude of 4 meters. The front wall of the cottage must accommodate a sliding door positioned as shown in the figure.



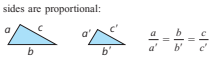
(A) Express the area $A(w)$ of the door as a function of the width w and state the domain of this function. [See the hint for Problem 105.]

(B) A provision of the building code requires that doorways must have an area of at least 4.2 square meters. Find the width of the doorways that satisfy this provision.

(C) A second provision of the building code requires all doorways to be at least 2 meters high. Discuss the effect of this requirement on the answer to part B.

107. TRANSPORTATION A delivery truck leaves a warehouse and travels north to factory A. From factory A the truck travels east to factory B and then returns directly to the warehouse (see the figure on the next page). The driver recorded the truck's odometer reading at the warehouse at both the beginning and the end of the trip and also at factory B, but forgot to record it at factory A (see the table on the next page). The driver does recall that it was farther from the warehouse to factory A than it was from factory A to factory B. Because delivery charges are based

**Euclid's theorem:* If two triangles are similar, their corresponding sides are proportional:



$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

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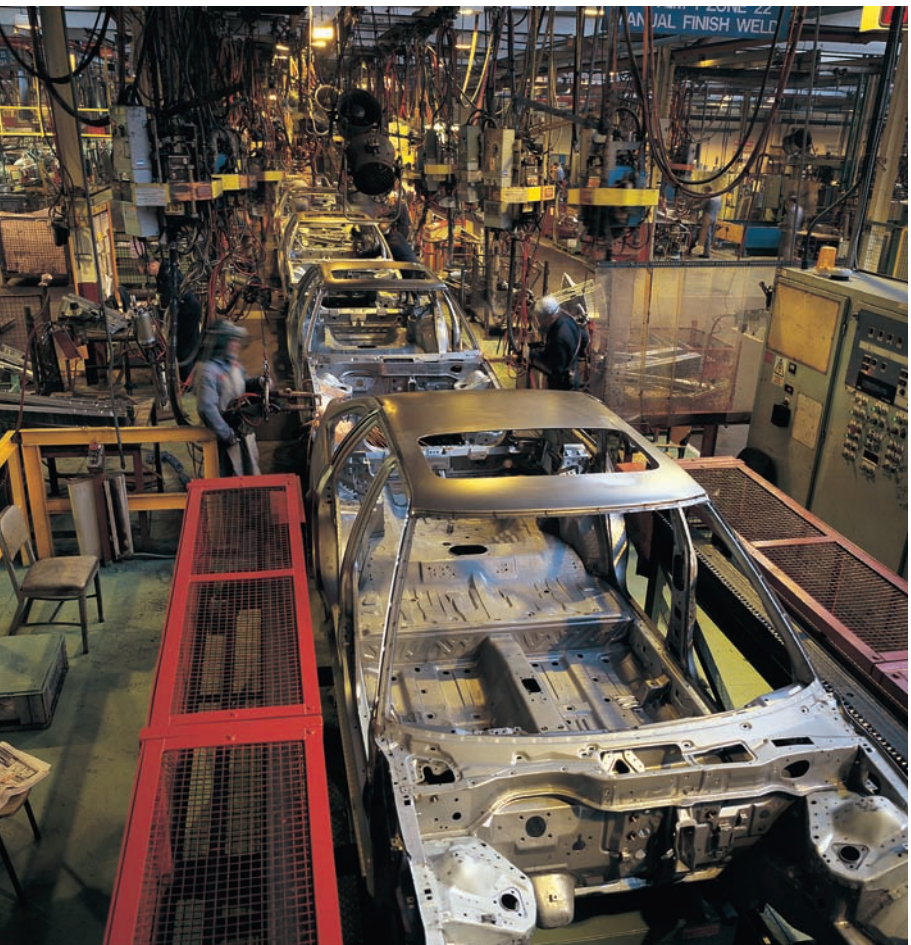
Functions, Graphs, and Models

THE function concept is one of the most important ideas in mathematics. To study math beyond the elementary level, you absolutely need to have a solid understanding of functions and their graphs. In this chapter, you'll learn the fundamentals of what functions are all about, and how to use them. In subsequent chapters, this will pay off as you study particular types of functions in depth. In the first section of this chapter, we discuss the techniques involved in using an electronic graphing device like a graphing calculator. In the remaining sections, we introduce the concept of functions and discuss general properties of functions and their graphs. Everything you learn in this chapter will increase your chance of success in this course, and in almost any other course you may take that involves mathematics.



CHAPTER

1

OUTLINE

1-1 Using Graphing Calculators

1-2 Functions

1-3 Functions: Graphs and Properties

1-4 Functions: Graphs and Transformations

1-5 Operations on Functions; Composition

1-6 Inverse Functions

Chapter 1 Review

Chapter 1 Group Activity: Mathematical Modeling: Choosing a Cell Phone Provider

1-1 Using Graphing Calculators

- Using Graphing Calculators
- Understanding Screen Coordinates
- Using the Trace, Zoom, and Intersect Commands
- Mathematical Modeling

The use of technology to aid in drawing and analyzing graphs is revolutionizing mathematics education and is the primary motivation for this book. Your ability to interpret mathematical concepts and to discover patterns of behavior will be greatly increased as you become proficient with an electronic graphing device. In this section we introduce some of the basic features of electronic graphing devices. Additional features will be introduced as the need arises. If you have already used an electronic graphing device in a previous course, you can use this section to quickly review basic concepts. If you need to refresh your memory about a particular feature, consult the Technology Index at the end of this book to locate the textbook discussion of that particular feature.

Using Graphing Calculators

We will begin with the use of electronic graphing devices to graph equations. We will refer to any electronic device capable of displaying graphs as a **graphing utility.** The two most common graphing utilities are handheld graphing calculators and computers with appropriate software. It's essential that you have such a device handy as you proceed through this book.

Since many different brands and models exist, we will discuss graphing calculators only in general terms. Refer to your manual for specific details relative to your own graphing calculator.*

An image on the screen of a graphing calculator is made up of darkened rectangles called **pixels** (Fig. 1). The pixel rectangles are the same size, and don't change in size during any application. Graphing calculators use pixel-by-pixel plotting to produce graphs.

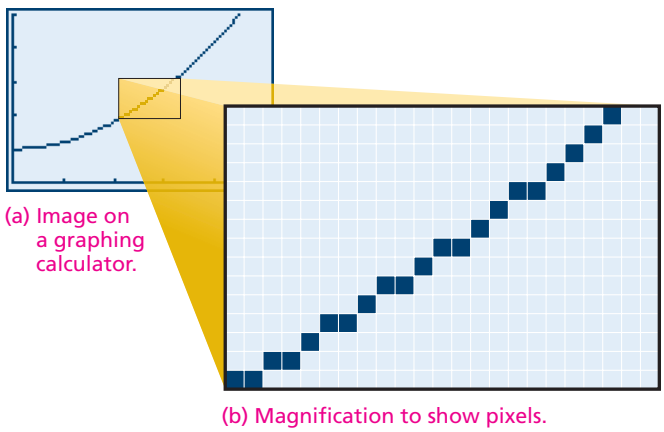
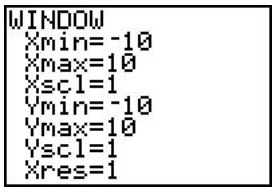
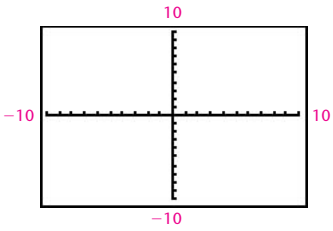


Figure 1 Pixel-by-pixel plotting on a graphing calculator.

*Manuals for most brands of graphing calculators are readily available on the Internet.



(a) Standard window variable values



(b) Standard viewing window

› **Figure 2** A standard viewing window and its dimensions.

The accuracy of the graph depends on the resolution of the graphing calculator. Most graphing calculators have screen resolutions of between 50 and 75 pixels per inch, which results in fairly rough but very useful graphs. Some computer systems can print very high quality graphs with resolutions greater than 1,000 pixels per inch.

Most graphing calculator screens are rectangular. The graphing screen on a graphing calculator represents a portion of the plane in the rectangular coordinate system. But this representation is an approximation, because pixels are not really points, as is clearly shown in Figure 1. Points are geometric objects without dimensions (you can think of them as "infinitely small"), whereas a pixel has dimensions. The coordinates of a pixel are usually taken at the center of the pixel and represent all the infinitely many geometric points within the pixel. Fortunately, this does not cause much of a problem, as we will see.

The portion of a rectangular coordinate system displayed on the graphing screen is called a **viewing window** and is determined by assigning values to six **window variables:** the lower limit, upper limit, and scale for the x axis and the lower limit, upper limit, and scale for the y axis. Figure 2(a) illustrates the names and values of **standard window variables,** and Figure 2(b) shows the resulting **standard viewing window.**

The names **Xmin, Xmax, Xscl, Ymin, Ymax,** and **Yscl** will be used for the six window variables. Xscl and Yscl determine the distance between tick marks on the x and y axes, respectively. **Xres** is a seventh window variable on some graphing calculators that controls the screen resolution; we will always leave this variable set to the default value 1. The window variables may be displayed slightly differently by your graphing calculator. In this book, when a viewing window of a graphing calculator is pictured in a figure, the values of Xmin, Xmax, Ymin, and Ymax are indicated by labels to make the graph easier to read [see Fig. 2(b)]. These labels are always centered on the sides of the viewing window, regardless of the location of the axes.

REMARK: We think it's important that actual output from existing graphing calculators be used in this book. The majority of the graphing calculator images in this book are screen dumps from a Texas Instruments TI-84 graphing calculator. Occasionally we use screen dumps from a TI-86 graphing calculator, which has a wider screen. You may not always be able to produce an exact replica of a figure on your graphing calculator, but the differences will be relatively minor.

We now turn to the use of a graphing calculator to graph equations that can be written in the form

$$y = (\text{some expression in } x) \tag{1}$$

Graphing an equation of the type shown in equation (1) using a graphing calculator is a simple three-step process:

› GRAPHING EQUATIONS USING A GRAPHING CALCULATOR

Step 1. Enter the equation.

Step 2. Enter values for the window variables. (A good rule of thumb for choosing Xscl and Yscl, unless there are reasons to the contrary, is to choose each about one-tenth the width of the corresponding variable range.)

Step 3. Press the GRAPH command.

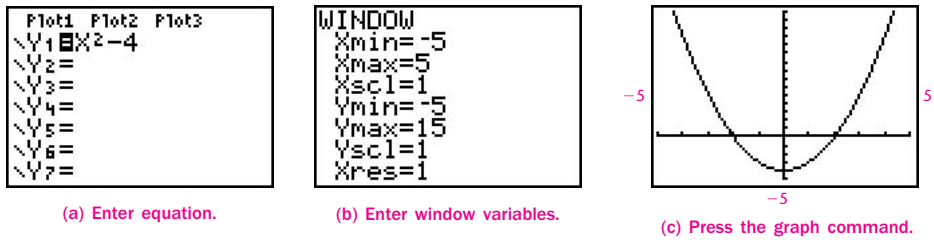
The following example illustrates this procedure for graphing the equation $y = x^2 - 4$. (See Example 1 of Appendix B, Section B-2 for a hand-drawn sketch of this equation.)

EXAMPLE 1 Graphing an Equation with a Graphing Calculator

Use a graphing calculator to graph $y = x^2 - 4$ for $-5 \le x \le 5$ and $-5 \le y \le 15$.

SOLUTION

Press the Y= key to display the equation editor and enter the equation [Fig. 3(a)]. Press WINDOW to display the window variables and enter the given values for these variables [Fig. 3(b)]. Press GRAPH to obtain the graph in Figure 3(c). (The form of the screens in Figure 3 may differ slightly, depending on the graphing calculator used.)



(a) Enter equation. (b) Enter window variables. (c) Press the graph command.

› **Figure 3** Graphing is a three-step process.

MATCHED PROBLEM 1*

Use a graphing calculator to graph $y = 8 - x^2$ for $-5 \le x \le 5$ and $-10 \le y \le 10$.

REMARK: For Example 1, we displayed a screen shot for each step in the graphing procedure. Generally, we will show only the final results, as illustrated in Figure 3(c).

Often, it is helpful to think about an appropriate viewing window before starting to graph an equation. This can help save time, as well as increase your odds of seeing the whole graph.

*Answers to matched problems in a given section are found near the end of the section, before the exercise set.

EXAMPLE 2 Finding an Appropriate Viewing Window

Find an appropriate viewing window in which to graph the equation $y = \sqrt{x - 13}$ with a graphing calculator.

SOLUTIONS

Algebraic Solution

We begin by thinking about reasonable x values for this equation. Since y values are determined by an expression under a root, only x values that result in $x - 13$ being nonnegative will have an associated y value. So we write and solve the inequality

$$x - 13 \geq 0$$
$$x \geq 13$$

This tells us that there will be points on the graph only for x values 13 or greater.

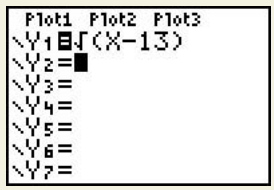
Next, we make a table of values for selected x values to see what y values are appropriate. Note that we chose x values that make it easy to compute y.

x	13	14	17	22	29	38
y	0	1	2	3	4	5

To clearly display all of these points and leave some space around the edges, we choose $X_{\min} = 10$, $X_{\max} = 40$, $Y_{\min} = -1$, and $Y_{\max} = 10$.

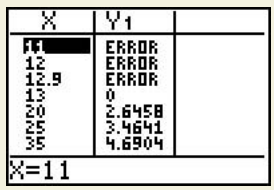
Graphical Solution

We first enter the equation $y_1 = \sqrt{x - 13}$ in a graphing calculator (Fig. 4).



› Figure 4

We then make a table of values for selected x values after trying a variety of choices for x (Fig. 5).*



› Figure 5

We find that there are no points on the graph for x values less than 13, and there appear to be points for all x values greater than or equal to 13.

To clearly display all of these points and leave some space around the edges, we choose $X_{\min} = 10$, $X_{\max} = 40$, $Y_{\min} = -1$, and $Y_{\max} = 10$.

MATCHED PROBLEM 2

Find an appropriate viewing window in which to graph the equation $y = 2 + \sqrt{x + 15}$ with a graphing calculator.

*Many graphing calculators can construct a table of values like the one in Figure 5 using the TBLSET and TABLE commands.

The next example illustrates how a graphing calculator can be used as an aid to sketching the graph of an equation by hand. The example illustrates the use of algebraic, numeric, and graphic approaches; an understanding of all three approaches will be a big help in problem solving.

EXAMPLE 3 Using a Graphing Calculator as an Aid to Hand Graphing—Net Cash Flow

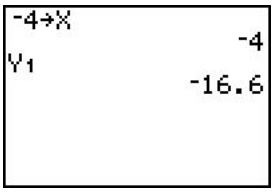
The net cash flow y in millions of dollars of a small high-tech company from 1998–2006 is given approximately by the equation

$$y = 0.4x^3 - 2x + 1 \qquad -4 \le x \le 4 \tag{2}$$

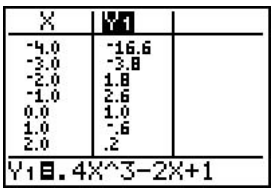
where x represents the number of years before or after 2002, when the board of directors appointed a new CEO.

(A) Construct a table of values for equation (2) for each year starting with 1998 and ending with 2006. Compute y to one decimal place.

(B) Obtain a graph of equation (2) in the viewing window of your graphing calculator. Plot the table values from part A by hand on graph paper, then join these points with a smooth curve using the graph in the viewing window as an aid.



(a)



(b)

› Figure 6

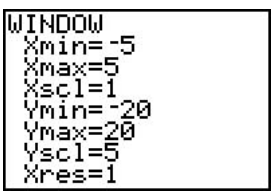
SOLUTIONS

(A) After entering the given equation as y_1, we can find the value of y for a given value of x by storing the value of x in the variable X and simply displaying y_1, as shown in Figure 6(a). To speed up this process, we can compute an entire table of values directly, as shown in Figure 6(b). We organize these results in Table 1.

Recall that x represents years before or after 2002, and y represents cash flow in millions of dollars.

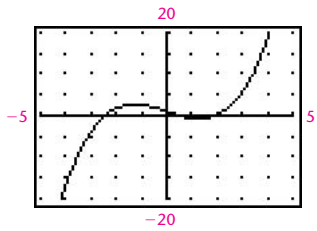
Table 1 Net Cash Flow

Year	1998	1999	2000	2001	2002	2003	2004	2005	2006
x	−4	−3	−2	−1	0	1	2	3	4
y (million \$)	−16.6	−3.8	1.8	2.6	1	−0.6	0.2	5.8	18.6

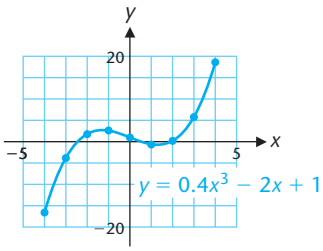


› Figure 7

(B) To create a graph of equation (2) in the viewing window of a graphing calculator, we select values for the viewing window variables that cover a little more than the values shown in Table 1, as shown in Figure 7. We add a grid to the viewing window to obtain the graphing calculator graph shown in Figure 8(a). (On many graphing calculators, this option is on the FORMAT screen.) The corresponding hand sketch is shown in Figure 8(b).



(a) Graphing calculator graph



(b) Hand sketch

› **Figure 8** Net cash flow.

REMARKS:

1. In applied problems, it's a great idea to begin by writing down what each variable in the problem represents.
2. Table 1 is useful for analyzing this data since it gives us specific detail; the equation and graph are also useful because they give us a quick overview of cash flow. Each viewpoint has its specific value.

MATCHED PROBLEM 3

The company in Example 3 is in competition with another company whose net cash flow y in millions of dollars from 1998 to 2006 is given approximately by the equation

$$y = 1 + 1.9x - 0.2x^3 \qquad -4 \le x \le 4 \tag{3}$$

where x represents the number of years before or after 2002.

(A) Construct a table of values for equation (3) for each year starting with 1998 and ending with 2006. Compute y to one decimal place.

(B) Plot the points corresponding to the table by hand, then hand sketch the graph of the equation with the help of a graphing calculator.

››› EXPLORE-DISCUSS 1

The choice of the viewing window has a pronounced effect on the shape of a graph. Graph $y = -x^3 + 2x$ in each of the following viewing windows:

(A) $-1 \le x \le 1, -1 \le y \le 1$

(B) $-10 \le x \le 10, -10 \le y \le 10$

(C) $-100 \le x \le 100, -100 \le y \le 100$

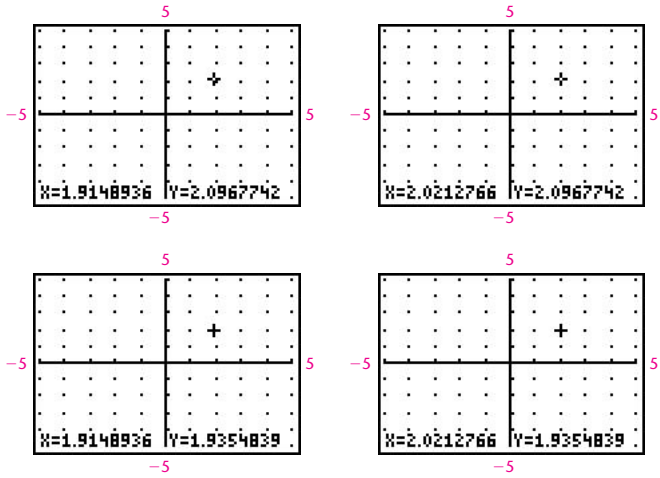
Which window gives the best view of the graph of this equation, and why?

› Understanding Screen Coordinates

We now take a closer look at *screen coordinates* of pixels. Earlier we indicated that the coordinates of the center point of a pixel are usually used as the **screen coordinates of the pixel,** and these coordinates represent all points within the pixel. As you might expect, screen coordinates of pixels change as you change values of window variables.

To find screen coordinates of various pixels, move a *cursor* around the viewing window and observe the coordinates displayed on the screen. A **cursor** is a special symbol, such as a plus (+) or times (×) sign, that locates one pixel on the screen at a time. As the cursor is moved around the screen, it moves from pixel to pixel. To see this, set the window variables in your graphing calculator so that $-5 \le x \le 5$ and $-5 \le y \le 5$, and activate a grid for the screen. Move the cursor as close as you can to the point (2, 2) and observe what happens. Figure 9 shows the screen coordinates

of the four pixels that are closest to (2, 2). The coordinates displayed on your screen may vary slightly from these, depending on the graphing calculator used.



› **Figure 9** Screen coordinates of pixels near (2, 2).

Any of the four pixels in Figure 9 can be used to approximate the point (2, 2), but it turns out that it is not possible to find a pixel in this viewing window whose screen coordinates are exactly (2, 2). Try to repeat this exercise with different window variables, say, $-7 \le x \le 7$ and $-7 \le y \le 7$.

› Using the Trace, Zoom, and Intersect Commands

When analyzing the graph of an equation, it's often useful to find the coordinates of certain points on the graph. Using the **TRACE** command on a graphing calculator is one way to accomplish this. The trace feature places a cursor directly on the graph and only permits movement left and right along the graph. The coordinates displayed during the tracing movement are coordinates of points that satisfy the equation. In most cases, these coordinates are not the same as the pixel screen coordinates displayed using the unrestricted cursor movement that we discussed earlier. Instead, they are the *exact* coordinates of points on the graph.

››› EXPLORE-DISCUSS 2

Graph the equation $y = x$ in a standard viewing window.

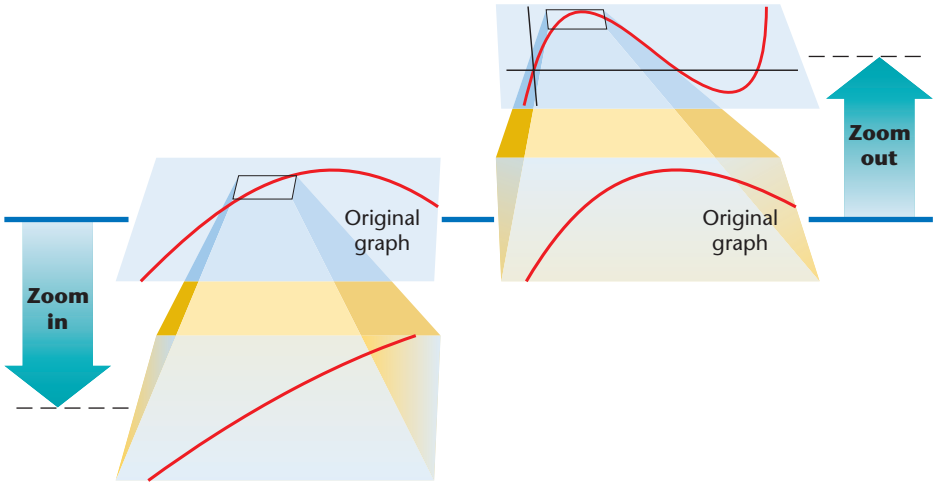
(A) Without selecting the TRACE command, move the cursor to a point on the screen that appears to lie on the graph of $y = x$ and is as close to (5, 5) as possible. Record these coordinates. Do these coordinates satisfy the equation $y = x$?

(B) Now select the TRACE command and move the cursor along the graph of $y = x$ to a point that has the same x coordinate found in part A. Is the y coordinate of this point the same as you found in part A? Do the coordinates of the point using trace satisfy the equation $y = x$?

(C) Explain the difference in using trace along a curve and trying to use unrestricted movement of a cursor along a curve.

Most graphing calculators have a **ZOOM** command. In general, **zooming in** on a graph reduces the window variables and magnifies the portion of the graph visible in the viewing window [Fig. 10(a)]. **Zooming out** enlarges the window variables so that more of the graph is visible in the viewing window [Fig. 10(b)].

› **Figure 10** The zoom operation.



(a) Zooming in (b) Zooming out

› **Figure 11** The ZOOM menu on a TI-84.

››› EXPLORE-DISCUSS 3

Figure 11 shows the ZOOM menu on a TI-84.* We want to explore the effects of some of these options on the graph of $y = x$. Enter this equation in the equation editor and select ZStandard from the ZOOM menu. What are the window variables? In each of the following, position the cursor at the origin and select the indicated zoom option. Observe the changes in the window variables and examine the coordinates displayed by tracing along the curve.

(A) ZSquare **(B)** ZDecimal **(C)** ZInteger **(D)** ZoomFit

Another command found on most graphing calculators is **INTERSECT**† or **ISECT.** This command enables the user to find the point(s) where two curves intersect without using trace or zoom.

The use of trace, zoom, and intersect is best illustrated by examples.

*The ZOOM menu on other graphing calculators may look quite different from the one on the TI-84.
†On the TI-84, INTERSECT is found on the CALC (2nd-TRACE) menu.

EXAMPLE 4 **Using Trace, Zoom, and Intersect**

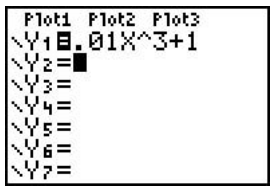
Let $y = 0.01x^3 + 1$.

(A) Use the TRACE command to find y when $x = 5$.
(B) Use the TRACE and ZOOM commands to find x when $y = 5$.
(C) Use the INTERSECT command to find x when $y = 5$.

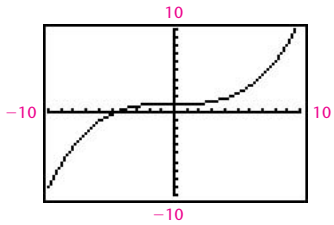
Round answers to two decimal places.

SOLUTIONS

(A) Enter $y_1 = 0.01x^3 + 1$ in a graphing calculator (Fig. 12). In Example 3, we discussed two ways to find the value of y_1 for a given value of x. Now we want to discuss a third way, the TRACE command. Graph y_1 in the standard viewing window (Fig. 13).

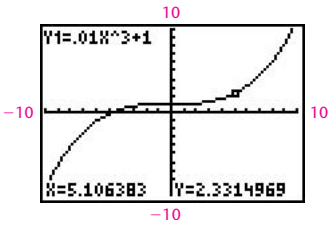


› Figure 12

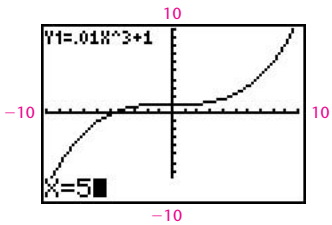


› Figure 13

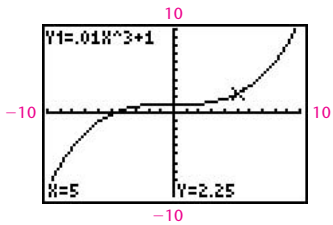
Select the TRACE command and move the cursor as close to $x = 5$ as possible (Fig. 14). This shows that $y = 2.33$ when $x = 5.11$, not quite what we want. However, we can direct the trace command to use the exact value of $x = 5$ by simply entering 5 (Fig. 15) and pressing ENTER (Fig. 16).



› Figure 14



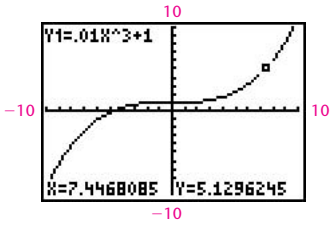
› Figure 15



› Figure 16

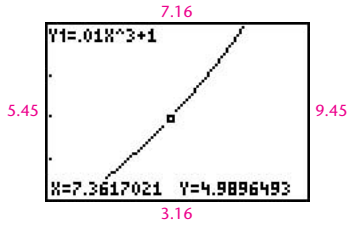
From Figure 16 we see that $y = 2.25$ when $x = 5$.

(B) Select the TRACE command and move the cursor as close to $y = 5$ as possible (Fig. 17). This shows that $x = 7.45$ when $y = 5.13$, again not exactly what we want. We cannot direct the TRACE command to use the exact value $y = 5$ since it only allows us to input x values. Instead we press the ZOOM command and select Zoom In to obtain more accuracy (Fig. 18). Then we select the TRACE command and move the cursor as close to $y = 5$ as possible (Fig. 19).



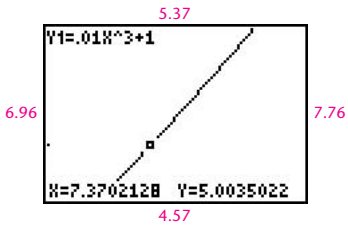
› Figure 17

› Figure 18

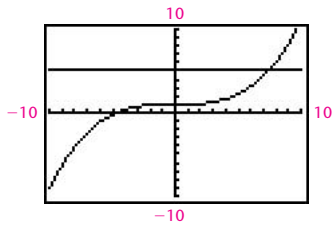


› Figure 19

Now we see that $x = 7.36$ when $y = 4.99$. This is an improvement, but we can do better. Repeating the Zoom In command and tracing along the curve (Fig. 20), we see that $x = 7.37$ when $y = 5.00$.

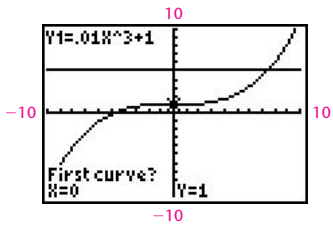


› Figure 20

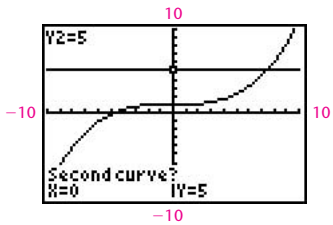


› Figure 21

(C) Enter $y_2 = 5$ in the graphing calculator and graph y_1 and y_2 in the standard viewing window (Fig. 21). Now there are two curves displayed on the graph. The horizontal line is the graph of $y_2 = 5$, and the other curve is the familiar graph of y_1. The coordinates of the intersection point of the two curves must satisfy both equations. Clearly, the y coordinate of this intersection point is 5. The x coordinate is the value we are looking for. We can use the INTERSECT command to find the coordinates of the intersection point in Figure 21. When we select the INTERSECT command, we are asked to make three choices: the first curve, the second curve, and a guess. When the desired equation is displayed at the top of the screen, press ENTER to select it (Figs. 22 and 23). It doesn't matter which of the two graphs is designated as the first curve. (If there are more than two curves, use the up and down arrows to select the desired equation, then press ENTER.)

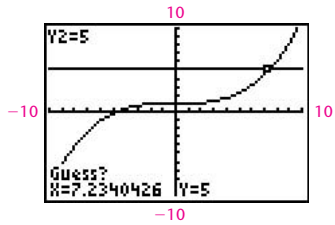


› Figure 22

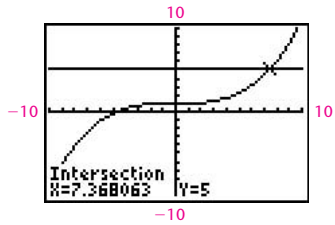


› Figure 23

To enter a guess, move the cursor close to the intersection point (Fig. 24) and press ENTER (Fig. 25). On many models, you can enter an x value close to the intersection rather than moving the cursor. (We will see a more important use of entering a guess in Example 5.) Examining Figure 25, we see that $x = 7.37$ when $y = 5$.



› Figure 24



› Figure 25

MATCHED PROBLEM 4

Repeat Example 4 for $y = 1 - 0.02x^3$.

EXAMPLE 5 **Solving an Equation with Multiple Solutions**

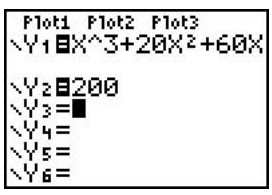
Use a graphing calculator to solve the equation

$$x^3 + 20x^2 + 60x = 200$$

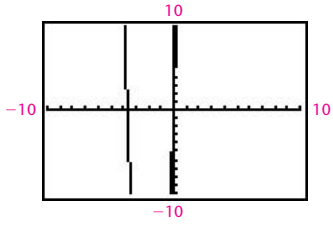
Round answers to two decimal places.

SOLUTION

We will solve this equation by graphing both sides in the same viewing window and finding the intersection points. First we enter $y_1 = x^3 + 20x^2 + 60x$ and $y_2 = 200$ in the graphing calculator (Fig. 26) and graph in the standard viewing window (Fig. 27).



› Figure 26



› Figure 27

A lot of students will always start with the standard viewing window, but in this case it is a poor choice. Because we are seeking the values of x that make the left side of the equation equal to 200, we need a value for Ymax that is larger than 200. Changing Ymax to 300 and Yscl to 30 produces a new graph (Fig. 28). The two curves

intersect twice in this window. The x coordinates of these points are the solutions we are looking for. But first, could there be other solutions that are not visible in this window? To find out, we must investigate the behavior outside this window. A table is the most convenient way to do this (Figs. 29 and 30).

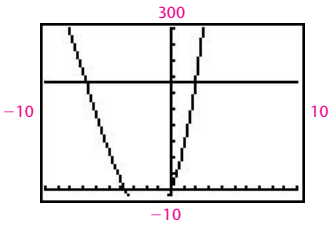


Figure 28

X	Y1	Y2
11	4411	200
12	5328	200
13	6357	200
14	7504	200
15	8775	200
16	10176	200
17	11713	200

Y1=X^3+20X²+60X

Figure 29

X	Y1	Y2
-17	-153	200
-16	64	200
-15	225	200
-14	336	200
-13	403	200
-12	432	200
-11	429	200

Y1=X^3+20X²+60X

Figure 30

Examining Figure 29, we see that the values of y_1 are getting very large. It's unlikely that there will be additional solutions to the equation $y_1 = 200$ for larger values of x. Examining Figure 30, we see that there are values of y_1 that are on both sides of 200, so there's a good chance that there will be additional solutions for more negative values of x. Based on the table values in Figure 30, we make the following changes in the window variables: Xmin $= -20$, Xscl $= 5$, Ymax $= 500$, Ymin $= -200$, Yscl $= 50$. This produces the graph in Figure 31. Examining the values of y_1 for values of x to the left of this window (Fig. 32), we conclude that there are no other intersection points.

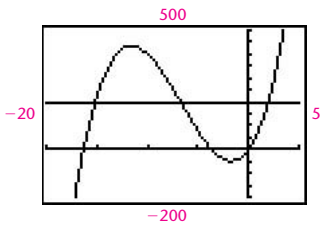


Figure 31

X	Y1	Y2
-27	-6723	200
-26	-5616	200
-25	-4625	200
-24	-3744	200
-23	-2967	200
-22	-2288	200
-21	-1701	200

Y1=X^3+20X²+60X

Figure 32

Now we use the INTERSECT command to find the x coordinates of the three intersection points in Figure 31. We select the two equations as before (Figs. 33 and 34). This time, the guess is very important. We have to specify which intersection point to find. To do this, we make a guess that is close to the desired point. We first select the leftmost intersection point by moving the cursor to that point (Fig. 35) and pressing ENTER.

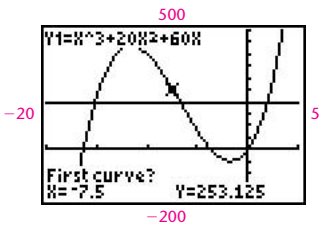


Figure 33

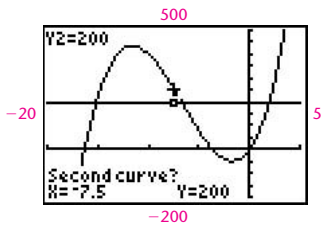


Figure 34

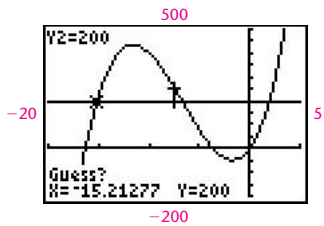
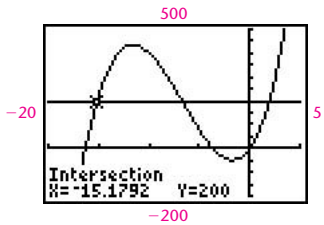
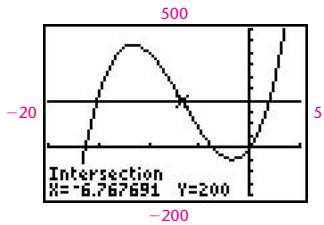


Figure 35

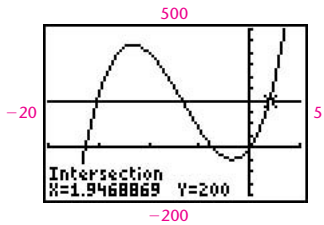
The coordinates of the leftmost intersection point are displayed at the bottom of the screen (Fig. 36). To find the other two intersection points, we repeat the entire process. When we get to the screen that asks for a guess, we place the cursor near the point we are looking for and press ENTER (Figs. 37 and 38).



› Figure 36



› Figure 37



› Figure 38

Thus, we see that the solutions to $x^3 + 20x^2 + 60x = 200$ are $x = -15.18$, $x = -6.77$, and $x = 1.95$.

MATCHED PROBLEM 5

Solve $x^3 + 10x^2 - 100x = 100$. Round answers to two decimal places.

In the solution to Example 5, we had to rely on examining tables and our intuition to conclude that the two graphs intersected only three times. One of the major objectives of this course is to broaden our knowledge of graphs and equations so that we can be more definitive in our reasoning. For example, in Chapter 3 we will show that any equation like the one in Example 5 can have no more than three solutions.

Examples 1 through 5 dealt with a variety of methods for finding the value of y that corresponds to a given value of x and the value(s) of x that correspond to a given value of y. These methods are summarized in the following box.

› FINDING SOLUTIONS TO AN EQUATION

Assume the equation is entered in a graphing calculator as $y_1 =$ (expression with variable x).

To find solutions (x, y) given $x =$ some number a, use any of the following methods:

Method 1. Store a in X on the graphing calculator and display y_1 on the home screen.

Method 2. Set TBLSTART to a and display the table.

Method 3. Graph y_1, select the TRACE command, and enter a.

To find solutions (x, y) given $y =$ some number b, use either of the following methods:

Method 1. Graph y_1 and use TRACE and ZOOM.

Method 2. Graph y_1 and $y_2 = b$ and use INTERSECT.

Mathematical Modeling

Now that we are able to solve equations on a graphing calculator, there are many applications that we can investigate. The term **mathematical modeling** refers to the process of using an equation or equations to describe data from the real world. The next example develops a mathematical model for manufacturing a box to certain specifications.

EXAMPLE 6 Manufacturing

A packaging company plans to manufacture open boxes from 11- by 17-inch sheets of cardboard by cutting x- by x-inch squares out of the corners and folding up the sides, as shown in Figure 39.

(A) Write an equation for the volume y of the resulting box in terms of the length x of the sides of the squares that are cut out. Indicate appropriate restrictions on x.

(B) Graph the equation for appropriate values of x. Adjust the window variables for y to include the entire graph of interest.

(C) Find the smallest square that can be cut out to produce a box with a volume of 150 cubic inches.

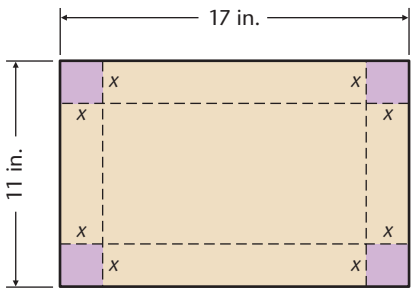


Figure 39 Template for boxes.

SOLUTIONS

(A) The dimensions of the box are expressed in terms of x in Figure 40(a), and the box is shown in Figure 40(b) with the sides folded up and dimensions added. From this figure we can write an equation of the volume in terms of x and establish restrictions on x.

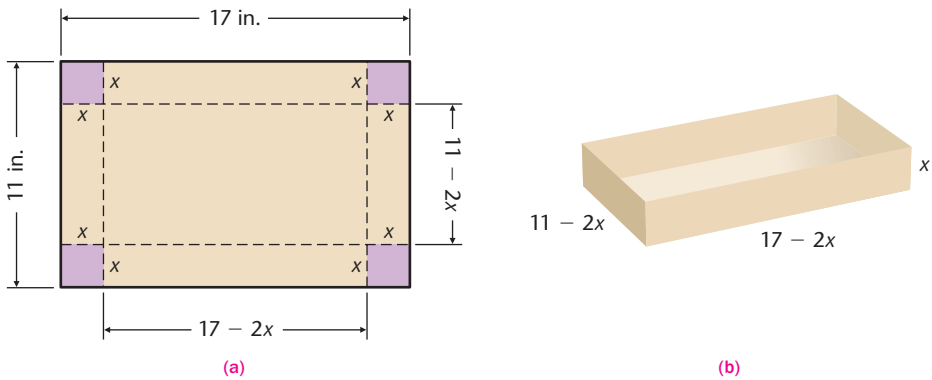


Figure 40 Box with dimensions added.

Note that the three dimensions of the box are x, $17 - 2x$, and $11 - 2x$. Of course, all of these dimensions have to be positive, so $x > 0$, $11 - 2x > 0$, and $17 - 2x > 0$. Now we solve the two latter inequalities:

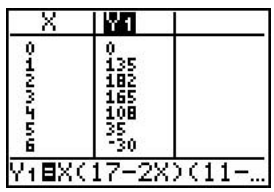
$$\begin{array}{rrl} 11 - 2x > 0 & 17 - 2x > 0 & \text{Add } 2x \text{ to each side.} \\ 11 > 2x & 17 > 2x & \text{Divide both sides by 2.} \\ 5.5 > x & 8.5 > x & \text{If } x < 5.5, \text{ then it is also less than 8.5.} \end{array}$$

We find that x has to be greater than zero and less than 5.5. (Inequalities are reviewed in Appendix B, Section B-1.)

The volume of a rectangular box is the product of its three dimensions, so the volume of the box is given by

$$y = x(17 - 2x)(11 - 2x) \qquad 0 < x < 5.5 \tag{4}$$

(B) Entering this equation in a graphing calculator (it doesn't need to be multiplied out) and evaluating it for several integers between 0 and 5 (Fig. 41), it appears that a good choice for the window dimensions for y is $0 \le y \le 200$. This choice can easily be changed if there is too much space above the graph or if part of the graph we are interested in is out of the viewing window. Figure 42 shows the graph of equation (4) in the selected viewing window.



› Figure 41

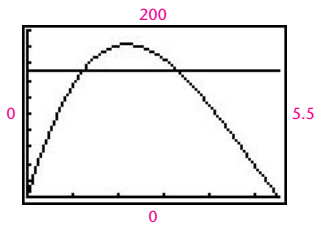
› Figure 42

(C) We want to find the smallest value of x for which $y = 150$. That is, we want to solve the equation

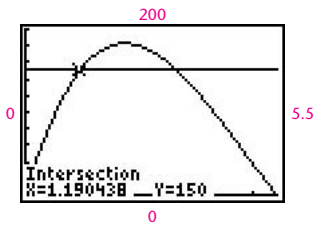
$$x(17 - 2x)(11 - 2x) = 150$$

We will solve this equation using the INTERSECT command. Entering $y_2 = 150$ in the graphing calculator and pressing GRAPH produces the two curves shown in Figure 43. The curves intersect twice. Because we were asked for the smallest value of x that satisfies the equation, we want the intersection point on the left (Fig. 44).

From Figure 44, we see that y is 150 when x is 1.19, so the smallest square that can be cut out to produce a box with a volume of 150 cubic inches is 1.19 inches.



› Figure 43



› Figure 44

››› CAUTION ›››

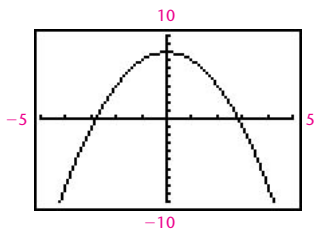
When providing an answer to application problems, it's almost always essential to attach appropriate units to your answer, in this case inches.

MATCHED PROBLEM 6

Refer to Example 6. Approximate to two decimal places the size of the largest square that can be cut out to produce a volume of 150 cubic inches.

ANSWERS TO MATCHED PROBLEMS

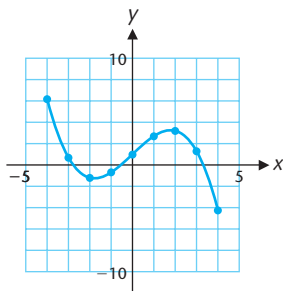
1.



2. One possible choice is Xmin = −20, Xmax = 10, Ymin = −1, Ymax = 10

3.

Year	1998	1999	2000	2001	2002	2003	2004	2005	2006
x	−4	−3	−2	−1	0	1	2	3	4
y	6.2	0.7	−1.2	−0.7	1	2.7	3.2	1.3	−4.2



4. (A) −1.5 (B) −5.85 (C) −5.85

5. −15.90, −0.92, 6.82

6. 3.32 in.

1-1 Exercises

In Problems 1–6, determine if the indicated point lies in the viewing window defined by

$$\text{Xmin} = -7, \text{Xmax} = 9, \text{Ymin} = -4, \text{Ymax} = 11$$

1. (0, 0) **2.** (0, 10) **3.** (10, 0)

4. (−3, −5) **5.** (−5, −3) **6.** (−8, 12)

7. Consider the points in the following table:

x	3	6	−7	−4	0
y	−9	−4	14	0	2

(A) Find the smallest rectangle in a Cartesian coordinate system that will contain all the points in the table. State your answer in terms of the window variables Xmin, Xmax, Ymin, and Ymax.

(B) Enter the window variables you determined in part A and display the corresponding viewing window. Can you use the cursor to display the coordinates of the points in the table on the graphing calculator screen? Discuss the differences between the rectangle in the plane and the pixels displayed on the screen.

8. Repeat Problem 7 for the following table.

x	−4	0	−2	7	4
y	2	−4	0	−2	3

9. Explain the significance of Xmin, Xmax, and Xscl when using a graphing calculator.

10. Explain the significance of Ymin, Ymax, and Yscl when using a graphing calculator.

In Problems 11–16, graph each equation in a standard viewing window.

11. $y = -x$ **12.** $y = 0.5x$

13. $y = 9 - 0.4x^2$ **14.** $y = 0.3x^2 - 4$

15. $y = 2\sqrt{x + 5}$ **16.** $y = -2\sqrt{x + 5}$

In Problems 17–20, find an appropriate viewing window in which to graph the given equation with a graphing calculator.

17. $y = \sqrt{x + 18}$ **18.** $y = \sqrt{x - 17}$

19. $y = 13 - \sqrt{10 - 2x}$ **20.** $y = \sqrt{4x + 36} - 17$

For each equation in Problems 21–26, use the TABLE command on a graphing calculator to construct a table of values over the indicated interval, computing y values to the nearest tenth of a unit. Plot these points on graph paper, then with the aid of a graph on a graphing calculator, complete the hand sketch of the graph.

21. $y = 4 + 4x - x^2$, $-2 \le x \le 6$ (Use even integers for the table.)

22. $y = 2x^2 + 12x + 5$, $-7 \le x \le 1$ (Use odd integers for the table.)

23. $y = 2\sqrt{2x + 10}$, $-5 \le x \le 5$ (Use odd integers for the table.)

24. $y = \sqrt{8 - 2x}$, $-4 \le x \le 4$ (Use even integers for the table.)

25. $y = 0.5x(4 - x)(x + 2)$, $-3 \le x \le 5$ (Use odd integers for the table.)

26. $y = 0.5x(x + 3.5)(2.8 - x)$, $-4 \le x \le 4$ (Use even integers for the table.)

In Problems 27–30, graph the equation in a standard viewing window. Approximate to two decimal places the x coordinates of the points in this window that are on the graph of the equation and have the indicated y coordinates. First use TRACE and ZOOM, then INTERSECT.

27. $y = 4 - 3\sqrt[3]{x + 4}$
(A) $(x, 8)$ (B) $(x, -1)$

28. $y = 3 + 4\sqrt[3]{x - 4}$
(A) $(x, 8)$ (B) $(x, -6)$

29. $y = 3 + x + 0.1x^3$
(A) $(x, 4)$ (B) $(x, -7)$

30. $y = 2 - 0.5x - 0.1x^3$
(A) $(x, 7)$ (B) $(x, -5)$

The graphs of each pair of equations in Problems 31–40 intersect in exactly two points. Find a viewing window that clearly shows both points of intersection (there are many windows that will do this). Then use INTERSECT to find the coordinates of each intersection point to two decimal places.

31. $y = x^2 - 10x, y = 12 - 5x$

32. $y = x^2 + 15x, y = 15 + 10x$

33. $y = 15x - x^2, y = 10 - 4x$

34. $y = -x^2 - 20x, y = 10x + 15$

35. $y = 0.4x^2 - 5x + 10, y = 5 + 9x - 0.3x^2$

36. $y = 0.2x^2 + 7x - 15, y = 9 - 7x - 0.1x^2$

37. $y = \sqrt{9x + 20}, y = 0.2x + 10$

38. $y = 20 - \sqrt{5x + 5}, y = 14 - 0.1x$

39. $y = \sqrt{x + 10}, y = 0.1x^2 - 5x - 10$

40. $y = \sqrt{5 - x}, y = 0.1x^2 + 5x$

41. (A) Sketch the graph of $x^2 + y^2 = 9$ by hand and identify the curve.*
(B) Graph $y_1 = \sqrt{9 - x^2}$ and $y_2 = -\sqrt{9 - x^2}$ in the standard viewing window of a graphing calculator. How do these graphs compare to the graph you drew in part A?
(C) Apply each of the following ZOOM options to the graphs in part B and determine which options produce a curve that looks like the curve you drew in part A: ZDecimal, ZSquare, ZoomFit

42. (A) Sketch the graph of $x^2 + y^2 = 4$ by hand and identify the curve.
(B) Graph $y_1 = \sqrt{4 - x^2}$ and $y_2 = -\sqrt{4 - x^2}$ in the standard viewing window of a graphing calculator. How do these graphs compare to the graph you drew in part A?
(C) Apply each of the following ZOOM options to the graphs in part B and determine which options produce a curve that looks like the curve you drew in part A: ZDecimal, ZSquare, ZoomFit

In Problems 43–46, use the INTERSECT command on a graphing calculator to solve each equation for the indicated values of b. Round answers to two decimal places.

43. $0.1x^3 - x^2 - 5x + 100 = b$
(A) $b = 25$ (B) $b = 75$ (C) $b = 125$

44. $0.1x^3 + x^2 - 7x - 100 = b$
(A) $b = -125$ (B) $b = -75$ (C) $b = 75$

45. $0.01x^4 + 4x + 50 = b$
(A) $b = 25$ (B) $b = 75$

46. $0.01x^4 - 3x + 50 = b$
(A) $b = 25$ (B) $b = 75$

In Problems 47–50, use the INTERSECT command on a graphing calculator to solve each equation for the indicated values of b. Round answers to two decimal places.

47. $1{,}200x - x^3 = b$
(A) $b = 12{,}000$ (B) $b = 16{,}000$ (C) $b = 20{,}000$

48. $x^3 - 300x = b$
(A) $b = 1{,}000$ (B) $b = 2{,}000$ (C) $b = 3{,}000$

49. $x^4 + 4{,}000x = b$
(A) $b = 30{,}000$ (B) $b = -30{,}000$ (C) $b = -40{,}000$

50. $800x^2 - x^4 = b$
(A) $b = 100{,}000$ (B) $b = 160{,}000$ (C) $b = 200{,}000$

In Problems 51–56, use the INTERSECT command on a graphing calculator to solve each equation. Round answers to two decimal places. (Hint: See Exercises 31–40)

51. $0.2x^4 + x^3 = \frac{1}{2}x + 2$

52. $\sqrt{x + 9} + 3 = 9 + x - 5x^2$

53. $x^2 - 3x + 1 = \sqrt{2x - 7}$

54. $\sqrt[3]{x^2 + 2} = 2x^4 - 7x^2 + 11$

55. $0.05(x + 4)^3 + 4 = 17 - x$

56. $3x - 29 = 5\sqrt[3]{-5 - 2x}$

57. The point $(\sqrt{2}, 2)$ is on the graph of $y = x^2$. Use TRACE and ZOOM to approximate $\sqrt{2}$ to four decimal places. Compare your result with the direct calculator evaluation of $\sqrt{2}$.

58. The point $(\sqrt[3]{4}, 4)$ is on the graph of $y = x^3$. Use TRACE and ZOOM to approximate $\sqrt[3]{4}$ to four decimal places. Compare your result with the direct calculator evaluation of $\sqrt[3]{4}$.

59. In a few sentences, discuss the difference between the mathematical coordinates of a point and the screen coordinates of a pixel.

60. In a few sentences, discuss the difference between the coordinates displayed during unrestricted cursor movement and those displayed during the trace procedure.

APPLICATIONS

61. **PROFIT** The monthly profit in dollars for a small consulting firm can be modeled by the equation $y = -16.7x^3 - 400x^2 + 6{,}367x - 7{,}000$, where x is the number of new clients acquired during that month. For tax purposes, the owner asks the manager to try to make the December profit as close as possible to $11,000. How many new clients should the manager try bring in?

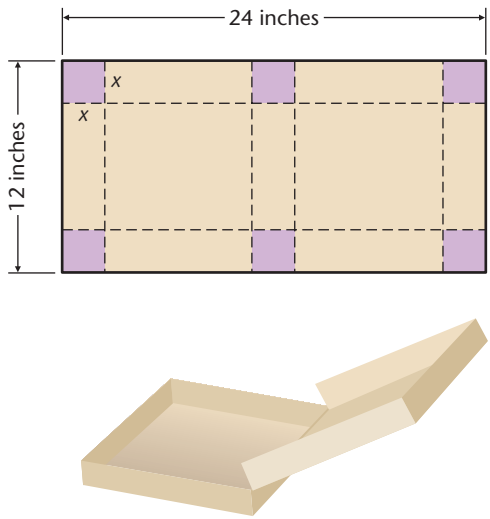
62. **METEOROLOGY** The average monthly high temperature in degrees Fahrenheit in Cincinnati for the first 10 months of the year can be modeled very accurately by the equation $y = -0.21x^3 + 2.24x^2 + 2.33x + 33.67$, where x is the month, with $x = 1$ corresponding to January. A wedding planner in Cincinnati is asked to set a date in a month where the high temperature is most likely to be close to 75 degrees. What month would be the best choice?

*Graphs of equations of this form are reviewed in Appendix B, Section B-3.

63. **MANUFACTURING** A rectangular open-top box is to be constructed out of an 8.5-inch by 11-inch sheet of thin cardboard by cutting x-inch squares out of each corner and bending the sides up, as in Figures 39 and 40 in Example 6. What size squares to two decimal places should be cut out to produce a box with a volume of 55 cubic inches? Give the dimensions to two decimal places of all possible boxes with the given volume.

64. **MANUFACTURING** A rectangular open-top box is to be constructed out of a 9-inch by 12-inch sheet of thin cardboard by cutting x-inch squares out of each corner and bending the sides up as shown in Figures 39 and 40 in Example 6. What size squares to two decimal places should be cut out to produce a box with a volume of 72 cubic inches? Give the dimensions to two decimal places of all possible boxes with the given volume.

65. **MANUFACTURING** A box with a lid is to be cut out of a 12-inch by 24-inch sheet of thin cardboard by cutting out six x-inch squares and folding as indicated in the figure. What are the dimensions to two decimal places of all possible boxes that will have a volume of 100 cubic inches?



66. **MANUFACTURING** A box with a lid is to be cut out of a 10-inch by 20-inch sheet of thin cardboard by cutting out six x-inch squares and folding as indicated in the figure. What are the dimensions to two decimal places of all possible boxes that will have a volume of 75 cubic inches? (Refer to the figure for Problem 65.)

67. **MANUFACTURING** An oil tank in the shape of a right circular cylinder* has a volume of 40,000 cubic feet. If regulations for such tanks require that the radius plus the height must be 50 feet, find the radius and the height to two decimal places.

*Geometric formulas can be found in Appendix C.

68. **MANUFACTURING** A drinking container in the shape of a right circular cone* has a volume of 50 cubic inches. If the radius plus the height of the cone is 8 inches, find the radius and the height to two decimal places.

69. **PRICE AND DEMAND** A nationwide office supply company sells high-grade paper for laser printers. The price per case y (in dollars) and the weekly demand x for this paper are related approximately by the equation

$$y = 100 - 0.6\sqrt{x} \qquad 5{,}000 \le x \le 20{,}000$$

(A) Complete the following table. Approximate each value of x to the nearest hundred cases.

x			
y	20	25	30

(B) Does the demand increase or decrease if the price is increased from \$25 to \$30? By how much?
(C) Does the demand increase or decrease if the price is decreased from \$25 to \$20? By how much?

70. **PRICE AND DEMAND** Refer to the relationship between price and demand given in Problem 69.
(A) Complete the following table. Approximate each value of x to the nearest hundred cases.

x			
y	35	40	45

(B) Does the demand increase or decrease if the price is increased from \$40 to \$45? By how much?
(C) Does the demand increase or decrease if the price is decreased from \$40 to \$35? By how much?

71. **PRICE AND REVENUE** Refer to Problem 69. The revenue from the sale of x cases of paper at \$$y$ per case is given by the product $R = xy$.
(A) Use the results from Problem 69 to complete the following table of revenues.

y	20	25	30
R			

(B) Does the revenue increase or decrease if the price is increased from \$25 to \$30? By how much?
(C) Does the revenue increase or decrease if the price is decreased from \$25 to \$20? By how much?
(D) If the current price of paper is \$25 per case and the company wants to increase revenue, should it raise the price \$5, lower the price \$5, or leave the price unchanged?

72. PRICE AND REVENUE Refer to Problem 70. The revenue from the sale of x cases of paper at \$$y$ per case is given by the product $R = xy$.

(A) Use the results from Problem 70 to complete the following table of revenues.

y	35	40	45
R			

(B) Does the revenue increase or decrease if the price is increased from \$40 to \$45? By how much?

(C) Does the revenue increase or decrease if the price is decreased from \$40 to \$35? By how much?

(D) If the current price of paper is \$40 per case and the company wants to increase revenue, should it raise the price \$5, lower the price \$5, or leave the price unchanged?

1-2 Functions

- Defining Relations and Functions
- Defining Functions by Equations
- Finding the Domain of a Function
- Using Function Notation
- Modeling and Data Analysis
- A Brief History of the Function Concept

The idea of correspondence plays a really important role in understanding the concept of functions, which is almost certainly the most important idea in this course. The good news is that you have already had years of experience with correspondences in everyday life. For example,

For every person, there is a corresponding age.
For every item in a store, there is a corresponding price.
For every season, there is a corresponding Super Bowl champion.
For every circle, there is a corresponding area.
For every number, there is a corresponding cube.

One of the most basic and important ways that math can be applied to other areas of study is the establishment of correspondences among various types of phenomena. In many cases, once a correspondence is known, it can be used to make important decisions and predictions. An engineer can use a formula to predict the weight capacity of a stadium grandstand. A political operative decides how many resources to allocate to a race given current polling results. A computer scientist can use formulas to compare the efficiency of algorithms for sorting data stored on a computer. An economist would like to be able to predict interest rates, given the rate of change of the money supply. And the list goes on and on . . . .

Defining Relations and Functions

What do all of the preceding examples have in common? Each of them describes the matching of elements in one set with elements in a second set. There is a special name for such a correspondence.

DEFINITION 1 Definition of Relation

A **relation** is a correspondence that matches up two sets of objects. The first set is called the **domain,** and the set of all corresponding elements is called the **range.**

REMARK: Some graphing calculators use the term *range* to refer to the window variables. In this book, *range* will always refer to the range of a relation.

Two examples of relations are provided below.

Table 1 Manufacturers of the Five Top Selling Cars in America, 1/1/06–8/31/06

Manufacturer	Model
Toyota	Camry Corolla
Honda	Accord Civic
Chevrolet	Impala

Source: Forbes.com

Table 2 Sales of the Five Top Selling Cars in America, 1/1/06–8/31/06

Model	U.S. Sales
Camry	302,636
Corolla	274,074
Accord	250,663
Civic	225,212
Impala	197,304

Table 1 specifies a relation with domain {Toyota, Honda, Chevrolet}, and range {Camry, Corolla, Accord, Civic, Impala}. Table 2 specifies a relation with domain {Camry, Corolla, Accord, Civic, Impala} and range {302,636, 274,074, 250,663, 225,212, 197,304}.

Notice that in Table 1, two of the domain elements (Toyota and Honda) correspond to more than one range element. But in Table 2, each domain element corresponds to a unique range element. We will give relations like the latter a special name.

› DEFINITION 2 Definition of Function

A **function** is a relation in which every element in the domain corresponds to one and only one element in the range.

Note that Table 2 defines a function, while Table 1 does not; both Toyota and Honda (domain elements) correspond to more than one range element.

››› EXPLORE-DISCUSS 1

(A) Consider the set of students enrolled in a college and the set of faculty members of that college. Define a relation between the two sets by saying that a student corresponds to a faculty member if the student is currently enrolled in a course taught by that faculty member. Is this relation a function? Discuss.

(B) Write an example of a function based on a real-life situation, then write a similar example of a relation that is not a function.

We now turn to an alternative way to represent relations. Every relation can be specified using ordered pairs of elements, where the first component represents a domain element and the second component represents the corresponding range element. Using this approach, the relations specified in Tables 1 and 2 can be written as follows:

$$F = \{(\text{Toyota, Camry}), (\text{Toyota, Corolla}), (\text{Honda, Accord}), (\text{Honda, Civic}), (\text{Chevrolet, Impala})\}$$
$$G = \{(\text{Camry, 302,636}), (\text{Corolla, 274,074}), (\text{Accord, 250,663}), (\text{Civic, 225,212}), (\text{Impala, 197,304})\}$$

(If this notation reminds you of points on a graph, good for you! You already have a leg up on the next section.)

In relation G, notice that no two ordered pairs have the same first component and different second components. This tells us that relation G is a function. On the other hand, in relation F, the first two ordered pairs have the same first component and different second components. This means that relation F is not a function.

This ordered pair approach suggests an alternative (but equivalent) way of defining the concept of a function.

› **DEFINITION 3 Set Form of the Definition of Function**

A **function** is a set of ordered pairs with the property that no two ordered pairs have the same first component and different second components.

The set of all first components in a function is called the **domain** of the function, and the set of all second components is called the **range.**

EXAMPLE 1 Determining If Ordered Pairs Define a Function

State the domain and range of each relation, then determine whether the relation is a function.

(A) $S = \{(1, 4), (2, 3), (3, 2), (4, 3), (5, 4)\}$
(B) $T = \{(1, 4), (2, 3), (3, 2), (2, 4), (1,5)\}$

SOLUTIONS

(A) The domain and range are

$$\text{Domain} = \{1, 2, 3, 4, 5\}$$
$$\text{Range} = \{2, 3, 4\}$$

Because all of the ordered pairs in S have distinct first components, this relation is a function.

(B) The domain and range are

$$\text{Domain} = \{1, 2, 3\}$$
$$\text{Range} = \{2, 3, 4, 5\}$$

Because there are ordered pairs in relation T with the same first component and different second components [for example, (1, 4) and (1, 5)], this relation is not a function.

MATCHED PROBLEM 1

State the domain and range of each relation, then determine whether the relation is a function.

(A) $S = \{(-2, 1), (-1, 2), (0, 0), (-1, 1), (-2, 2)\}$
(B) $T = \{(-2, 1), (-1, 2), (0, 0), (1, 2), (2, 1)\}$

REMARK: Notice that in Relation S of Example 1, the range values 3 and 4 each correspond to two different domain values. This does not violate the definition of function, since no domain value appears more than once in the list of ordered pairs.

Defining Functions by Equations

So far, we have described a particular function in various ways: (1) by a verbal description; (2) by a table; and (3) by a set of ordered pairs. We will see that if the domain and range are sets of numbers, we can also define a function by an equation, or by a graph.

If the domain of a function is a large or infinite set, it may be impractical or impossible to actually list all of the ordered pairs that belong to the function, or to display the function in a table. Such a function can often be defined by a verbal description of the "rule of correspondence" that clearly specifies the element of the range that corresponds to each element of the domain. For example, "to each real number corresponds its square." When the domain and range are sets of numbers, the algebraic and graphical analogs of the verbal description are the equation and graph, respectively. We will find it valuable to be able to view a particular function from multiple perspectives—algebraic (in terms of an equation), graphical (in terms of a graph), and numeric (in terms of a table or ordered pairs).

Both versions of our definition of function are very general. The objects in the domain and range can be pretty much anything, and there is no restriction on the number of elements in each.

In this text, we are primarily interested, however, in functions with real number domains and ranges. Unless otherwise indicated, **the domain and range of a function will be sets of real numbers.** For such a function we often use an equation in two variables to specify both the rule of correspondence and the set of ordered pairs.

Consider the equation

$$y = x^2 + 2x \qquad x \text{ any real number} \tag{1}$$

This equation assigns to each domain value x exactly one range value y. For example,

$$\begin{aligned} &\text{If } x = 4, && \text{then} && y = (4)^2 + 2(4) = 24 \\ &\text{If } x = -\tfrac{1}{3}, && \text{then} && y = (-\tfrac{1}{3})^2 + 2(-\tfrac{1}{3}) = -\tfrac{5}{9} \end{aligned}$$

Thus, we can view equation (1) as a function with rule of correspondence

$$y = x^2 + 2x \qquad x \text{ corresponds to } x^2 + 2x$$

or, equivalently, as a function with set of ordered pairs

$$\{(x, y) \mid y = x^2 + 2x, x \text{ a real number}\}$$

This is called set-builder notation and is read as "The set of all ordered pairs (x, y) such that $y = x^2 + 2x$, where x is a real number." The variable x is called an *independent variable,* indicating that values can be assigned "independently" to x from the domain. The variable y is called a *dependent variable,* indicating that the value of y "depends" on the value assigned to x and on the given equation. In general, any variable used as a placeholder for domain values is called an **independent variable;** any variable used as a placeholder for range values is called a **dependent variable.**

We often refer to a value of the independent variable as the input of the function, and the corresponding value of the dependent variable as the associated output. In this regard, a function can be thought of as a process that accepts an input from the domain and outputs an appropriate range element.

Next, we address the question of which equations can be used to define functions.

› FUNCTIONS DEFINED BY EQUATIONS

In an equation in two variables, if to each value of the independent variable there corresponds exactly one value of the dependent variable, then the equation defines a function.

If there is any value of the independent variable to which there corresponds more than one value of the dependent variable, then the equation does not define a function.

We have already decided that equation (1) defines a function. Now we will look at an example of an equation that does not define a function,

$$y^2 = x, \text{ where } x \text{ is any real number.} \tag{2}$$

For $x = 9$, for example, there are two associated values of the dependent variable, $y = 3$ and $y = -3$. Thus, equation (2) does not define a function.

Notice that we have used the phrase "an equation defines a function" rather than "an equation is a function." This is a somewhat technical distinction, but it is employed consistently in mathematical literature and we will adhere to it in this text.

››› EXPLORE-DISCUSS 2

(A) Graph $y = x^2 - 4$ for $-5 \le x \le 5$ and $-5 \le y \le 5$ and trace along this graph. Discuss the relationship between the coordinates displayed while tracing and the function defined by this equation.

(B) The graph of the equation $x^2 + y^2 = 16$ is a circle. Because most graphing calculators will accept only equations that have been solved for y, we must graph both of the equations $y_1 = \sqrt{16 - x^2}$ and $y_2 = -\sqrt{16 - x^2}$ to produce a graph of the circle. Graph these equations for $-5 \le x \le 5$ and $-5 \le y \le 5$. Then try different values for Xmin and Xmax until the graph looks more like a circle. Use the TRACE command to find two points on this circle with the same x coordinate and different y coordinates.

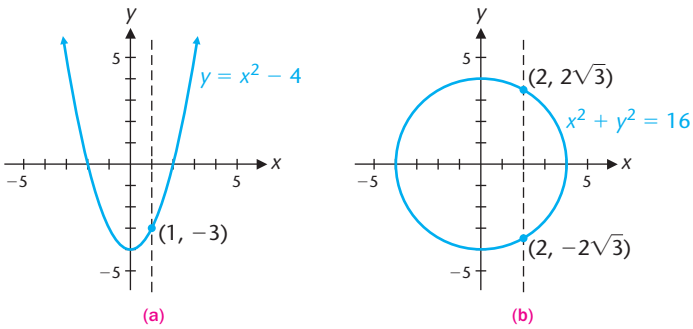
(C) Is it possible to graph a single equation of the form $y =$ (expression in x) on your graphing calculator and obtain a graph that is not the graph of a function? Explain your answer.

REMARK: If we want the graph of a circle to actually appear to be circular, we must choose window variables so that a unit length on the x axis is the same number of pixels as a unit length on the y axis. Such a window is often referred to as a *squared viewing window*. Most graphing calculators have an option under the zoom menu that does this automatically.

One way to determine if an equation defines a function is to examine its graph. The graphs of the equations

$$y = x^2 - 4 \qquad \text{and} \qquad x^2 + y^2 = 16$$

are shown in Figure 1.



› **Figure 1** Graphs of equations and the vertical line test.

The graph in Figure 1(a) is a parabola and the graph in Figure 1(b) is a circle.* Each vertical line intersects the parabola in exactly one point. This shows that to each value of the independent variable x there corresponds exactly one value of the dependent variable y. For example, to the x value 1 there corresponds only the y value -3 [Fig. 1(a)]. Thus, the equation $y = x^2 - 4$ defines a function. On the other hand, there are vertical lines that intersect the circle in Figure 1(b) in two points. For example, the vertical line through $x = 2$ intersects the circle in the points $(2, 2\sqrt{3})$ and $(2, -2\sqrt{3})$ [Fig. 1(b)]. Thus, to the x value 2 there correspond two y values, $2\sqrt{3}$ and $-2\sqrt{3}$. Consequently, the equation $x^2 + y^2 = 16$ does not define a function. These observations are generalized in Theorem 1.

› THEOREM 1 Vertical Line Test for a Function

An equation defines a function if each vertical line in the rectangular coordinate system passes through at most one point on the graph of the equation.

If any vertical line passes through two or more points on the graph of an equation, then the equation does not define a function.

EXAMPLE 2 Determining If an Equation Defines a Function

Determine if each equation defines a function with independent variable x.

(A) $y = 2x^2 - 3$

(B) $2x - 1 = y^2$

*Parabolas and circles are discussed extensively later in the book.

SOLUTIONS

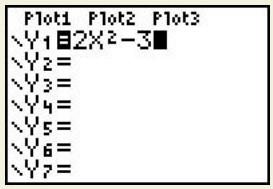
Algebraic Solutions

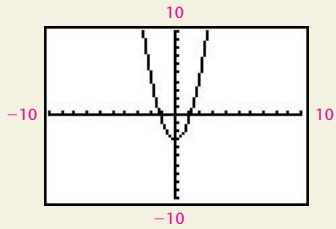
(A) For any real number x, the square of x is a unique number. When you multiply the square by 2 and then subtract 3, the result is still a unique number. Thus, for any input x, there is a unique output y, and the equation defines a function.

(B) For any real number x, as long as $2x - 1$ is positive, there will be two associated values of y whose square will equal $2x - 1$, one positive and one negative. Thus, the equation has more than one output for certain inputs, and does not define a function.

Graphical Solutions

(A) We enter the equation into a graphing calculator, then graph in a standard viewing window.





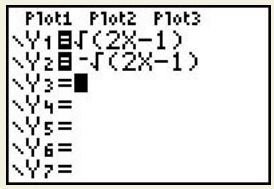
Any vertical line will pass through the graph only once, so the equation defines a function.

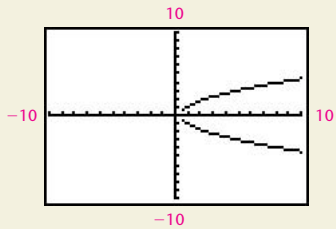
(B) To enter this equation in a graphing calculator, we must first solve for y:

$$2x - 1 = y^2$$

$$y = \sqrt{2x - 1} \quad \text{or} \quad y = -\sqrt{2x - 1}$$

We enter these two equations as Y_1 and Y_2 and graph in a standard viewing window.





Almost any vertical line that intersects the graph at all will actually intersect it twice, so we conclude that the equation does not define a function.

MATCHED PROBLEM 2

Determine if each equation defines a function with independent variable x.

(A) $y^2 + 4 = 3x$
(B) $y = 7 - x^2$

Finding the Domain of a Function

Sometimes when a function is defined by an equation, a domain is specified, as in

$$y = 2x^2 + 5, \quad x > 0$$

The "$x > 0$" tells us that the domain is all positive real numbers.

More often, a function is defined by an equation with no domain specified. Refer to Figure 1 on page 27. Because the expression $x^2 - 4$ represents a real number for any real number input x, the function defined by the equation $y = x^2 - 4$ is defined for all real numbers. Thus, its domain is the set of all real numbers, often denoted by the letter R or the interval* $(-\infty, \infty)$. On the other hand, the expression $\sqrt{16 - x^2}$ represents a real number only if the expression under the root, $16 - x^2$, is nonnegative. This occurs only for x values from -4 to 4. Thus, the domain of the function $y = \sqrt{16 - x^2}$ is $\{x \mid -4 \le x \le 4\}$ or $[-4, 4]$. Unless a domain is specified, we will adhere to the following convention regarding domains and ranges for functions defined by equations.

AGREEMENT ON DOMAINS AND RANGES

If a function is defined by an equation and the domain is not indicated, then we assume that the domain is the set of all real number replacements of the independent variable that produce *real values* for the dependent variable. The range is the set of all values of the dependent variable corresponding to these domain values.

EXAMPLE 3 Finding the Domain of a Function

Find the domain of the function defined by the equation $y = 4 + \sqrt{x + 2}$, assuming x is the independent variable.

*See Appendix B, Section B-1, for a discussion of interval notation and inequalities.

SOLUTION

For $\sqrt{x+2}$ to be real, $x + 2$ must be greater than or equal to 0. This occurs when x is greater than or equal to -2.

$$\text{Domain} = \{x \mid x \geq -2\} \quad \text{or} \quad [-2, \infty)$$

Note that in many cases we will dispense with set notation and simply write $x \geq -2$ instead of $\{x \mid x \geq -2\}$.

MATCHED PROBLEM 3

Find the domain of the function defined by the equation $y = 3 + \sqrt{-x}$, assuming x is the independent variable.

Using Function Notation

We will use letters to name functions and to provide a very important and convenient notation for defining functions. For example, if f is the name of the function defined by the equation $y = 2x + 1$, we could use the formal representations

$$f: y = 2x + 1 \qquad \text{Rule of correspondence}$$

or

$$f: \{(x, y) \mid y = 2x + 1\} \qquad \text{Set of ordered pairs}$$

But instead, we will simply write

$$f(x) = 2x + 1 \qquad \text{Function notation}$$

The symbol $f(x)$ is read "f of x," "f at x," or "the value of f at x" and represents the number in the range of the function f (the output) that is paired with the domain value x (the input).

››› CAUTION ›››

The symbol "$f(x)$" should *never* be read as "f times x." The notation does not represent a product. It tells us that the function named f has independent variable x.

$f(x)$ is the value of the function f at x.
$2(x) = 2x$ is algebraic multiplication.

Using function notation, $f(3)$ is the output for the function f associated with the input 3.

We find this range value by replacing x with 3 wherever x occurs in the function definition

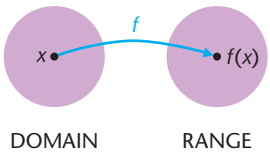
$$f(x) = 2x + 1$$

and evaluating the right side,

$$\begin{aligned} f(\mathbf{3}) &= 2 \cdot \mathbf{3} + 1 \\ &= 6 + 1 \\ &= 7 \end{aligned}$$

The statement $f(3) = 7$ indicates in a concise way that the function f assigns the range value 7 to the domain value 3 or, equivalently, that the ordered pair (3, 7) belongs to f.

The symbol $f: x \rightarrow f(x)$, read "f maps x into $f(x)$," is also used to denote the relationship between the domain value x and the range value $f(x)$ (Fig. 2).



The function f "maps" the domain value x into the range value $f(x)$.

› **Figure 2** Function notation.

In many cases, when defining functions with equations, we will use the symbols y and $f(x)$ interchangeably. Whenever we write $y = f(x)$, we assume that x is an independent variable and that y and $f(x)$ both represent the dependent variable.

Letters other than f and x can be used to represent functions and independent variables. For example,

$$g(t) = t^2 - 3t + 7$$

defines g as a function of the independent variable t. To find $g(-2)$, we replace t by -2 wherever t occurs in

$$g(t) = t^2 - 3t + 7$$

and evaluate the right side:

$$\begin{aligned} g(\mathbf{-2}) &= (\mathbf{-2})^2 - 3(\mathbf{-2}) + 7 \\ &= 4 + 6 + 7 \\ &= 17 \end{aligned}$$

Thus, the function g assigns the range value 17 to the domain value -2; the ordered pair $(-2, 17)$ belongs to g.

It is important to understand and remember the definition of the symbol $f(x)$:

› DEFINITION 4 The Symbol $f(x)$

The symbol $\boldsymbol{f(x)}$, read "f of x," represents the real number in the range of the function f corresponding to the domain value x. The symbol $f(x)$ is also called the *value of the function f at x*. The ordered pair $(x, f(x))$ belongs to the function f. If x is a real number that is not in the domain of f, then f is *undefined* at x and $f(x)$ *does not exist*.

EXAMPLE 4 Evaluating Functions

For

$$f(x) = \frac{15}{x - 3} \qquad g(x) = 16 + 3x - x^2 \qquad h(x) = \frac{6}{\sqrt{x} - 1}$$

find:

(A) $f(6)$ (B) $g(-7)$ (C) $h(-2)$ (D) $f(0) + g(4) - h(16)$

SOLUTIONS

In each case, the independent variable is replaced with the given domain value.

(A) $f(\mathbf{6}) = \dfrac{15}{\mathbf{6} - 3} = \dfrac{15}{3} = 5$*

(B) $g(\mathbf{-7}) = 16 + 3(\mathbf{-7}) - (\mathbf{-7})^2 = 16 - 21 - 49 = -54$

(C) $h(\mathbf{-2}) = \dfrac{6}{\sqrt{\mathbf{-2}} - 1}$

But $\sqrt{-2}$ is not a real number. Because we have agreed to restrict the domain of a function to values of x that produce real values for the function, -2 is not in the domain of h and $h(-2)$ does not exist.

(D) $f(\mathbf{0}) + g(\mathbf{4}) - h(\mathbf{16})$

$$= \frac{15}{\mathbf{0} - 3} + [16 + 3(\mathbf{4}) - \mathbf{4}^2] - \frac{6}{\sqrt{\mathbf{16}} - 1}$$

$$= \frac{15}{-3} + 12 - \frac{6}{3}$$

$$= -5 + 12 - 2 = 5$$

*The dashed "think boxes" are used to enclose steps that may be performed mentally.

MATCHED PROBLEM 4

Use the functions in Example 4 to find

(A) $f(-2)$ (B) $g(6)$ (C) $h(-8)$ (D) $\dfrac{f(8)}{h(9)}$

EXAMPLE 5 Finding the Domain of a Function

Find the domain of $g(t) = 5 + 2t - 3t^2$.

SOLUTION

Because $5 + 2t - 3t^2$ represents a real number for all replacements of t by real numbers, the domain of g is R, the set of all real numbers. To express this domain in interval notation, we write

$$\text{Domain of } g = (-\infty, \infty)$$

MATCHED PROBLEM 5

Find the domain of $h(w) = 3w^2 + 2w - 9$.

The reasoning used in Example 5 can be applied to any polynomial: **The domain of any polynomial is *R*, the set of real numbers.** (We will study polynomials extensively in Chapter 3.)

EXAMPLE 6 Finding the Domain of a Function

Find the domain of $F(w) = \dfrac{5}{w^2 - 9}$.

SOLUTION

Because division by 0 is not defined, we must exclude all values of w that would make the denominator 0. Factoring the denominator, we can write

$$F(w) = \frac{5}{w^2 - 9} = \frac{5}{(w - 3)(w + 3)} \qquad a^2 - b^2 = (a - b)(a + b)$$

Thus, we see that $w = 3$ and $w = -3$ both make the denominator zero and must be excluded from the domain of F. That is,

$$\begin{aligned} \text{Domain of } F &= \{w \mid w \neq 3, w \neq -3\} && \text{Set notation} \\ &= (-\infty, -3) \cup (-3, 3) \cup (3, \infty) && \text{Interval notation} \end{aligned}$$

We often simplify this by writing

$$F(w) = \frac{5}{w^2 - 9} \qquad w \neq 3, w \neq -3$$

MATCHED PROBLEM 6

Find the domain of $G(w) = \dfrac{5}{w^2 - 1}$.

EXAMPLE 7 Finding the Domain of a Function

Find the domain of $s(x) = \sqrt{3 - x}$.

SOLUTIONS

Algebraic Solution

Because $\sqrt{3 - x}$ is not a real number when $3 - x$ is a negative real number, we must restrict the domain of s to the real numbers x for which $3 - x$ is nonnegative:

$$\begin{aligned} 3 - x &\geq 0 \\ 3 &\geq x \end{aligned}$$

Thus, we have

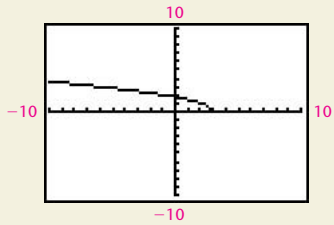
$$\begin{aligned} \text{Domain of } s &= \{x \mid x \leq 3\} \\ &= (-\infty, 3] \end{aligned}$$

or, more informally,

$$s(x) = \sqrt{3 - x} \qquad x \leq 3$$

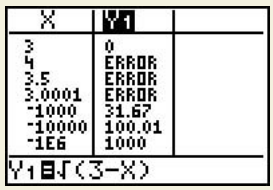
Graphical Solution

Entering $y_1 = \sqrt{3 - x}$ in the equation editor and graphing in a standard viewing window produces Figure 3.



› Figure 3

Using a table (Fig. 4), we see that evaluating s at any number greater than 3 produces an error message, whereas evaluating s for large negative values produces no errors.



› Figure 4

We conclude that the domain of s is $(-\infty, 3]$.

MATCHED PROBLEM 7

Find the domain of $r(t) = \sqrt{t - 5}$.

EXAMPLE 8 Finding the Domain of a Function

Find the domain of $f(x) = \dfrac{2}{2 - \sqrt{x}}$.

SOLUTIONS

Algebraic Solution

Because $\sqrt{x}$ is not a real number for negative real numbers x, x must be a nonnegative real number. Because division by 0 is not defined, we must exclude any values of x that make the denominator 0. To find x values that make the denominator zero, we solve

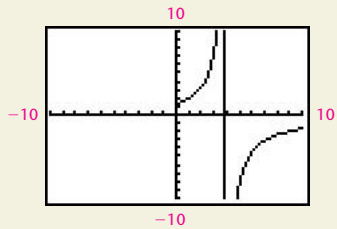
$$\begin{aligned} 2 - \sqrt{x} &= 0 \\ 2 &= \sqrt{x} \\ 4 &= x \end{aligned}$$

and conclude that the domain of f is all nonnegative real numbers except 4. This can be written as

$$\begin{aligned} \text{Domain of } f &= \{x \mid x \geq 0, x \neq 4\} \\ &= [0, 4) \cup (4, \infty) \end{aligned}$$

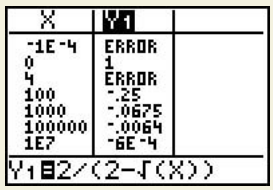
Graphical Solution

Figure 5 shows the graph of $y = 2/(2 - \sqrt{x})$ in a standard viewing window.



› Figure 5

The curve appears to start at $x = 0$, indicating that f is not defined for $x < 0$. Evaluating f at a small negative number confirms this (Fig. 6). The vertical line on the graph indicates some strange behavior at $x = 4$. Evaluating f at $x = 4$ (Fig. 6) shows that f is not defined at $x = 4$. Evaluating f at large positive numbers produces no errors.



› Figure 6

We conclude that

$$\begin{aligned} \text{Domain of } f &= \{x \mid x \geq 0, x \neq 4\} \\ &= [0, 4) \cup (4, \infty) \end{aligned}$$

MATCHED PROBLEM 8

Find the domain of $g(x) = \dfrac{1}{\sqrt{x} - 1}$.

Refer to Figure 5. What do you think caused the break in this graph? Explore-Discuss 3 will help you find out.

››› EXPLORE-DISCUSS 3

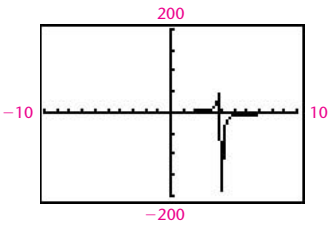
Graph $y = 2/(2 - \sqrt{x})$ in a standard viewing window (see Fig. 5).

(A) Press TRACE and move the cursor as close to $x = 4$ on the left side of 4 as possible. What is the y coordinate of this point?

(B) Now move the cursor as close to $x = 4$ on the right side of 4 as possible. What is the y coordinate of this point?

(C) Change Ymax to a value greater than the y coordinate in (A) and Ymin to a value less than the y coordinate in (B).

(D) Redraw the graph in the window from part (C) and discuss the result.



› Figure 7

Depending on the model of graphing calculator you have, your efforts in Explore-Discuss 3 may have produced a graph similar to Figure 7. The nearly vertical line is produced by connecting the last point on the left of $x = 4$ with the first point on the right of $x = 4$.

In addition to evaluating functions at specific numbers, it is important to be able to evaluate functions at expressions that involve one or more variables. For example, the **difference quotient**

$$\frac{f(x + h) - f(x)}{h} \quad \text{for } x \text{ and } x + h \text{ in the domain of } f,\ h \neq 0$$

is studied extensively in a calculus course.

››› EXPLORE-DISCUSS 4

Let x and h be any real numbers.

(A) If $f(x) = 3x + 2$, which of the following is correct?

(i) $f(x + h) = 3x + 2 + h$

(ii) $f(x + h) = 3x + 3h + 2$

(iii) $f(x + h) = 3x + 3h + 4$

(B) If $f(x) = x^2$, which of the following is correct?

(i) $f(x + h) = x^2 + h^2$

(ii) $f(x + h) = x^2 + h$

(iii) $f(x + h) = x^2 + 2xh + h^2$

(C) If $f(x) = x^2 + 3x + 2$, write a verbal description of the operations that must be performed to evaluate $f(x + h)$.

EXAMPLE 9 Evaluating and Simplifying a Difference Quotient

For $f(x) = x^2 + 4x + 5$, find and simplify:

(A) $f(2)$ (B) $f(2 + h)$ (C) $\dfrac{f(2 + h) - f(2)}{h}$

(D) $f(x + h)$ (E) $\dfrac{f(x + h) - f(x)}{h}$

SOLUTIONS

(A) $f(2) = 2^2 + 4(2) + 5 = 17$

(B) To find $f(2 + h)$, replace x with $2 + h$ everywhere it occurs in the equation that defines f, then simplify:

$$\begin{aligned} f(2 + h) &= (2 + h)^2 + 4(2 + h) + 5 \\ &= 4 + 4h + h^2 + 8 + 4h + 5 \\ &= h^2 + 8h + 17 \end{aligned}$$

(C) Using parts (A) and (B), we have

$$\frac{f(2 + h) - f(2)}{h} = \frac{(h^2 + 8h + 17) - (17)}{h}$$

$$= \frac{h^2 + 8h}{h} = \boxed{\frac{h(h + 8)}{h}} = h + 8$$

(D) To find $f(x + h)$, we replace x with $x + h$ everywhere it appears in the equation that defines f and simplify:

$$\begin{aligned} f(x + h) &= (x + h)^2 + 4(x + h) + 5 \\ &= x^2 + 2xh + h^2 + 4x + 4h + 5 \end{aligned}$$

(E) Using the result of part (D), we get

$$\frac{f(x + h) - f(x)}{h} = \frac{(x^2 + 2xh + h^2 + 4x + 4h + 5) - (x^2 + 4x + 5)}{h}$$

$$= \frac{\cancel{x^2} + 2xh + h^2 + \cancel{4x} + 4h + \cancel{5} - \cancel{x^2} - \cancel{4x} - \cancel{5}}{h}$$

$$= \frac{2xh + h^2 + 4h}{h} \quad \boxed{= \frac{h(2x + h + 4)}{h}} \quad = 2x + h + 4$$

MATCHED PROBLEM 9

Repeat Example 9 for $f(x) = x^2 + 3x + 7$.

*The symbol denotes problems that are related to calculus.

››› CAUTION ›››

1. If f is a function, then the symbol $f(x + h)$ represents the value of f at the number $x + h$ and must be evaluated by replacing the independent variable in the equation that defines f with the expression $x + h$, as we did in Example 9. Do not confuse this notation with the familiar algebraic notation for multiplication:

$$f(x + h) \neq fx + fh \qquad f(x + h) \text{ is function notation.}$$
$$4(x + h) = 4x + 4h \qquad 4(x + h) \text{ is algebraic multiplication notation.}$$

2. There is another common incorrect interpretation of the symbol $f(x + h)$. If f is an arbitrary function, then

$$f(x + h) \neq f(x) + f(h)$$

It is possible to find some particular functions for which $f(x + h) = f(x) + f(h)$ is a true statement, but in general these two expressions are not equal.

3. Finally, note that even though both may be read aloud as "f of x plus h," $f(x + h)$ is *not* the same as $f(x) + h$. (See Explore-Discuss 4.)

› Modeling and Data Analysis

The next example explores the relationship between the *algebraic* definition of a function, the *numeric* values of the function, and a *graphical* representation of the function. The interplay between the algebraic, numeric, and graphical aspects of a function is one of the central themes of this book. In this example, we also see how a function can be used to describe data from the real world, a process that we referred to in Section 1-1 as mathematical modeling.

EXAMPLE 10 Consumer Debt

Revolving-credit debt (in billions of dollars) in the United States over a 25-year period is given in Table 3. A financial analyst used statistical techniques to produce a mathematical model for this data:

$$f(x) = 0.7x^2 + 14.6x + 45.9$$

where $x = 0$ corresponds to 1980.

Table 3 Revolving-Credit Debt

Year	Total Debt (Billions)
1980	$58.5
1985	$128.9
1990	$234.8
1995	$443.2
2000	$663.8
2005	$848.3

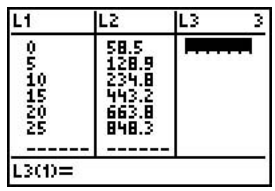
Source: Federal Reserve System.

Compare the data and the model both numerically and graphically. Use the modeling function f to estimate the debt to the nearest tenth of a billion in 2006 and in 2010.

SOLUTION

Most graphing calculators have the ability to manipulate a list of numbers, such as the total debt in Table 3. The relevant commands are usually found by pressing STAT* (Fig. 8). Then select EDIT and enter the data. Enter the years as L1 and the debt as L2 (Fig. 9). Unlike the TABLE command, which computes a y value for each x value you enter, the EDIT command does not assume any correspondence between the numbers in two different lists. It is your responsibility to make sure that each pair on the same line in Figure 9 corresponds to a line in Table 3.

› Figure 8



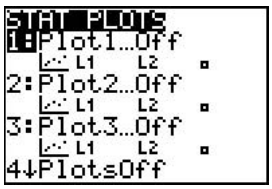
› Figure 9

A graph of a finite data set is called a **scatter plot.** To form a scatter plot for the data in Table 3, first press STAT PLOT (Fig. 10) and select Plot1 (Fig. 11). The Plot1 screen contains a number of options, some of which you select by placing the cursor over the option and pressing ENTER. First, select ON to activate the plot. Then select the type of plot you want. The darkened choice in Figure 11 produces a scatter plot. Next use the 2nd key to enter L1 for the Xlist and L2 for the Ylist. Finally, select the mark you want to use for the plot.

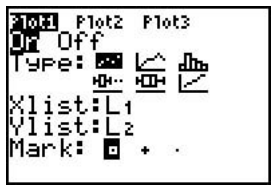
Before graphing the data, we must enter values for the window variables that will produce a window that contains the points in the scatter plot. Examining the data in Figure 9, we see that Xmin $= -5$, Xmax $= 30$, Xscl $= 5$, Ymin $= 0$, Ymax $= 1000$, and Yscl $= 100$ should provide a viewing window that contains all of these points.

*We used a TI-84 to produce the screen shots in this section. If you are using a different graphing calculator, consult your manual for the appropriate commands.

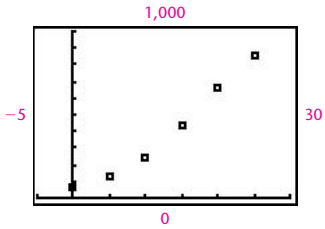
(We chose Xmin = −5 to clearly show the point at $x = 0$.) Pressing GRAPH displays the scatter plot (Fig. 12).



› Figure 10

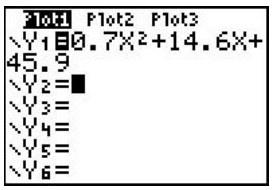


› Figure 11

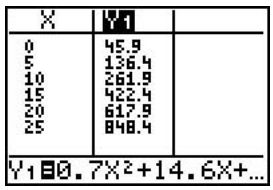


› Figure 12

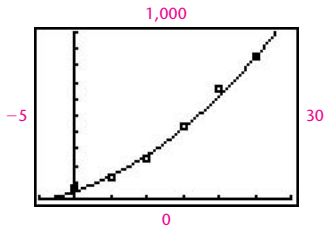
Now we enter the modeling function in the equation editor (Fig. 13) and use the TABLE command to evaluate the function (Fig. 14). To compare the data and the function numerically, we enter the data from Figure 9 and the values from Figure 14 in Table 4. To compare them graphically, we press GRAPH to graph both the model and the scatter plot (Fig. 15).



› Figure 13



› Figure 14



› Figure 15

Table 4

x	0	5	10	15	20	25
Debt	58.5	128.9	234.8	443.2	663.8	848.3
$f(x)$	45.9	136.4	261.9	422.4	617.9	848.4

To estimate the debt in 2006 and 2010, we evaluate $f(x)$ at 26 and 30.

$$f(26) = 898.7 \qquad f(30) = 1{,}113.9$$

So the revolving-credit debt should be \$898.7 billion in 2006 and \$1,113.9 billion (or \$1.1139 trillion) in 2010.

Table 5

Year	Total Debt (Billions)
1980	44.1
1985	74.0
1990	91.6
1995	131.9
2000	184.4
2005	228.6

MATCHED PROBLEM 10

Credit union debt (in billions of dollars) in the United States is given in Table 5. Repeat Example 10 using these data and the modeling function

$$y = 0.15x^2 + 3.6x + 45.9$$

REMARKS:

1. Modeling functions like the function f in Example 10 provide reasonable and useful representations of the given data, but they don't always correctly predict future behavior. For example, the model in Example 10 indicates that the revolving-credit debt for 2006 should be about \$898.7 billion. But the actual debt for 2006 turned out to be \$875.2 billion, which differs from the predicted value by more than \$23 billion. (As the old government saying goes, "A billion here and a billion there, and pretty soon you're talking about some real money!") Getting the most out of mathematical modeling requires both an understanding of the techniques used to develop the model and frequent reevaluation, modification, and interpretation of the results produced by the model.
2. The tricky part in mathematical modeling is usually finding a function that models a given set of data. We will discuss methods for doing this in Chapter 2. It turns out that this is fairly easy to do with a graphing calculator.

A Brief History of the Function Concept

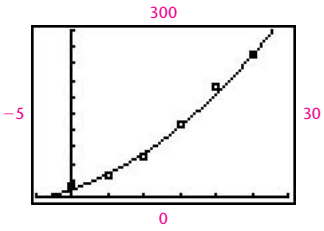
The history of the use of functions in mathematics illustrates the tendency of mathematicians to extend and generalize a concept. The word *function* appears to have been first used by Leibniz in 1694 to stand for any quantity associated with a curve. By 1718, Johann Bernoulli considered a function any expression made up of constants and a variable. Later in the same century, Euler came to regard a function as any equation made up of constants and variables. Euler made extensive use of the extremely important notation $f(x)$, although its origin is generally attributed to Clairaut (1734).

The form of the definition of function that had been used until well into the twentieth century (many texts still contain this definition) was formulated by Dirichlet (1805–1859). He stated that, if two variables x and y are so related that for each value of x there corresponds exactly one value of y, then y is said to be a (single-valued) function of x. He called x, the variable to which values are assigned at will, the independent variable, and y, the variable whose values depend on the values assigned to x, the dependent variable. He called the values assumed by x the domain of the function, and the corresponding values assumed by y the range of the function.

Now, because set concepts permeate almost all mathematics, we have the more general definition of function presented in this section in terms of sets of ordered pairs of elements.

ANSWERS TO MATCHED PROBLEMS

1. (A) Domain $= \{-2, -1, 0\}$, Range $= \{0, 1, 2\}$; S is not a function.
(B) Domain $= \{-2, -1, 0, 1, 2\}$, Range $= \{0, 1, 2\}$; T is a function.
2. (A) No (B) Yes **3.** $x \le 0$ **4.** (A) -3 (B) -2 (C) Does not exist (D) 1
5. All real numbers or $(-\infty, \infty)$
6. All real numbers except -1 and 1 or $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
7. $t \ge 5$ or $[5, \infty)$ **8.** $x \ge 0, x \ne 1$ or $[0, 1) \cup (1, \infty)$
9. (A) 17 (B) $h^2 + 7h + 17$ (C) $h + 7$ (D) $x^2 + 2xh + h^2 + 3x + 3h + 7$
(E) $2x + h + 3$



10.

x	0	5	10	15	20	25
Debt	44.1	74.0	91.6	131.9	184.4	228.6
$f(x)$	45.9	67.7	97.2	134.2	178.7	230.8

The credit union debt should be \$242.1 billion in 2006, and \$290.4 billion in 2010.

1-2 Exercises

1. True or False: Every relation is a function. Explain your answer.

2. True or False: Every function is a relation. Explain your answer.

3. Explain the difference between $f(x + h)$ and $f(x) + h$.

4. Explain the difference between $f(x)$, where f represents a function, and $2(x)$.

Indicate whether each relation in Problems 5–10 defines a function, then write each as a set of ordered pairs.

5.

Domain	Range
−1	1
0	2
1	3

6.

Domain	Range
2	1
4	3
6	5

7.

Domain	Range
1	3
3	5
5	7
	9

8.

Domain	Range
−1	0
−2	5
−3	8

9.

Domain	Range
−1	3
0	
1	
2	

10.

Domain	Range
2	8
3	
4	9
5	

In Problems 11–16, write the domain and range of each relation, then indicate whether the relation defines a function.

11. {(2, 4), (3, 6), (4, 8), (5, 10)}

12. {(−1, 4), (0, 3), (1, 2), (2, 1)}

13. {(10, −10), (5, −5), (0, 0), (5, 5), (10, 10)}

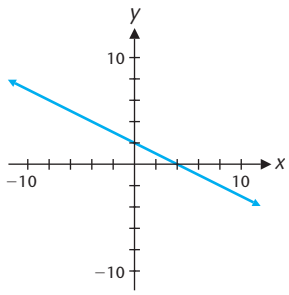
14. {(−10, 10), (−5, 5), (0, 0), (5, 5), (10, 10)}

15. {(0, 1), (1, 1), (2, 1), (3, 2), (4, 2), (5, 2)}

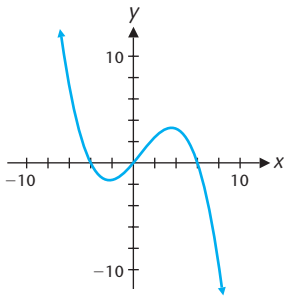
16. {(1, 1), (2, 1), (3, 1), (1, 2), (2, 2), (3, 2)}

**Indicate whether each graph in Problems 17–22 is the graph of a function.*

17.



18.



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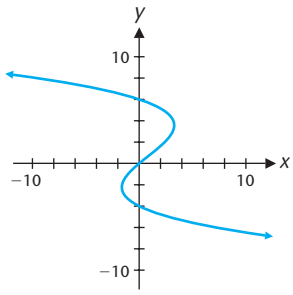

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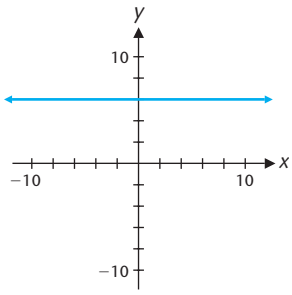
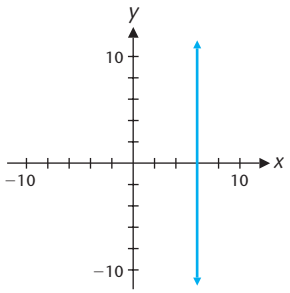
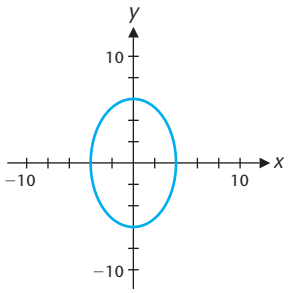


*The symbol denotes problems that require graphical interpretation.

21.



22.



In Problems 23 and 24, which of the indicated correspondences define functions? Explain.

23. Let W be the set of residents of Wisconsin and let R and S be the set of members of the U.S. House of Representatives and the set of members of the U.S. Senate, respectively, elected by the residents of Wisconsin.
(A) A resident corresponds to the congressperson representing the resident's congressional district.
(B) A resident corresponds to the senators representing the resident's state.

24. Let P be the set of patients in a hospital and let D be the set of doctors and N the set of nurses, respectively, that are on the staff of the hospital.
(A) A patient corresponds to a doctor if that doctor admitted the patient to the hospital.
(B) A patient corresponds to a nurse if that nurse cares for the patient.

In Problems 25–28, determine if the indicated equation defines a function. Justify your answer.

25. $x + y = 4$ **26.** $x^2 + y = 4$

27. $x + y^2 = 4$ **28.** $x^2 + y^2 = 4$

Problems 29–40 refer to the functions

$f(x) = 3x - 5$ $g(t) = 4 - t$

$F(m) = 3m^2 + 2m - 4$ $G(u) = u - u^2$

Evaluate as indicated.

29. $f(-1)$ **30.** $g(6)$

31. $G(-2)$ **32.** $F(-3)$

33. $F(-1) + f(3)$ **34.** $G(2) - g(-3)$

35. $2F(-2) - G(-1)$ **36.** $3G(-2) + 2F(-1)$

37. $\dfrac{f(0) \cdot g(-2)}{F(-3)}$ **38.** $\dfrac{g(4) \cdot f(2)}{G(1)}$

39. $\dfrac{f(4) - f(2)}{2}$ **40.** $\dfrac{g(5) - g(3)}{2}$

In Problems 41–56, find the domain of the indicated function. Express answers informally using inequalities, then formally using interval notation.

41. $f(x) = 4 - 9x + 3x^2$ **42.** $g(x) = 1 + 7x - x^2$

43. $h(z) = \dfrac{2}{4 - z}$ **44.** $k(z) = \dfrac{z}{z - 3}$

45. $g(t) = \sqrt{t - 4}$ **46.** $h(t) = \sqrt{6 - t}$

47. $k(w) = \sqrt{7 + 3w}$ **48.** $j(w) = \sqrt{9 + 4w}$

49. $H(u) = \dfrac{u}{u^2 + 4}$ **50.** $G(u) = \dfrac{u}{u^2 - 4}$

51. $L(v) = \dfrac{v + 2}{v^2 - 16}$ **52.** $K(v) = \dfrac{v + 8}{v^2 + 16}$

53. $M(x) = \dfrac{\sqrt{x + 4}}{x - 1}$ **54.** $N(x) = \dfrac{\sqrt{x - 3}}{x + 2}$

55. $s(t) = \dfrac{1}{3 - \sqrt{t}}$ **56.** $r(t) = \dfrac{1}{\sqrt{t} - 4}$

In Problems 57–60, find a function f that makes all three equations true. [Hint: There may be more than one possible answer, but there is one obvious answer suggested by the pattern illustrated in the equations.]

57. $f(1) = 2(1) - 3$ **58.** $f(1) = 5(1)^2 - 6$
$f(2) = 2(2) - 3$ $f(2) = 5(2)^2 - 6$
$f(3) = 2(3) - 3$ $f(3) = 5(3)^2 - 6$

59. $f(1) = 4(1)^2 - 2(1) + 9$
$f(2) = 4(2)^2 - 2(2) + 9$
$f(3) = 4(3)^2 - 2(3) + 9$

60. $f(1) = -8 + 5(1) - 2(1)^2$
$f(2) = -8 + 5(2) - 2(2)^2$
$f(3) = -8 + 5(3) - 2(3)^2$

61. If $F(s) = 3s + 15$, find $\dfrac{F(2 + h) - F(2)}{h}$.

62. If $K(r) = 7 - 4r$, find $\dfrac{K(1 + h) - K(1)}{h}$.

63. If $g(x) = 2 - x^2$, find $\dfrac{g(3 + h) - g(3)}{h}$.

64. If $P(m) = 2m^2 + 3$, find $\dfrac{P(2 + h) - P(2)}{h}$.

65. If $L(w) = -2w^2 + 3w - 1$, find $\dfrac{L(-2 + h) - L(-2)}{h}$.

66. If $D(p) = -3p^2 - 4p + 9$, find $\dfrac{D(-1+h) - D(-1)}{h}$.

The verbal statement "function f multiplies the square root of the domain element by 2 and then subtracts 5" and the algebraic statement $f(x) = 2\sqrt{x} - 5$ define the same function. In Problems 67–70, translate each verbal definition of the function into an algebraic definition.

67. Function g multiplies the domain element by 3 and then adds 1.

68. Function f multiplies the domain element by 7 and then adds the product of 5 and the cube of the domain element.

69. Function F divides the domain element by the sum of 8 and the square root of the domain element.

70. Function G takes the square root of the sum of 4 and the square of the domain element.

In Problems 71–74, translate each algebraic definition of the function into a verbal definition.

71. $f(x) = 2x - 3x^2$ **72.** $g(x) = 5x^3 - 8x$

73. $F(x) = \sqrt{x^4 + 9}$ **74.** $G(x) = \dfrac{x}{3x - 6}$

In Problems 75–78, use the given information to write a verbal description of the function f and then find the equation for f(x).

75. $f(x + h) = 2(x + h)^2 - 4(x + h) + 6$

76. $f(x + h) = -7(x + h)^2 + 8(x + h) + 5$

77. $f(x + h) = 4(x + h) - 3\sqrt{x + h} + 9$

78. $f(x + h) = 2\sqrt[3]{x + h} - 6(x + h) - 5$

In Problems 79–86, for each given function f find and simplify:

(A) $\dfrac{f(x+h) - f(x)}{h}$ (B) $\dfrac{f(x) - f(a)}{x - a}$

79. $f(x) = 3x - 4$ **80.** $f(x) = -2x + 5$

81. $f(x) = x^2 - 1$ **82.** $f(x) = x^2 + x - 1$

83. $f(x) = -3x^2 + 9x - 12$ **84.** $f(x) = -x^2 - 2x - 4$

85. $f(x) = x^3$ **86.** $f(x) = x^3 + x$

In Problems 87 and 88, the domain of the function f is all real numbers x except $x = 1$. Describe the behavior of f for x values very close to 1, but not equal to 1. Support your conclusions with information obtained by exploring the graph of f near $x = 1$, by examining the numerical values of f near $x = 1$, and by algebraically simplifying the expression used to define f.

87. $f(x) = \dfrac{x^2 - 1}{x - 1}$ **88.** $f(x) = \dfrac{x^3 - 1}{x - 1}$

APPLICATIONS

89. BOILING POINT OF WATER At sea level, water boils when it reaches a temperature of 212°F. At higher altitudes, the atmospheric pressure is lower and so is the temperature at which water boils. The boiling point $B(x)$ in degrees Fahrenheit at an altitude of x feet is given approximately by

$$B(x) = 212 - 0.0018x$$

(A) Complete the following table.

x	0	5,000	10,000	15,000	20,000	25,000	30,000
$B(x)$							

(B) Based on the information in the table, write a brief verbal description of the relationship between altitude and the boiling point of water.

90. AIR TEMPERATURE As dry air moves upward, it expands and cools. The air temperature $A(x)$ in degrees Celsius at an altitude of x kilometers is given approximately by

$$A(x) = 25 - 9x$$

(A) Complete the following table.

x	0	1	2	3	4	5
$A(x)$						

(B) Based on the information in the table, write a brief verbal description of the relationship between altitude and air temperature.

91. PHYSICS—RATE The distance in feet that an object falls in the absence of air resistance is given by $s(t) = 16t^2$, where t is time in seconds.

(A) Find $s(0)$, $s(1)$, $s(2)$, and $s(3)$.

(B) Find and simplify $\dfrac{s(2+h) - s(2)}{h}$.

(C) Evaluate the expression in part B for $h = \pm 1, \pm 0.1, \pm 0.01, \pm 0.001$.

(D) What happens in part C as h gets closer and closer to 0? What do you think this tells us about the motion of the object? [*Hint:* Think about what each of the numerator and denominator represents.]

92. PHYSICS—RATE An automobile starts from rest and travels along a straight and level road. The distance in feet traveled by the automobile is given by $s(t) = 10t^2$, where t is time in seconds.
(A) Find $s(8)$, $s(9)$, $s(10)$, and $s(11)$.
(B) Find and simplify $\dfrac{s(11 + h) - s(11)}{h}$.
(C) Evaluate the expression in part B for $h = \pm 1, \pm 0.1, \pm 0.01, \pm 0.001$.
(D) What happens in part C as h gets closer and closer to 0? What do you think this tells us about the motion of the object? [*Hint:* Think about what each of the numerator and denominator represents.]

93. CAR RENTAL A car rental agency computes daily rental charges for compact cars with the function

$$D(x) = 20 + 0.25x$$

where $D(x)$ is the daily charge in dollars and x is the daily mileage. Translate this algebraic statement into a verbal statement that can be used to explain the daily charges to a customer.

94. INSTALLATION CHARGES A telephone store computes charges for phone installation with the function

$$S(x) = 15 + 0.7x$$

where $S(x)$ is the installation charge in dollars and x is the time in minutes spent performing the installation. Translate this algebraic statement into a verbal statement that can be used to explain the installation charges to a customer.

MODELING AND DATA ANALYSIS

Table 6 contains the average price of admission (in dollars) to a motion picture and the total box office gross (in millions of dollars) for all theaters in the United States.

Table 6 Selected Financial Data for the Motion Picture Industry

Year	1997	1999	2001	2003	2005
Average Price of Admission ($)	4.59	5.08	5.66	6.03	6.41
Box Office Gross ($ in millions)	6,360	7,450	8,410	9,490	8,990

95. DATA ANALYSIS A mathematical model for the average price of admission to a motion picture is

$$A(t) = 0.23t + 4.6$$

where t represents time in years and $t = 0$ corresponds to 1997.
(A) Compare the model and the data graphically and algebraically.
(B) Estimate (to the nearest cent) the average price of admission in 2006 and 2007.

96. REVENUE ANALYSIS A mathematical model for the total box office gross is given by

$$G(t) = -54.6t^2 + 802t + 6240$$

where t represents time in years and $t = 0$ corresponds to 1997.
(A) Compare the model and the data graphically and algebraically.
(B) Estimate (to three significant digits) the total box office gross in 2006 and 2007.

Merck & Co., Inc. is the world's largest pharmaceutical company. Problems 97–100 refer to the data in Table 7 taken from the company's 2005 annual report.

Table 7 Selected Financial Data for Merck & Co., Inc.

($ in Billions)	1997	1999	2001	2003	2005
Sales	14.0	17.3	21.2	22.5	22.0
R & D Expenses	1.7	2.1	2.5	3.2	3.8
Net Income	4.6	5.9	7.3	6.8	4.6

97. SALES ANALYSIS A mathematical model for Merck's sales is given by

$$S(t) = -0.18t^2 + 2.5t + 14$$

where t is time in years and $t = 0$ corresponds to 1997.
(A) Compare the model and the data graphically and numerically.
(B) Estimate (to two significant digits) Merck's sales in 2006 and in 2008.
(C) Write a brief verbal description of Merck's sales from 1997 to 2005.

98. INCOME ANALYSIS A mathematical model for Merck's net income is given by

$$I(t) = -0.16t^2 + 1.3t + 4.4$$

where t is time in years and $t = 0$ corresponds to 1997.
(A) Compare the model and the data graphically and numerically.
(B) Estimate (to two significant digits) Merck's net income in 2006 and in 2008.
(C) Write a brief verbal description of Merck's net income from 1997 to 2005.

99. **RESEARCH AND DEVELOPMENT ANALYSIS** A mathematical model for Merck's sales as a function of research and development (R & D) expenses is given by

$$S(r) = -3.54r^2 + 23.3r - 15.5$$

where r represents R & D expenditures.
(A) Compare the model and the data graphically and numerically.
(B) Estimate (to two significant digits) Merck's sales if the company spends \$1.2 billion on research and development and if the company spends \$4.2 billion.

100. **RESEARCH AND DEVELOPMENT ANALYSIS** A mathematical model for Merck's net income as a function of R & D expenses is given by

$$I(r) = -2.47r^2 + 13.7r - 11.6$$

where r represents R & D expenditures.
(A) Compare the model and the data graphically and numerically.
(B) Estimate (to two significant digits) Merck's net income if the company spends \$1.2 billion on research and development and if the company spends \$4.2 billion.

1-3 Functions: Graphs and Properties

- Basic Concepts
- Identifying Increasing and Decreasing Functions
- Finding Local Maxima and Minima
- Mathematical Modeling
- Defining Functions Piecewise

One of the key goals of this course is to provide you with a set of mathematical tools that can be used to analyze graphs. In many cases, these graphs will arise naturally from real-world situations. In fact, studying functions by analyzing their graphs is one of the biggest reasons that a graphing calculator is useful. In this section, we will discuss some basic concepts that are commonly used to describe graphs of functions.

Basic Concepts

In the previous section, we saw that one way to describe a function is in terms of ordered pairs. Based on your earlier experience with graphing, this definition of function may have reminded you of points on a graph, which are also described with an ordered pair of numbers. This simple connection between graphs and functions is the basis for the study of graphs of functions.

Every function that has a real-number domain and range has a graph, which is simply a pictorial representation of the ordered pairs of real numbers that make up the function. When functions are graphed, domain values are usually associated with the horizontal axis and range values with the vertical axis. In this regard, the **graph of a function *f*** is the same as the graph of the equation $y = f(x)$, where x is the independent variable and the first coordinate, or **abscissa,** of a point on the graph of f. The variables y and $f(x)$ can both be used to represent the dependent variable, and either one is the second coordinate, or **ordinate,** of a point on the graph of f (Fig. 1).

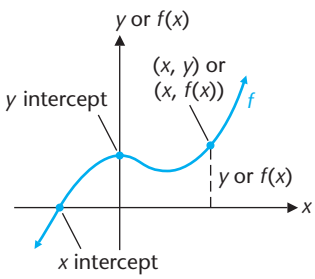


Figure 1 Graph of a function.

To reflect this typical usage of variables x and y, we often refer to the abscissa of a point as the ***x* coordinate,** and the ordinate as the ***y* coordinate.**

An x coordinate of a point where the graph of a function intersects the x axis is called an ***x* intercept** of the function.

Since the height of the graph, and consequently the value of the function, at such a point is zero, the x intercepts are often referred to as **real zeros** of the function. They are also solutions or **roots** of the equation $f(x) = 0$.

The y coordinate of a point where the graph of a function crosses the y axis is called the ***y* intercept** of the function. The y intercept is given by $f(0)$, provided 0 is in the domain of f. Note that a function can have more than one x intercept but can never have more than one y intercept—otherwise it would fail the vertical line test discussed in the last section and consequently fail to be a function.

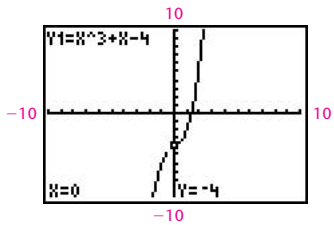
In the first section of this chapter, we solved equations of the form $f(x) = c$ with a graphing calculator by graphing both sides of the equation and using the INTERSECT command. Most graphing calculators also have a **ZERO** or **ROOT** command that finds the x intercepts of a function directly from the graph of the function. Example 1 illustrates the use of this command.

EXAMPLE 1 Finding x and y Intercepts

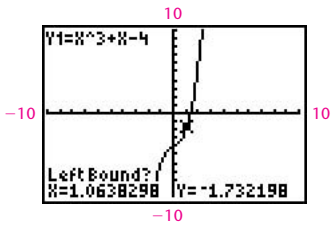
Find the x and y intercepts (correct to three decimal places) of $f(x) = x^3 + x - 4$.

SOLUTION

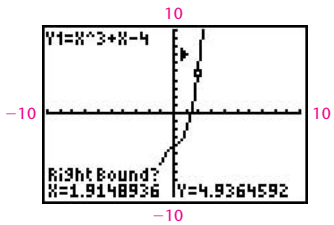
The y intercept occurs at the point where $x = 0$, and from the graph of f in Figure 2, we see that it is $f(0) = -4$. We also see that there appears to be an x intercept between $x = 1$ and $x = 2$. We will use the ZERO command to find this intercept. First we are asked to select a **left bound** (Fig. 3). This is a value of x to the left of the x intercept. Next we are asked to find a **right bound** (Fig. 4). This is a value of x to the right of the x intercept. If a function has more than one x intercept, you should select the left and right bounds so that there is only one intercept between the bounds.



› Figure 2

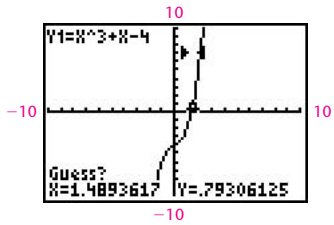


› Figure 3

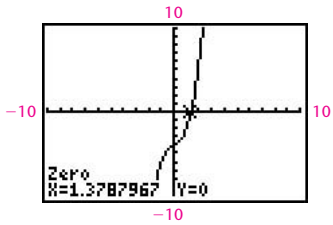


› Figure 4

Finally, we are asked to select a **guess.** The guess must be between the bounds and should be close to the intercept (Fig. 5). Figure 6 shows that the x intercept (to three decimal places) is 1.379.



› Figure 5



› Figure 6

MATCHED PROBLEM 1

Find the x and y intercepts (correct to three decimal places) of $f(x) = x^3 + x + 5$.

››› EXPLORE-DISCUSS 1

Let $f(x) = x^2 - 2x - 5$.

(A) Use the ZERO command on a graphing calculator to find the x intercepts of f.

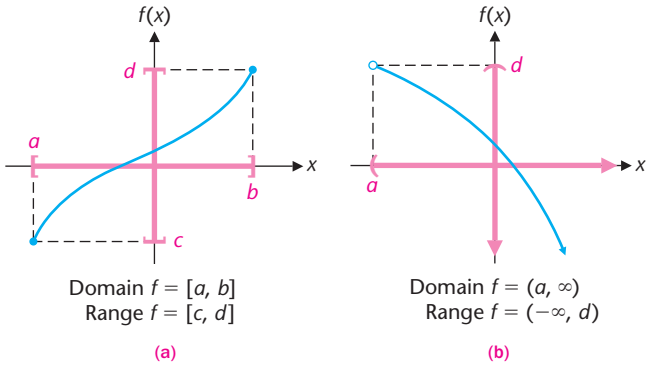
(B) Find all solutions to the equation $x^2 - 2x - 5 = 0$.

(C) Discuss the differences between the graph of f, the x intercepts, and the solutions to the equation $f(x) = 0$.

The domain of a function is the set of all the x coordinates of points on the graph of the function and the range is the set of all the y coordinates. It is very useful to view the domain and range as subsets of the coordinate axes as in Figure 7. Note the effective use of interval notation* in describing the domain and range of the functions in this figure. In Figure 7(a) a solid dot is used to indicate that a point is on the graph of the function and in Figure 7(b) an open dot to indicate that

*Interval notation is reviewed in Appendix B, Section B-1.

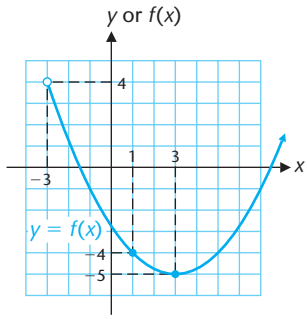
a point is not on the graph of the function. An open or solid dot at the end of a graph indicates that the graph terminates there, whereas an arrowhead indicates that the graph continues indefinitely beyond the portion shown with no significant changes of direction [see Fig. 7(b) and note that the arrowhead indicates that the domain extends infinitely far to the right, and the range extends infinitely far downward].



› **Figure 7** Domain and range.

EXAMPLE 2 Finding the Domain and Range from a Graph

(A) Find the domain and range of the function f whose graph is shown in Figure 8.
(B) Find $f(1)$, $f(3)$, and $f(5)$.



› **Figure 8**

SOLUTIONS

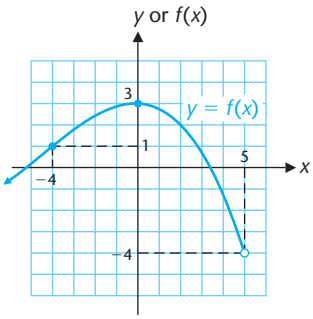
(A) The dot at the left end of the graph indicates that the graph terminates at that point, while the arrowhead on the right end indicates that the graph continues infinitely far to the right. So the x coordinates on the graph go from -3 to ∞. The open dot at $(-3, 4)$ indicates that -3 is not in the domain of f.

$$\text{Domain: } -3 < x < \infty \quad \text{or} \quad (-3, \infty)$$

The least y coordinate on the graph is -5, and there is no greatest y coordinate. (The arrowhead tells us that the graph continues infinitely far upward.) The closed dot at $(3, -5)$ indicates that -5 is in the range of f.

$$\text{Range: } -5 \le y < \infty \quad \text{or} \quad [-5, \infty)$$

(B) The point on the graph with x coordinate 1 is $(1, -4)$, so $f(1) = -4$. Likewise, $(3, -5)$ and $(5, -4)$ are on the graph, so $f(3) = -5$ and $f(5) = -4$.



› Figure 9

MATCHED PROBLEM 2

(A) Find the domain and range of the function f given by the graph in Figure 9.

(B) Find $f(-4)$, $f(0)$, and $f(2)$.

››› CAUTION †††

When using interval notation to describe domain and range, make sure that you always write the least number first! You should find the domain by working left to right along the x axis, and find the range by working bottom to top along the y axis.

› Identifying Increasing and Decreasing Functions

››› EXPLORE-DISCUSS 2

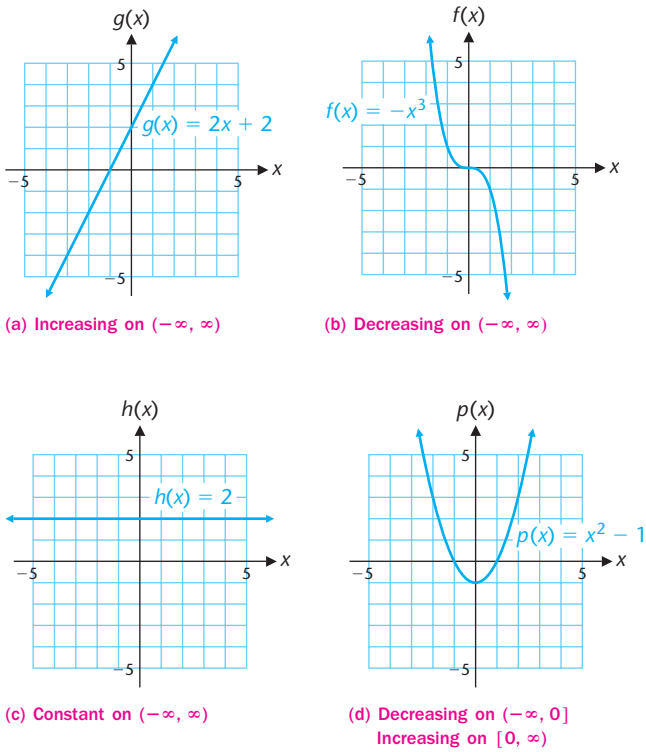
Graph each function in the standard viewing window, then write a verbal description of the behavior exhibited by the graph as x moves from left to right.

(A) $f(x) = 2 - x$ **(B)** $f(x) = x^3$

(C) $f(x) = 5$ **(D)** $f(x) = 9 - x^2$

We will now take a look at *increasing* and *decreasing* properties of functions. Informally, a function is increasing over an interval if its graph rises as the x coordinate

increases (moves from left to right) over that interval. A function is decreasing over an interval if its graph falls as the x coordinate increases over that interval. A function is *constant* on an interval if its graph is horizontal over that interval (Fig. 10).



(a) Increasing on $(-\infty, \infty)$

(b) Decreasing on $(-\infty, \infty)$

(c) Constant on $(-\infty, \infty)$

(d) Decreasing on $(-\infty, 0]$
Increasing on $[0, \infty)$

› **Figure 10** Increasing, decreasing, and constant functions.

More formally, we define increasing, decreasing, and constant functions as follows:

› **DEFINITION 1 Increasing, Decreasing, and Constant Functions**

Let I be an interval in the domain of function f. Then,

1. f is **increasing** on I and the graph of f is **rising** on I if $f(a) < f(b)$ whenever $a < b$ in I.
2. f is **decreasing** on I and the graph of f is **falling** on I if $f(a) > f(b)$ whenever $a < b$ in I.
3. f is **constant** on I and the graph of f is **horizontal** on I if $f(a) = f(b)$ whenever $a < b$ in I.

Refer to Figure 10(a) on page 51. As x moves from left to right, the values of g increase and the graph of g rises. In Figure 10(b), as x moves from left to right, the values of f decrease and the graph of f falls.

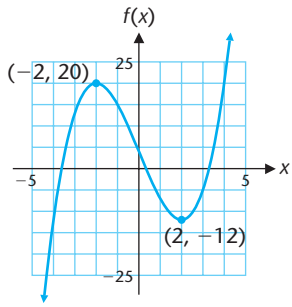
In Figure 10(c), the value of f doesn't change (remains constant), and the graph stays at the same height. In Figure 10(d), moving from left to right, the graph falls as x increases from $-\infty$ to 0, then rises from 0 to ∞.

EXAMPLE 3 Describing a Graph

The graph of

$$f(x) = x^3 - 12x + 4$$

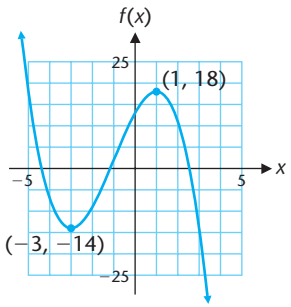
is shown in Figure 11. Use the terms *increasing, decreasing, rising,* and *falling* to write a verbal description of this graph.



› **Figure 11** $f(x) = x^3 - 12x + 4$.

SOLUTION

The values of f increase and the graph of f rises as x increases from $-\infty$ to -2. The values of f decrease and the graph of f falls as x increases from -2 to 2. Finally, the values of f increase and the graph of f rises as x increases from 2 to ∞.



› **Figure 12**

MATCHED PROBLEM 3

The graph of

$$f(x) = -x^3 - 3x^2 + 9x + 13$$

is shown in Figure 12. Use the terms *increasing, decreasing, rising,* and *falling* to write a verbal description of this graph.

››› CAUTION ›››

The arrow on the left edge of the graph in Figure 11 does not indicate that the graph is "moving" downward. It simply tells us that there are no significant changes in direction for x values less than -5. The graph is increasing on that portion.

› Finding Local Maxima and Minima

The graph of $f(x) = x^3 - 12x + 4$ from Example 3 (displayed in Figure 11) will help us to define some very important terms. Notice that at the point $(-2, 20)$, the function changes from increasing to decreasing. This means that the function value $f(-2) = 20$ is greater than any of the nearby values of the function. We will call such a point a *local maximum*. At the point $(2, -12)$, the function changes from decreasing to increasing. This means that the function value $f(2) = -12$ is less than any nearby values of the function. We will call such a point a *local minimum*.

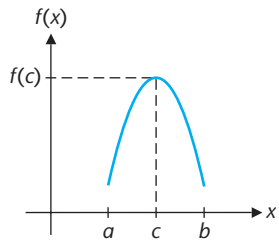
Local maxima and minima* play a crucial role in the study of graphs of functions. For many graphs, they are the key points that determine the shape of the graph. We will also see that maxima and minima are very useful in application problems. When a function represents some quantity of interest (the profit made by a business, for example), finding the largest or smallest that quantity can get is usually very helpful.

The concepts of local maxima and minima are made more formal in the following definition:

› DEFINITION 2 Local Maxima and Local Minima

The function value $f(c)$ is called a **local maximum** if there is an interval (a, b) containing c such that

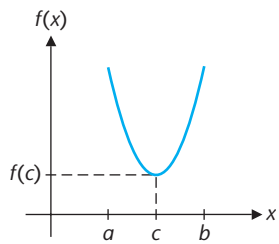
$$f(x) \le f(c) \text{ for all } x \text{ in } (a, b)$$



Local maximum

The function value $f(c)$ is called a **local minimum** if there is an interval (a, b) containing c such that

$$f(x) \ge f(c) \text{ for all } x \text{ in } (a, b)$$



Local minimum

The function value $f(c)$ is called a **local extremum** if it is either a local maximum or a local minimum.

*Maxima and minima are the plural forms of maximum and minimum, respectively.

››› EXPLORE-DISCUSS 3

Plot the points $A = (0, 0)$, $B = (3, 4)$, $C = (7, 1)$, and $D = (10, 5)$ in a coordinate plane. Draw a curve that satisfies each of the following conditions.

(A) Passes through A and B and is always increasing.

(B) Passes through A, B, and C with a local maximum at $x = 3$.

(C) Passes through A, B, C, and D with a local maximum at $x = 3$ and a local minimum at $x = 7$.

What does this tell you about the connection between local extrema and increasing/decreasing properties of functions?

Since finding maximum or minimum values of functions is so important, most graphing calculators have commands that approximate local maxima and minima. Examples 4 and 5 illustrate the use of these commands.

EXAMPLE 4 Finding Local Extrema

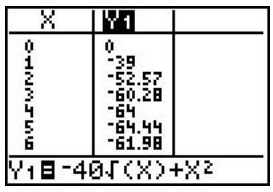
Find the domain, any local extrema, and the range of

$$f(x) = x^2 - 40\sqrt{x}$$

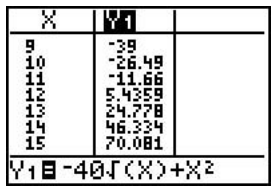
Round answers to two decimal places.

SOLUTION

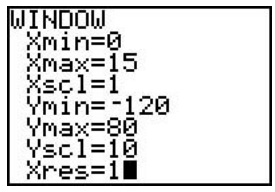
Because $\sqrt{x}$ represents a real number only if $x \geq 0$, the domain of f is $[0, \infty)$. First we must select a viewing window. Because the domain of f is $[0, \infty)$, we choose Xmin $= 0$. We will construct a table of values on a graphing calculator (Figs. 13 and 14) to help select the remaining window variables. From Figure 13 we see that Ymin should be less than -64.44. Figure 14 indicates that Xmax $= 15$ and Ymax greater than 70.081 should produce a reasonable view of the graph. Our choice for the window variables is shown in Figure 15.



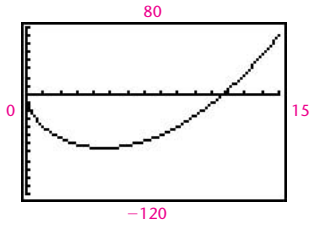
› Figure 13



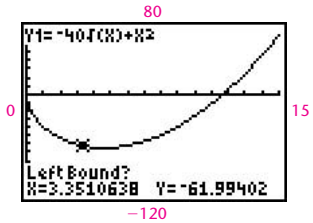
› Figure 14



› Figure 15



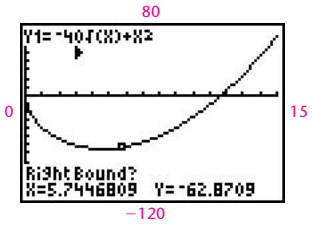
› Figure 16



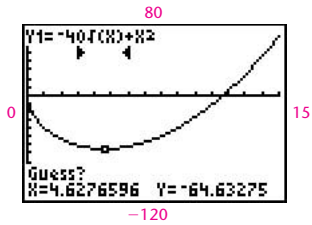
› Figure 17

The graph of f is shown in Figure 16. Notice that we adjusted Ymin to provide space at the bottom of the screen for the text that the graphing calculator will display.

Both the table in Figure 13 and the graph in Figure 16 indicate that f has a local minimum near $x = 5$. After selecting the **MINIMUM** command on our graphing calculator, we are asked to select a left bound (Fig. 17), a right bound (Fig. 18), and a guess (Fig. 19). Note the arrowheads that mark the right and left boundaries.

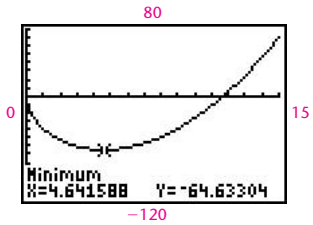


› Figure 18

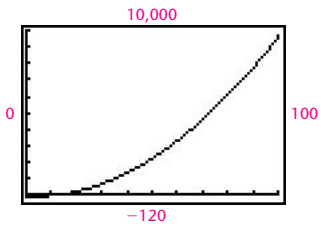


› Figure 19

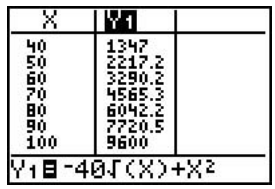
The final graph (Fig. 20) shows that, to two decimal places, f has a local minimum value of -64.63 at $x = 4.64$. The curve in Figure 20 suggests that as x increases to the right without bound, the values of $f(x)$ also increase without bound. The graph in Figure 21 and the table in Figure 22 support this suggestion. Thus, we conclude that there are no other local extrema and that the range of f is $[-64.63, \infty)$.



› Figure 20



› Figure 21



› Figure 22

Summarizing our results, we have

$$\text{Domain of } f = [0, \infty)$$
$$\text{Range of } f = [-64.63, \infty)$$
$$\text{Local minimum: } f(4.64) = -64.63$$

MATCHED PROBLEM 4

Find the domain, any local extrema, and the range of

$$f(x) = x - 5\sqrt{x}$$

Round answers to two decimal places.

In Example 4, we used numerical and graphical evidence to conclude that f continues to increase as x increases to the right. It turns out that calculus techniques are required to be absolutely certain. Without calculus, we have to rely on intuitive arguments involving graphical and numerical techniques, as we did in Example 4. As we broaden our experience and become familiar with a larger variety of functions and their graphs, we'll be able to draw conclusions with a greater degree of certainty. This is one of the major objectives of this book.

› Mathematical Modeling

In Example 5, we use the MAXIMUM command to find the maximum value of a revenue function.

EXAMPLE 5 Maximizing Revenue

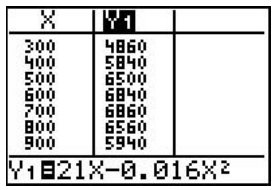
The revenue (in dollars) from the sale of x bicycle locks is given by

$$R(x) = 21x - 0.016x^2 \qquad 0 \le x \le 1{,}300$$

Find the number of locks that must be sold to maximize the revenue. What is the maximum revenue, to the nearest dollar?

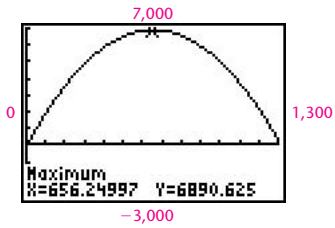
SOLUTION

We begin by entering the revenue function as Y_1 and constructing a table of values for the revenue (Fig. 23). From the limits given in the problem, we select Xmin = 0 and Xmax = 1,300. The table in Figure 23 suggests that Ymax = 7,000 is a good choice and, as before, we select Ymin so the text displayed by the graphing calculator does not cover any important portions of the graph. We enter the window variables (Fig. 24) and use the MAXIMUM command (Fig. 25). (The MAXIMUM command works just like the MINIMUM command used in Example 4. The details of selecting the bounds and the initial guess are omitted.)

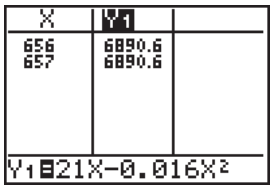


› Figure 23

› Figure 24



› Figure 25



› Figure 26

The results in Figure 25 show that, to two decimal places, the maximum revenue is \$6,890.63 when $x = 656.25$ locks. But this cannot be the answer to the problem. It is not reasonable to sell one-fourth of a lock. Examining the values of R at $x = 656$ and $x = 657$ (Fig. 26), we conclude that the maximum revenue, to the nearest dollar, is \$6,891 when either 656 locks or 657 locks are sold. ◉

MATCHED PROBLEM 5

The profit (in dollars) from the sale of x T-shirts is given by

$$P(x) = 17.5x - 0.016x^2 - 2{,}000 \qquad 0 \le x \le 1{,}300$$

Find the number of shirts that must be sold to maximize the profit. What is the maximum profit, to the nearest dollar?

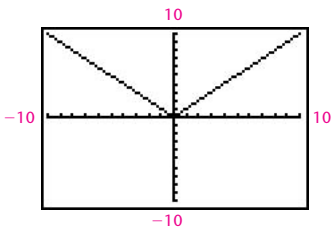
Example 5 illustrates an important step in the mathematical modeling process. Solutions obtained from a model must be interpreted in terms of the original real-world problem. In the case of Example 5, the revenue function R is defined only for integer values of x, $x = 0, 1, 2, \ldots, 1{,}300$. However, for the purposes of mathematical analysis and as an aid in visualizing the behavior of the function R, we assume that the revenue function is defined for all x, $0 \le x \le 1{,}300$. After finding that the maximum value of the revenue function occurs at $x = 656.25$, we must remember to interpret this solution to mean either $x = 656$ or $x = 657$.

› Defining Functions Piecewise

You probably think of finding the absolute value of a number as a simple process: do nothing if the number is zero or greater, and throw away the negative sign if it's less than zero. In fact, we can think of absolute value as a function, and define it by a pair of formulas.

$$f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \ge 0 \end{cases} \quad \begin{matrix} \text{For example, } |-4| = -(-4) = 4 \\ \text{For example, } |3| = 3 \end{matrix}$$

The output is the same as the input (x) for $x \ge 0$, and has the opposite sign as the input ($-x$) for $x < 0$.



› **Figure 27** Graph of $f(x) = |x| = \text{abs}(x)$.

The graph of $|x|$ is shown in Figure 27. Most graphing calculators use abs or ABS to denote this function, and the graph is produced directly using $y_1 = \text{abs}(x)$.

Obviously, the absolute value function is defined by different formulas for different parts of its domain. Functions whose definitions involve more than one formula are called **piecewise-defined functions.** Notice that the graph of the absolute value function has a **sharp corner** at (0, 0), a common characteristic of piecewise-defined functions. As Example 6 illustrates, piecewise-defined functions occur naturally in many applications.

EXAMPLE 6 **Rental Charges**

A car rental agency charges \$0.25 per mile if the mileage does not exceed 100. If the total mileage exceeds 100, the agency charges \$0.25 per mile for the first 100 miles and \$0.15 per mile for any additional mileage.

(A) If x represents the number of miles a rented vehicle is driven, express the mileage charge $C(x)$ as a function of x.

(B) Complete the following table.

x	0	50	100	150	200
$C(x)$					

(C) Sketch the graph of $y = C(x)$ by hand, using a graphing calculator as an aid, and indicate the points in the table on the graph with solid dots.

SOLUTIONS

(A) If $0 \le x \le 100$, then

$$C(x) = 0.25x \quad \text{25 cents per mile up to a hundred}$$

If $x > 100$, then $x - 100$ represents the portion of the mileage over 100, and $0.15(x - 100)$ is the charge for this mileage. So if $x > 100$,

Charge for the first 100 miles; Charge for the additional mileage

$$\begin{aligned} C(x) &= 0.25(100) + 0.15(x - 100) \\ &= 25 + 0.15x - 15 \\ &= 10 + 0.15x \end{aligned}$$

Thus, we see that C is a piecewise-defined function:

$$C(x) = \begin{cases} 0.25x & \text{if } 0 \le x \le 100 \\ 10 + 0.15x & \text{if } x > 100 \end{cases}$$

(B) Piecewise-defined functions are evaluated by first determining which formula applies and then using the appropriate formula to find the value of the function. To begin, we enter both formulas in a graphing calculator and use the TABLE command (Fig. 28). To complete the table, we use the values of $C(x)$ from the y_1 column if $0 \le x \le 100$, and from the y_2 column if $x > 100$. In essence, we are evaluating f using the top formula when $x \le 100$, and the bottom formula when $x > 100$.

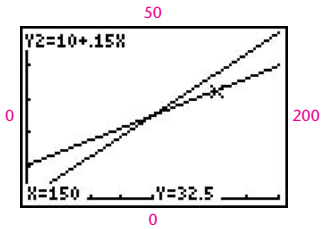
x	0	50	100	150	200
$C(x)$	\$0	\$12.50	\$25	\$32.50	\$40

(C) Using a graph of both rules in the same viewing window as an aid (Fig. 29), we sketch the graph of $y = C(x)$ and add the points from the table to produce Figure 30. ◉

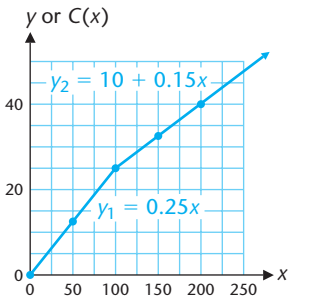
X	Y1	Y2
0	0	10
50	12.5	17.5
100	25	25
150	37.5	32.5
200	50	40
250	62.5	47.5
300	75	55

Y1 = .25X

› **Figure 28**
$y_1 = 0.25x$, $y_2 = 10 + 0.15x$.



› **Figure 29**
$y_1 = 0.25x$, $y_2 = 10 + 0.15x$.



Notice the sharp corner at (100, 25).

› **Figure 30** Hand sketch of the graph of $y = C(x)$.

MATCHED PROBLEM 6

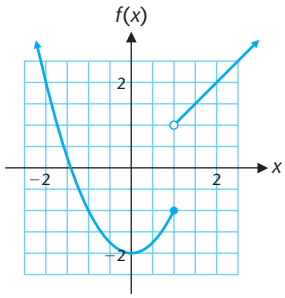
Another car rental agency charges \$0.30 per mile when the total mileage does not exceed 75, and \$0.30 per mile for the first 75 miles plus \$0.20 per mile for the additional mileage when the total mileage exceeds 75.

(A) If x represents the number of miles a rented vehicle is driven, express the mileage charge $C(x)$ as a function of x.

(B) Complete the following table.

x	0	50	75	100	150
$C(x)$					

(C) Sketch the graph of $y = C(x)$ by hand, using a graphing calculator as an aid, and indicate the points in the table on the graph with solid dots.

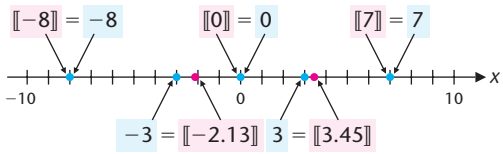


› **Figure 31** Graph of $f(x) = \begin{cases} x^2 - 2 & \text{for } x \le 1 \\ x & \text{for } x > 1 \end{cases}$.

Refer to Figures 28 and 30 in the solution to Example 6. Notice that the two formulas in the definition of C produce the same value at $x = 100$ and that the graph of C contains no breaks. Informally, a graph (or portion of a graph) is said to be **continuous** if it contains no breaks or gaps and can be drawn without lifting a pen from the paper. A graph is **discontinuous** at any points where there is a break or a gap. For example, the graph of the function in Figure 31 is discontinuous at $x = 1$. The entire graph cannot be drawn without lifting a pen from the paper. (A formal presentation of continuity can be found in calculus texts.)

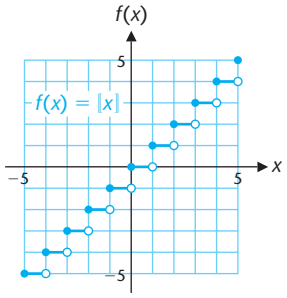
We conclude this section with a discussion of a particular piecewise-defined function that plays in important role in computer science. It is called the *greatest integer function.*

The greatest integer of a real number x, denoted by $[\![x]\!]$ is the largest integer that is less than or equal to x. That is, $[\![x]\!]$ is the integer n such that $n \le x < n + 1$ (Fig. 32).



› **Figure 32**

The **greatest integer function** f is defined by the equation $f(x) = [\![x]\!]$. A piecewise definition of f for $-2 \le x < 3$ is shown on page 60 and a sketch of the graph of f for $-5 \le x \le 5$ is shown in Figure 33. Since the domain of f is all real numbers, the piecewise definition continues indefinitely in both directions, as does the stairstep pattern in the figure. Thus, the range of f is the set of all integers.



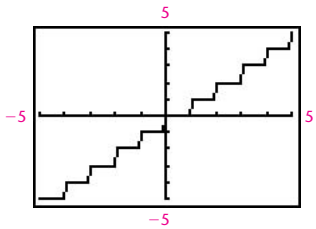
› **Figure 33** Greatest integer function.

$$f(x) = [\![x]\!] = \begin{cases} \vdots & \\ -2 & \text{if } -2 \le x < -1 \\ -1 & \text{if } -1 \le x < 0 \\ 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x < 2 \\ 2 & \text{if } 2 \le x < 3 \\ \vdots & \end{cases}$$

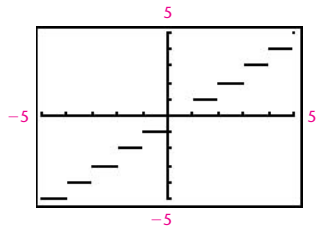
Notice in Figure 33 that at each integer value of x there is a break in the graph, and between integer values of x there is no break. Thus, the greatest integer function is discontinuous at each integer n and continuous on each interval of the form $[n, n + 1)$.

Most graphing calculators will graph the greatest integer function, usually denoted by int, but these graphs require careful interpretation. Comparing the sketch of $y = [\![x]\!]$ in Figure 33 with the graph of $y = \text{int}(x)$ in Figure 34(a), we see that the graphing calculator has connected the endpoints of the horizontal line segments. This gives the appearance that the graph is continuous when it is not. To obtain a correct graph, consult the manual to determine how to change the graphing mode on your graphing calculator from **connected mode** to **dot mode** [Fig. 34(b)].

› **Figure 34** Greatest integer function on a graphing calculator.



(a) Graph of $y = \text{int}(x)$ in the connected mode.



(b) Graph of $y = \text{int}(x)$ in the dot mode.

››› CAUTION ›››

When in connected mode, your graphing calculator will connect portions of a graph where a break should occur. To avoid misleading graphs, use the dot mode when graphing a function with discontinuities.

EXAMPLE 7 Computer Science

Let

$$f(x) = \frac{[\![10x + 0.5]\!]}{10}$$

Find

(A) $f(6)$ (B) $f(1.8)$ (C) $f(3.24)$ (D) $f(4.582)$ (E) $f(-2.68)$

What operation does this function perform?

SOLUTIONS

(A) $f(6) = \dfrac{[\![60.5]\!]}{10} = \dfrac{60}{10} = 6$

(B) $f(1.8) = \dfrac{[\![18.5]\!]}{10} = \dfrac{18}{10} = 1.8$

(C) $f(3.24) = \dfrac{[\![32.9]\!]}{10} = \dfrac{32}{10} = 3.2$

(D) $f(4.582) = \dfrac{[\![46.32]\!]}{10} = \dfrac{46}{10} = 4.6$

(E) $f(-2.68) = \dfrac{[\![-26.3]\!]}{10} = \dfrac{-27}{10} = -2.7$

x	$f[x]$
6	6
1.8	1.8
3.24	3.2
4.582	4.6
−2.68	−2.7

Comparing the values of x and $f(x)$ in the table, we conclude that this function rounds decimal fractions to the nearest tenth.

The greatest integer function is used in programming (in spreadsheet programs, for example) to round numbers to a specified accuracy.

MATCHED PROBLEM 7

Let $f(x) = [\![x + 0.5]\!]$. Find

(A) $f(6)$ (B) $f(1.8)$ (C) $f(3.24)$ (D) $f(-4.3)$ (E) $f(-2.69)$

What operation does this function perform?

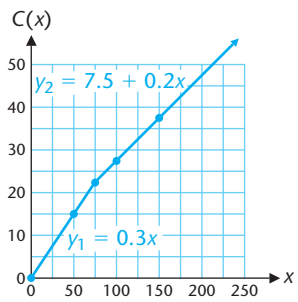
ANSWERS MATCHED PROBLEMS

1. x intercept: -1.516; y intercept: 5
2. (A) Domain: $-4 < x < 5$ or $(-4, 5)$
   Range: $-4 < y \le 3$ or $(-4, 3]$
   (B) $f(-4) = 1; f(0) = 3; f(2) = 2$
3. The values of f decrease and the graph of f is falling on $(-\infty, -3)$ and $(1, \infty)$. The values of f increase and the graph of f is rising on $(-3, 1)$.
4. Domain: $[0, \infty)$; range: $[-6.25, \infty)$; local minimum: $f(6.25) = -6.25$
5. The maximum profit of \$2,785 occurs when 547 shirts are sold.
6. (A) $C(x) = \begin{cases} 0.3x & \text{if } 0 \le x \le 75 \\ 7.5 + 0.2x & \text{if } x > 75 \end{cases}$

(B)

x	0	50	75	100	150
$C(x)$	\$0	\$15	\$22.50	\$27.50	\$37.50

(C)



7. (A) 6 (B) 2 (C) 3 (D) -4 (E) -3; f rounds decimal fractions to the nearest integer.

1-3 Exercises

1. Describe in your own words what the graph of a function is.

2. Explain how to find the domain and range of a function from its graph.

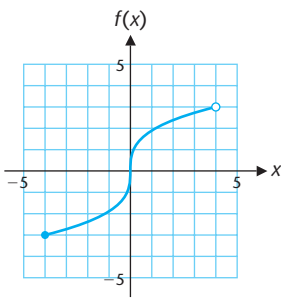
3. How many y intercepts can a function have? What about x intercepts? Explain.

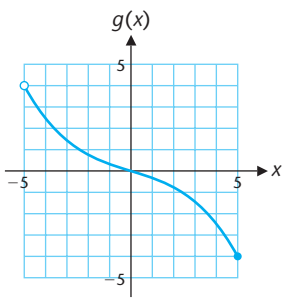
4. True or false: On any interval in its domain, every function is either increasing or decreasing. Explain.

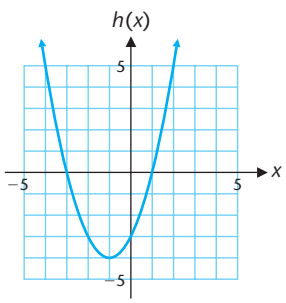
5. Explain in your own words what it means to say that a function is increasing on an interval.

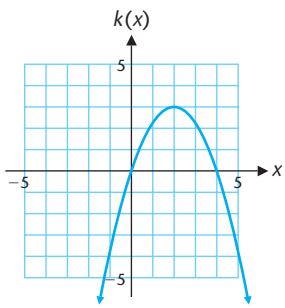
6. Explain in your own words what it means to say that a function is decreasing on an interval.

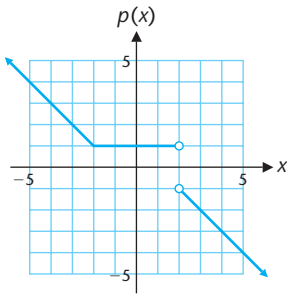
Problems 7–18 refer to functions f, g, h, k, p, and q given by the following graphs. (Assume the graphs continue as indicated beyond the parts shown.)

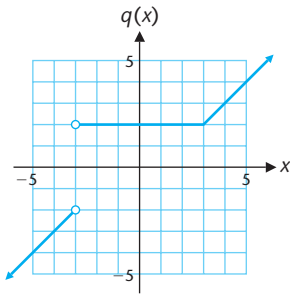












7. For the function f, find
(A) Domain
(B) Range
(C) x intercepts
(D) y intercept
(E) Intervals over which f is increasing
(F) Intervals over which f is decreasing

(G) Intervals over which f is constant
(H) Any points of discontinuity

8. Repeat Problem 7 for the function g.

9. Repeat Problem 7 for the function h.

10. Repeat Problem 7 for the function k.

11. Repeat Problem 7 for the function p.

12. Repeat Problem 7 for the function q.

13. Find $f(-4)$, $f(0)$, and $f(4)$.

14. Find $g(-5)$, $g(0)$, and $g(5)$.

15. Find $h(-3)$, $h(0)$, and $h(2)$.

16. Find $k(0)$, $k(2)$, and $k(4)$.

17. Find $p(-2)$, $p(2)$, and $p(5)$.

18. Find $q(-4)$, $q(-3)$, and $q(1)$.

In Problems 19–26, examine the graph of the function to determine the intervals over which the function is increasing, the intervals over which the function is decreasing, and the intervals over which the function is constant. Approximate the endpoints of the intervals to the nearest integer.

19. $f(x) = |x + 2| - 5$

20. $k(x) = |x - 2| - x$

21. $j(x) = 0.05x^3 + 0.25x^2 - 1.5x - 1.5$

22. $g(x) = -0.02x^3 - 0.14x^2 + 0.35x + 2.5$

23. $m(x) = |x - 3| - |x + 4|$

24. $q(x) = |x + 2| + |x - 4|$

25. $r(x) = |x + 4| - |x| + |x - 4|$

26. $s(x) = |x| - |x + 5| - |x - 3|$

In Problems 27–32, use a graphing calculator to find the x intercepts, y intercept, and any local extrema. Round answers to three decimal places.

27. $f(x) = x^2 - 5x - 9$

28. $g(x) = -x^2 + 7x + 14$

29. $h(x) = -x^3 + 4x + 25$

30. $k(x) = x^3 + 3x^2 - 15$

31. $m(x) = \sqrt{|x^2 - 12|}$

32. $n(x) = \sqrt{|x^3 - 12|}$

In Problems 33–38, sketch by hand the graph of a continuous function f over the interval $[-5, 5]$ that is consistent with the given information.

33. The function f is increasing on $[-5, -2]$, constant on $[-2, 2]$, and decreasing on $[2, 5]$.

34. The function f is decreasing on $[-5, -2]$, constant on $[-2, 2]$, and increasing on $[2, 5]$.

35. The function f is decreasing on $[-5, -2]$, constant on $[-2, 2]$, and decreasing on $[2, 5]$.

36. The function f is increasing on $[-5, -2]$, constant on $[-2, 2]$, and increasing on $[2, 5]$.

37. The function f is decreasing on $[-5, -2]$, increasing on $[-2, 2]$, and decreasing on $[2, 5]$.

38. The function f is increasing on $[-5, -2]$, decreasing on $[-2, 2]$, and increasing on $[2, 5]$.

39. Review the informal description of the term increasing on page 51. Explain how the formal definition of increasing in Definition 1 really says the same thing.

40. Review the informal description of the term decreasing on page 51. Explain how the formal definition of decreasing in Definition 1 really says the same thing.

In Problems 41–46, sketch the graph of $y = f(x)$ and evaluate $f(-2)$, $f(-1)$, $f(1)$, and $f(2)$.

41. $f(x) = \begin{cases} x + 1 & \text{if } x \le 0 \\ -x + 1 & \text{if } x > 0 \end{cases}$

42. $f(x) = \begin{cases} x & \text{if } x < 1 \\ -x + 2 & \text{if } x \ge 1 \end{cases}$

43. $f(x) = \begin{cases} x + 2 & \text{if } x < -1 \\ x - 2 & \text{if } x \ge -1 \end{cases}$

44. $f(x) = \begin{cases} -x - 1 & \text{if } x \le 2 \\ -x + 5 & \text{if } x > 2 \end{cases}$

45. $f(x) = \begin{cases} x^2 + 1 & \text{if } x < 0 \\ -x^2 - 1 & \text{if } x > 0 \end{cases}$

46. $f(x) = \begin{cases} -x^2 - 2 & \text{if } x < 0 \\ x^2 + 2 & \text{if } x > 0 \end{cases}$

In Problems 47–54, use the graph of each function to find the domain, range, y intercept, and x intercepts. Round answers to two decimal places.

47. $m(x) = x^3 + 45x^2 - 30$

48. $f(x) = x^3 - 35x^2 + 25$

49. $n(x) = 200 + 200x^2 - x^4$

50. $g(x) = 200 - 40x^3 - x^4$

51. $h(x) = 8\sqrt{x} - x$

52. $s(x) = x - 18\sqrt{x}$

53. $k(x) = x^2 - 50\sqrt{x + 5}$

54. $t(x) = 30\sqrt{20 - x} - x^2$

In Problems 55–60, write a verbal description of the graph of the given function using the terms increasing, decreasing, rising, and falling, and indicate any local maximum and

minimum values. Approximate the coordinates of any points used in your description to two decimal places.

55. $f(x) = x^3 - 12x^2 + 3x - 10$

56. $h(x) = x^3 + 15x^2 + 5x + 15$

57. $m(x) = 24|x| - x^2$ **58.** $n(x) = 15\sqrt{|x|} - x$

59. $g(x) = |x^2 - 5x - 300|$ **60.** $k(x) = |x^2 - 4x - 480|$

In Problems 61–66, write a verbal description of the graph of the given function using increasing and decreasing terminology, and indicating any local maximum and minimum values. Approximate the coordinates of any points used in your description to two decimal places.

61. $f(x) = x^2 + 4.3x - 32$ **62.** $g(x) = -x^2 + 6.9x + 25$

63. $h(x) = x^3 - x^2 - 74x + 60$

64. $k(x) = -x^3 + x^2 + 82x - 25$

65. $p(x) = |x^2 - x - 18|$ **66.** $q(x) = |x^2 - 2x - 30|$

In Problems 67–72, sketch the graph of a function f by hand that is continuous on the interval [−5, 5], except as noted, and is consistent with the given information.

67. The function f is increasing on $[-5, 0)$, discontinuous at $x = 0$, increasing on $(0, 5]$, $f(-2) = 0$, and $f(2) = 0$.

68. The function f is decreasing on $[-5, 0)$, discontinuous at $x = 0$, decreasing on $(0, 5]$, $f(-3) = 0$, and $f(3) = 0$.

69. The function f is discontinuous at $x = 0, f(-3) = -2$ is a local maximum, and $f(2) = 3$ is a local minimum.

70. The function f is discontinuous at $x = 0, f(-3) = 2$ is a local minimum, and $f(2) = -3$ is a local maximum.

71. The function f is discontinuous at $x = -2$ and $x = 2$, $f(-3) = -2$ and $f(3) = -2$ are local maxima, and $f(0) = 0$ is a local minimum.

72. The function f is discontinuous at $x = -2$ and $x = 2$, $f(-3) = 2$ and $f(3) = 2$ are local minima, and $f(0) = 0$ is a local maximum.

In Problems 73–78, graph $y = f(x)$ in a standard viewing window. Assuming that the graph continues as indicated beyond the part shown in this viewing window, find the domain, range, and any points of discontinuity. (Use the dot mode on your graphing calculator.)

73. $f(x) = \dfrac{|5x - 10|}{x - 2}$ **74.** $f(x) = \dfrac{4x + 12}{|x + 3|}$

75. $f(x) = x + \dfrac{|4x - 4|}{x - 1}$ **76.** $f(x) = x + \dfrac{|2x + 2|}{x + 1}$

77. $f(x) = |x| - \dfrac{|9 - 3x|}{x - 3}$ **78.** $f(x) = |x| + \dfrac{|2x + 4|}{x + 2}$

In Problems 79–84, write a piecewise definition of f and sketch the graph of f, by hand using a graphing calculator as an aid. Include sufficient intervals to clearly illustrate both the definition and the graph. Find the domain, range, and any points of discontinuity.

79. $f(x) = [\![x/2]\!]$ **80.** $f(x) = [\![x/3]\!]$ **81.** $f(x) = [\![3x]\!]$

82. $f(x) = [\![2x]\!]$ **83.** $f(x) = x - [\![x]\!]$ **84.** $f(x) = [\![x]\!] - x$

85. The function f is continuous and increasing on the interval $[1, 9]$ with $f(1) = -5$ and $f(9) = 4$.
(A) Sketch a graph of f that is consistent with the given information.
(B) How many times does your graph cross the x axis? Could the graph cross more times? Fewer times? Support your conclusions with additional sketches and/or verbal arguments.

86. Repeat Problem 85 if the function does not have to be continuous.

87. The function f is continuous on the interval $[-5, 5]$ with $f(-5) = -4, f(1) = 3$, and $f(5) = -2$.
(A) Sketch a graph of f that is consistent with the given information.
(B) How many times does your graph cross the x axis? Could the graph cross more times? Fewer times? Support your conclusions with additional sketches and/or verbal arguments.

88. Repeat Problem 87 if f is continuous on $[-8, 8]$ with $f(-8) = -6, f(-4) = 3, f(3) = -2$, and $f(8) = 5$.

89. The function f is continuous on $[0, 10]$, $f(5) = -5$ is a local minimum, and f has no other local extrema on this interval.
(A) Sketch a graph of f that is consistent with the given information.
(B) How many times does your graph cross the x axis? Could the graph cross more times? Fewer times? Support your conclusions with additional sketches and/or verbal arguments.

90. Repeat Problem 89 if $f(5) = 1$ and all other information is unchanged.

APPLICATIONS

91. **REVENUE** The revenue (in dollars) from the sale of x car seats for infants is given by

$$R(x) = 60x - 0.035x^2 \qquad 0 \le x \le 1{,}700$$

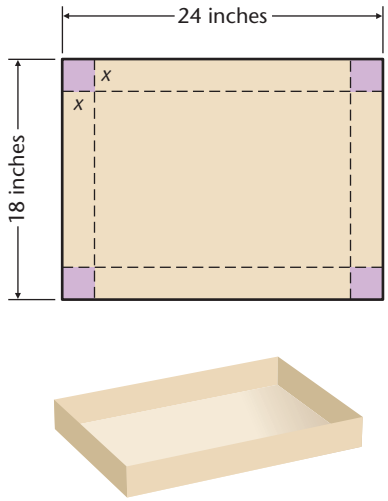
Find the number of car seats that must be sold to maximize the revenue. What is the maximum revenue (to the nearest dollar)?

92. **PROFIT** The profit (in dollars) from the sale of x car seats for infants is given by

$$P(x) = 38x - 0.035x^2 - 4{,}000 \qquad 0 \le x \le 1{,}700$$

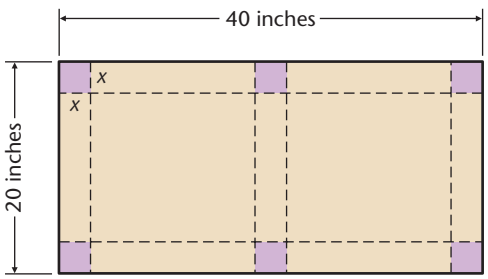
Find the number of car seats that must be sold to maximize the profit. What is the maximum profit (to the nearest dollar)?

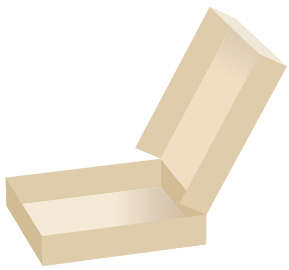
93. **MANUFACTURING** A box is to be made out of a piece of cardboard that measures 18 by 24 inches. Squares, x inches on a side, will be cut from each corner and then the ends and sides will be folded up (see the figure).



Find the size of the cutout squares that will make the maximum volume. What is the maximum volume? Round answers to two decimal places.

94. **MANUFACTURING** A box with a hinged lid is to be made out of a piece of cardboard that measures 20 by 40 inches. Six squares, x inches on a side, will be cut from each corner and the middle of the sides, and then the ends and sides will be folded up to form the box and its lid (see the figure).



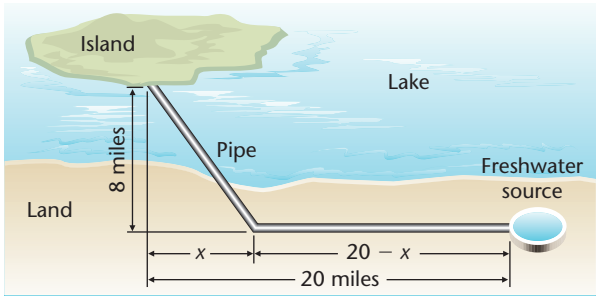


Find the size of the cutout squares that will make the maximum volume. What is the maximum volume? Round answers to two decimal places.

95. **CONSTRUCTION** A freshwater pipe is to be run from a source on the edge of a lake to a small resort community on an island 8 miles offshore, as indicated in the figure. It costs \$10,000 per mile to lay the pipe on land and \$16,000 per mile to lay the pipe in the lake. The total cost $C(x)$ in thousands of dollars of laying the pipe is given by

$$C(x) = 10(20 - x) + 16\sqrt{x^2 + 64} \qquad 0 \le x \le 20$$

Find the length (to two decimal places) of the land portion of the pipe that will make the production costs minimum. Find the minimum cost to the nearest thousand dollars.



96. **TRANSPORTATION** The construction company laying the freshwater pipe in Problem 95 uses an amphibious vehicle to travel down the beach and then out to the island. The vehicle travels at 30 miles per hour on land and 7.5 miles per hour in water. The total time $T(x)$ in minutes for a trip from the freshwater source to the island is given by

$$T(x) = 2(20 - x) + 8\sqrt{x^2 + 64} \qquad 0 \le x \le 20$$

Find (to two decimal places) the length of the land portion of the trip that will make the time minimum. Find the minimum time to the nearest minute.

97. **COMPUTER SCIENCE** Let $f(x) = 10[\![0.5 + x/10]\!]$. Evaluate f at 4, −4, 6, −6, 24, 25, 247, −243, −245, and −246. What operation does this function perform?

98. **COMPUTER SCIENCE** Let $f(x) = 100[\![0.5 + x/100]\!]$. Evaluate f at 40, −40, 60, −60, 740, 750, 7,551, −601, −649, and −651. What operation does this function perform?

99. **COMPUTER SCIENCE** Use the greatest integer function to define a function f that rounds real numbers to the nearest hundredth.

100. **COMPUTER SCIENCE** Use the greatest integer function to define a function f that rounds real numbers to the nearest thousandth.

MODELING AND DATA ANALYSIS

Table 1 contains daily automobile rental rates from a New Jersey firm.

Table 1

Vehicle Type	Daily Charge	Included Miles	Mileage Charge*
Compact	$32.00	100/Day	$0.16/mile
Midsize	$41.00	200/Day	$0.18/mile

*Mileage charge does not apply to included miles.
Source: www.gogelauto.com

101. **AUTOMOBILE RENTAL** Use the data in Table 1 to construct a piecewise-defined model for the daily rental charge for a compact automobile that is driven x miles.

102. **AUTOMOBILE RENTAL** Use the data in Table 1 to construct a piecewise-defined model for the daily rental charge for a midsize automobile that is driven x miles.

103. **DELIVERY CHARGES** A nationwide package delivery service charges $15 for overnight delivery of packages weighing 1 pound or less. Each additional pound (or fraction thereof) costs an additional $3. Let $C(x)$ be the charge for overnight delivery of a package weighing x pounds.
(A) Write a piecewise definition of C for $0 < x \le 6$ and sketch the graph of C by hand.
(B) Can the function f defined by $f(x) = 15 + 3[\![x]\!]$ be used to compute the delivery charges for all x, $0 < x \le 6$? Justify your answer.

104. **TELEPHONE CHARGES** Calls to 900 numbers are charged to the caller. A 900 number hot line for tips and hints for video games charges $4 for the first minute of the call and $2 for each additional minute (or fraction thereof). Let $C(x)$ be the charge for a call lasting x minutes.
(A) Write a piecewise definition of C for $0 < x \le 6$ and sketch the graph of C by hand.
(B) Can the function f defined by $f(x) = 4 + 2[\![x]\!]$ be used to compute the charges for all x, $0 < x \le 6$? Justify your answer.

105. **SALES COMMISSIONS** An appliance salesperson receives a base salary of $200 a week and a commission of 4% on all sales over $3,000 during the week. In addition, if the weekly sales are $8,000 or more, the salesperson receives a $100 bonus. If x represents weekly sales (in dollars), express the weekly earnings $E(x)$ as a function of x, and sketch its graph. Identify any points of discontinuity. Find $E(5{,}750)$ and $E(9{,}200)$.

106. **SERVICE CHARGES** On weekends and holidays, an emergency plumbing repair service charges $2.00 per minute for the first 30 minutes of a service call and $1.00 per minute for each additional minute. If x represents the duration of a service call in minutes, express the total service charge $S(x)$ as a function of x, and sketch its graph. Identify any points of discontinuity. Find $S(25)$ and $S(45)$.

Table 2 contains income tax rates for Minnesota in a recent year.

Table 2

Status	Taxable Income Over	But Not Over	Tax Is	Of the Amount Over
Single	$0	$19,890	5.35%	$0
	19,890	65,330	$1,064 + 7.05%	19,890
	65,330	. . .	4,268 + 7.85%	65,330
Married	0	29,070	5.35%	0
	29,070	115,510	1,555 + 7.05%	29,070
	115,510	. . .	7,649 + 7.85%	115,510

107. **STATE INCOME TAX** Use the schedule in Table 2 to construct a piecewise-defined model for the taxes due for a single taxpayer with a taxable income of x dollars. Find the tax on the following incomes: $10,000, $30,000, $100,000.

108. **STATE INCOME TAX** Use the schedule in Table 2 to construct a piecewise-defined model for the taxes due for a married taxpayer with a taxable income of x dollars. Find the tax on the following incomes: $20,000, $60,000, $200,000.

109. **TIRE MILEAGE** An automobile tire manufacturer collected the data in the table relating tire pressure x, in pounds per square inch (lb/in.^{2}), and mileage in thousands of miles.

x	28	30	32	34	36
Mileage	45	52	55	51	47

A mathematical model for these data is given by

$$f(x) = -0.518x^2 + 33.3x - 481$$

(A) Compare the model and the data graphically and numerically.
(B) Find (to two decimal places) the mileage for a tire pressure of 31 lb/in.^{2} and for 35 lb/in.^{2}.
(C) Write a brief description of the relationship between tire pressure and mileage, using the terms *increasing, decreasing, local maximum,* and *local minimum* where appropriate.

110. STOCK PRICES The table lists the closing price of stock in Carnival Cruise Lines on the first trading day of each year from 2003 to 2007.

Year	03	04	05	06	07
Price	26.05	39.82	57.31	54.57	50.95

A mathematical model for Carnival's stock price is given by

$$f(x) = -3.93x^2 + 22.2x + 24.9$$

where $x = 0$ corresponds to 2003.

(A) Compare the model and the data graphically and numerically.

(B) What does the model predict the stock price will be at the beginning of 2008 and 2009? Round to the nearest cent.

(C) Write a brief verbal description of Carnival's stock price from 2003 to 2007, using *increasing, decreasing, local maximum,* and *local minimum* where appropriate.

1-4 Functions: Graphs and Transformations

- A Library of Elementary Graphs
- Shifting Graphs Horizontally and Vertically
- Stretching and Shrinking Graphs
- Reflecting Graphs in the x and y Axes
- Even and Odd Functions

We have seen that the graph of a function can provide valuable insight into the information provided by that function. But there is a seemingly endless variety of functions out there, and it seems like an insurmountable task to learn about so many different graphs. In this section, we will see that relationships between the formulas for certain functions lead to relationships between their graphs as well. For example, the functions

$$g(x) = x^2 + 2 \qquad h(x) = (x + 2)^2 \qquad k(x) = 2x^2$$

can be expressed in terms of the function $f(x) = x^2$ as follows:

$$g(x) = f(x) + 2 \qquad h(x) = f(x + 2) \qquad k(x) = 2f(x)$$

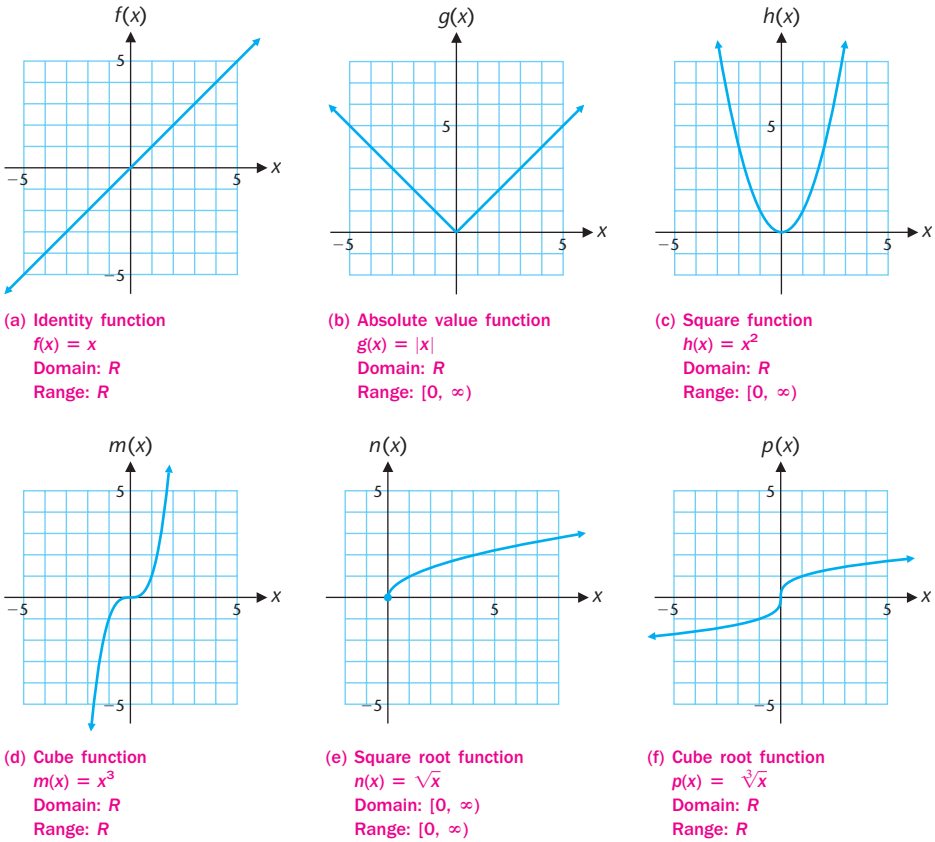
We will see that the graphs of functions g, h, and k are closely related to the graph of function f.

Once we understand these relationships, knowing the graph of a very simple function like $f(x) = x^2$ will enable us to learn about the graphs of many related functions.

A Library of Elementary Graphs

As you progress through this book, you will encounter a number of basic functions that you will want to add to your library of elementary functions. Figure 1 on the next page shows six basic functions that you will encounter frequently. You should know the definition, domain, and range of each of these functions, and be able to recognize

their graphs. To help you become familiar with these graphs, it would be a good idea to graph each basic function in Figure 1 on your graphing calculator.



(a) Identity function
$f(x) = x$
Domain: R
Range: R

(b) Absolute value function
$g(x) = |x|$
Domain: R
Range: $[0, \infty)$

(c) Square function
$h(x) = x^2$
Domain: R
Range: $[0, \infty)$

(d) Cube function
$m(x) = x^3$
Domain: R
Range: R

(e) Square root function
$n(x) = \sqrt{x}$
Domain: $[0, \infty)$
Range: $[0, \infty)$

(f) Cube root function
$p(x) = \sqrt[3]{x}$
Domain: R
Range: R

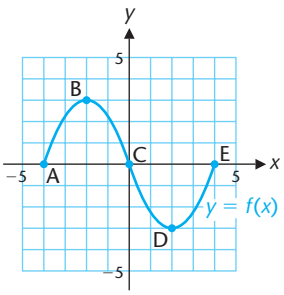
› **Figure 1** Some basic functions and their graphs.

[*Note*: Letters used to designate these functions may vary from context to context; R represents the set of all real numbers.]

Most graphing calculators allow you to define a number of functions, usually denoted by $y_1, y_2, y_3, \ldots$. You can graph all of these functions simultaneously, or you can select certain functions for graphing and suppress the graphs of the others. Consult your manual to determine how many functions can be stored in your graphing calculator at one time and how to select particular functions for graphing. Many of our investigations in this section will involve graphing two or more functions at the same time.

› Shifting Graphs Horizontally and Vertically

If a new function is formed by performing an operation on a given function, then the graph of the new function is called a **transformation** of the graph of the original function. For example, if we add a constant k to $f(x)$, then the graph of $y = f(x)$ is transformed into the graph of $y = f(x) + k$.



› Figure 2

Table 1

	x	f(x)
A	−4	0
B	−2	3
C	0	0
D	2	−3
E	4	0

››› EXPLORE-DISCUSS 1

The following activities refer to the graph of f shown in Figure 2 and the corresponding points on the graph shown in Table 1.

(A) Use the points in Table 1 to construct a similar table and then sketch a graph for each of the following functions: $y = f(x) + 2, y = f(x) - 3$. Describe the relationship between the graph of $y = f(x)$ and the graph of $y = f(x) + k$ for any real number k.

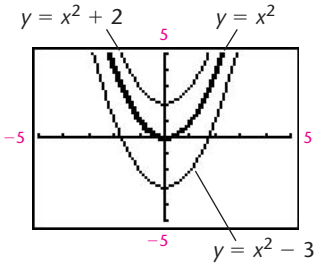
(B) Use the points in Table 1 to construct a similar table and then sketch a graph for each of the following functions: $y = f(x + 2), y = f(x - 3)$. [*Hint:* Choose values of x so that $x + 2$ or $x - 3$ is in Table 1.] Describe the relationship between the graph of $y = f(x)$ and the graph of $y = f(x + h)$ for any real number h.

(C) Make a conjecture as to the relationship between the graph of $y = f(x)$ and the graph of $y = f(x + h) + k$ for any real numbers h and k. Then check your conjecture by constructing a table and sketching the graph for $y = f(x + 2) - 3$ and $y = f(x - 3) + 2$.

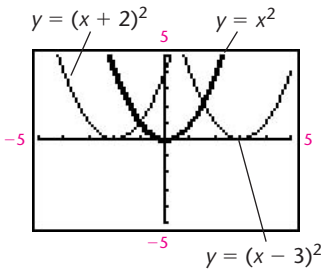
EXAMPLE 1 Vertical and Horizontal Shifts

(A) How are the graphs of $y = x^2 + 2$ and $y = x^2 - 3$ related to the graph of $y = x^2$? Confirm your answer by graphing all three functions simultaneously in the same viewing window.

(B) How are the graphs of $y = (x + 2)^2$ and $y = (x - 3)^2$ related to the graph of $y = x^2$? Confirm your answer by graphing all three functions simultaneously in the same viewing window.



› Figure 3 Vertical shifts.



› Figure 4 Horizontal shifts.

SOLUTIONS

(A) Note that the output of $y = x^2 + 2$ is always exactly two more than the output of $y = x^2$. Consequently, the graph of $y = x^2 + 2$ is the same as the graph of $y = x^2$ shifted upward two units, and the graph of $y = x^2 - 3$ is the same as the graph of $y = x^2$ shifted downward three units. Figure 3 confirms these conclusions. (It appears that the graph of $y = f(x) + k$ is the graph of $y = f(x)$ shifted up if k is positive and down if k is negative.)

(B) Note that the output of $y = (x + 2)^2$ is zero for $x = -2$, while the output of $y = x^2$ is zero for $x = 0$. This suggests that the graph of $y = (x + 2)^2$ is the same as the graph of $y = x^2$ shifted to the left two units, and the graph of $y = (x - 3)^2$ is the same as the graph of $y = x^2$ shifted to the right three units. Figure 4 confirms these conclusions. It appears that the graph of $y = f(x + h)$ is the graph of $y = f(x)$ shifted right if h is negative and left if h is positive.

MATCHED PROBLEM 1

(A) How are the graphs of $y = \sqrt{x} + 3$ and $y = \sqrt{x} - 1$ related to the graph of $y = \sqrt{x}$? Confirm your answer by graphing all three functions simultaneously in the same viewing window.

(B) How are the graphs of $y = \sqrt{x + 3}$ and $y = \sqrt{x - 1}$ related to the graph of $y = \sqrt{x}$? Confirm your answer by graphing all three functions simultaneously in the same viewing window.

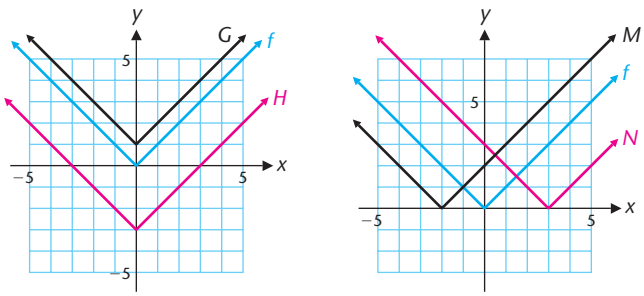
Comparing the graph of $y = f(x) + k$ with the graph of $y = f(x)$, we see that the graph of $y = f(x) + k$ can be obtained from the graph of $y = f(x)$ by **vertically translating** (shifting) the graph of the latter upward k units if k is positive and downward $|k|$ units if k is negative. Comparing the graph of $y = f(x + h)$ with the graph of $y = f(x)$, we see that the graph of $y = f(x + h)$ can be obtained from the graph of $y = f(x)$ by **horizontally translating** (shifting) the graph of the latter h units to the left if h is positive and $|h|$ units to the right if h is negative.

››› CAUTION ›››

It may seem intuitive that a positive value of h for $y = f(x + h)$ should shift the graph of f to the right, but in fact it shifts it left. Likewise, a negative value of h shifts the graph to the right, the opposite of the more obvious guess.

EXAMPLE 2 Vertical and Horizontal Translations [Shifts]

The graphs in Figure 5 are either horizontal or vertical shifts of the graph of $f(x) = |x|$. Write appropriate equations for functions G, H, M, and N in terms of f.



› **Figure 5** Vertical and horizontal shifts.

SOLUTION

The graphs of functions H and G are 3 units lower and 1 unit higher, respectively, than the graph of f, so H and G are vertical shifts given by

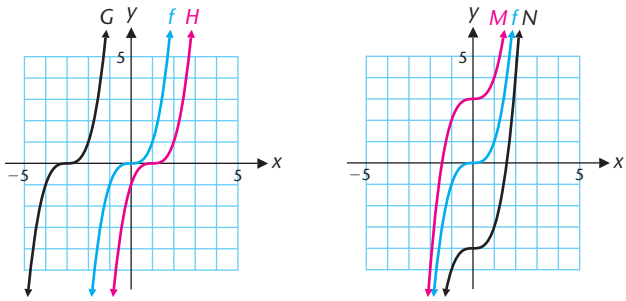
$$H(x) = |x| - 3 \qquad G(x) = |x| + 1$$

The graphs of functions M and N are 2 units to the left and 3 units to the right, respectively, of the graph of f, so M and N are horizontal shifts given by

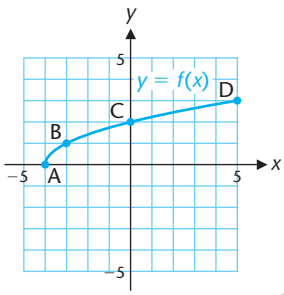
$$M(x) = |x + 2| \qquad N(x) = |x - 3|$$

MATCHED PROBLEM 2

The graphs in Figure 6 are either horizontal or vertical shifts of the graph of $f(x) = x^3$. Write appropriate equations for functions H, G, M, and N in terms of f.



› **Figure 6** Vertical and horizontal shifts.



› **Figure 7**

› Stretching and Shrinking Graphs

We will now investigate how the graph of $y = f(x)$ is related to the graph of $y = Af(x)$ and to the graph of $y = f(Ax)$ for different positive real numbers A.

››› EXPLORE-DISCUSS 2

The following activities refer to the graph of f shown in Figure 7 and the corresponding points on the graph shown in Table 2.

(A) Construct a similar table and then sketch a graph for each of the following functions: $y = 2f(x), y = \frac{1}{2}f(x)$. Describe the relationship between the graph of $y = f(x)$ and the graph of $y = Af(x)$ for A any positive real number.

(B) Construct a similar table and then sketch a graph for each of the following functions: $y = f(2x), y = f(\frac{1}{2}x)$. [*Hint:* Select the x values for your table so that the multiples of x are in Table 2.] Describe the relationship between the graph of $y = f(x)$ and the graph of $y = f(Ax)$ for A any positive real number.

Table 2

	x	$f(x)$
A	−4	0
B	−3	1
C	0	2
D	5	3

Comparing the graph of $y = Af(x)$ in Explore-Discuss 2 with the graph of $y = f(x)$, we see that the graph of $y = Af(x)$ can be obtained from the graph of $y = f(x)$ by multiplying each y coordinate of the latter by A. If $A > 1$, multiplying by A makes the y coordinates bigger; we say that this **vertically stretches** the graph of f. If $0 < A < 1$, multiplying by A makes the y coordinates smaller; this **vertically shrinks** the graph of f. Note that in each case, points on the x axis remain fixed, while all other points are "pulled away" from, or "pushed toward" the x axis.

Likewise, comparing the graph of $y = f(Ax)$ with the graph of $y = f(x)$, we see that the graph of $y = f(Ax)$ can be obtained from the graph of $y = f(x)$ by multiplying each x coordinate of the latter by $\frac{1}{A}$. This **horizontally stretches** the graph of $y = f(x)$ if $0 < A < 1$ and **horizontally shrinks** the graph of $y = f(x)$ if $A > 1$.

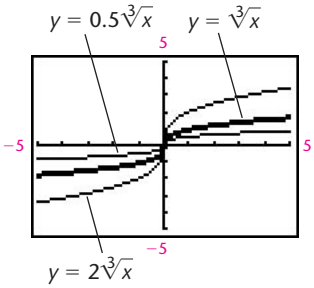
››› CAUTION ›››

As with horizontal shifts, horizontally stretching and shrinking work opposite the way you might expect. When $A > 1$, $f(Ax)$ is shrunk toward the y axis, and when $0 < A < 1$, $f(Ax)$ is stretched away from the y axis.

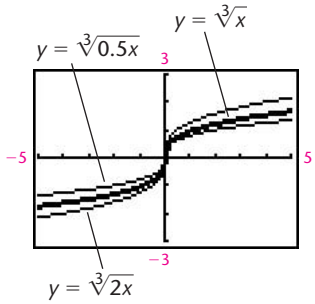
EXAMPLE 3 Stretches and Shrinks

(A) How are the graphs of $y = 2\sqrt[3]{x}$ and $y = 0.5\sqrt[3]{x}$ related to the graph of $y = \sqrt[3]{x}$? Confirm your answer by graphing all three functions simultaneously in the same viewing window.

(B) How are the graphs of $y = \sqrt[3]{2x}$ and $y = \sqrt[3]{0.5x}$ related to the graph of $y = \sqrt[3]{x}$? Confirm your answer by graphing all three functions simultaneously in the same viewing window.



› **Figure 8** Vertical stretching and shrinking.



› **Figure 9** Horizontal stretching and shrinking.

SOLUTIONS

(A) The graph of $y = 2\sqrt[3]{x}$ can be obtained from the graph of $y = \sqrt[3]{x}$ by multiplying each y value by 2. This stretches the graph vertically by a factor of 2. The graph of $y = 0.5\sqrt[3]{x}$ can be obtained from the graph of $y = \sqrt[3]{x}$ by multiplying each y value by 0.5. This shrinks the graph vertically by a factor of 0.5 (Fig. 8).

(B) The graph of $y = \sqrt[3]{2x}$ can be obtained from the graph of $y = \sqrt[3]{x}$ by multiplying each x value by $\frac{1}{2}$. This shrinks the graph horizontally by a factor of $\frac{1}{2}$. The graph of $y = \sqrt[3]{0.5x}$ can be obtained from the graph of $y = \sqrt[3]{x}$ by multiplying each x value by 2. This stretches the graph horizontally by a factor of 2 (Fig. 9).

MATCHED PROBLEM 3

(A) How are the graphs of $y = 2x^3$ and $y = 0.5x^3$ related to the graph of $y = x^3$? Confirm your answer by graphing all three functions simultaneously in the same viewing window.

(B) How are the graphs of $y = (2x)^3$ and $y = (0.5x)^3$ related to the graph of $y = x^3$? Confirm your answer by graphing all three functions simultaneously in the same viewing window.

Reflecting Graphs in the x and y Axes

Next, we will investigate how the graphs of $y = -f(x)$ and $y = f(-x)$ are related to the graph of $y = f(x)$.

EXPLORE-DISCUSS 3

The following activities refer to the graph of f shown in Figure 10 and the corresponding points on the graph shown in Table 3.

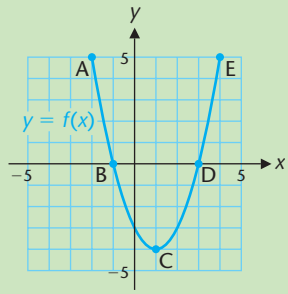


Figure 10

Table 3

	x	$f(x)$
A	−2	5
B	−1	0
C	1	−4
D	3	0
E	4	5

(A) Construct a similar table and then sketch a graph for $y = -f(x)$. Describe the relationship between the graph of $y = f(x)$ and the graph of $y = -f(x)$.

(B) Construct a similar table and then sketch a graph for $y = f(-x)$. [*Hint:* Choose x values so that $-x$ is in Table 3.] Describe the relationship between the graph of $y = f(x)$ and the graph of $y = f(-x)$.

(C) Construct a similar table and then sketch a graph for $y = -f(-x)$. [*Hint:* Choose x values so that $-x$ is in Table 3.] Describe the relationship between the graph of $y = f(x)$ and the graph of $y = -f(-x)$.

The graph of $y = -f(x)$ can be obtained from the graph of $y = f(x)$ by changing the sign of each y coordinate. This has the effect of moving every point on the graph to the opposite side of the x axis. We call this a **reflection in the x axis.** In other words, the graph of $y = -f(x)$ is a mirror image of the graph of $y = f(x)$ on the opposite side of the x axis.

Similarly, the graph of $y = f(-x)$ can be obtained from the graph of $y = f(x)$ by changing the sign of every x coordinate. This has the effect of moving every point on the graph to the opposite side of the y axis, which we call a **reflection in the y axis.**

Finally, the graph of $y = -f(-x)$ can be obtained from the graph of $y = f(x)$ by changing the sign of each x coordinate *and* each y coordinate. This is called a **reflection in the origin,** and is equivalent to reflecting first in one axis, then in the other. Figure 11 illustrates these reflections for $f(x) = \sqrt{x}$.

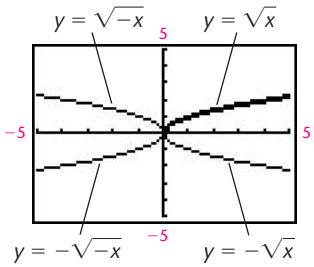


Figure 11 Reflections of the graph of $y = \sqrt{x}$.

The transformations we've studied so far are summarized in a box for easy reference.

SUMMARY OF GRAPH TRANSFORMATIONS

Vertical Translation [Fig. 12(a)]:

$y = f(x) + k$

$\begin{cases} k > 0 & \text{Shift graph of } y = f(x) \text{ up } k \text{ units} \\ k < 0 & \text{Shift graph of } y = f(x) \text{ down } |k| \text{ units} \end{cases}$

Horizontal Translation [Fig. 12(b)]:

$y = f(x + h)$

$\begin{cases} h > 0 & \text{Shift graph of } y = f(x) \text{ left } h \text{ units} \\ h < 0 & \text{Shift graph of } y = f(x) \text{ right } |h| \text{ units} \end{cases}$

Vertical Stretch and Shrink [Fig. 12(c)]:

$y = Af(x)$

$\begin{cases} A > 1 & \text{Vertically stretch the graph of } y = f(x) \text{ by multiplying each } y \text{ value by } A \\ 0 < A < 1 & \text{Vertically shrink the graph of } y = f(x) \text{ by multiplying each } y \text{ value by } A \end{cases}$

Horizontal Stretch and Shrink [Fig. 12(d)]:

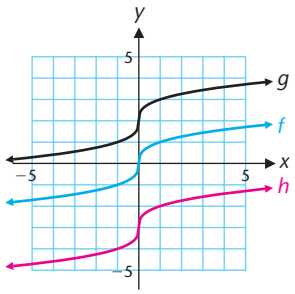
$y = f(Ax)$

$\begin{cases} A > 1 & \text{Horizontally shrink the graph of } y = f(x) \text{ by multiplying each } x \text{ value by } \frac{1}{A} \\ 0 < A < 1 & \text{Horizontally stretch the graph of } y = f(x) \text{ by multiplying each } x \text{ value by } \frac{1}{A} \end{cases}$

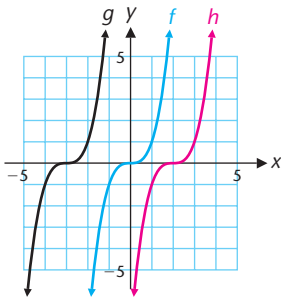
Reflection [Fig. 12(e)]:

$y = -f(x)$ Reflect the graph of $y = f(x)$ in the x axis
$y = f(-x)$ Reflect the graph of $y = f(x)$ in the y axis
$y = -f(-x)$ Reflect the graph of $y = f(x)$ in the origin

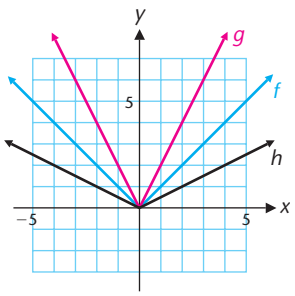
› **Figure 12** Graph transformations.



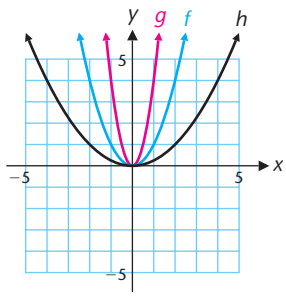
(a)
Vertical translation
$g(x) = f(x) + 2$
$h(x) = f(x) - 3$



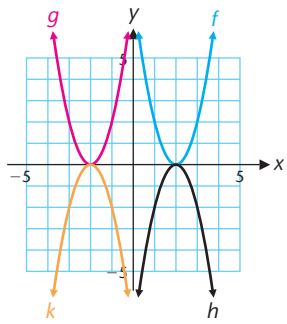
(b)
Horizontal translation
$g(x) = f(x + 3)$
$h(x) = f(x - 2)$



(c)
Vertical stretch and shrink
$g(x) = 2f(x)$
$h(x) = 0.5f(x)$



(d)
Horizontal stretch and shrink
$g(x) = f(2x)$
$h(x) = f(0.5x)$



(e)
Reflection
$g(x) = f(-x)$
$h(x) = -f(x)$
$k(x) = -f(-x)$

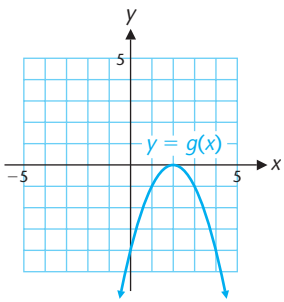
››› EXPLORE-DISCUSS 4

(A) Use a graphing calculator to explore the graph of $y = A(x + h)^2 + k$ for various values of the constants A, h, and k. Discuss how the graph of $y = A(x + h)^2 + k$ is related to the graph of $y = x^2$.

(B) The graph of $y = A(x + h)^2 + k$ has both shifts and a stretch or shrink. Based on the graphs you explored in part A, which come first, shifts or a stretch/shrink? Can you think of an algebraic reason why this should be the case?

EXAMPLE 4 Combining Graph Transformations

The graph of $y = g(x)$ in Figure 13 is a transformation of the graph of $y = x^2$. Find an equation for the function g.

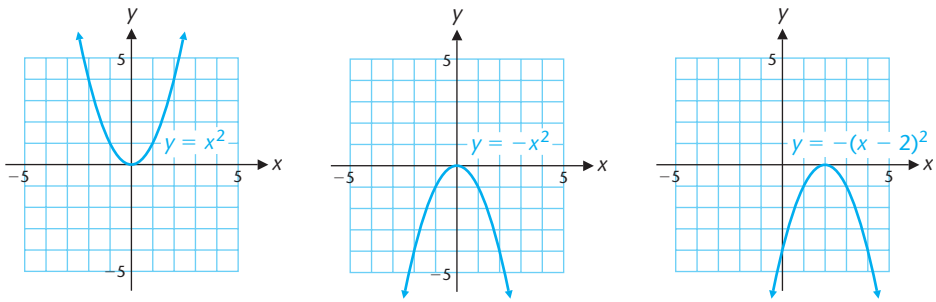


› Figure 13

SOLUTION

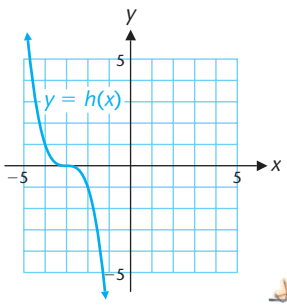
To transform the graph of $y = x^2$ [Fig. 14(a)] into the graph of $y = g(x)$, we first reflect the graph of $y = x^2$ in the x axis [Fig. 14(b)], then shift it to the right two units [Fig. 14(c)]. Thus, an equation for the function g is

$$g(x) = -(x - 2)^2$$



(a) $y = x^2$ (b) $y = -x^2$ (c) $y = -(x - 2)^2$

› Figure 14



› Figure 15

MATCHED PROBLEM 4

The graph of $y = h(x)$ in Figure 15 is a transformation of the graph of $y = x^3$. Find an equation for the function h.

Even and Odd Functions

Certain transformations leave the graphs of some functions unchanged. Review the graphs of $y = x^2$ and $y = x^3$ at the beginning of this section. Reflecting the graph of $y = x^2$ in the y axis does not change the graph. Functions with this property are called *even functions*. Similarly, reflecting the graph of $y = x^3$ in the x axis and then in the y axis does not change the graph. Functions with this property are called *odd functions*. More formally, we have the following definitions.

EVEN AND ODD FUNCTIONS

If $f(x) = f(-x)$ for all x in the domain of f, then f is an **even function.**

If $f(-x) = -f(x)$ for all x in the domain of f, then f is an **odd function.**

The graph of an even function is said to be **symmetric with respect to the y axis** and the graph of an odd function is said to be **symmetric with respect to the origin** (Fig. 16).

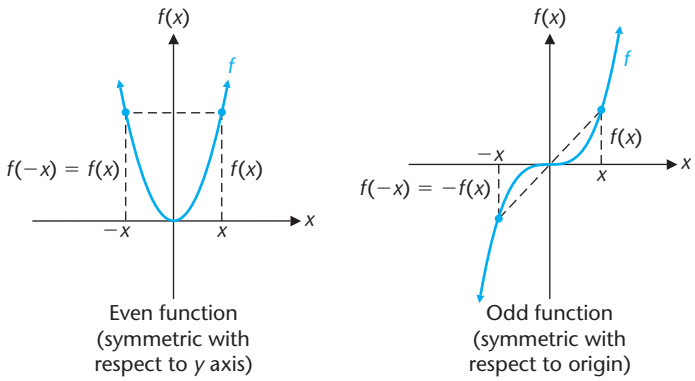


Figure 16 Even and odd functions.

Look again at the graphs of the basic functions in Figure 1. These graphs show that the square and absolute value functions are even functions, and the identity, cube, and cube root functions are odd functions. Notice in Figure 1(e) that the square root function is not symmetric with respect to the y axis or the origin. Thus, the square root function is neither even nor odd.

EXAMPLE 5 Testing for Even and Odd Functions

Determine whether the functions f, g, and h are even, odd, or neither.

(A) $f(x) = x^4 + 1$ (B) $g(x) = x^3 + 1$ (C) $h(x) = x^5 + x$

SOLUTIONS

(A) Algebraic Solution

We substitute $(-x)$ for x, then simplify.

$$\begin{aligned} f(x) &= x^4 + 1 \\ f(-x) &= (-x)^4 + 1 \\ &= [(-1)x]^4 + 1 \\ &= (-1)^4 x^4 + 1 \\ &= x^4 + 1 \end{aligned}$$

The result is identical to $f(x)$, so f is even.

(B) Algebraic Solution

$$\begin{aligned} g(x) &= x^3 + 1 \\ g(-x) &= (-x)^3 + 1 \\ &= [(-1)x]^3 + 1 \\ &= (-1)^3 x^3 + 1 \\ &= -x^3 + 1 \\ -g(x) &= -(x^3 + 1) \\ &= -x^3 - 1 \end{aligned}$$

Because $g(-x) \neq g(x)$ and $g(-x) \neq -g(x)$, g is neither even nor odd.

(C) Algebraic Solution

$$\begin{aligned} h(x) &= x^5 + x \\ h(-x) &= (-x)^5 + (-x) \\ &= -x^5 - x \\ &= -(x^5 + x) \end{aligned}$$

This is the negative of h, so h is odd.

(A) Graphical Solution

Enter $y_1 = x^4 + 1$ and $y_2 = y_1(-x)$ (Fig. 17), draw the graph (Fig. 18), and use the TRACE command or a table to see that the graphs are identical. Therefore f is even.

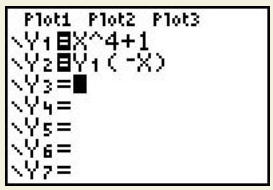


Figure 17

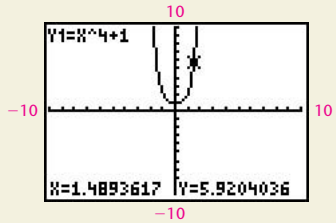


Figure 18

(B) Graphical Solution

Enter $y_1 = x^3 + 1$, $y_2 = y_1(-x)$, and $y_3 = -y_1(x)$ (Fig. 19), graph (Fig. 20), and observe that no two of these functions are identical. Therefore, g is neither even nor odd.

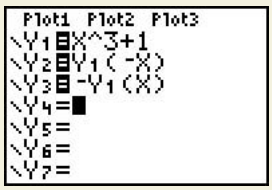


Figure 19

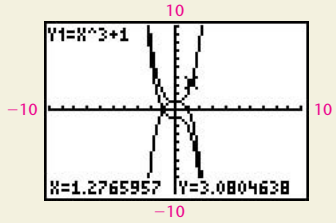


Figure 20

(C) Graphical Solution

Enter $y_1 = x^5 + x$, $y_2 = y_1(-x)$, and $y_3 = -y_1(x)$ (Fig. 21), graph (Fig. 22), and use the TRACE command or a table to show that y_2 and y_3 are identical. Therefore, h is an odd function.

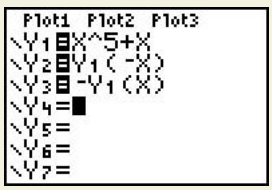


Figure 21

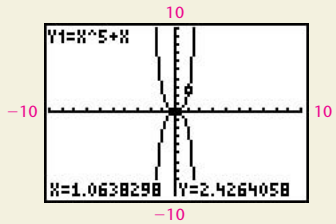


Figure 22

MATCHED PROBLEM 5

Determine whether the functions F, G, and H are even, odd, or neither:

(A) $F(x) = x^3 - 2x$ (B) $G(x) = x^2 + 1$ (C) $H(x) = 2x + 4$

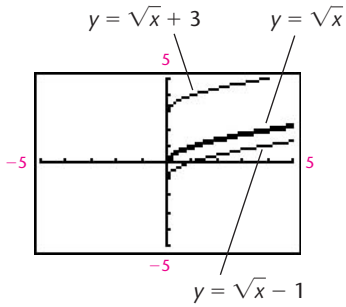
In the solution of Example 5, notice that we used the fact that

$$(-x)^n = \begin{cases} x^n & \text{if } n \text{ is an even integer} \\ -x^n & \text{if } n \text{ is an odd integer} \end{cases}$$

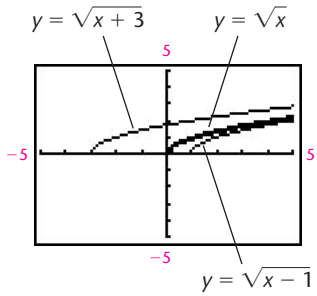
It is this property that explains the use of the terms *even* and *odd* when describing symmetry properties of the graphs of functions. In addition to being an aid to graphing, certain problems and developments in calculus and more advanced mathematics are simplified if we can recognize when a function is even or odd.

ANSWERS TO MATCHED PROBLEMS

1. (A) The graph of $y = \sqrt{x} + 3$ is the same as the graph of $y = \sqrt{x}$ shifted upward three units, and the graph of $y = \sqrt{x} - 1$ is the same as the graph of $y = \sqrt{x}$ shifted downward one unit. The figure confirms these conclusions.

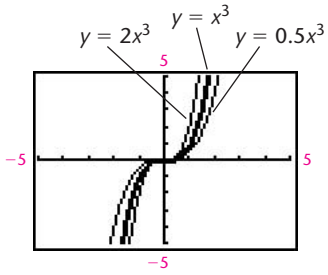


(B) The graph of $y = \sqrt{x + 3}$ is the same as the graph of $y = \sqrt{x}$ shifted to the left three units, and the graph of $y = \sqrt{x - 1}$ is the same as the graph of $y = \sqrt{x}$ shifted to the right one unit. The figure confirms these conclusions.

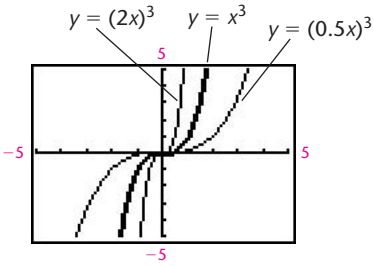


2. $G(x) = (x + 3)^3$, $H(x) = (x - 1)^3$, $M(x) = x^3 + 3$, $N(x) = x^3 - 4$

3. (A) The graph of $y = 2x^3$ is a vertical stretch of the graph of $y = x^3$ by a factor of 2, and the graph of $y = 0.5x^3$ is a vertical shrink of the graph of $y = x^3$ by a factor of 1/2. The figure confirms these conclusions.



(B) The graph of $y = (2x)^3$ is a horizontal shrink of the graph of $y = x^3$ by a factor of 1/2, and the graph of $y = (0.5x)^3$ is a horizontal stretch of the graph of $y = x^3$ by a factor of 2. The figure confirms these conclusions.



4. The graph of function h is a reflection in the x axis and a horizontal translation of three units to the left of the graph of $y = x^3$. An equation for h is $h(x) = -(x + 3)^3$.

5. (A) Odd (B) Even (C) Neither

1-4 Exercises

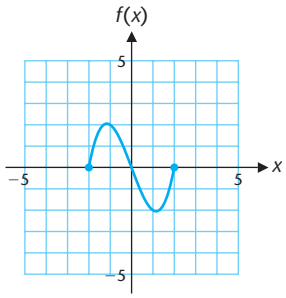
1. Explain why the graph of $y = f(x) + k$ is the same as the graph of $y = f(x)$ moved upward k units when k is positive.

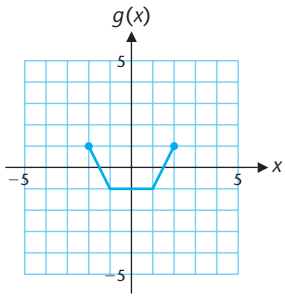
2. Explain why the graph of $y = Af(x)$ is a vertical stretch of the graph of $y = f(x)$ if $A > 1$.

3. Is every function either even or odd? Explain.

4. Explain how the informal description of even and odd functions in terms of reflections on page 76 really says the same thing as the formal definition in the box on that page.

Problems 5–20 refer to the functions f and g given by the graphs below (the domain of each function is [−2, 2]). Use the graph of f or g, as required, to graph each given function.





5. $f(x) + 2$

6. $g(x) - 1$

7. $g(x) + 2$

8. $f(x) - 1$

9. $f(x - 2)$

10. $g(x - 1)$

11. $g(x + 2)$

12. $f(x - 1)$

13. $-f(x)$

14. $-g(x)$

15. $2g(x)$

16. $\frac{1}{2}f(x)$

17. $g(2x)$

18. $f(\frac{1}{2}x)$

19. $f(-x)$

20. $-g(-x)$

Indicate whether each function in Problems 21–30 is even, odd, or neither.

21. $g(x) = x^3 + x$

22. $f(x) = x^5 - x$

23. $m(x) = x^4 + 3x^2$

24. $h(x) = x^4 - x^2$

25. $F(x) = x^5 + 1$

26. $f(x) = x^5 - 3$

27. $G(x) = x^4 + 2$

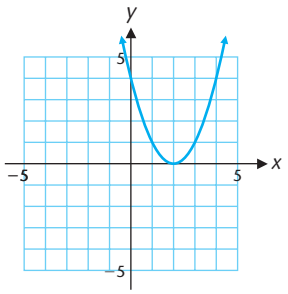
28. $P(x) = x^4 - 4$

29. $q(x) = x^2 + x - 3$

30. $n(x) = 2x - 3$

Each graph in Problems 31–38 is the result of applying a transformation to the graph of one of the six basic functions in Figure 1. Identify the basic function, describe the transformation verbally, and find an equation for the given graph. Check by graphing the equation on a graphing calculator.

31.



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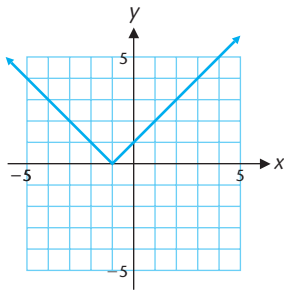

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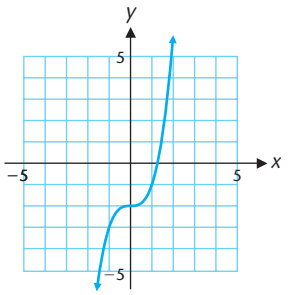

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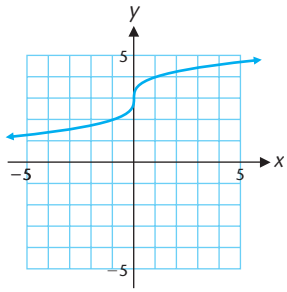

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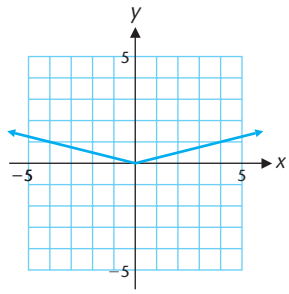

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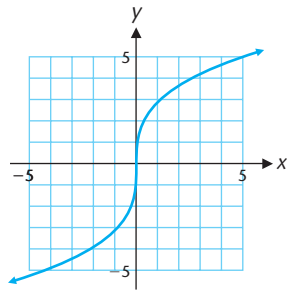

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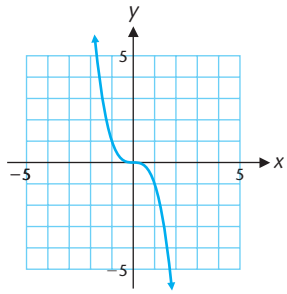
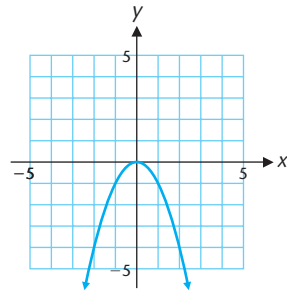


38.



In Problems 39–46, the graph of the function g is formed by applying the indicated sequence of transformations to the given function f. Find an equation for the function g. Check your work by graphing f and g in a standard viewing window.

39. The graph of $f(x) = \sqrt[3]{x}$ is shifted four units to the left and five units down.

40. The graph of $f(x) = x^3$ is shifted five units to the right and four units up.

41. The graph of $f(x) = \sqrt{x}$ is shifted six units up, reflected in the x axis, and vertically shrunk by a factor of 0.5.

42. The graph of $f(x) = \sqrt{x}$ is shifted two units down, reflected in the x axis, and vertically stretched by a factor of 4.

43. The graph of $f(x) = x^2$ is reflected in the x axis, vertically stretched by a factor of 2, shifted four units to the left, and shifted two units down.

44. The graph of $f(x) = |x|$ is reflected in the x axis, vertically shrunk by a factor of 0.5, shifted three units to the right, and shifted four units up.

45. The graph of $f(x) = \sqrt{x}$ is horizontally stretched by a factor of 0.5, reflected in the y axis, and shifted two units to the left.

46. The graph of $f(x) = \sqrt[3]{x}$ is horizontally shrunk by a factor of 2, shifted three units up, and reflected in the y axis.

In Problems 47–54, indicate how the graph of each function is related to the graph of one of the six basic functions in Figure 1, then graph the function.

47. $f(x) = (x + 7)^2 + 9$

48. $g(x) = (x - 4)^2 - 6$

49. $h(x) = -|x - 8|$

50. $k(x) = -|x + 5|$

51. $p(x) = 3 - \sqrt{x}$

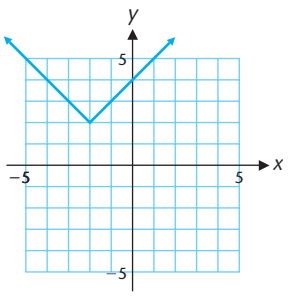
52. $q(x) = -2 + \sqrt{x + 3}$

53. $r(x) = -4x^2$

54. $s(x) = -0.5|x|$

Each graph in Problems 55–62 is the result of applying a sequence of transformations to the graph of one of the six basic functions in Figure 1. Find an equation for the given graph. Check by graphing the equation on a graphing calculator.

55.



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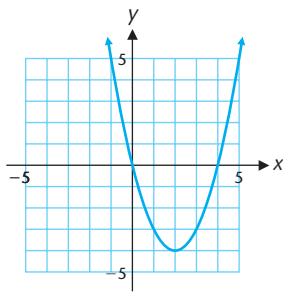

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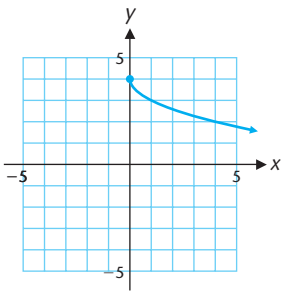

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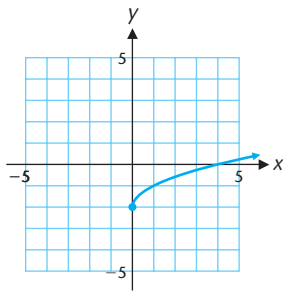

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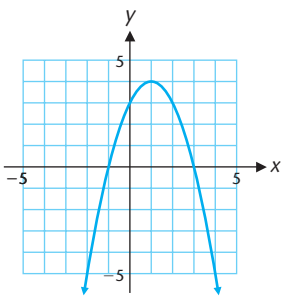

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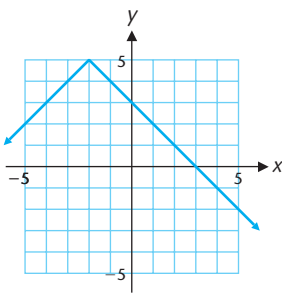

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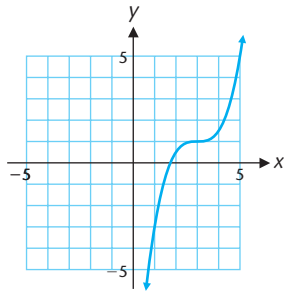

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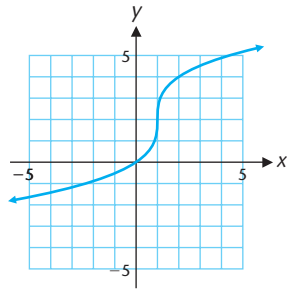


63. Consider the graphs of $f(x) = \sqrt[3]{8x}$ and $g(x) = 2\sqrt[3]{x}$.
(A) Describe each as a stretch or shrink of $y = \sqrt[3]{x}$.
(B) Graph both functions in the same viewing window on a graphing calculator. What do you notice?
(C) Rewrite the formula for f algebraically to show that f and g are in fact the same function. (This shows that for some functions, a horizontal stretch or shrink can also be interpreted as a vertical stretch or shrink.)

64. Consider the graphs of $f(x) = (3x)^3$ and $g(x) = 27x^3$.
(A) Describe each as a stretch or shrink of $y = x^3$.
(B) Graph both functions in the same viewing window on a graphing calculator. What do you notice?
(C) Rewrite the formula for f algebraically to show that f and g are in fact the same function. (This shows that for some functions, a horizontal stretch or shrink can also be interpreted as a vertical stretch or shrink.)

65. (A) Starting with the graph of $y = x^2$, apply the following transformations.
(i) Shift downward 5 units, then reflect in the x axis.
(ii) Reflect in the x axis, then shift downward 5 units.
What do your results indicate about the significance of order when combining transformations?
(B) Write a formula for the function corresponding to each of the above transformations. Discuss the results of part A in terms of order of operations.

66. (A) Starting with the graph of $y = |x|$, apply the following transformations.
(i) Stretch vertically by a factor of 2, then shift upward 4 units.
(ii) Shift upward 4 units, then stretch vertically by a factor of 2.
What do your results indicate about the significance of order when combining transformations?
(B) Write a formula for the function corresponding to each of the above transformations. Discuss the results of part A in terms of order of operations.

Changing the order in a sequence of transformations may change the final result. Investigate each pair of transformations in Problems 67–72 to determine if reversing their order can produce a different result. Support your conclusions with specific examples and/or mathematical arguments.

67. Vertical shift, horizontal shift

68. Vertical shift, reflection in y axis

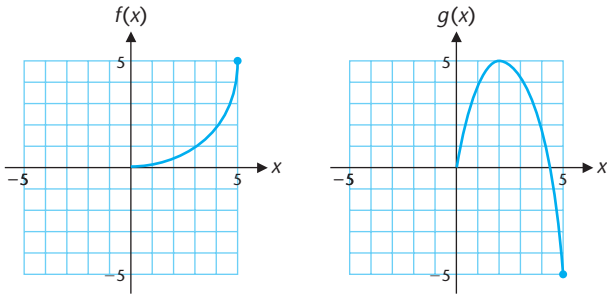
69. Vertical shift, reflection in x axis

70. Vertical shift, stretch

71. Horizontal shift, reflection in x axis

72. Horizontal shift, shrink

Problems 73–76 refer to two functions f and g with domain $[-5, 5]$ and partial graphs as shown below.



73. Complete the graph of f over the interval $[-5, 0]$, given that f is an even function.

74. Complete the graph of f over the interval $[-5, 0]$, given that f is an odd function.

75. Complete the graph of g over the interval $[-5, 0]$, given that g is an odd function.

76. Complete the graph of g over the interval $[-5, 0]$, given that g is an even function.

77. Let f be any function with the property that $-x$ is in the domain of f whenever x is in the domain of f, and let E and O be the functions defined by

$$E(x) = \tfrac{1}{2}[f(x) + f(-x)]$$

and

$$O(x) = \tfrac{1}{2}[f(x) - f(-x)]$$

(A) Show that E is always even.
(B) Show that O is always odd.
(C) Show that $f(x) = E(x) + O(x)$. What is your conclusion?

78. Let f be any function with the property that $-x$ is in the domain of f whenever x is in the domain of f, and let $g(x) = xf(x)$.
(A) If f is even, is g even, odd, or neither?
(B) If f is odd, is g even, odd, or neither?

In Problems 79–82, graph $f(x)$, $|f(x)|$, and $-|f(x)|$ in a standard viewing window. For purposes of comparison, it will be helpful to graph each function separately and make a hand sketch.

79. $f(x) = 0.2x^2 - 5$ **80.** $f(x) = 4 - 0.25x^2$

81. $f(x) = 4 - 0.1(x + 2)^3$ **82.** $f(x) = 0.25(x - 1)^3 - 1$

83. Describe the relationship between the graphs of $f(x)$ and $|f(x)|$ in Problems 79–82.

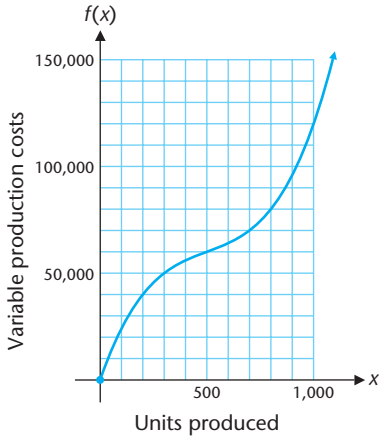
84. Describe the relationship between the graphs of $f(x)$ and $-|f(x)|$ in Problems 79–82.

APPLICATIONS

85. **PRODUCTION COSTS** Total production costs for a product can be broken down into fixed costs, which do not depend on the number of units produced, and variable costs, which do depend on the number of units produced. Thus, the total cost of producing x units of the product can be expressed in the form

$$C(x) = K + f(x)$$

where K is a constant that represents the fixed costs and $f(x)$ is a function that represents the variable costs. Use the graph of the variable-cost function $f(x)$ shown in the figure to graph the total cost function if the fixed costs are \$30,000.



86. **COST FUNCTIONS** Refer to the variable-cost function $f(x)$ in Problem 85. Suppose construction of a new production facility results in a 25% decrease in the variable cost at all levels of output. If F is the new variable-cost function, use the graph of f to graph $y = F(x)$, then graph the total cost function for fixed costs of \$30,000.

87. **TIMBER HARVESTING** To determine when a forest should be harvested, forest managers often use formulas to estimate the number of board feet a tree will produce. A board foot equals

1 square foot of wood, 1 inch thick. Suppose that the number of board feet y yielded by a tree can be estimated by

$$y = f(x) = C + 0.004(x - 10)^3$$

where x is the diameter of the tree in inches measured at a height of 4 feet above the ground and C is a constant that depends on the species being harvested. Graph $y = f(x)$ for $C = 10, 15,$ and 20 simultaneously in the viewing window with Xmin $= 10$, Xmax $= 25$, Ymin $= 10$, and Ymax $= 35$. Write a brief verbal description of this collection of functions.

88. SAFETY RESEARCH If a person driving a vehicle slams on the brakes and skids to a stop, the speed v in miles per hour at the time the brakes are applied is given approximately by

$$v = f(x) = C\sqrt{x}$$

where x is the length of the skid marks and C is a constant that depends on the road conditions and the weight of the vehicle. The table lists values of C for a midsize automobile and various road conditions. Graph $v = f(x)$ for the values of C in the table simultaneously in the viewing window with Xmin $= 0$, Xmax $= 100$, Ymin $= 0$, and Ymax $= 60$. Write a brief verbal description of this collection of functions.

Road Condition	C
Wet (concrete)	3.5
Wet (asphalt)	4
Dry (concrete)	5
Dry (asphalt)	5.5

89. FAMILY OF CURVES In calculus, solutions to certain types of problems often involve an unspecified constant. For example, consider the equation

$$y = \frac{1}{C}x^2 - C$$

where C is a positive constant. The collection of graphs of this equation for all permissible values of C is called a **family of curves.** Graph the members of this family corresponding to $C = 2, 3, 4,$ and 5 simultaneously in a standard viewing window. Write a brief verbal description of this family of functions.

90. FAMILY OF CURVES A family of curves is defined by the equation

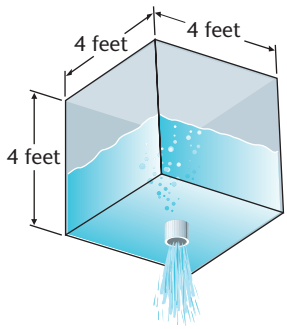
$$y = 2C - \frac{5}{C}x^2$$

where C is a positive constant. Graph the members of this family corresponding to $C = 1, 2, 3,$ and 4 simultaneously in a standard viewing window. Write a brief verbal description of this family of functions.

91. FLUID FLOW A cubic tank is 4 feet on a side and is initially full of water. Water flows out an opening in the bottom of the tank at a rate proportional to the square root of the depth (see the figure). Using advanced concepts from mathematics and physics, it can be shown that the volume of the water in the tank t minutes after the water begins to flow is given by

$$V(t) = \frac{64}{C^2}(C - t)^2 \qquad 0 \le t \le C$$

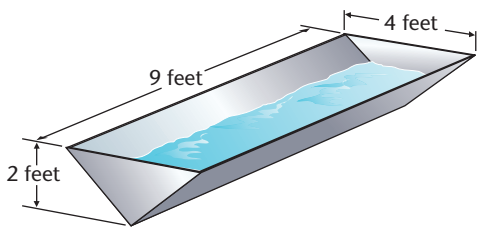
where C is a constant that depends on the size of the opening. Sketch by hand the graphs of $y = V(t)$ for $C = 1, 2, 4,$ and 8. Write a brief verbal description of this collection of functions. Based on the graphs, do larger values of C correspond to a larger or smaller opening?



92. EVAPORATION A water trough with triangular ends is 9 feet long, 4 feet wide, and 2 feet deep (see the figure). Initially, the trough is full of water, but due to evaporation, the volume of the water in the trough decreases at a rate proportional to the square root of the volume. Using advanced concepts from mathematics and physics, it can be shown that the volume after t hours is given by

$$V(t) = \frac{1}{C^2}(t + 6C)^2 \quad 0 \le t \le 6|C|$$

where C is a constant. Sketch by hand the graphs of $y = V(t)$ for $C = -4, -5,$ and -6. Write a brief verbal description of this collection of functions. Based on the graphs, do values of C with a larger absolute value correspond to faster or slower evaporation?



1-5 Operations on Functions; Composition

- Performing Operations on Functions
- Composing Functions
- Mathematical Modeling

Perhaps the most basic thing you've done in math classes is operations on numbers: things like addition, subtraction, multiplication, and division. In this section, we will explore the concept of operations on functions. In many cases, combining functions will enable us to model more complex and useful situations.

If two functions f and g are both defined at some real number x, then $f(x)$ and $g(x)$ are both real numbers, so it makes sense to perform the four basic arithmetic operations with $f(x)$ and $g(x)$. Furthermore, if $g(x)$ is a number in the domain of f, then it is also possible to evaluate f at $g(x)$. We will see that operations on the outputs of the functions can be used to define operations on the functions themselves.

Performing Operations on Functions

The functions f and g given by

$$f(x) = 2x + 3 \text{ and } g(x) = x^2 - 4$$

are both defined for all real numbers. Note that $f(3) = 9$ and $g(3) = 5$, so it would seem reasonable to assign the value $9 + 5$, or 14, to a new function $(f + g)(x)$. Based on this idea, for any real x we can perform the operation

$$f(x) + g(x) = (2x + 3) + (x^2 - 4) = x^2 + 2x - 1$$

Similarly, we can define other operations on functions:

$$f(x) - g(x) = (2x + 3) - (x^2 - 4) = -x^2 + 2x + 7$$
$$f(x)g(x) = (2x + 3)(x^2 - 4) = 2x^3 + 3x^2 - 8x - 12$$

For $x \neq \pm 2$ (to avoid zero in the denominator) we can also form the quotient

$$\frac{f(x)}{g(x)} = \frac{2x + 3}{x^2 - 4} \qquad x \neq \pm 2$$

Notice that the result of each operation is a new function. Thus, we have

$$(f + g)(x) = f(x) + g(x) = x^2 + 2x - 1 \quad \text{Sum}$$
$$(f - g)(x) = f(x) - g(x) = -x^2 + 2x + 7 \quad \text{Difference}$$
$$(fg)(x) = f(x)g(x) = 2x^3 + 3x^2 - 8x - 12 \quad \text{Product}$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x + 3}{x^2 - 4} \qquad x \neq \pm 2 \quad \text{Quotient}$$

The sum, difference, and product functions are defined for all values of x, as were the original functions f and g, but the domain of the quotient function must be restricted to exclude those values where $g(x) = 0$.

› DEFINITION 1 Operations on Functions

The **sum, difference, product,** and **quotient** of the functions f and g are the functions defined by

$$(f + g)(x) = f(x) + g(x) \qquad \text{Sum function}$$

$$(f - g)(x) = f(x) - g(x) \qquad \text{Difference function}$$

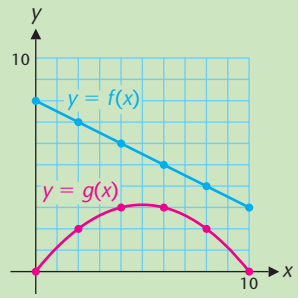
$$(fg)(x) = f(x)g(x) \qquad \text{Product function}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad g(x) \neq 0 \qquad \text{Quotient function}$$

The **domain** of each function consists of all elements in the domains of *both* f and g, with the exception that the values of x where $g(x) = 0$ must be excluded from the domain of the quotient function.

››› EXPLORE-DISCUSS 1

The following activities refer to the graphs of f and g shown in Figure 1 and the corresponding points on the graph shown in Table 1.



› Figure 1

Table 1

x	$f(x)$	$g(x)$
0	8	0
2	7	2
4	6	3
6	5	3
8	4	2
10	3	0

For each of the following functions, construct a table of values, sketch a graph, and state the domain and range.

(A) $(f + g)(x)$ **(B)** $(f - g)(x)$ **(C)** $(fg)(x)$ **(D)** $\left(\frac{f}{g}\right)(x)$

EXAMPLE 1 Finding the Sum of Two Functions

Let $f(x) = \sqrt{4 - x}$ and $g(x) = \sqrt{3 + x}$. Find $f + g$ and find its domain.

SOLUTIONS

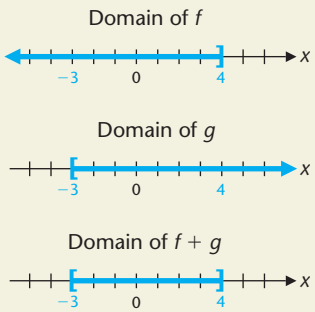
Algebraic Solution

$$(f + g)(x) = f(x) + g(x)$$
$$= \sqrt{4 - x} + \sqrt{3 + x}$$

The domains of f and g are

Domain of f: $x \le 4$ or $(-\infty, 4]$
Domain of g: $x \ge -3$ or $[-3, \infty)$

Any x value that makes either of f or g undefined will make $f + g$ undefined as well, so only x values in the domains of *both* f and g are in the domain of $f + g$. In other words, the domain of $f + g$ is the intersection* of the first two sets in Figure 2:

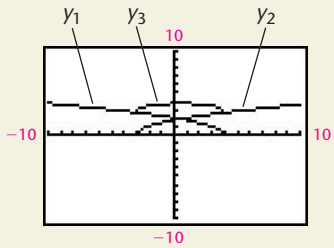


› Figure 2

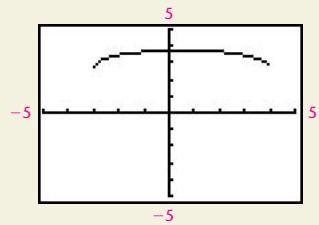
The domain of $f + g$ is $[-3, 4]$.

Graphical Solution

We begin by entering $y_1 = \sqrt{4 - x}$, $y_2 = \sqrt{3 + x}$ in the equation editor of a graphing calculator. Note that $(f + g)(x) = \sqrt{4 - x} + \sqrt{3 + x}$. For convenience, we can enter this as $y_3 = y_1 + y_2$. The graph 5 of all three functions are shown in a standard viewing window in Figure 3. To get a better look at y_3, we turn off the graphs of y_1 and y_2, and change the viewing window (Fig. 4).

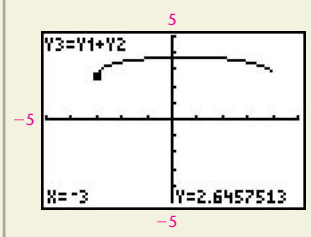


› Figure 3 Graphs of y_1, y_2, and y_3.

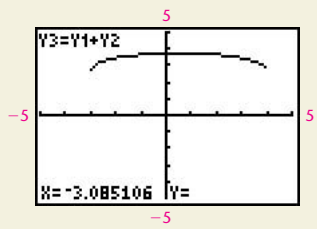


› Figure 4 Graph of y_3.

Next we press TRACE and enter -3 (Fig. 5). Pressing the left arrow suggests that y_3 is not defined for $x < -3$ (Note the lack of y value in Fig. 6).



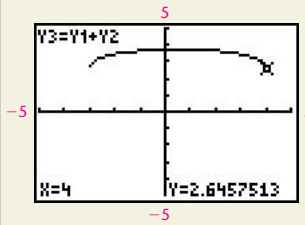
› Figure 5



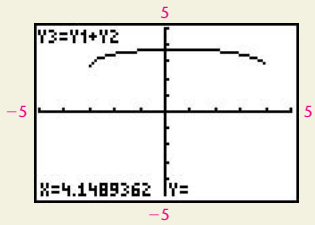
› Figure 6

*Intersection of intervals is discussed in Appendix B, Section B-1.

Figures 7 and 8 suggest that y_3 is not defined for $x > 4$. Thus, the domain of $y_3 = f + g$ is $[-3, 4]$.



› Figure 7



› Figure 8

MATCHED PROBLEM 1

Let $f(x) = \sqrt{x}$ and $g(x) = \sqrt{10 - x}$. Find $f - g$ and find its domain.

EXAMPLE 2 **Finding the Quotient of Two Functions**

Let $f(x) = \dfrac{x}{x-1}$ and $g(x) = \dfrac{x-4}{x+3}$. Find the function $\dfrac{f}{g}$ and find its domain.

SOLUTION

Because division by 0 must be excluded, the domain of f is all x except $x = 1$ and the domain of g is all x except $x = -3$. Now we find f/g.*

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\dfrac{x}{x-1}}{\dfrac{x-4}{x+3}}$$

To divide by a fraction, multiply by the reciprocal.

$$= \frac{x}{x-1} \cdot \frac{x+3}{x-4}$$

Multiply numerators and denominators.

$$= \frac{x(x+3)}{(x-1)(x-4)} \tag{1}$$

The fraction in equation (1) indicates that 1 and 4 must be excluded from the domain of f/g to avoid division by 0. But, equation (1) does not indicate that -3 must be excluded also. Although the fraction in equation (1) is defined at $x = -3$, -3 was excluded from the domain of g, so it must be excluded from the domain of f/g also.

The domain of f/g is all real numbers x except -3, 1, and 4 or $\{x \mid x \neq -3, 1, 4\}$.

*Operations on fractions are discussed in Appendix A, Section A-1.

MATCHED PROBLEM 2

Let $f(x) = \dfrac{1}{x+2}$ and $g(x) = \dfrac{x-5}{x}$. Find the function $\dfrac{f}{g}$ and find its domain.

Composing Functions

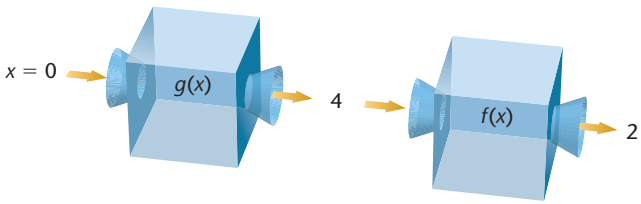
Consider the functions f and g given by

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = 4 - 2x$$

Note that $g(0) = 4 - 2(0) = 4$ and $f(4) = \sqrt{4} = 2$. So if we apply these two functions *consecutively*, we get

$$f(g(0)) = f(4) = 2$$

In a diagram, this would look like



When two functions are applied consecutively, we call the result the **composition** of functions. We will use the symbol $f \circ g$ to represent the composition of f and g, which we formally define now.

DEFINITION 2 Composition of Functions

The composition of a function f with another function g is denoted by $f \circ g$ (read "f composed with g") and is defined by

$$(f \circ g)(x) = f(g(x))$$

EXAMPLE 3 Computing Composition From a Table

Functions f and g are defined by Table 2. Find $(f \circ g)(2)$, $(f \circ g)(5)$, and $(f \circ g)(-3)$.

Table 2

x	$f(x)$	$g(x)$
-5	-8	11
-3	-6	2
0	-1	-6
2	5	-3
5	12	0

SOLUTION

We will use the formula provided by Definition 2.

$$(f \circ g)(2) = f(g(2)) = f(-3) = -6$$
$$(f \circ g)(5) = f(g(5)) = f(0) = -1$$
$$(f \circ g)(-3) = f(g(-3)) = f(2) = 5$$

MATCHED PROBLEM 3

Functions h and k are defined by Table 3. Find $(h \circ k)(10)$, $(h \circ k)(-8)$, and $(h \circ k)(0)$.

Table 3

x	$h(x)$	$k(x)$
-8	12	0
-4	18	22
0	40	-4
10	52	-8
20	70	-30

››› CAUTION ›››

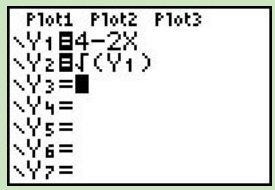
When computing $f \circ g$, it's important to keep in mind that the first function that appears in the notation (f, in this case) is actually the second function that is applied. For this reason, some people read $f \circ g$ as "f following g."

››› EXPLORE-DISCUSS 2

For $f(x) = \sqrt{x}$ and $g(x) = 4 - 2x$, complete Table 4.

Table 4

x	$g(x)$	$h(x) = f(g(x))$
0	$g(0) = 4$	$h(0) = f(g(0)) = f(4) = 2$
1		
2		
3		
4		



› Figure 9

Now enter g and h in the equation editor of a graphing calculator as shown in Figure 9 and check your table. The domain of f is $\{x \mid x \geq 0\}$ and the domain of g is the set of all real numbers. What is the domain of h? Support your conclusion both algebraically and graphically.

So far, we have looked at composition on a point-by-point basis. Using algebra, we can find a formula for the composition of two functions.

EXAMPLE 4 **Finding the Composition of Two Functions**

Find $(f \circ g)(x)$ for $f(x) = x^2 - x$ and $g(x) = 3 + 2x$.

SOLUTION

We again use the formula in Definition 2.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(3 + 2x) \\ &= (3 + 2x)^2 - (3 + 2x) \\ &= 9 + 12x + 4x^2 - 3 - 2x \\ &= 4x^2 + 10x + 6\end{aligned}$$

MATCHED PROBLEM 4

Find $(h \circ k)(x)$ for $h(x) = 11 + x^2$ and $k(x) = 4x - 1$.

››› EXPLORE-DISCUSS 3

(A) For $f(x) = x - 10$ and $g(x) = 3 + 7x$, find $(f \circ g)(x)$ and $(g \circ f)(x)$. Based on this result, what do you think is the relationship between $f \circ g$ and $g \circ f$ in general?

(B) Repeat for $f(x) = 2x + 1$ and $g(x) = \dfrac{x - 1}{2}$. Does this change your thoughts on the relationship between $f \circ g$ and $g \circ f$?

Explore-Discuss 3 tells us that order is important in composition. Sometimes $f \circ g$ and $g \circ f$ are equal, but more often they are not.

Finding the domain of a composition of functions can sometimes be a bit tricky. Based on the definition $(f \circ g)(x) = f(g(x))$, we can see that for an x value to be in the domain of $f \circ g$, two things must occur. First, x must be in the domain of g so that $g(x)$ is defined. Second, $g(x)$ must be in the domain of f, so that $f(g(x))$ is defined.

EXAMPLE 5 Finding the Composition of Two Functions

Find $(f \circ g)(x)$ and $(g \circ f)(x)$ and their domains for $f(x) = x^{10}$ and $g(x) = 3x^4 - 1$.

SOLUTION

$$(f \circ g)(x) = f(g(x)) = f(3x^4 - 1) = (3x^4 - 1)^{10}$$
$$(g \circ f)(x) = g(f(x)) = g(x^{10}) = 3(x^{10})^4 - 1 = 3x^{40} - 1$$

Note that the functions f and g are both defined for all real numbers. If x is any real number, then x is in the domain of g, so $g(x)$ is a real number. This then tells us that $g(x)$ is in the domain of f, which means that $f(g(x))$ is a real number. In other words, every real number is in the domain of $f \circ g$. Using similar reasoning, we can conclude that the domain of $g \circ f$ is also the set of all real numbers.

MATCHED PROBLEM 5

Find $(f \circ g)(x)$ and $(g \circ f)(x)$ and their domains for $f(x) = \sqrt[3]{x}$ and $g(x) = 7x + 5$.

The line of reasoning used in Example 5 can be used to deduce the following fact:

If two functions are both defined for all real numbers, then so is their composition.

››› EXPLORE-DISCUSS 4

Verify that if $f(x) = 1/(1 - 2x)$ and $g(x) = 1/x$, then $(f \circ g)(x) = x/(x - 2)$. Because division by 0 is not defined, $f \circ g$ is not defined at $x = 2$. Are there any other values of x where $f \circ g$ is not defined? Explain.

If either function in a composition is not defined for some real numbers, then, as Example 6 illustrates, the domain of the composition may not be what you first think it should be.

EXAMPLE 6 Finding the Composition of Two Functions

Find $(f \circ g)(x)$ for $f(x) = \sqrt{4 - x^2}$ and $g(x) = \sqrt{3 - x}$. Find the domain algebraically and check graphically.

SOLUTION

We begin by stating the domains of f and g, which is a good idea in any composition problem:

$$\text{Domain } f\text{: } -2 \le x \le 2 \quad \text{or} \quad [-2, 2]$$
$$\text{Domain } g\text{: } x \le 3 \quad \text{or} \quad (-\infty, 3]$$

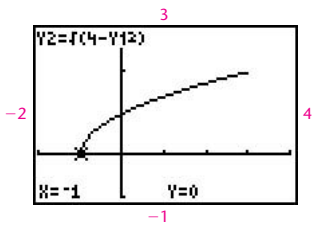
Next we find the composition:

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(\sqrt{3 - x}) && \text{Substitute } \sqrt{3 - x} \text{ for } g(x). \\ &= \sqrt{4 - (\sqrt{3 - x})^2} && \text{Square } (\sqrt{t})^2 = t \text{ as long as } t \ge 0. \\ &= \sqrt{4 - (3 - x)} && \text{Subtract.} \\ &= \sqrt{1 + x}\end{aligned}$$

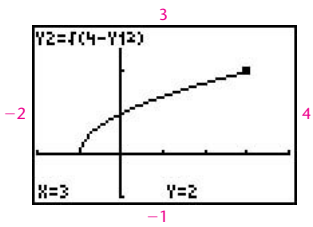
Although $\sqrt{1 + x}$ is defined for all $x \ge -1$, we must restrict the domain of $f \circ g$ to those values that also are in the domain of g. Thus,

$$\text{Domain } f \circ g\text{: } x \ge -1 \text{ and } x \le 3 \quad \text{or} \quad [-1, 3]$$

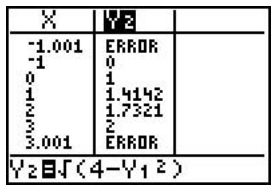
To check this, enter $y_1 = \sqrt{3 - x}$ and $y_2 = \sqrt{4 - y_1^2}$. This defines y_2 as the composition $f \circ g$. Graph y_2 and use TRACE or a table to verify that $[-1, 3]$ is the domain of $f \circ g$ (Figs. 10–12).



› Figure 10



› Figure 11



› Figure 12

MATCHED PROBLEM 6

Find $f \circ g$ for $f(x) = \sqrt{9 - x^2}$ and $g(x) = \sqrt{x - 1}$. Find the domain of $f \circ g$ algebraically and check graphically.

››› CAUTION ›››

The domain of $f \circ g$ cannot always be determined simply by examining the final form of $(f \circ g)(x)$. Any numbers that are excluded from the domain of g must also be excluded from the domain of $f \circ g$.

››› EXPLORE-DISCUSS 5

Here is another way to enter the composition of two functions in a graphing calculator. Refer to Example 6. Enter $y_1 = \sqrt{3 - x}$, $y_2 = \sqrt{4 - x^2}$, and $y_3 = y_2(y_1)$ in the equation editor of your graphing calculator and graph y_3. Does this graph agree with the graph we found in Example 6? Does your graphing calculator seem to handle this composition correctly? (Not all do!)

In calculus, it is not only important to be able to find the composition of two functions, but also to recognize when a given function is the composition of simpler functions.

EXAMPLE 7 Recognizing Composition Forms

Express h as a composition of two simpler functions for

$$h(x) = \sqrt{1 + 3x^4}$$

SOLUTION

If we were to evaluate this function for some x value, say, $x = 1$, we would do so in two stages. First, we would find the value of $1 + 3(1)^4$, which is 4. Then we would apply the square root to get 2. This shows that h can be thought of as two consecutive functions: First, $g(x) = 1 + 3x^4$, then $f(x) = \sqrt{x}$. So $h(x) = f(g(x))$, and we have written h as $f \circ g$.

MATCHED PROBLEM 7

Express h as the composition of two simpler functions for $h(x) = (4x^3 - 7)^4$.

The answers to Example 7 and Matched Problem 7 are not unique. For example, if $f(x) = \sqrt{1 + 3x}$ and $g(x) = x^4$, then

$$f(g(x)) = \sqrt{1 + 3g(x)} = \sqrt{1 + 3x^4} = h(x)$$

› Mathematical Modeling

The operations discussed in this section can be applied in many different situations. The next example shows how they are used to construct a model in economics.

EXAMPLE 8 **Modeling Profit**

The research department for an electronics firm estimates that the weekly demand for a certain brand of headphones is given by

$$x = f(p) = 20{,}000 - 1{,}000p \qquad 0 \le p \le 20 \qquad \text{Demand function}$$

This function describes the number x of pairs of headphones retailers are likely to buy per week at p dollars per pair. The research department also has determined that the total cost (in dollars) of producing x pairs per week is given by

$$C(x) = 25{,}000 + 3x \qquad \text{Cost function}$$

and the total weekly revenue (in dollars) obtained from the sale of these headphones is given by

$$R(x) = 20x - 0.001x^2 \qquad \text{Revenue function}$$

Express the firm's weekly profit as a function of the price p and find the price that produces the largest profit.

SOLUTION

The basic economic principle we are using is that profit is revenue minus cost. Thus, the profit function P is the difference of the revenue function R and the cost function C.

$$\begin{aligned} P(x) &= (R - C)(x) \\ &= R(x) - C(x) \\ &= (20x - 0.001x^2) - (25{,}000 + 3x) \\ &= 17x - 0.001x^2 - 25{,}000 \end{aligned}$$

This is a function of the demand x. We were asked to find the profit P as a function of the price p; we can accomplish this using composition, because $x = f(p)$.

$$\begin{aligned} (P \circ f)(p) &= P(f(p)) \\ &= P(20{,}000 - 1{,}000p) \\ &= 17(20{,}000 - 1{,}000p) - 0.001(20{,}000 - 1{,}000p)^2 - 25{,}000 \\ &= 340{,}000 - 17{,}000p - 400{,}000 + 40{,}000p - 1{,}000p^2 - 25{,}000 \\ &= -85{,}000 + 23{,}000p - 1{,}000p^2 \end{aligned}$$

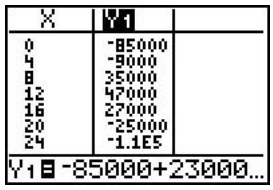
Technically, $P \circ f$ and P are different functions, because the first has independent variable p and the second has independent variable x. However, because both functions represent the same quantity (the profit), it is customary to use the same symbol to name each function. Thus,

$$P(p) = -85{,}000 + 23{,}000p - 1{,}000p^2$$

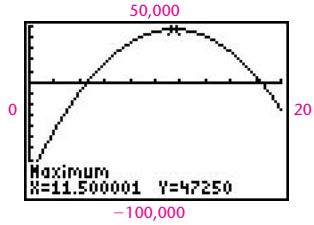
expresses the weekly profit P as a function of price p. To find the price that produces the largest profit, we examine the graph of P. To do this, we will enter the function into our graphing calculator, using y_1 and x in place of P and p. Define

$$y_1 = -85{,}000 + 23{,}000x - 1{,}000x^2 \qquad 0 \le x \le 20$$

The limits for x were given in the statement of the problem. Examining a table (Fig. 13) suggests that reasonable limits on y_1 are $-100{,}000 \le y_1 \le 50{,}000$. Graphing y_1 and using the MAXIMUM command (Fig. 14) shows that the largest profit occurs when the price of a pair of headphones is \$11.50.



› Figure 13



› Figure 14

MATCHED PROBLEM 8

Repeat Example 8 for the functions

$$x = f(p) = 10{,}000 - 1{,}000p \qquad 0 \le p \le 10$$
$$C(x) = 10{,}000 + 2x \qquad R(x) = 10x - 0.001x^2$$

ANSWERS TO MATCHED PROBLEMS

1. $(f - g)(x) = \sqrt{x} - \sqrt{10 - x}$; domain: $[0, 10]$
2. $\left(\frac{f}{g}\right)(x) = \frac{x}{(x + 2)(x - 5)}$; domain: all real numbers x except -2, 0, and 5
3. $(h \circ k)(10) = 12$; $(h \circ k)(-8) = 40$; $(h \circ k)(0) = 18$
4. $(h \circ k)(x) = 16x^2 - 8x + 12$
5. $(f \circ g)(x) = \sqrt[3]{7x + 5}$, domain: $(-\infty, \infty)$
   $(g \circ f)(x) = 7\sqrt[3]{x + 5}$, domain: $(-\infty, \infty)$
6. $(f \circ g)(x) = \sqrt{10 - x}$; domain: $x \ge 1$ and $x \le 10$ or $[1, 10]$
7. $h(x) = (f \circ g)(x)$ where $f(x) = x^4$ and $g(x) = 4x^3 - 7$
8. $P(p) = -30{,}000 + 12{,}000p - 1{,}000p^2$
   The largest profit occurs when the price is \$6.

1-5 Exercises

1. Explain how to find the sum of two functions.

2. Explain how to find the product of two functions.

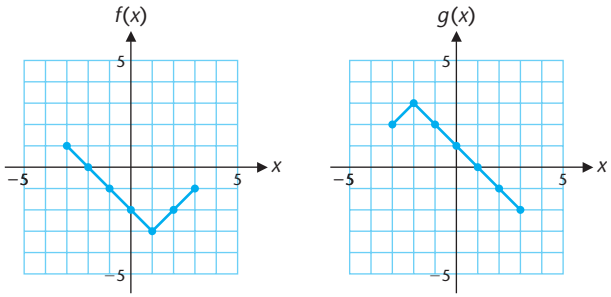
3. Is the domain of f/g always the intersection of the domains of f and g? Explain.

4. Is the domain of fg always the intersection of the domains of f and g? Explain.

5. Is composition of functions a commutative* process? Explain.

6. Which of addition, subtraction, multiplication, and division of functions is commutative*? Explain.

Problems 7–18 refer to functions f and g whose graphs are shown below.



In Problems 7–10 use the graphs of f and g to construct a table of values and sketch the graph of the indicated function.

7. $(f + g)(x)$ **8.** $(g - f)(x)$

9. $(fg)(x)$ **10.** $(f - g)(x)$

In Problems 11–18, use the graphs of f and g to find each of the following:

11. $(f \circ g)(-1)$ **12.** $(f \circ g)(2)$

13. $(g \circ f)(-2)$ **14.** $(g \circ f)(3)$

15. $f(g(1))$ **16.** $f(g(0))$

17. $g(f(2))$ **18.** $g(f(-3))$

19. Functions f and g are defined by Table 5. Find $(f \circ g)(-7)$, $(f \circ g)(0)$, and $(f \circ g)(4)$.

*You can find the definition of commutative in Appendix A, Section A-1.

20. Functions h and k are defined by Table 6. Find $(h \circ k)(-15)$, $(h \circ k)(-10)$, and $(h \circ k)(15)$.

Table 5

x	$f(x)$	$g(x)$
−7	5	4
−2	9	10
0	0	−2
4	3	6
6	−10	−3

Table 6

x	$h(x)$	$k(x)$
−20	−100	30
−15	−200	5
−10	−300	15
5	−150	8
15	−90	−10

In Problems 21–26, for the indicated functions f and g, find the functions $f + g$, $f - g$, fg, and f/g, and find their domains.

21. $f(x) = 4x$; $g(x) = x + 1$

22. $f(x) = 3x$; $g(x) = x - 2$

23. $f(x) = 2x^2$; $g(x) = x^2 + 1$

24. $f(x) = 3x$; $g(x) = x^2 + 4$

25. $f(x) = 3x + 5$; $g(x) = x^2 - 1$

26. $f(x) = 2x - 7$; $g(x) = 9 - x^2$

In Problems 27–32, for the indicated functions f and g, find the functions $f \circ g$, and $g \circ f$, and find their domains.

27. $f(x) = x^3$; $g(x) = x^2 - x + 1$

28. $f(x) = x^2$; $g(x) = x^3 + 2x + 4$

29. $f(x) = |x + 1|$; $g(x) = 2x + 3$

30. $f(x) = |x - 4|$; $g(x) = 3x + 2$

31. $f(x) = x^{1/3}$; $g(x) = 2x^3 + 4$

32. $f(x) = x^{2/3}$; $g(x) = 8 - x^3$

In Problems 33–36, find $f \circ g$ and $g \circ f$. Graph f, g, $f \circ g$, and $g \circ f$ in a squared viewing window and describe any apparent symmetry between these graphs.

33. $f(x) = \frac{1}{2}x + 1$; $g(x) = 2x - 2$

34. $f(x) = 3x + 2$; $g(x) = \frac{1}{3}x - \frac{2}{3}$

35. $f(x) = -\frac{2}{3}x - \frac{5}{3}$; $g(x) = -\frac{3}{2}x - \frac{5}{2}$

36. $f(x) = -2x + 3$; $g(x) = -\frac{1}{2}x + \frac{3}{2}$

In Problems 37–42, for the indicated functions f and g, find the functions $f + g, f - g, fg$, and f/g, and find their domains.

37. $f(x) = \sqrt{2 - x}$; $g(x) = \sqrt{x + 3}$

38. $f(x) = \sqrt{x + 4}$; $g(x) = \sqrt{3 - x}$

39. $f(x) = \sqrt{x} + 2$; $g(x) = \sqrt{x} - 4$

40. $f(x) = 1 - \sqrt{x}$; $g(x) = 2 - \sqrt{x}$

41. $f(x) = \sqrt{x^2 + x - 6}$; $g(x) = \sqrt{7 + 6x - x^2}$

42. $f(x) = \sqrt{8 + 2x - x^2}$; $g(x) = \sqrt{x^2 - 7x + 10}$

In Problems 43–48, for the indicated functions f and g, find the functions $f \circ g$ and $g \circ f$, and find their domains.

43. $f(x) = \sqrt{x}$; $g(x) = x - 4$

44. $f(x) = \sqrt{x}$; $g(x) = 2x + 5$

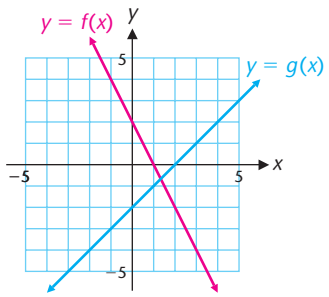
45. $f(x) = x + 2$; $g(x) = \dfrac{1}{x}$

46. $f(x) = x - 3$; $g(x) = \dfrac{1}{x^2}$

47. $f(x) = |x|$; $g(x) = \dfrac{1}{x - 1}$

48. $f(x) = |x - 1|$; $g(x) = \dfrac{1}{x}$

Use the graphs of functions f and g shown below to match each function in Problems 49–52 with one of graphs (a)–(d).

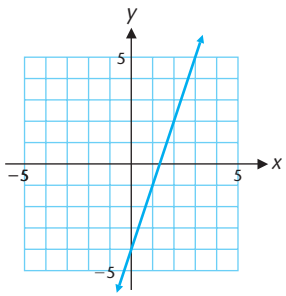


49. $(f + g)(x)$

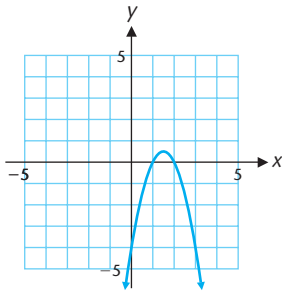
50. $(f - g)(x)$

51. $(g - f)(x)$

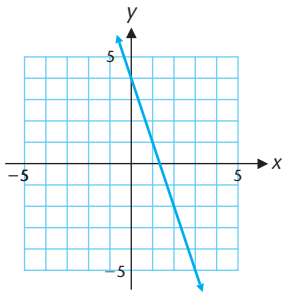
52. $(fg)(x)$



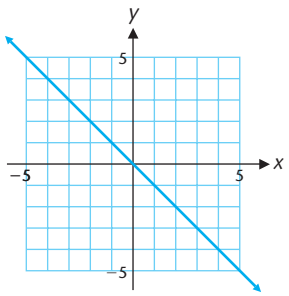
(a)



(b)



(c)



(d)

In Problems 53–60, express h as a composition of two simpler functions f and g.

53. $h(x) = (2x - 7)^4$

54. $h(x) = (3 - 5x)^7$

55. $h(x) = \sqrt{4 + 2x}$

56. $h(x) = \sqrt{3x - 11}$

57. $h(x) = 3x^7 - 5$

58. $h(x) = 5x^6 + 3$

59. $h(x) = \dfrac{4}{\sqrt{x}} + 3$

60. $h(x) = -\dfrac{2}{\sqrt{x}} + 1$

61. Is there a function g that satisfies $f \circ g = g \circ f = f$ for all functions f? If so, what is it?

62. Is there a function g that satisfies $fg = gf = f$ for all functions f? If so, what is it?

In Problems 63–66, for the indicated functions f and g, find the functions $f + g, f - g, fg$, and f/g, and find their domains.

63. $f(x) = x + \dfrac{1}{x}$; $g(x) = x - \dfrac{1}{x}$

64. $f(x) = x - 1$; $g(x) = x - \dfrac{6}{x - 1}$

65. $f(x) = 1 - \dfrac{x}{|x|}$; $g(x) = 1 + \dfrac{x}{|x|}$

66. $f(x) = x + |x|$; $g(x) = x - |x|$

In Problems 67–72, for the indicated functions f and g, find the functions $f \circ g$ and $g \circ f$, and find their domains.

67. $f(x) = \sqrt{4 - x}$; $g(x) = x^2$

68. $f(x) = \sqrt{x - 1}$; $g(x) = x^2$

69. $f(x) = \dfrac{x + 5}{x}$; $g(x) = \dfrac{x}{x - 2}$

70. $f(x) = \dfrac{x}{x - 1}$; $g(x) = \dfrac{2x - 4}{x}$

71. $f(x) = \sqrt{25 - x^2}$; $g(x) = \sqrt{9 + x^2}$

72. $f(x) = \sqrt{x^2 - 9}$; $g(x) = \sqrt{x^2 + 25}$

In Problems 73–78, enter the given expression for $(f \circ g)(x)$ exactly as it is written and graph on a graphing calculator for $-10 \le x \le 10$. Then simplify the expression, enter the result, and graph in a new viewing window, again for $-10 \le x \le 10$. Find the domain of $f \circ g$. Which is the correct graph of $f \circ g$?

73. $f(x) = \sqrt{5 - x^2}$; $g(x) = \sqrt{3 - x}$;
$(f \circ g)(x) = \sqrt{5 - (\sqrt{3 - x})^2}$

74. $f(x) = \sqrt{6 - x^2}$; $g(x) = \sqrt{x - 1}$;
$(f \circ g)(x) = \sqrt{6 - (\sqrt{x - 1})^2}$

75. $f(x) = \sqrt{x^2 + 5}$; $g(x) = \sqrt{x^2 - 4}$;
$(f \circ g)(x) = \sqrt{(\sqrt{x^2 - 4})^2 + 5}$

76. $f(x) = \sqrt{x^2 + 5}$; $g(x) = \sqrt{4 - x^2}$;
$(f \circ g)(x) = \sqrt{(\sqrt{4 - x^2})^2 + 5}$

77. $f(x) = \sqrt{x^2 + 7}$; $g(x) = \sqrt{9 - x^2}$;
$(f \circ g)(x) = \sqrt{(\sqrt{9 - x^2})^2 + 7}$

78. $f(x) = \sqrt{x^2 + 7}$; $g(x) = \sqrt{x^2 - 9}$;
$(f \circ g)(x) = \sqrt{(\sqrt{x^2 - 9})^2 + 7}$

APPLICATIONS

79. **MARKET RESEARCH** The demand x and the price p (in dollars) for a certain product are related by

$$x = f(p) = 4{,}000 - 200p \qquad 0 \le p \le 20$$

The revenue (in dollars) from the sale of x units is given by

$$R(x) = 20x - \frac{1}{200}x^2$$

and the cost (in dollars) of producing x units is given by

$$C(x) = 2x + 8{,}000$$

Express the profit as a function of the price p and find the price that produces the largest profit.

80. **MARKET RESEARCH** The demand x and the price p (in dollars) for a certain product are related by

$$x = f(p) = 5{,}000 - 100p \qquad 0 \le p \le 50$$

The revenue (in dollars) from the sale of x units and the cost (in dollars) of producing x units are given, respectively, by

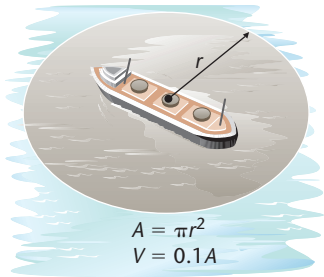
$$R(x) = 50x - \frac{1}{100}x^2 \quad \text{and} \quad C(x) = 20x + 40{,}000$$

Express the profit as a function of the price p and find the price that produces the largest profit.

81. **ENVIRONMENTAL SCIENCE** An oil tanker aground on a reef is leaking oil that forms a circular oil slick about 0.1 foot thick (see the figure). The radius of the slick (in feet) t minutes after the leak first occurred is given by

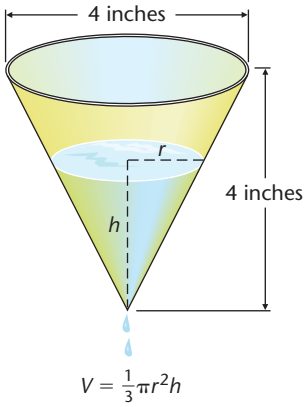
$$r(t) = 0.4t^{1/3}$$

Express the volume of the oil slick as a function of t.



82. **METEOROLOGY** A weather balloon is rising vertically. An observer is standing on the ground 100 meters from the point where the weather balloon was released.
(A) Express the distance d between the balloon and the observer as a function of the balloon's distance h above the ground.
(B) If the balloon's distance above the ground after t seconds is given by $h = 5t$, express the distance d between the balloon and the observer as a function of t.

83. **FLUID FLOW** A conical paper cup with diameter 4 inches and height 4 inches is initially full of water. A small hole is made in the bottom of the cup and the water begins to flow out of the cup. Let h and r be the height and radius, respectively, of the water in the cup t minutes after the water begins to flow.

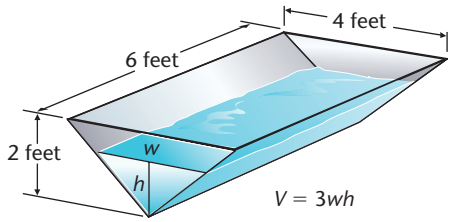


(A) Express r as a function of h.
(B) Express the volume V as a function of h.
(C) If the height of the water after t minutes is given by

$$h(t) = 0.5\sqrt{t}$$

express V as a function of t.

84. EVAPORATION A water trough with triangular ends is 6 feet long, 4 feet wide, and 2 feet deep. Initially, the trough is full of water, but due to evaporation, the volume of the water is decreasing. Let h and w be the height and width, respectively, of the water in the tank t hours after it began to evaporate.



(A) Express w as a function of h.
(B) Express V as a function of h.
(C) If the height of the water after t hours is given by

$$h(t) = 2 - 0.2\sqrt{t}$$

express V as a function of t.

1-6 Inverse Functions

- One-to-One Functions
- Finding the Inverse of a Function
- Mathematical Modeling
- Graphing Inverse Functions

We have seen that many important mathematical relationships can be expressed in terms of functions. For example,

$C = \pi d$	The circumference of a circle is a function of the diameter d.
$V = s^3$	The volume of a cube is a function of length s of the edges.
$d = 1{,}000 - 100p$	The demand for a product is a function of the price p.
$F = \frac{9}{5}C + 32$	Temperature measured in °F is a function of temperature in °C.

In many cases, we are interested in *reversing* the correspondence determined by a function. For our examples,

$$d = \frac{C}{\pi}$$ The diameter of a circle is a function of the circumference C.

$$s = \sqrt[3]{V}$$ The length of the edge of a cube is a function of the volume V.

$$p = 10 - \frac{1}{100}d$$ The price of a product is a function of the demand d.

$$C = \frac{5}{9}(F - 32)$$ Temperature measured in °C is a function of temperature in °F.

As these examples illustrate, reversing the correspondence between two quantities often produces a new function. This new function is called the *inverse* of the original function. Later in this text we will see that many important functions are actually defined as the inverses of other functions.

In this section, we develop techniques for determining whether the inverse of a function exists, some general properties of inverse functions, and methods for finding the rule of correspondence that defines the inverse function. A review of function basics in the second section of this chapter would be very helpful at this point.

One-to-One Functions

Recall the set form of the definition of function:

A function is a set of ordered pairs with the property that no two ordered pairs have the same first component and different second components.

However, it is possible that two ordered pairs in a function could have the same second component and different first components. If this does not happen, then we call the function a *one-to-one function.*

In other words, a function is one-to-one if there are no duplicates among the second components. It turns out that one-to-one functions are the only functions that have inverse functions.

DEFINITION 1 One-to-One Function

A function is **one-to-one** if no two ordered pairs in the function have the same second component and different first components.

››› EXPLORE-DISCUSS 1

Given the following sets of ordered pairs:

$$f = \{(0, 1), (0, 2), (1, 1), (1, 2)\}$$
$$g = \{(0, 1), (1, 1), (2, 2), (3, 2)\}$$
$$h = \{(0, 1), (1, 2), (2, 3), (3, 0)\}$$

(A) Which of these sets represent functions?

(B) Which of the functions are one-to-one functions?

(C) For each set that is a function, form a new set by reversing each ordered pair in the set. Which of these new sets represent functions?

(D) What do these results tell you about the result of reversing the ordered pairs for functions that are one-to-one, and for functions that are not one-to-one?

EXAMPLE 1 Determining Whether a Function Is One-to-One

Determine whether f is a one-to-one function for

(A) $f(x) = x^2$ (B) $f(x) = 2x - 1$

SOLUTIONS

(A) To show that a function is *not* one-to-one, all we have to do is find two different ordered pairs in the function with the same second component and different first components. Because

$$f(2) = 2^2 = 4 \quad \text{and} \quad f(-2) = (-2)^2 = 4$$

the ordered pairs $(2, 4)$ and $(-2, 4)$ both belong to f, and f is not one-to-one. Note that there's nothing special about 2 and -2 here: Any real number and its negative can be used in the same way.

(B) To show that a function *is* one-to-one, we have to show that no two ordered pairs have the same second component and different first components. To do this, we'll show that if any two ordered pairs $(a, f(a))$ and $(b, f(b))$ in f have the same second components, then the first components must also be the same. That is, we show that $f(a) = f(b)$ implies $a = b$. We proceed as follows:

$$\begin{aligned} f(a) &= f(b) && \text{Assume second components are equal. Evaluate } f(a) \text{ and } f(b). \\ 2a - 1 &= 2b - 1 \\ 2a &= 2b && \text{Simplify.} \\ a &= b && \text{Conclusion: } f \text{ is one-to-one.} \end{aligned}$$

By Definition 1, f is a one-to-one function. ◉

MATCHED PROBLEM 1

Determine whether f is a one-to-one function for

(A) $f(x) = 4 - x^2$ (B) $f(x) = 4 - 2x$

The methods used in the solution of Example 1 can be stated as a theorem.

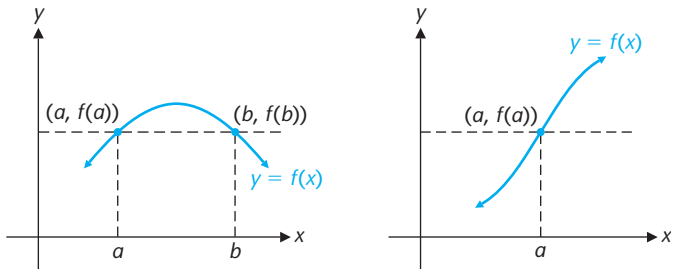
› THEOREM 1 One-to-One Functions

1. If $f(a) = f(b)$ for at least one pair of domain values a and b, $a \neq b$, then f is not one-to-one.
2. If the assumption $f(a) = f(b)$ always implies that the domain values a and b are equal, then f is one-to-one.

Applying Theorem 1 is not always easy—try testing $f(x) = x^3 + 2x + 3$, for example. (Good luck!) However, the graph of a function can help us develop a simple procedure for determining if a function is one-to-one. If any horizontal line intersects the graph in more than one point [as shown in Fig. 1(a)], then there is a second component (height) that corresponds to two different first components (x values). This shows that the function is not one-to-one.

On the other hand, if every horizontal line intersects the graph in just one point or not at all [as shown in Fig. 1(b)], the function is one-to-one. These observations form the basis of the *horizontal line test*.

› **Figure 1** Intersections of graphs and horizontal lines.



$f(a) = f(b)$ for $a \neq b$
f is not one-to-one
(a)

Only one point has second component $f(a)$; f is one-to-one
(b)

› THEOREM 2 Horizontal Line Test

A function is one-to-one if and only if every horizontal line intersects the graph of the function in at most one point.

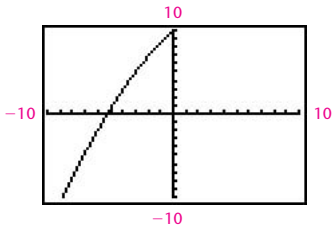
EXAMPLE 2 Using the Horizontal Line Test

Use the horizontal line test to determine if each function is one-to-one.

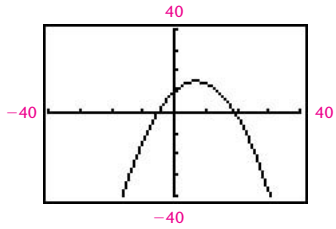
(A) $f(x) = -0.1x^2 + 1.4x + 10.1$ (B) $g(x) = 0.2x^3 + x$

SOLUTIONS

(A) We first use a graphing calculator to graph $f(x)$ in a standard viewing window (Fig. 2). It appears to pass the horizontal line test. But when we zoom out (Fig. 3) we see that there are many heights where a horizontal line will intersect the graph twice, so $f(x)$ is not one-to-one.

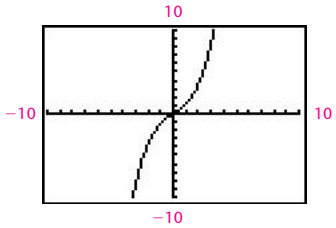


› Figure 2

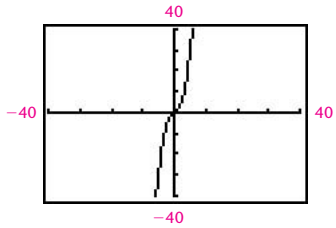


› Figure 3

(B) In a standard viewing window (Fig. 4), $g(x)$ appears to be one-to-one. We again zoom out (Fig. 5) and see that there don't appear to be any changes of direction in the graph. No horizontal line intersects the graph more than once, so $g(x)$ is one-to-one.



› Figure 4



› Figure 5

MATCHED PROBLEM 2

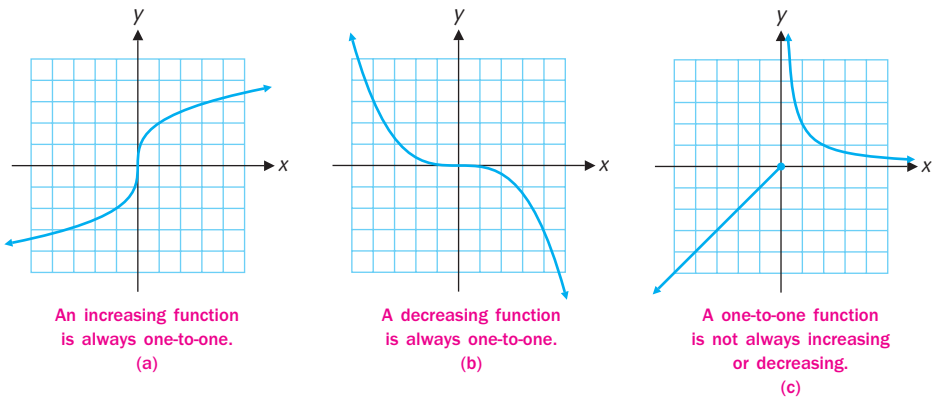
Use the horizontal line test to determine if each function is one-to-one.

(A) $f(x) = \dfrac{11 - 3x}{5}$ (B) $g(x) = x^2 + 20x + 80$

››› CAUTION ›››

When using the horizontal line test to determine if a function is one-to-one, you have to be very careful about viewing windows. Make sure that you explore the graph enough to be reasonably certain that you're not missing some important features that are off the screen.

Example 2 suggests that a function fails to be one-to-one exactly when its graph changes direction somewhere. We'll see shortly that this is not true, but it *is* true that a function that is either increasing or decreasing throughout its domain will always pass the horizontal line test [Figs. 6(a) and 6(b)]. Thus, we have the following theorem.



› **Figure 6** Increasing, decreasing, and one-to-one functions.

› THEOREM 3 Increasing and Decreasing Functions

If a function f is increasing throughout its domain or decreasing throughout its domain, then f is a one-to-one function.

The opposite of Theorem 3 is false. Consider the function whose graph is in Figure 6(c). This function is increasing on $(-\infty, 0]$ and decreasing on $(0, \infty)$, but the graph still passes the horizontal line test. So there are one-to-one functions that are neither increasing nor decreasing functions.

› Finding the Inverse of a Function

Now we want to see how we can form a new function by reversing the correspondence determined by a given function. Let g be the function defined as follows:

$$g = \{(-3, 9), (0, 0), (3, 9)\}$$ g is not one-to-one.

Notice that g is not one-to-one because the domain elements -3 and 3 both correspond to the range element 9. We can reverse the correspondence determined by function g simply by reversing the components in each ordered pair in g, producing the following set:

$$G = \{(9, -3), (0, 0), (9, 3)\} \quad G \text{ is not a function.}$$

But the result is not a function because the domain element 9 corresponds to two different range elements, -3 and 3. On the other hand, if we reverse the ordered pairs in the function

$$f = \{(1, 2), (2, 4), (3, 9)\} \quad f \text{ is one-to-one.}$$

we obtain

$$F = \{(2, 1), (4, 2), (9, 3)\} \quad F \text{ is a function.}$$

This time f is a one-to-one function, and the set F turns out to be a function also. This new function F, formed by reversing all the ordered pairs in f, is called the *inverse* of f and is usually denoted by f^{-1}. In this case,

$$f^{-1} = \{(2, 1)\,(4, 2), (9, 3)\} \quad \text{The inverse of } f$$

Notice that f^{-1} is also a one-to-one function and that the following relationships hold:

$$\text{Domain of } f^{-1} = \{2, 4, 9\} = \text{Range of } f$$
$$\text{Range of } f^{-1} = \{1, 2, 3\} = \text{Domain of } f$$

Reversing all the ordered pairs in a one-to-one function forms a new one-to-one function and reverses the domain and range in the process. But reversing the ordered pairs from a function that is not one-to-one results in a new relation that is not a function. We are now ready to present a formal definition of the inverse of a function.

› DEFINITION 2 Inverse of a Function

If f is a one-to-one function, then the **inverse** of f, denoted f^{-1}, is the function formed by reversing all the ordered pairs in f. That is,

$$f^{-1} = \{(y, x) \mid (x, y) \text{ is in } f\}$$

If f is not one-to-one, then ***f does not have an inverse function*** and f^{-1} ***does not exist.***

››› CAUTION ›››

Do not confuse inverse notation and reciprocal notation:

$$2^{-1} = \frac{1}{2} \quad \text{Reciprocal notation for numbers}$$

$$(f(x))^{-1} = \frac{1}{f(x)} \quad \text{Reciprocal notation for functions}$$

$$f^{-1}(x) \neq \frac{1}{f(x)} \quad \text{Inverse notation is not reciprocal notation.}$$

Make sure that you read the symbol f^{-1} as "the inverse of f" or "f inverse," but *never* "f to the negative one power."

EXAMPLE 3 Finding the Inverse of a Function

Find the inverse if it exists.

(A) $f = \{(-3, 10), (-1, 8), (2, 5), (5, 8)\}$
(B) $g = \{(10, 15), (15, 20), (20, 25), (25, 30)\}$

SOLUTIONS

(A) Note that f has a second component, 8, that appears twice. So f is not one-to-one, and f^{-1} does not exist.
(B) Function g is one-to-one, so g^{-1} exists. We find it by reversing all of the ordered pairs.

$$g^{-1} = \{(15, 10), (20, 15), (25, 20), (30, 25)\}$$

MATCHED PROBLEM 3

Find the inverse of each function, if it exists.

(A) $F = \{(3/2, -5), (5/2, -7), (7/2, -9), (9/2, -11)\}$
(B) $G = \{(-10, 3), (10, -3), (-20, 3), (20, -3)\}$

The following properties of inverse functions follow directly from the definition.

› THEOREM 4 Properties of Inverse Functions

For a given function f, if f^{-1} exists, then

1. f^{-1} is a one-to-one function.
2. The domain of f^{-1} is the range of f.
3. The range of f^{-1} is the domain of f.

››› EXPLORE-DISCUSS 2

(A) For the function $f = \{(3, 5), (7, 11), (11, 17)\}$, find f^{-1}.

(B) What do you think would be the result of composing f with f^{-1}? Justify your answer using Definition 2.

(C) Check your conjecture from part B by finding both $f \circ f^{-1}$ and $f^{-1} \circ f$. Were you correct?

Explore-Discuss 2 brings up an important point: if you apply a function to any number in its domain, then apply the inverse of that function to the result, you'll get right back where you started. This leads to the following theorem.

› THEOREM 5 Inverse Functions and Composition

If f^{-1} exists, then

1. $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.
2. $f^{-1}(f(x)) = x$ for all x in the domain of f.

If f and g are one-to-one functions satisfying

$$f(g(x)) = x \text{ for all } x \text{ in the domain of } g \text{ and}$$
$$g(f(x)) = x \text{ for all } x \text{ in the domain of } f$$

then f and g are inverses of one another.

We can use Theorem 5 to see if two functions defined by equations are inverses.

EXAMPLE 4 **Deciding If Two Functions Are Inverses**

Use Theorem 5 to decide if these two functions are inverses.

$$f(x) = 3x - 7 \qquad g(x) = \frac{x + 7}{3}$$

SOLUTION

The domain of both functions is all real numbers. For any x,

$$\begin{aligned} f(g(x)) &= f\left(\frac{x + 7}{3}\right) && \text{Substitute into } f(x). \\ &= 3\left(\frac{x + 7}{3}\right) - 7 && \text{Multiply.} \\ &= x + 7 - 7 && \text{Add.} \\ &= x \\ g(f(x)) &= g(3x - 7) && \text{Substitute into } g(x). \\ &= \frac{3x - 7 + 7}{3} && \text{Add.} \\ &= \frac{3x}{3} && \text{Simplify.} \\ &= x \end{aligned}$$

By Theorem 5, f and g are inverses.

MATCHED PROBLEM 4

Use Theorem 5 to decide if these two functions are inverses.

$$f(x) = \frac{2}{5}(11 - x) \qquad g(x) = -\frac{5}{2}x + 11$$

There is one obvious question that remains: when a function is defined by an equation, how can we find the inverse? Given a function $y = f(x)$, the first coordinates of points on the graph are represented by x, and the second coordinates are represented by y. Finding the inverse by reversing the order of the coordinates would then correspond to switching the variables x and y. This leads us to the following procedure, which can be applied whenever it is possible to solve $y = f(x)$ for x in terms of y.

› Finding the Inverse of a Function *f*

Step 1. Find the domain of f and verify that f is one-to-one. If f is not one-to-one, then stop, because f^{-1} does not exist.

Step 2. If the function is written with function notation, like $f(x)$, replace the function symbol with the letter y. Then interchange x and y.

Step 3. Solve the resulting equation for y. The result is $f^{-1}(x)$.

Step 4. Find the domain of f^{-1}. Remember, the domain of f^{-1} must be the same as the range of f.

You can check your work using Theorem 5.

EXAMPLE 5 Finding the Inverse of a Function

Find f^{-1} for $f(x) = \sqrt{x - 1}$.

SOLUTION

Step 1. Find the domain of f and verify that f is one-to-one. The domain of f is $[1, \infty)$. The graph of f in Figure 7 shows that f is one-to-one, so f^{-1} exists.

Step 2. Replace $f(x)$ with y, then interchange x and y.

$$y = \sqrt{x - 1} \qquad \text{Interchange } x \text{ and } y.$$
$$x = \sqrt{y - 1}$$

Step 3. Solve the equation for y.

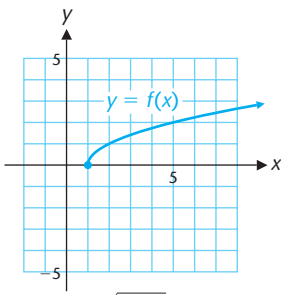
$$x = \sqrt{y - 1} \qquad \text{Square both sides.}$$
$$x^2 = y - 1 \qquad \text{Add 1 to each side.}$$
$$x^2 + 1 = y$$

The inverse is $f^{-1}(x) = x^2 + 1$.

Step 4. Find the domain of f^{-1}.

The equation we found for f^{-1} is defined for all x, but the domain should be the range of f. From Figure 7, we see that the range of f is $[0, \infty)$ so that is the domain of f^{-1}. Therefore,

$$f^{-1}(x) = x^2 + 1 \qquad x \geq 0$$



› Figure 7

CHECK

Find the composition of f with the alleged inverse (in both orders!).

For x in $[1, \infty)$, the domain of f, we have

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}(\sqrt{x-1}) && \text{Substitute } \sqrt{x-1} \text{ into } f^{-1}. \\ &= (\sqrt{x-1})^2 + 1 && \text{Square } \sqrt{x-1}. \\ &= x - 1 + 1 && \text{Add.} \\ &\overset{\checkmark}{=} x \end{aligned}$$

For x in $[0, \infty)$, the domain of f^{-1}, we have

$$\begin{aligned} f(f^{-1}(x)) &= f(x^2 + 1) && \text{Substitute } x^2 + 1 \text{ into } f. \\ &= \sqrt{(x^2 + 1) - 1} && \text{Add.} \\ &= \sqrt{x^2} && \sqrt{x^2} = |x| \text{ for any real number } x. \\ &= |x| && |x| = x \text{ for } x \geq 0. \\ &\overset{\checkmark}{=} x \end{aligned}$$

MATCHED PROBLEM 5

Find f^{-1} for $f(x) = \sqrt{x + 2}$.

The technique of finding an inverse by interchanging x and y leads to the following property of inverses that comes in very handy later in the course.

› THEOREM 6 A Property of Inverses

If f^{-1} exists, then $x = f^{-1}(y)$ if and only if $y = f(x)$.

››› EXPLORE-DISCUSS 3

Most basic arithmetic operations can be reversed by performing a second operation: subtraction reverses addition, division reverses multiplication, squaring reverses taking the square root, and so on. Viewing a function as a sequence of reversible operations gives insight into the inverse function concept. For example, the function $f(x) = 2x - 1$ can be described verbally as a function that multiplies each domain element by 2 and then subtracts 1. Reversing this sequence describes a function g that adds 1 to each domain element and then divides by 2, or $g(x) = (x + 1)/2$, which is the inverse of the function f. For each of the following functions, write a verbal description of the function, reverse your description, and write the resulting algebraic equation. Verify that the result is the inverse of the original function.

(A) $f(x) = 3x + 5$ **(B)** $f(x) = \sqrt{x - 1}$ **(C)** $f(x) = \dfrac{1}{x + 1}$

Explore-Discuss 3 emphasizes that the inverse of a function is another function that performs the opposite steps in the opposite order as the original function. This helps us to understand the concept of inverses, but it also provides an alternative method for finding the inverse of some functions. We will explore this in the exercises.

› Mathematical Modeling

Example 6 shows how an inverse function is used in constructing a revenue model. See Example 8 in the last section.

EXAMPLE 6 **Modeling Revenue**

The research department for an electronics firm estimates that the weekly demand for a certain brand of headphones is given by

$$x = f(p) = 20{,}000 - 1{,}000p \qquad 0 \le p \le 20 \qquad \text{Demand function}$$

where x is the number of pairs of headphones retailers are likely to buy per week at p dollars per pair. Express the revenue as a function of the demand x.

SOLUTION

If x pairs of headphones are sold at p dollars each, the total revenue is

$$\begin{aligned}\text{Revenue} &= (\text{Number of pairs of headphones})(\text{price of each pair})\\ &= xp\end{aligned}$$

To express the revenue as a function of the demand x, we must express the price in terms of x. That is, we must find the inverse of the demand function.

$$\begin{aligned} x &= 20{,}000 - 1{,}000p && \text{Subtract 20,000 from each side.}\\ x - 20{,}000 &= -1{,}000p && \text{Divide both sides by } -1{,}000.\\ \frac{x - 20{,}000}{-1{,}000} &= p && \text{Distribute.}\\ -0.001x + 20 &= p \end{aligned}$$

The inverse of the demand function is

$$p = f^{-1}(x) = 20 - 0.001x$$

and the revenue is given by

$$\begin{aligned} R &= xp\\ R(x) &= x(20 - 0.001x)\\ &= 20x - 0.001x^2 \end{aligned}$$

MATCHED PROBLEM 6

Repeat Example 6 for the demand function

$$x = f(p) = 10{,}000 - 1{,}000p \qquad 0 \le p \le 10$$

The demand function in Example 6 was defined with independent variable p and dependent variable x. When we found the inverse function, we did not rewrite it with independent variable p. Because p represents price and x represents number of pairs of headphones, to interchange these variables would be confusing. In most applications, the variables have specific meaning and should not be interchanged as part of the inverse process.

Example 6 illustrates an important application of inverse functions. We were given a function with input price and output demand, but we needed one with input demand and output price. This is exactly what inverse functions are all about!

Graphing Inverse Functions

EXPLORE-DISCUSS 4

The following activities refer to the graph of f in Figure 8 and Tables 1 and 2.

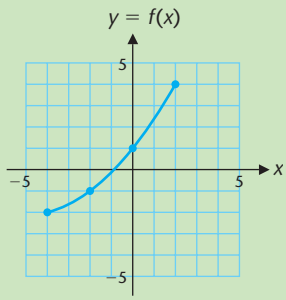


Table 1

x	$f(x)$
−4	
−2	
0	
2	

Table 2

x	$f^{-1}(x)$

Figure 8

(A) Complete the second column in Table 1.

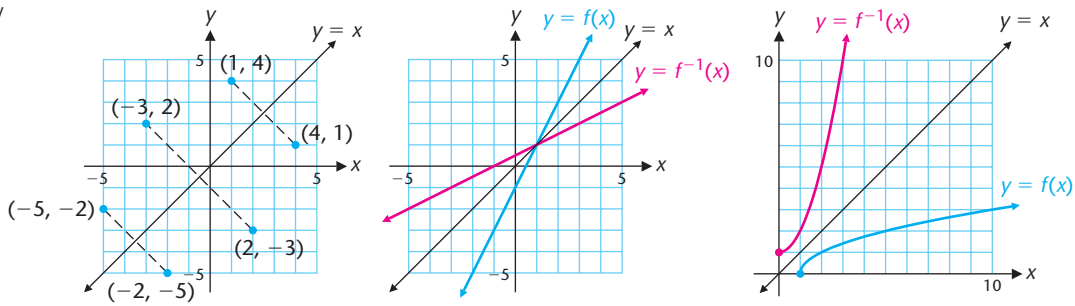
(B) Reverse the ordered pairs in Table 1 and list the results in Table 2.

(C) Add the points in Table 2 to Figure 8 (or a copy of the figure) and sketch the graph of f^{-1}.

(D) Discuss any symmetry you observe between the graphs of f and f^{-1}.

There is an important relationship between the graph of any function and its inverse that is based on the following observation: In a rectangular coordinate system, switching the order of the coordinates moves a point to the opposite side of the line $y = x$. Specifically, any point P is moved to the point Q such that $y = x$ is the perpendicular bisector of segment PQ [Fig. 9(a)]. In other words, the points (a, b) and (b, a) are symmetric with respect to the line $y = x$. Theorem 7 is an immediate consequence of this observation.

› **Figure 9** Symmetry with respect to the line $y = x$.



(a, b) and (b, a) are symmetric with respect to the line $y = x$
(a)

$f(x) = 2x - 1$
$f^{-1}(x) = \frac{1}{2}x + \frac{1}{2}$
(b)

$f(x) = \sqrt{x - 1}$
$f^{-1}(x) = x^2 + 1, x \geq 0$
(c)

› **THEOREM 7 Symmetry Property for the Graphs of f and f^{-1}**

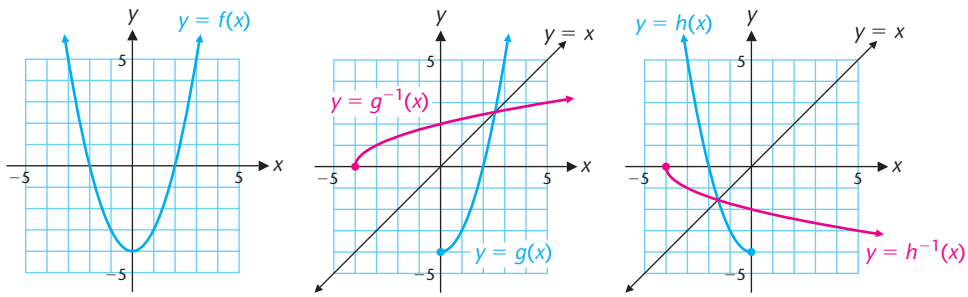
The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetric with respect to the line $y = x$.

Knowledge of this symmetry property allows us to graph f^{-1} if the graph of f is known, and vice versa. Figures 9(b) and 9(c) illustrate this property for the two inverse functions we found earlier.

If a function is not one-to-one, we usually can restrict the domain of the function to produce a new function that is one-to-one. Then we can find an inverse for the restricted function. Suppose we start with $f(x) = x^2 - 4$. Because f is not one-to-one, f^{-1} does not exist [Fig. 10(a)]. But there are many ways the domain of f can be restricted to obtain a one-to-one function. Figures 10(b) and 10(c) illustrate two such restrictions.

In essence, we are "forcing" the function to be one-to-one by throwing out a portion of the graph that would make it fail the horizontal line test.

› **Figure 10** Restricting the domain of a function.



$f(x) = x^2 - 4$
f^{-1} does not exist
(a)

$g(x) = x^2 - 4, x \geq 0$
$g^{-1}(x) = \sqrt{x + 4}, x \geq -4$
(b)

$h(x) = x^2 - 4, x \leq 0$
$h^{-1}(x) = -\sqrt{x + 4}, x \geq -4$
(c)

Recall from Theorem 3 that increasing and decreasing functions are always one-to-one. This provides the basis for a convenient method of restricting the domain of a function:

If the domain of a function f is restricted to an interval on the x axis over which f is increasing (or decreasing), then the new function determined by this restriction is one-to-one and has an inverse.

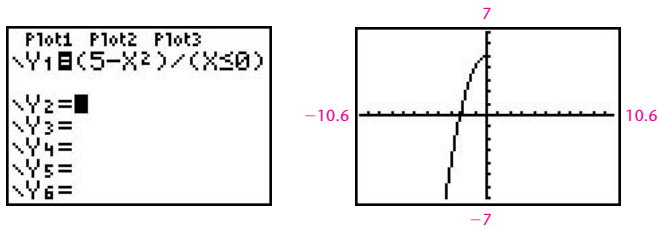
We used this method to form the one-to-one functions g and h in Figure 10.

EXAMPLE 7 Finding the Inverse of a Function

Find the inverse of $f(x) = 5 - x^2$, $x \le 0$. Graph f, f^{-1}, and $y = x$ in a squared viewing window on a graphing calculator. Then sketch the graph by hand, adding appropriate labels.

SOLUTION

Step 1. Find the domain of f and verify that f is one-to-one. We are given a domain of $(-\infty, 0]$. Note the syntax we used in Figure 11(a). Dividing $5 - x^2$ by the expression $(x \le 0)$* restricts the graph to the domain we were given [Fig. 11(b)]. This graph shows that f is one-to-one.



(a) (b)

› Figure 11

Step 2. Replace $f(x)$ with y, then interchange x and y.

$$y = 5 - x^2$$
$$x = 5 - y^2$$

Step 3. Solve the equation for y.

$$x = 5 - y^2 \qquad \text{Add } y^2 \text{ to each side.}$$
$$y^2 + x = 5 \qquad \text{Subtract } x \text{ from each side.}$$
$$y^2 = 5 - x \qquad \text{Apply the square root to each side.}$$
$$y = \pm\sqrt{5 - x}$$

*The graphing calculator assigns this expression a value of 1 for $x \le 0$, and a value of 0 for $x > 0$. This makes y_1 undefined for positive x values.

The restricted domain of f tells us which solution to use. Because $x \le 0$ for f, we must have $y \le 0$ for f^{-1}. We should choose the negative square root. The inverse is

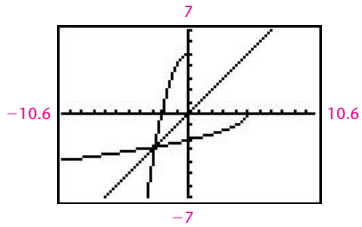
$$f^{-1}(x) = -\sqrt{5 - x}$$

Step 4. Find the domain of f^{-1}. The equation $f^{-1}(x) = -\sqrt{5 - x}$ is defined only for $x \le 5$. From the graph in Figure 11(b), the range of f is also $(-\infty, 5]$, so

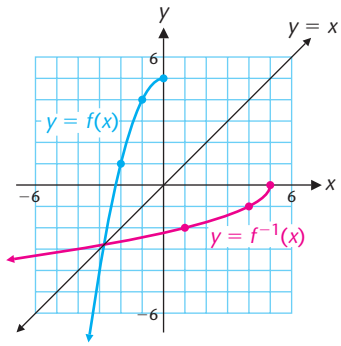
$$f^{-1}(x) = -\sqrt{5 - x} \qquad x \le 5$$

The check is left for the reader.

The graphs of f, f^{-1}, and $y = x$ on a graphing calculator are shown in Figure 12 and a hand sketch is shown in Figure 13. Note that we plotted several points on the graph of f and their reflections on the graph of f^{-1} to aid in preparing the hand sketch.



› Figure 12



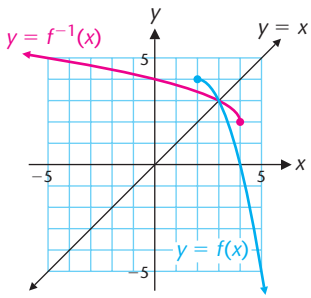
› Figure 13

MATCHED PROBLEM 7

Find the inverse of $f(x) = 5 - x^2, x \ge 0$. Graph f, f^{-1}, and $y = x$ in the same coordinate system.

ANSWERS TO MATCHED PROBLEMS

1. (A) Not one-to-one (B) One-to-one **2.** (A) One-to-one (B) Not one-to-one
3. (A) $F^{-1} = \{(-5, 3/2), (-7, 5/2), (-9, 7/2), (-11, 9/2)\}$ (B) G^{-1} does not exist.
4. f and g are inverses. **5.** $f^{-1}(x) = x^2 - 2, x \ge 0$ **6.** $R(x) = 10x - 0.001x^2$
7. $f^{-1}(x) = \sqrt{5 - x}, x \le 5$



1-6 Exercises

For each set of ordered pairs in Problems 1–6, determine if the set is a function, a one-to-one function, or neither. Reverse all the ordered pairs in each set and determine if this new set is a function, a one-to-one function, or neither.

1. $\{(1, 2), (2, 1), (3, 4), (4, 3)\}$

2. $\{(-1, 0), (0, 1), (1, -1), (2, 1)\}$

3. $\{(5, 4), (4, 3), (3, 3), (2, 4)\}$

4. $\{(5, 4), (4, 3), (3, 2), (2, 1)\}$

5. $\{(1, 2), (1, 4), (-3, 2), (-3, 4)\}$

6. $\{(0, 5), (-4, 5), (-4, 2), (0, 2)\}$

Which of the functions in Problems 7–18 are one-to-one?

7.

Domain	Range
−2	−4
−1	−2
0	0
1	1
2	5

8.

Domain	Range
−2	−3
−1	
0	7
1	9
2	

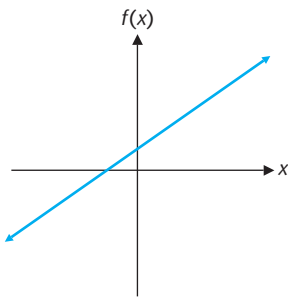
9.

Domain	Range
1	7
2	
3	
4	
5	

10.

Domain	Range
1	5
2	3
3	1
4	2
5	4

11.



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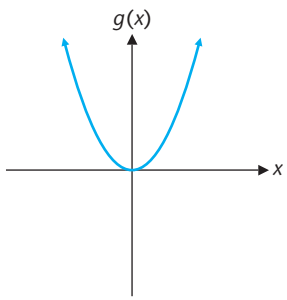

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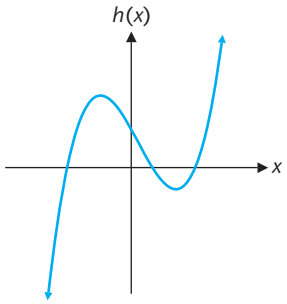

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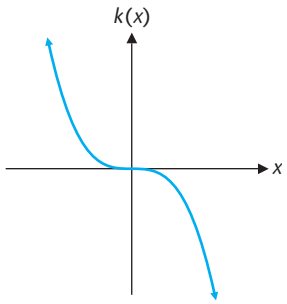

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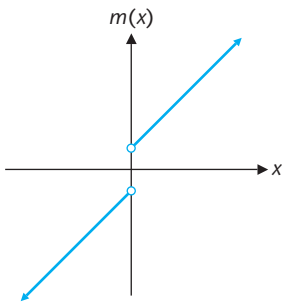

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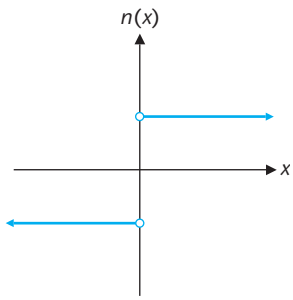

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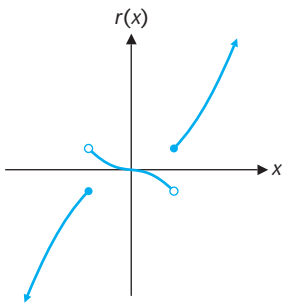

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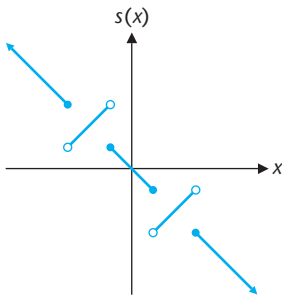


In Problems 19–24, find the inverse of each function, if it exists.

19. $f = \{(2, 3), (3, 4), (4, 5), (5, 6)\}$

20. $g = \{(-5, 10), (-10, 20), (-20, 40), (-40, 80)\}$

21. $h = \{(7, -3), (0, 0), (-7, -3)\}$

22. $k = \{(3, -8), (0, 8), (-3, -8)\}$

23. $F = \{(a, 7), (c, 11), (e, -9), (g, -13)\}$

24. $G = \{(1, x), (3, y), (-5, z), (-7, w)\}$

25. When a function is defined by ordered pairs, how can you tell if it is one-to-one?

26. When you have the graph of a function, how can you tell if it is one-to-one?

27. Why does a function fail to have an inverse if it is not one-to-one? Give an example using ordered pairs to illustrate your answer.

28. True or False: Any function whose graph changes direction is not one-to-one. Explain.

29. What is the result of composing a function with its inverse? Why does this make sense?

30. What is the relationship between the graphs of two functions that are inverses?

In Problems 31–36, use Theorem 1 to determine which functions are one-to-one.

31. $F(x) = \frac{1}{2}x + 2$

32. $G(x) = -\frac{1}{3}x + 1$

33. $H(x) = 4 - x^2$

34. $K(x) = \sqrt{4 - x}$

35. $M(x) = \sqrt{x + 1}$

36. $N(x) = x^2 - 1$

In Problems 37–44, use the horizontal line test (Theorem 2) to determine which functions are one-to-one.

37. $f(x) = \dfrac{7}{x}$

38. $g(x) = \dfrac{-11}{x^3}$

39. $f(x) = 0.3x^3 - 3.8x^2 + 12x + 11$

40. $g(x) = -0.4x^3 + 0.1x^2 + 2x - 20$

41. $f(x) = \dfrac{x^2 + |x|}{x}$

42. $f(x) = \dfrac{x^2 - |x|}{x}$

43. $f(x) = \dfrac{x^2 - 4}{|x - 2|}$

44. $f(x) = \dfrac{1 - x^2}{|x + 1|}$

In Problems 45–50, use Theorem 5 to determine if g is the inverse of f.

45. $f(x) = 3x + 5; \quad g(x) = \frac{1}{3}x - \frac{5}{3}$

46. $f(x) = 2x - 4; \quad g(x) = \frac{1}{2}x - 2$

47. $f(x) = 2 - (x + 1)^3; \quad g(x) = \sqrt[3]{3 - x} - 1$

48. $f(x) = (x - 3)^3 + 4; \quad g(x) = \sqrt[3]{x - 4} + 3$

49. $f(x) = \dfrac{2x - 3}{x + 4}; \quad g(x) = \dfrac{3 + 4x}{2 - x}$

50. $f(x) = \dfrac{x + 1}{2x - 3}; \quad g(x) = \dfrac{3x + 1}{2x + 1}$

Because the inverse of a function reverses the action of the original function, if you can write a step-by-step description of what a function does to numbers in its domain, you can reverse those steps to find the inverse. In Problems 51–56, write a step-by-step description of the given function, then reverse that description and use your result to write an equation for the inverse function.

51. $h(x) = 3x - 7$

52. $k(x) = 6 - 9x$

53. $m(x) = \sqrt[3]{x + 11}$

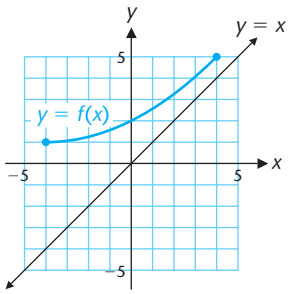
54. $n(x) = \sqrt[5]{2 + x}$

55. $s(x) = (3x + 17)^5$

56. $t(x) = (2x + 7)^3$

In Problems 57–60, use the graph of the one-to-one function f to sketch the graph of f^{-1}. State the domain and range of f^{-1}.

57.



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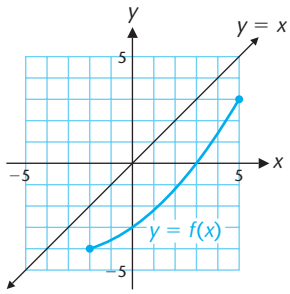

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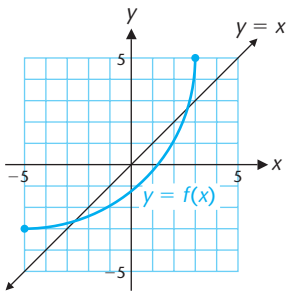
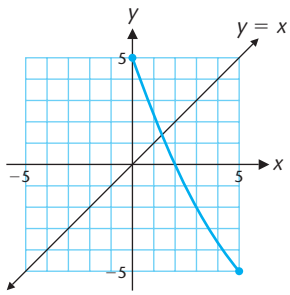


60.



In Problems 61–66, verify that g is the inverse of the one-to-one function f. Sketch the graphs of f, g, and y = x in the same coordinate system and identify each graph.

61. $f(x) = 3x + 6; \quad g(x) = \frac{1}{3}x - 2$

62. $f(x) = -\frac{1}{2}x + 2; \quad g(x) = -2x + 4$

63. $f(x) = 4 + x^2, x \geq 0; \quad g(x) = \sqrt{x - 4}$

64. $f(x) = \sqrt{x + 2}; \quad g(x) = x^2 - 2, x \geq 0$

65. $f(x) = -\sqrt{x - 2}; \quad g(x) = x^2 + 2, x \leq 0$

66. $f(x) = 6 - x^2, x \leq 0; \quad g(x) = -\sqrt{6 - x}$

The functions in Problems 67–86 are one-to-one. Find f^{-1}.

67. $f(x) = 3x$

68. $f(x) = \frac{1}{2}x$

69. $f(x) = 4x - 3$

70. $f(x) = -\frac{1}{3}x + \frac{5}{3}$

71. $f(x) = \frac{1}{10}x + \frac{3}{5}$

72. $f(x) = -2x - 7$

73. $f(x) = \dfrac{2}{x - 1}$

74. $f(x) = \dfrac{3}{x + 4}$

75. $f(x) = \dfrac{x}{x + 2}$

76. $f(x) = \dfrac{x - 3}{x}$

77. $f(x) = \dfrac{2x + 5}{3x - 4}$

78. $f(x) = \dfrac{5 - 3x}{7 - 4x}$

79. $f(x) = x^3 + 1$

80. $f(x) = x^5 - 2$

81. $f(x) = 4 - \sqrt[5]{x + 2}$

82. $f(x) = \sqrt[3]{x + 3} - 2$

83. $f(x) = \frac{1}{2}\sqrt{16 - x}$

84. $f(x) = -\frac{1}{3}\sqrt{36 - x}$

85. $f(x) = 3 - \sqrt{x - 2}$

86. $f(x) = 4 + \sqrt{5 - x}$

87. How are the x and y intercepts of a function and its inverse related?

88. Does a constant function have an inverse? Explain.

89. Are the functions $f(x) = x^2$ and $g(x) = \sqrt{x}$ inverses? Why or why not?

90. Are the functions $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ inverses? Why or why not?

The functions in Problems 91–94 are one-to-one. Find f^{-1}.

91. $f(x) = (x - 1)^2 + 2, x \geq 1$

92. $f(x) = 3 - (x - 5)^2, x \leq 5$

93. $f(x) = x^2 + 2x - 2, x \leq -1$

94. $f(x) = x^2 + 8x + 7, x \geq -4$

In Problems 95–100, find f^{-1}, find the domain and range of f^{-1}, sketch the graphs of f, f^{-1}, and $y = x$ in the same coordinate system, and identify each graph.

95. $f(x) = -\sqrt{9 - x^2}, 0 \leq x \leq 3$

96. $f(x) = \sqrt{9 - x^2}, 0 \leq x \leq 3$

97. $f(x) = \sqrt{9 - x^2}, -3 \leq x \leq 0$

98. $f(x) = -\sqrt{9 - x^2}, -3 \leq x \leq 0$

99. $f(x) = 1 + \sqrt{1 - x^2}, 0 \leq x \leq 1$

100. $f(x) = 1 - \sqrt{1 - x^2}, 0 \leq x \leq 1$

In Problems 101–104, the function f is not one-to-one. Find the inverses of the functions formed by restricting the domain of f as indicated. Check by graphing f, f^{-1}, and the line $y = x$ in a squared viewing window on a graphing calculator. [Hint: To restrict the graph of $y = f(x)$ to an interval of the form $a \leq x \leq b$, enter $y = f(x)/((a \leq x)*(x \leq b))$.]

101. $f(x) = (2 - x)^2$: (A) $x \leq 2$ (B) $x \geq 2$

102. $f(x) = (1 + x)^2$: (A) $x \leq -1$ (B) $x \geq -1$

103. $f(x) = \sqrt{4x - x^2}$: (A) $0 \leq x \leq 2$ (B) $2 \leq x \leq 4$

104. $f(x) = \sqrt{6x - x^2}$: (A) $0 \leq x \leq 3$ (B) $3 \leq x \leq 6$

APPLICATIONS

105. **PRICE AND DEMAND** The number q of CD players consumers are willing to buy per week from a retail chain at a price of $\$p$ is given approximately by

$$q = d(p) = \frac{3{,}000}{0.2p + 1} \qquad 10 \leq p \leq 70$$

(A) Find the range of d.
(B) Find $p = d^{-1}(q)$, and find its domain and range.
(C) Should you interchange p and q in part B? Explain.

106. **PRICE AND SUPPLY** The number q of CD players a retail chain is willing to supply at a price of $\$p$ is given approximately by

$$q = s(p) = \frac{900p}{p + 20} \qquad 10 \leq p \leq 70$$

(A) Find the range of s.
(B) Find $p = s^{-1}(q)$, and find its domain and range.
(C) Should you interchange p and q in part B? Explain.

107. **REVENUE** The demand x and the price p (in dollars) for a certain product are related by

$$x = f(p) = 2{,}000 - 40p \qquad 0 \leq p \leq 50$$

Express the revenue as a function of x.

108. **REVENUE** The demand x and the price p (in dollars) for a certain product are related by

$$x = f(p) = 3{,}000 - 30p \qquad 0 \leq p \leq 100$$

Express the revenue as a function of x.

CHAPTER 1 Review

1-1 Using Graphing Calculators

A **graphing utility** is any electronic device capable of displaying the graph of an equation. The smallest darkened rectangular area that a graphing calculator can display is called a **pixel.** The **window variables** for a **standard viewing window** are

$$\text{Xmin} = -10,\ \text{Xmax} = 10,\ \text{Xscl} = 1,\ \text{Ymin} = -10,$$
$$\text{Ymax} = 10,\ \text{Yscl} = 1$$

Other viewing windows can be defined by assigning different values to these variables. Most graphing calculators will construct a table of ordered pairs that satisfy an equation. A grid can be added to a graph to aid in reading the graph. A **cursor** is used to locate a single pixel on the screen. The coordinates of the pixel at the cursor location, called **screen coordinates,** approximate the mathematical coordinates of all the points close to the pixel. The **TRACE** command constrains cursor movement to the graph of an equation and displays coordinates of points that satisfy the equation. The **ZOOM** command enlarges or reduces the viewing window. The **INTERSECT** or **ISECT** command finds the intersection points of two curves.

Mathematical Modeling

The term **mathematical modeling** refers to the process of using an equation or equations to describe data from the real world.

1-2 Functions

A **relation** is a correspondence that matches up two sets of objects. The first set is called the **domain** and the set of all corresponding elements in the second set is called the **range.** A relation where every element in the domain gets matched with only one element of the range is called a **function**. Equivalently, a *function* is a set of ordered pairs with the property that no two ordered pairs have the same first component and different second components. The *domain* is the set of all first components and the *range* is the set of all second components. An equation in two variables defines a function if to each value of the independent variable, the placeholder for domain values, there corresponds exactly one value of the **dependent variable**, the placeholder for range values. A vertical line will intersect the graph of a function in at most one point. Unless otherwise specified, the domain of a function defined by an equation is assumed to be the set of all real number replacements for the independent variable that produce real values for the dependent variable. The symbol $f(x)$ represents the real number in the range of the function f that is paired with the domain value x. Equivalently, the ordered pair $(x, f(x))$ belongs to the function f. The **STAT** editor on a graphing calculator is used to enter data and the **STAT PLOT** command will produce a **scatter plot** of the data.

1-3 Functions: Graphs and Properties

The **graph of a function** f is the set of all points $(x, f(x))$, where x is in the domain of f and $f(x)$ is the associated output. This is also the same as the graph of the equation $y = f(x)$. The first coordinate of a point where the graph of a function intersects the x axis is called an ***x* intercept** or **real zero** of the function. The x intercept is also a real solution or **root** of the equation $f(x) = 0$. The second coordinate of a point where the graph of a function crosses the y axis is called the ***y* intercept** of the function. The y intercept is given by $f(0)$, provided 0 is in the domain of f. Most graphing utilities contain a built-in command, usually called **ROOT** or **ZERO,** for approximating x intercepts. A solid dot on a graph of a function indicates a point that belongs to the graph and an open dot indicates a point that does not belong to the graph. Dots are also used to indicate that a graph terminates at a point, and arrows are used to indicate that the graph continues indefinitely with no significant changes in direction.

Let I be an open interval in the domain of a function f. Then,

1. f is **increasing** on I and the graph of f is **rising** on I if $f(a) < f(b)$ whenever $a < b$ in I.
2. f is **decreasing** on I and the graph of f is **falling** on I if $f(a) > f(b)$ whenever $a < b$ in I.
3. f is **constant** on I and the graph of f is **horizontal** on I if $f(a) = f(b)$ whenever $a < b$ in I.

The functional value $f(c)$ is called a **local maximum** if there is an interval (a, b) containing c such that $f(x) \le f(c)$ for all x in (a, b) and a **local minimum** if there is an interval (a, b) containing c such that $f(x) \ge f(c)$ for all x in (a, b). The functional value $f(c)$ is called a **local extremum** if it is either a local maximum or a local minimum. Most graphing calculators have a **MAXIMUM** command and a **MINIMUM** command for finding local extrema.

A **piecewise-defined function** is a function that is defined by different formulas for different domain values. Graphs of piecewise-defined functions may have **sharp corners.** A function is **continuous** if its graph has no holes or breaks and **discontinuous** at any point where it has a hole or break. Intuitively, the graph of a continuous function can be sketched without lifting a pen from the paper. The **greatest**

integer of a real number x, denoted by $[\![x]\!]$, is the largest integer less than or equal to x; that is, $[\![x]\!] = n$, where n is an integer, $n \le x < n + 1$.

The **greatest integer function** f is defined by the equation $f(x) = [\![x]\!]$. Changing the mode on a graphing calculator from **connected mode** to **dot mode** makes discontinuities on some graphs more apparent.

1-4 Functions: Graphs and Transformations

The first six basic functions in our library of elementary functions are defined by $f(x) = x$ (identity function), $g(x) = |x|$ (absolute value function), $h(x) = x^2$ (square function), $m(x) = x^3$ (cube function), $n(x) = \sqrt{x}$ (square root function), and $p(x) = \sqrt[3]{x}$ (cube root function) (see Figure 1 in this section). Performing an operation on a function produces a transformation of the graph of the function. The basic transformations are the following:

Vertical Translation:

$$y = f(x) + k \begin{cases} k > 0 & \text{Shift graph of } y = f(x) \text{ up } k \text{ units} \\ k < 0 & \text{Shift graph of } y = f(x) \text{ down } |k| \text{ units} \end{cases}$$

Horizontal Translation:

$$y = f(x + h) \begin{cases} h > 0 & \text{Shift graph of } y = f(x) \text{ left } h \text{ units} \\ h < 0 & \text{Shift graph of } y = f(x) \text{ right } |h| \text{ units} \end{cases}$$

Vertical Stretch and Shrink:

$$y = Af(x) \begin{cases} A > 1 & \text{Vertically stretch the graph of } y = f(x) \text{ by multiplying each } y \text{ value by } A \\ 0 < A < 1 & \text{Vertically shrink the graph of } y = f(x) \text{ by multiplying each } y \text{ value by } A \end{cases}$$

Horizontal Stretch and Shrink:

$$y = f(Ax) \begin{cases} A > 1 & \text{Horizontally shrink the graph of } y = f(x) \text{ by multiplying each } x \text{ value by } \dfrac{1}{A} \\ 0 < A < 1 & \text{Horizontally stretch the graph of } y = f(x) \text{ by mltiplying each } x \text{ value by } \dfrac{1}{A} \end{cases}$$

Reflection:

$y = -f(x)$ Reflect the graph of $y = f(x)$ in the x axis
$y = f(-x)$ Reflect the graph of $y = f(x)$ in the y axis
$y = -f(-x)$ Reflect the graph of $y = f(x)$ in the origin

A function f is called an **even function** if $f(x) = f(-x)$ for all x in the domain of f and an **odd function** if $f(-x) = -f(x)$ for all x in the domain of f. The graph of an even function is said to be **symmetric with respect to the *y* axis** and the graph of an odd function is said to be **symmetric with respect to the origin.**

1-5 Operations on Functions; Composition

The **sum, difference, product,** and **quotient** of the functions f and g are defined by

$$(f + g)(x) = f(x) + g(x) \qquad (f - g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad g(x) \neq 0$$

The **domain** of each function is the intersection of the domains of f and g, with the exception that values of x where $g(x) = 0$ must be excluded from the domain of f/g.

The **composition** of functions f and g is defined by $(f \circ g)(x) = f(g(x))$. The **domain** of $f \circ g$ is the set of all real numbers x in the domain of g such that $g(x)$ is in the domain of f. The domain of $f \circ g$ is always a subset of the domain of g. Composition is not a commutative process: the order of functions is important.

1-6 Inverse Functions

A function is **one-to-one** if no two ordered pairs in the function have the same second component and different first components. A **horizontal line** will intersect the graph of a one-to-one function in at most one point. A function that is increasing (or decreasing) throughout its domain is one-to-one. The **inverse** of the one-to-one function f is the function f^{-1} formed by reversing all the ordered pairs in f.

If f is a one-to-one function, then:

1. f^{-1} is one-to-one.

2. Domain of f^{-1} = Range of f.

3. Range of f^{-1} = Domain of f.

4. $x = f^{-1}(y)$ if and only if $y = f(x)$.

5. $f^{-1}(f(x)) = x$ for all x in the domain of f.

6. $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.

7. To find f^{-1}, replace $f(x)$ with y, then interchange x and y and solve for y. (In some applications, it may not be appropriate to interchange variables, in which case you simply solve for the independent variable of f.)

8. The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetric with respect to the line $y = x$.

CHAPTER 1 Review Exercises

Work through all the problems in this review and check answers in the back of the book. Answers to most review problems are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

1. Find the smallest viewing window that will contain all the points in the table. State your answer in terms of the window variables.

x	-3	5	-4	0	9
y	2	-6	7	-5	1

2. Indicate whether each relation defines a function. Indicate whether any of the functions are one-to-one. Find the domain and range of each function. Find the inverse of any one-to-one functions. Find the domain and range of any inverse functions.

(A) $\{(1, 1), (2, 4), (3, 9)\}$

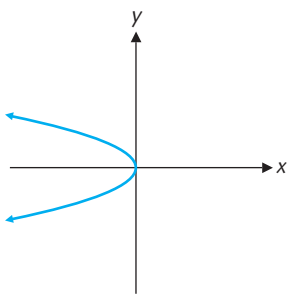
(B) $\{(1, 1), (1, -1), (2, 2), (2, -2)\}$

(C) $\{(-2, 2), (-1, 2), (0, 2), (1, 2), (2, 2)\}$

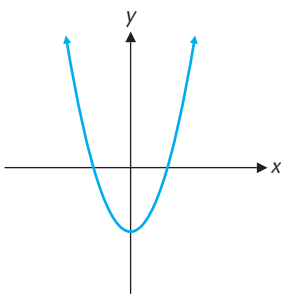
(D) $\{(-2, 2), (-1, 3), (0, -1), (1, -2), (2, 1)\}$

3. Indicate whether each graph specifies a function:

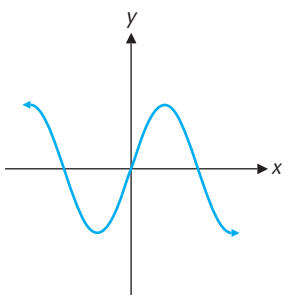
(A)



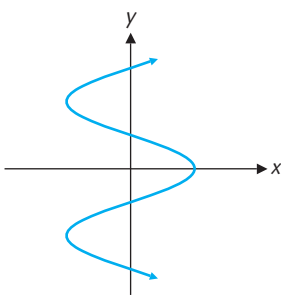
(B)



(C)



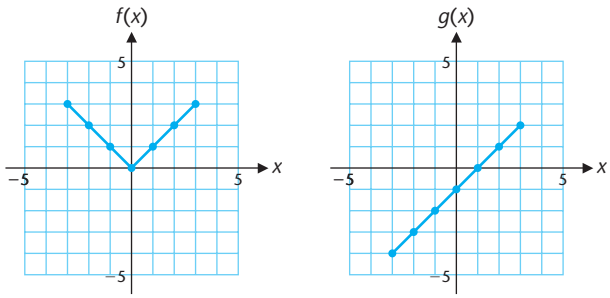
(D)



4. For $f(x) = x^2 - 2x$, find:

(A) $f(1)$ (B) $f(-4)$ (C) $f(2) \cdot f(-1)$ (D) $\dfrac{f(0)}{f(3)}$

Problems 5–12 refer to the graphs of f and g shown below.



5. Construct a table of values of $(f - g)(x)$ for $x = -3$, -2, -1, 0, 1, 2, and 3, and sketch the graph of $f - g$.

6. Construct a table of values of $(fg)(x)$ for $x = -3$, -2, -1, 0, 1, 2, and 3, and sketch the graph of fg.

In Problems 7–10, use the graphs of f and g to find:

7. $(f \circ g)(-1)$ **8.** $(g \circ f)(-2)$ **9.** $f(g(1))$

10. $g(f(-3))$ **11.** Is f a one-to-one function?

12. Is g a one-to-one function?

13. Functions f and g are defined by Table 1. Find $(f \circ g)(-11)$, $(f \circ g)(1)$, and $(f \circ g)(6)$.

Table 1

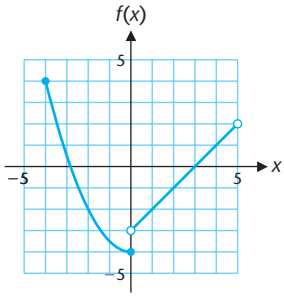
x	$f(x)$	$g(x)$
-11	12	-4
-4	8	16
1	-4	-11
6	-10	9
9	1	0

14. Indicate whether each function is even, odd, or neither:

(A) $f(x) = x^5 + 6x$ (B) $g(t) = t^4 + 3t^2$

(C) $h(z) = z^5 + 4z^2$

Problems 15–25 refer to the function f given by the following graph.



15. Find $f(-4), f(0), f(3)$, and $f(5)$.

16. Find all values of x for which $f(x) = -2$.

17. Find the domain and range of f.

18. Find the intervals over which f is increasing and decreasing.

19. Find any points of discontinuity.

Sketch the graph of each of the following,

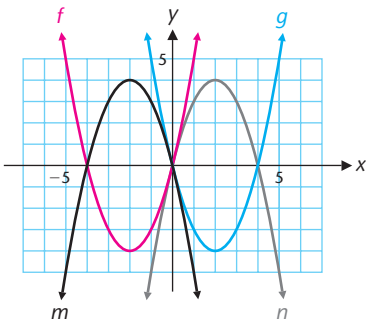
20. $f(x) + 1$ **21.** $f(x + 1)$ **22.** $-f(x)$

23. $0.5f(x)$ **24.** $f(2x)$ **25.** $-f(-x)$

26. Match each equation with a graph of one of the functions f, g, m, or n in the figure. Each graph is a graph of one of the equations.

(A) $y = (x - 2)^2 - 4$ (B) $y = -(x + 2)^2 + 4$

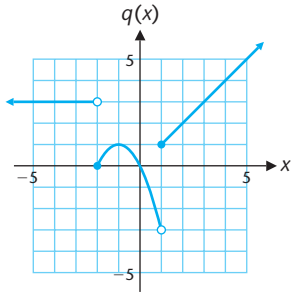
(C) $y = -(x - 2)^2 + 4$ (D) $y = (x + 2)^2 - 4$



27. Let $f(x) = x^2 - 4$ and $g(x) = x + 3$. Find each of the following functions and find their domains.

(A) f/g (B) g/f (C) $f \circ g$ (D) $g \circ f$

Problems 28–34 refer to the function q given by the following graph.



28. Find y to the nearest integer:

(A) $y = q(0)$ (B) $y = q(1)$

(C) $y = q(2)$ (D) $y = q(-2)$

29. Find x to the nearest integer:

(A) $q(x) = 0$ (B) $q(x) = 1$

(C) $q(x) = -3$ (D) $q(x) = 3$

30. Find the domain and range of q.

31. Find the intervals over which q is increasing.

32. Find the intervals over which q is decreasing.

33. Find the intervals over which q is constant.

34. Identify any points of discontinuity.

The graphs of each pair of equations in Problems 35 and 36 intersect in exactly two points. Find a viewing window that clearly shows both points of intersection. Use INTERSECT to find the coordinates of each intersection point to two decimal places.

35. $y = x^2 - 20x, y = 4x - 15$

36. $y = \sqrt{10x + 50}, y = 0.3x + 4$

37. Solve the following equation for the indicated values of b. Round answers to two decimal places.

$$0.1x^3 - 2x^2 - 6x + 80 = b$$

(A) $b = 0$ (B) $b = 100$

(C) $b = -50$ (D) $b = -150$

In Problems 38 and 39, determine if the indicated equation defines a function. Justify your answer.

38. $x + 2y = 10$ **39.** $x + 2y^2 = 10$

40. Find the domain of each of the following functions:

(A) $f(x) = x^2 - 4x + 5$ (B) $g(t) = \dfrac{t+2}{t-5}$

(C) $h(w) = 2 + 3\sqrt{w}$

41. If $g(t) = 2t^2 - 3t + 6$, find $\dfrac{g(2+h) - g(2)}{h}$.

42. The function f multiplies the cube of the domain element by 4 and then subtracts the square root of the domain element. Write an algebraic definition of f.

43. Write a verbal description of the function $f(x) = 3x^2 + 4x - 6$.

In Problems 44 and 45, find the x intercepts, y intercept, local extrema, domain, and range. Round answers to two decimal places.

44. $g(x) = 6\sqrt{x} - x^2$ **45.** $s(x) = x^3 + 27x^2 - 300$

46. Let $f(x) = \begin{cases} -x - 5 & \text{for } -4 \le x < 0 \\ 0.2x^2 & \text{for } 0 \le x \le 5 \end{cases}$

(A) Sketch the graph of $y = f(x)$.

(B) Find the domain and range.

(C) Find any points of discontinuity.

(D) Find the intervals over which f is increasing, decreasing, and constant.

47. Let $f(x) = 0.1x^3 - 6x + 5$. Write a verbal description of the graph of f using increasing and decreasing terminology and indicating any local maximum and minimum values. Approximate to two decimal places the coordinates of any points used in your description.

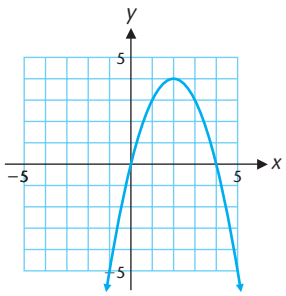
48. How are the graphs of the following related to the graph of $y = x^2$?

(A) $y = -x^2$ (B) $y = x^2 - 3$

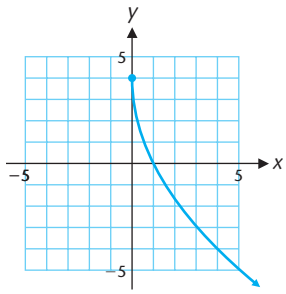
(C) $y = (x+3)^2$ (D) $y = (2x)^2$

49. Each of the following graphs is the result of applying one or more transformations to the graph of one of the six basic functions in Figure 1, Section 1.4. Find an equation for the graph. Check by graphing the equation on a graphing utility.

(A)



(B)



50. The graph of $f(x) = |x|$ is stretched vertically by a factor of 3, reflected in the x axis, shifted four units to the right and eight units up to form the graph of the function g. Find an equation for the function g and graph g.

51. The graph of $m(x) = x^2$ is stretched horizontally by a factor of 2, shifted two units to the left and four units down to form the graph of the function t. Find an equation for the function t and graph t.

52. Is $u(x) = 4x - 8$ the inverse of $v(x) = 0.25x + 2$?

53. Let $k(x) = x^3 + 5$. Write a verbal description of k, reverse your description, and write the resulting algebraic equation. Verify that the result is the inverse of the original function.

54. Find the domain of $f(x) = \dfrac{x}{\sqrt{x} - 3}$.

55. Given $f(x) = \sqrt{x} - 8$ and $g(x) = |x|$,

(A) Find $f \circ g$ and $g \circ f$.

(B) Find the domains of $f \circ g$ and $g \circ f$.

56. Which of the following functions are one-to-one?

(A) $f(x) = x^3$ (B) $g(x) = (x-2)^2$

(C) $h(x) = 2x - 3$ (D) $F(x) = (x+3)^2, x \ge -3$

(E) $f(x) = 0.3x^2 - 7x$

In Problems 57–59, find f^{-1}, find the domain and range of f^{-1}, sketch the graphs of f, f^{-1}, and $y = x$ in the same coordinate system, and identify each graph.

57. $f(x) = 3x - 7$ **58.** $f(x) = \sqrt{x-1}$

59. $f(x) = x^2 - 1, x \ge 0$

60. Sketch by hand the graph of a function that is consistent with the given information.

(A) The function f is continuous on $[-5, 5]$, increasing on $[-5, -3]$, decreasing on $[-3, 1]$, constant on $[1, 3]$, and increasing on $[3, 5]$.

(B) The function f is continuous on $[-5, 1)$ and $[1, 5]$, $f(-2) = -1$ is a local maximum, and $f(3) = 2$ is a local minimum.

61. Write a verbal description of the function g and then find an equation for $g(t)$.

$$g(t+h) = 2(t+h)^2 - 4(t+h) + 5$$

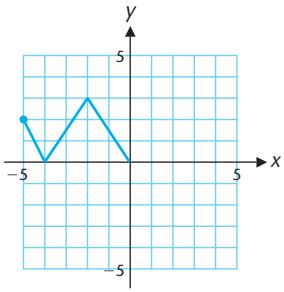
62. Graph in the standard viewing window:

$$f(x) = 0.1(x-2)^2 + \frac{|3x-6|}{x-2}$$

Assuming the graph continues as indicated beyond the part shown in this viewing window, find the domain, range, and any points of discontinuity. [*Hint:* Use the dot mode on your graphing calculator, if it has one.]

63. A partial graph of the function f is shown in the figure. Complete the graph of f over the interval $[0, 5]$ given that:

(A) f is an even function. (B) f is an odd function.



64. For $f(x) = 3x^2 - 5x + 7$, find and simplify:

(A) $\dfrac{f(x+h) - f(x)}{h}$ (B) $\dfrac{f(x) - f(a)}{x - a}$

65. The function f is decreasing on $[-5, 5]$ with $f(-5) = 4$ and $f(5) = -3$.

(A) If f is continuous on $[-5, 5]$, how many times can the graph of f cross the x axis? Support your conclusion with examples and/or verbal arguments.

(B) Repeat part A if the function does not have to be continuous.

66. Let $f(x) = [\![|x|]\!]$.

(A) Write a piecewise definition of f. Include sufficient intervals to clearly illustrate the definition.

(B) Sketch by hand the graph of $y = f(x)$, using a graphing calculator as an aid. Include sufficient intervals to clearly illustrate the graph.

(C) Find the range of f. (D) Find any points of discontinuity.

(E) Indicate whether f is even, odd, or neither.

APPLICATIONS

67. **PRICE AND DEMAND** The price $\$p$ per hot dog at which q hot dogs can be sold during a baseball game is given approximately by

$$p = g(q) = \frac{9}{1 + 0.002q} \qquad 1{,}000 \le q \le 4{,}000$$

(A) Find the range of g.

(B) Find $q = g^{-1}(p)$ and find its domain and range.

(C) Express the revenue as a function of p.

(D) Express the revenue as a function of q.

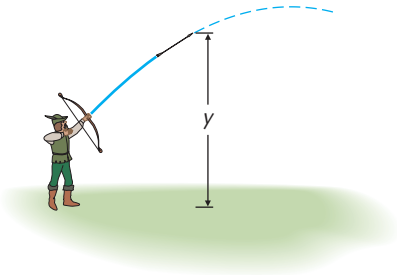
68. **MARKET RESEARCH** If x units of a product are produced each week and sold for a price of $\$p$ per unit, then the weekly demand, revenue, and cost equations are, respectively,

$$\begin{aligned} x &= 500 - 10p \\ R(x) &= 50x - 0.1x^2 \\ C(x) &= 10x + 1{,}500 \end{aligned}$$

Express the weekly profit as a function of the price p and find the price that produces the largest profit.

69. **PHYSICS: POSITION OF A MOVING OBJECT** In flight shooting distance competitions, archers are capable of shooting arrows 600 meters or more. An archer standing on the ground shoots an arrow. After x seconds, the arrow is y meters above the ground as given approximately by

$$y = 55x - 4.88x^2$$



(A) Find the time (to the nearest tenth of a second) the arrow is airborne.

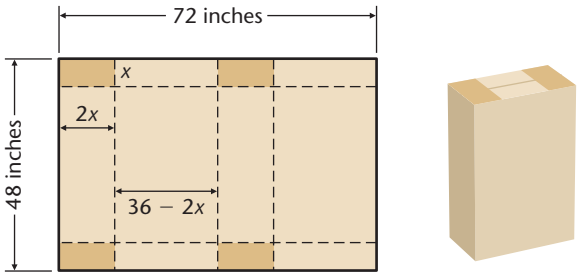
(B) Find the maximum altitude (to the nearest meter) the arrow reaches during its flight.

70. **PHYSICS: HEIGHT OF A BUNGEE JUMPER** The world's highest bungee jumping bridge is the Bloukrans River Bridge in South Africa, at a height of 708.7 feet. The height in feet of one jumper can be modeled by the function $h(x) = 9.5x^2 - 152x + 708.7$ $0 \le x \le 10$, where x is seconds after he jumps.

(A) How long does it take for the jumper to reach height 300 feet? Round to the nearest tenth of a second.

(B) How high above the ground is the jumper when he reaches the lowest point of the jump? How many seconds pass until he reaches the low point?

71. **MANUFACTURING** A box with four flaps on each end is to be made out of a piece of cardboard that measures 48 by 72 inches. The width of each flap is x inches and the length of one pair of opposite flaps is $2x$ inches to ensure that the other pair of flaps will meet when folded over to close the box (see the figure).



(A) Find the width of the flap (to two decimal places) that will produce a box with maximum volume. What is the maximum volume?

(B) How wide should the flap be if a manufacturer needs the box to have a volume of 9,200 cubic inches?

72. EFFECTS OF ALTITUDE The percentage of oxygen in the air depends on the altitude of a given location. This percentage can be modeled by the function $f(x) = -0.000767x + 21.6$ $0 \le x \le 8000$, where x is the elevation in feet.

(A) Complete the table of values for f.

Elevation	Oxygen Level (%)
0	
2,000	
4,000	
6,000	
8,000	

(B) Based on the information in the table, write a brief verbal description of the relationship between elevation and oxygen level.

(C) The elevation of the summit of Mt. Everest is 29,035 feet. According to the model, what is the oxygen level there? Is your answer reasonable? Why do you suppose this happened?

73. CELL PHONE CHARGES A local cell phone provider calculates monthly usage charges in dollars using the function $c(x) = 19 + 0.012x$, where x is minutes used. Translate this algebraic statement into a verbal description that a sales representative could use to explain the monthly charges to a potential customer.

74. REVENUE The revenue in dollars made by a banquet facility when hosting a wedding is modeled by the function $R(x) = C + 22.50x$, where x is the number of guests and C is the flat fee charged for hosting.

(A) Graph R for $C = 200$, 900, and 1,500 simultaneously in the viewing window Xmin = 0, Xmax = 600, Ymin = 0, Ymax = 15,000. Write a brief verbal description of this collection of functions.

(B) What is the maximum flat fee the facility can charge if they want their price to be no higher than a competing facility that offers a wedding with 300 guests for $7,500?

75. MEDICINE Proscar is a drug produced by Merck & Co., Inc., to treat symptomatic benign prostate enlargement. One of the long-term effects of the drug is to increase urine flow rate. Results from a 3-year study show that

$$f(x) = 0.00005x^3 - 0.007x^2 + 0.255x$$

is a mathematical model for the average increase in urine flow rate in cubic centimeters per second where x is time taking the drug in months.

(A) Graph this function for $0 \le x \le 36$.

(B) Write a brief verbal description of the graph using increasing, decreasing, local maximum, and local minimum as appropriate. Approximate to two decimal places the coordinates of any points used in your description.

76. COMPUTER SCIENCE In computer programming, it is often necessary to check numbers for certain properties (even, odd, perfect square, etc.). The greatest integer function provides a convenient method for determining some of these properties. Consider the function

$$f(x) = x - (\llbracket \sqrt{x} \rrbracket)^2$$

(A) Evaluate f for $x = 1, 2, \ldots, 16$.

(B) Find $f(n^2)$, where n is a positive integer.

(C) What property of x does this function determine?

MODELING AND DATA ANALYSIS

77. DATA ANALYSIS Winning times in the men's Olympic 400-meter freestyle event in minutes for selected years are given in Table 2 on the next page. A mathematical model for these data is

$$f(x) = -0.021x + 5.57$$

where x is years since 1900.

(A) Compare the model and the data graphically and numerically.

(B) Estimate (to three decimal places) the winning time in 2008.

Table 2

Year	Time
1912	5.41
1932	4.81
1952	4.51
1972	4.00
1992	3.75

Table 3 Virginia Tax Rate Schedule

Status	Taxable Income Over	But Not Over	Tax Is	Of the Amount Over
Single	$ 0	$ 3,000	2%	$ 0
	$ 3,000	$ 5,000	$ 60 + 3%	$ 3,000
	$ 5,000	$17,000	$120 + 5%	$ 5,000
	$17,000	.....	$720 + 5.75%	$17,000

78. Use the schedule in Table 3 to construct a piecewise-defined model for the taxes due for a single taxpayer in Virginia with a taxable income of x dollars. Find the tax on the following incomes: $2,000, $4,000, $10,000, $30,000.

CHAPTER 1

››› GROUP ACTIVITY Mathematical Modeling: Choosing a Cell Phone Provider

The number of companies offering cellular telephone service has grown rapidly in recent years. The plans they offer vary greatly and it can be difficult to select the plan that is best for you. Here are five typical plans:

Plan 1: A flat fee of $50 per month for unlimited calls.
Plan 2: A $30 per month fee for a total of 30 hours of calls and an additional charge of $0.01 per minute for all minutes over 30 hours.
Plan 3: A $5 per month fee and a charge of $0.04 per minute for all calls.
Plan 4: A $2 per month fee and a charge of $0.045 per minute for all calls; the fee is waived if the charge for calls is $20 or more.
Plan 5: A charge of $0.05 per minute for all calls; there are no additional fees.

(A) Construct a mathematical model for each plan that gives the total monthly cost in terms of the total number of minutes of calls placed in a month. Graph each model on a graphing calculator. You may find dividing by expressions like $(x > a)$ helpful in entering your model in a graphing calculator (see Example 7 in Section 1-6).
(B) Compare plans 1 and 2. Determine how many minutes per month would make plan 1 cheaper and how many would make plan 2 cheaper.
(C) Repeat part (B) for plans 1 and 3; plans 1 and 4; plans 1 and 5.
(D) Repeat part (B) for plans 2 and 3; plans 2 and 4; plans 2 and 5.
(E) Repeat part (B) for plans 3 and 4; plans 3 and 5.
(F) Repeat part (B) for plans 4 and 5.
(G) Is there one plan that is always better than all the others? Based on your personal calling history, which plan would you choose and why?

CHAPTER

2

Modeling with Linear and Quadratic Functions

IN Chapter 1, we investigated the general concept of functions using graphs, tables, and algebraic equations. Now it's time to get specific: most of the remainder of the book is devoted to studying particular categories of functions in detail. Our goal is to develop a library of functions that we can work with and understand comfortably. The types of functions we will study are used with great frequency in almost any place where mathematics is used: the physical, social, and life sciences; business; computers, engineering, and most technical fields; and of course in any math course you might take beyond this one.

In this chapter, we study two basic types of functions, the linear and quadratic functions. As you will see, many significant real-world problems can be represented by these functions.

OUTLINE

2-1 Linear Functions
2-2 Linear Equations and Models
2-3 Quadratic Functions
2-4 Complex Numbers
2-5 Quadratic Equations and Models
2-6 Additional Equation-Solving Techniques
2-7 Solving Inequalities
Chapter 2 Review
Chapter 2 Group Activity: Mathematical Modeling in Population Studies
Cumulative Review Exercises Chapters 1 and 2

2-1 Linear Functions

- Constant and Linear Functions
- Exploring the Graph of $Ax + By = C$
- Defining the Slope of a Line
- Using Special Forms of Linear Equations
- Recognizing Parallel and Perpendicular Lines
- Mathematical Modeling: Slope as a Rate of Change

The straight line is a very simple geometric object, but it is also an important tool in mathematical modeling. In this section, we will add linear functions to our library of functions and explore the relationship between graphs of linear functions and straight lines. We will also determine how to find the equation of a line, given information about that line. This will be a big help in modeling many quantities. We will conclude the section with a look at how slope is used to model quantities that have a constant rate of change.

Constant and Linear Functions

One of the elementary functions introduced in Section 1-4 was the identity function $f(x) = x$ (Fig. 1).

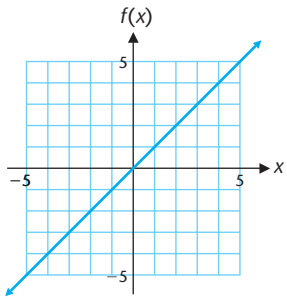


Figure 1 Identity function: $f(x) = x$.

EXPLORE-DISCUSS 1

Use the transformations discussed in Section 1-4 to describe verbally the relationship between the graph of $f(x) = x$ and each of the following functions. Graph each function.

(A) $g(x) = 3x + 1$ **(B)** $h(x) = 0.5x - 2$ **(C)** $k(x) = -x + 1$

If we apply a sequence of translations, reflections, expansions, and/or contractions to the identity function, the result is always a function whose graph is a straight line. Because of this, functions like g, h, and k in Explore-Discuss 1 are called *linear functions*. There's one thing that all such functions will have in common: the independent variable will appear only to the first power.

DEFINITION 1 Linear and Constant Functions

A function f is a **linear function** if

$$f(x) = mx + b \qquad m \neq 0$$

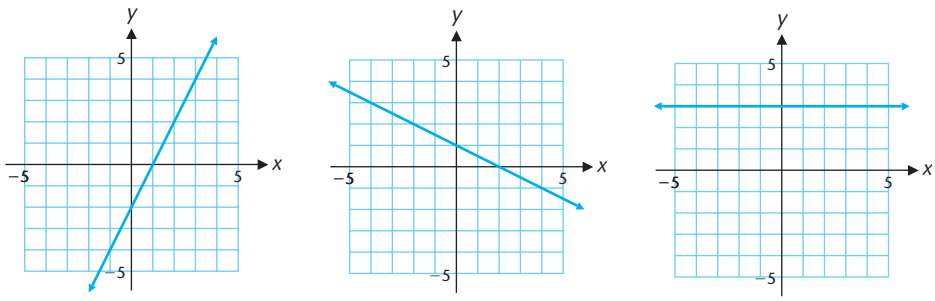
where m and b are real numbers. The domain is the set of all real numbers and the range is also the set of all real numbers.

If $m = 0$, then f is called a **constant function,**

$$f(x) = b$$

which has the set of all real numbers as its domain and the constant b as its range.

Figure 2 shows the graphs of two linear functions f and g, and a constant function h.



(a) $f(x) = 2x - 2$ (b) $g(x) = -0.5x + 1$ (c) $h(x) = 3$

Figure 2 Two linear functions and a constant function.

It can be shown that

The graph of a linear function is a straight line that is neither horizontal nor vertical. The graph of a constant function is a horizontal straight line.

What about vertical lines? Recall from Chapter 1 that the graph of a function cannot contain two points with the same x coordinate and different y coordinates. Because *all* the points on a vertical line have the same x coordinate, the graph of a function can never be a vertical line. Later in this section, we will discuss equations of vertical lines, but these equations never define functions.

Recall from Section 1-3 that the y intercept of a function f is $f(0)$, provided $f(0)$ exists, and the x intercepts are the solutions of the equation $f(x) = 0$.

››› EXPLORE-DISCUSS 2

(A) Is it possible for a linear function to have two x intercepts? No x intercept? If either of your answers is yes, give an example.

(B) Is it possible for a linear function to have two y intercepts? No y intercept? If either of your answers is yes, give an example.

(C) Discuss the possible numbers of x and y intercepts for a constant function.

EXAMPLE 1 Finding x and y Intercepts

Find the x and y intercepts for $f(x) = \frac{2}{3}x - 3$.

SOLUTION

The y intercept is $f(0) = -3$. The x intercept can be found algebraically using standard equation-solving techniques (Appendix B, Section B-1), or graphically using the ZERO command on a graphing calculator (Section 1-1).

Algebraic Solution

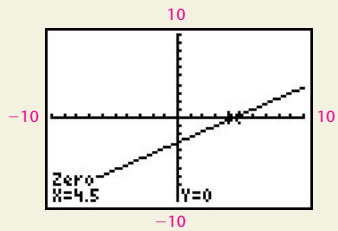
$$f(x) = 0$$

$$\frac{2}{3}x - 3 = 0 \qquad \text{Add 3 to both sides.}$$

$$\frac{2}{3}x = 3 \qquad \text{Multiply both sides by } \tfrac{3}{2}.$$

$$x = \frac{9}{2} = 4.5$$

Graphical Solution



MATCHED PROBLEM 1

Find the x and y intercepts of $g(x) = -\frac{4}{3}x + 5$.

Exploring the Graph of $Ax + By = C$

EXPLORE-DISCUSS 3

Graph each of the following cases of $Ax + By = C$ in the same coordinate system:

1. $3x + 2y = 6$

2. $0x - 3y = 12$

3. $2x + 0y = 10$

Which cases define functions? Explain why or why not.

Graph each case using a graphing utility (check your manual on how to graph vertical lines).

We will now investigate graphs of linear equations in two variables, like

$$Ax + By = C \tag{1}$$

where at least one of A and B is not zero. Keep in mind that x and y are variables in this setting, while A, B, and C are just numbers. Depending on the values of A and B, this equation can define a linear function, a constant function, or no function at all.

If A and B are both nonzero, then we can solve equation (1) for y:

$$\begin{aligned} Ax + By &= C && \text{Subtract } Ax \text{ from both sides.} \\ By &= C - Ax && \text{Divide both sides by } B. \\ y &= -\frac{A}{B}x + \frac{C}{B} \end{aligned}$$

This fits the form $f(x) = mx + B$ since $A/B \neq 0$. Based on Definition 1, this is a linear function.

If A is zero and B is nonzero, equation (1) becomes

$$\begin{aligned} 0x + By &= C && \text{Divide both sides by } B. \\ y &= \frac{C}{B} \end{aligned}$$

This fits the form $g(x) = b$ (remember, C/B is just a number), so it is a constant function.

If A is nonzero and B is zero, equation (1) becomes

$$\begin{aligned} Ax + 0y &= C && \text{Divide both sides by } A. \\ x &= \frac{C}{A} \end{aligned}$$

This equation specifies the same x value (the number C/A) for every possible y value. This tells us two things: The equation does not define a function, and the graph is a vertical line since every point has the same x coordinate.

The following theorem summarizes the preceding discussion:

› THEOREM 1 Graph of a Linear Equation in Two Variables

The graph of any equation of the form

$$Ax + By = C \qquad \text{Standard form} \qquad (2)$$

where A, B, and C are real numbers (A and B not both 0) is a straight line. Every straight line in a Cartesian coordinate system is the graph of an equation of this type. Vertical and horizontal lines are special cases of equation (2):

Horizontal line with y intercept b: $y = b$

Vertical line with x intercept a: $x = a$

To sketch the graph of an equation of the form

$$Ax + By = C \quad \text{or} \quad y = mx + b$$

all that is necessary is to plot any two points from the solution set and use a straightedge to draw a line through these two points. The x and y intercepts are often the easiest points to find, but any two points will do.

EXAMPLE 2 Sketching Graphs of Lines

(A) Describe the graphs of $x = -2$ and $y = 3$ verbally. Graph both equations in the same rectangular coordinate system by hand and in the same viewing window on a graphing calculator.

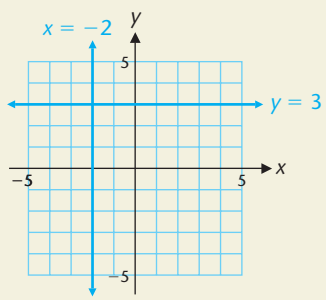
(B) Write the equations of the vertical and horizontal lines that pass through the point $(1, -4)$.

(C) Graph the equation $3x - 2y = 6$ by hand and on a graphing calculator.

SOLUTIONS

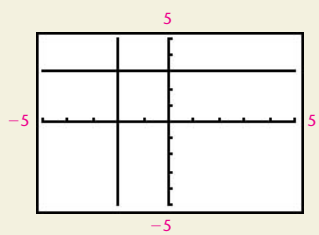
(A) The graph of $x = -2$ is a vertical line with x intercept -2 and the graph of $y = 3$ is a horizontal line with y intercept 3 (Fig. 3 and Fig. 4).

Hand-Drawn Solution



› Figure 3

Graphing Calculator Solution



› Figure 4

(B) The y coordinate of every point on a horizontal line through $(1, -4)$ is -4, so the equation is $y = -4$. The x coordinate of every point on a vertical line through $(1, -4)$ is 1, so the equation is $x = 1$.

(C) Hand-Drawn Solution

Find the x intercept by substituting $y = 0$ and solving for x, and then find the y intercept by substituting $x = 0$ and solving for y.

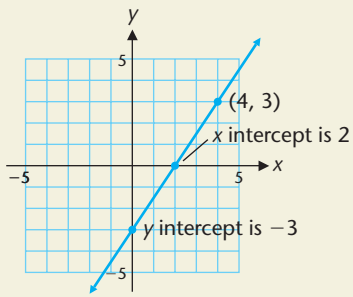
x intercept	y intercept
$3x - 2(0) = 6$	$3(0) - 2y = 6$
$3x = 6$	$-2y = 6$
$x = 2$	$y = -3$

To confirm our work, we'll also plot a third point and make sure that all three points appear to be on the same line.

Additional point:

$$3(4) - 2y = 6 \quad \text{Substitute } x = 4.$$
$$-2y = -6$$
$$y = 3$$
$$(4, 3)$$

Now we draw a line through all three points (Fig. 5).



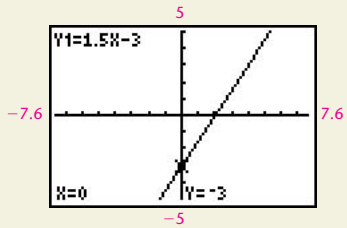
› Figure 5

(C) Graphing Calculator Solution

To enter the equation in the equation editor of a graphing calculator, we'll first need to solve for y.

$$3x - 2y = 6$$
$$-2y = -3x + 6$$
$$y = 1.5x - 3$$

Now enter the result in the equation editor, and graph (Fig. 6).



› Figure 6

Note that we used a *squared viewing window* in Figure 6 to produce units of the same length on both axes. This makes it easier to compare the hand sketch with the graphing calculator graph.

›››› CAUTION ››››

Even though finding two points is sufficient to graph a line, it's always a good idea to find three. If the three you find do not line up, you know you made a mistake somewhere.

MATCHED PROBLEM 2

(A) Describe the graphs of $x = 4$ and $y = -3$ verbally. Graph both equations in the same rectangular coordinate system by hand and in the same viewing window on a graphing calculator.

(B) Write the equations of the vertical and horizontal lines that pass through the point $(-7, 5)$.

(C) Graph the equation $4x + 3y = 12$ by hand and on a graphing calculator.

Defining the Slope of a Line

EXPLORE-DISCUSS 4

(A) For the linear function $f(x) = 3x - 5$, fill in the table of values.

x	-3	-2	-1	0	1	2	3
$f(x)$							

(B) Do you notice a pattern in the outputs as x gets bigger by 1 unit?

(C) Repeat for $g(x) = -3x - 5$. What do you notice?

x	-3	-2	-1	0	1	2	3
$g(x)$							

Explore-Discuss 4 illustrates the key feature of lines: the change in height is always the same for any 1-unit change in x. In fact, that's exactly why the graph is a straight line! This leads to the important concept of slope.

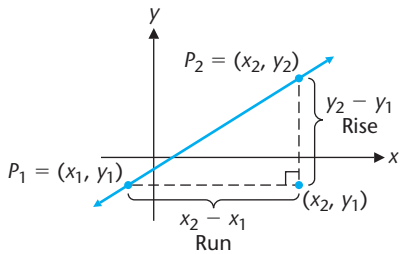
If we take two different points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ on a line, then the ratio of the change in y to the change in x as we move from point P_1 to point P_2 is called the **slope** of the line. Roughly speaking, slope is a measure of the "steepness" of a line. Steep lines have slopes with relatively large absolute values, and gradual lines have slopes that are near zero. Sometimes the change in x is called the **run** and the change in y the **rise.**

DEFINITION 2 Slope of a Line

If a line passes through two distinct points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, then its slope m is given by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad x_1 \neq x_2$$

$$= \frac{\text{Vertical change (rise)}}{\text{Horizontal change (run)}}$$



For a horizontal line, y doesn't change as x changes, so its slope is 0. For a vertical line, x doesn't change as y changes, so $x_1 = x_2$, the denominator in the slope formula is 0, and its slope is not defined. In general, the slope of a line can be positive, negative, zero, or not defined. Each case is illustrated geometrically in Table 1.

Table 1 Geometric Interpretation of Slope

Line	Slope	Example
Rising as x moves from left to right	Positive	
Falling as x moves from left to right	Negative	
Horizontal	0	
Vertical	Not defined	

In using the formula to find the slope of the line through two points, it doesn't matter which point is labeled P_1 or P_2, because changing the labeling will change the sign in both the numerator and denominator of the slope formula:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

For example, the slope of the line through the points (3, 2) and (7, 5) is

$$\frac{5 - 2}{7 - 3} = \frac{3}{4} \quad \text{or} \quad \frac{2 - 5}{3 - 7} = \frac{-3}{-4} = \frac{3}{4}$$

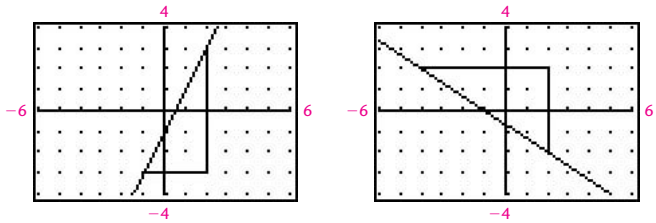
In addition, it is important to note that the definition of slope doesn't depend on the two points chosen on the line as long as they are distinct. Lines are straight exactly because the slope at every point on the line is the same.

EXAMPLE 3 Finding Slopes

For each line in Figure 7 on the next page, find the run, the rise, and the slope. (All the horizontal and vertical line segments have integer lengths.)

SOLUTION

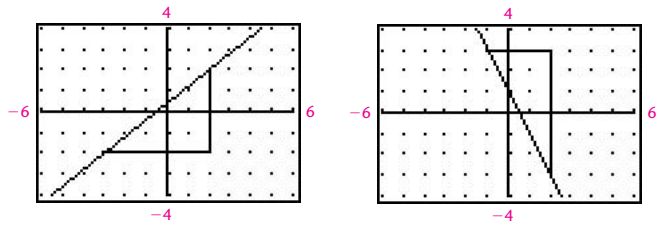
In Figure 7(a), the run is 3, the rise is 6 and the slope is $\frac{6}{3} = 2$. In Figure 7(b), the run is 6, the rise is -4 and the slope is $\frac{-4}{6} = -\frac{2}{3}$.



› Figure 7

MATCHED PROBLEM 3

For each line in Figure 8, find the run, the rise, and the slope. (All the horizontal and vertical line segments have integer lengths.)



› Figure 8

EXAMPLE 4 Finding Slopes

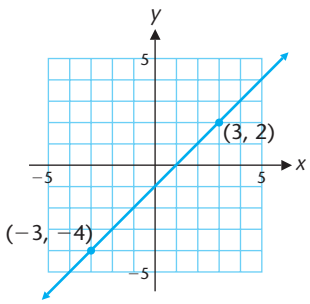
Sketch a line through each pair of points and find the slope of each line.

(A) $(-3, -4), (3, 2)$ (B) $(-2, 3), (1, -3)$

(C) $(-4, 2), (3, 2)$ (D) $(2, 4), (2, -3)$

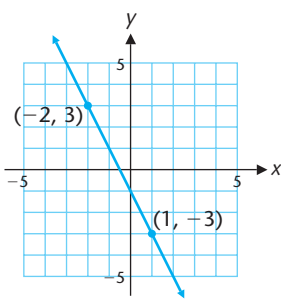
SOLUTIONS

(A)



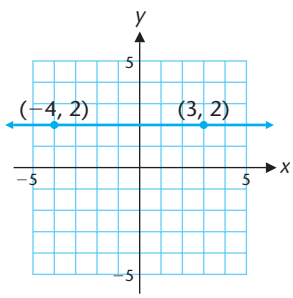
$$m = \frac{2-(-4)}{3-(-3)} = \frac{6}{6} = 1$$

(B)



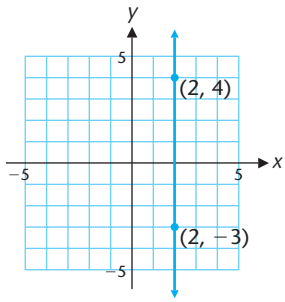
$$m = \frac{-3-3}{1-(-2)} = \frac{-6}{3} = -2$$

(C)



$$m = \frac{2-2}{3-(-4)} = \frac{0}{7} = 0$$

(D)



$$m = \frac{-3-4}{2-2} = \frac{-7}{0}$$

slope is not defined

MATCHED PROBLEM 4

Sketch a line through each pair of points and find the slope of each line.

(A) $(-3, -3), (2, -3)$ (B) $(-2, -1), (1, 2)$

(C) $(0, 4), (2, -4)$ (D) $(-3, 2), (-3, -1)$

The graphs in Example 4 serve to illustrate the summary in Table 2:

Table 2 Graph Properties of Linear and Constant Functions

Linear Functions		Constant Function
$f(x) = mx + b, m > 0$	$f(x) = mx + b, m < 0$	$f(x) = b$
Domain $= (-\infty, \infty)$	Domain $= (-\infty, \infty)$	Domain $= (-\infty, \infty)$
Range $= (-\infty, \infty)$	Range $= (-\infty, \infty)$	Range $= \{b\}$
Increasing on $(-\infty, \infty)$	Decreasing on $(-\infty, \infty)$	Constant on $(-\infty, \infty)$

› Using Special Forms of Linear Equations

Let's start by investigating why $y = mx + b$ is called the *slope–intercept form* for a line.

››› EXPLORE-DISCUSS 5

(A) Using a graphing calculator, graph $y = x + b$ for $b = -5, -3, 0, 3$, and 5 simultaneously in a standard viewing window. Verbally describe the geometric significance of b.

(B) Using a graphing calculator, graph $y = mx - 1$ for $m = -2, -1, 0, 1$, and 2 simultaneously in a standard viewing window. Verbally describe the geometric significance of m.

Both constants (m and b) in $y = mx + b$ have special geometric significance, which we now explicitly state.

Explore-Discuss 4 and 5 both suggest that for an equation of the form $y = mx + b$, the number m is the slope of the line. (See Exercise 84 for a proof.)

If we let $x = 0$, then $y = m \cdot 0 + b = b$ and the graph of $y = mx + b$ crosses the y axis at $(0, b)$. This tells us that the constant b is the y intercept. For example, the y intercept of the graph of $y = 2x - 7$ is -7.

To summarize:

› THEOREM 2 Slope–Intercept Form

The equation

$$y = mx + b$$

is called the **slope–intercept form** of the equation of a line. The slope is m, and the y intercept is b.

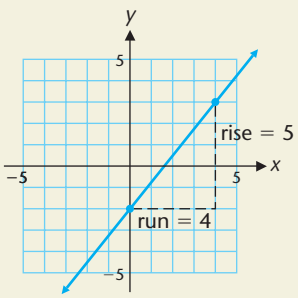
EXAMPLE 5 Using the Slope–Intercept Form

Graph the line with y intercept -2 and slope $\frac{5}{4}$.

SOLUTION

Hand-Drawn Solution

If we start at the point $(0, -2)$ and move four units to the right (run), then the y coordinate of a point on the line must move up five units (rise) to the point $(4, 3)$. Drawing a line through these two points produces the graph shown in Figure 9.



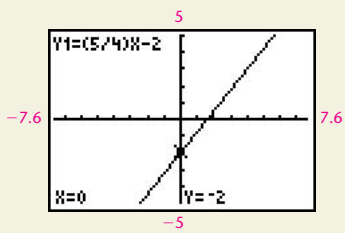
› Figure 9

Graphing Calculator Solution

To graph the line on a graphing calculator, we first use the slope–intercept form to find the equation of the line. The equation of a line with y intercept -2 and slope $\frac{5}{4}$ is

$$y = \frac{5}{4}x - 2$$

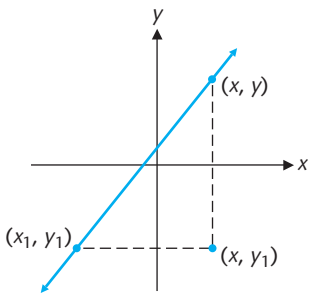
Graphing this equation on a graphing calculator produces the graph in Figure 10.



› Figure 10

MATCHED PROBLEM 5

Graph the line with y intercept 3 and slope $-\frac{3}{4}$ by hand and on a graphing calculator.



› Figure 11

Next, we will examine the *point–slope form* of a line.

Suppose a line has slope m and passes through the point (x_1, y_1). If (x, y) is any other point on the line (Fig. 11), then the slope formula gives us

$$\frac{y - y_1}{x - x_1} = m$$

Multiplying both sides by $(x - x_1)$, we get

$$y - y_1 = m(x - x_1) \tag{3}$$

Because the point (x_1, y_1) also satisfies equation (3), we can conclude that equation (3) is an equation of a line with slope m that passes through (x_1, y_1).

› THEOREM 3 Point–Slope Form

An equation of a line with slope m that passes through (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

which is called the **point–slope form** of an equation of a line.

Note that x and y are variables, while m, x_1, and y_1 are all numbers.

The point–slope form is especially useful because it provides a simple way to find the equation of a line. To do so, we need two pieces of information: the slope and any point on the line.

EXAMPLE 6 Point–Slope Form

(A) Find an equation for the line that has slope $\frac{2}{3}$ and passes through the point $(-2, 1)$. Write your answer in slope–intercept form.

(B) Find an equation for the line that passes through the two points $(4, -1)$ and $(-8, 5)$. Write your answer in slope–intercept form.

SOLUTIONS

(A) If $m = \frac{2}{3}$ and $(x_1, y_1) = (-2, 1)$, then

$$y - y_1 = m(x - x_1) \qquad \text{Substitute } m = \tfrac{2}{3},\ x_1 = -2, \text{ and } y_1 = 1.$$

$$y - 1 = \frac{2}{3}[x - (-2)] \qquad \text{Subtract.}$$

$$y - 1 = \frac{2}{3}(x + 2) \qquad \text{Distribute } \tfrac{2}{3}.$$

$$y - 1 = \frac{2}{3}x + \frac{4}{3} \qquad \text{Add 1 to each side.}$$

$$y = \frac{2}{3}x + \frac{7}{3}$$

(B) First use the slope formula to find the slope of the line:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-1)}{-8 - 4} = \frac{6}{-12} = -\frac{1}{2}$$

Now we choose $(x_1, y_1) = (4, -1)$ and proceed as in part A:

$$y - y_1 = m(x - x_1) \qquad \text{Substitute } m = -\tfrac{1}{2},\ x_1 = 4, \text{ and } y_1 = -1.$$

$$y - (-1) = -\frac{1}{2}(x - 4) \qquad \text{Subtract.}$$

$$y + 1 = -\frac{1}{2}(x - 4) \qquad \text{Distribute } -\tfrac{1}{2}.$$

$$y + 1 = -\frac{1}{2}x + 2 \qquad \text{Subtract 1 from each side.}$$

$$y = -\frac{1}{2}x + 1$$

You might want to verify that choosing the other given point $(x_1, y_1) = (-8, 5)$ produces the same equation.

MATCHED PROBLEM 6

(A) Find an equation for the line that has slope $-\frac{2}{5}$ and passes through the point $(3, -2)$. Write your answer in slope–intercept form.

(B) Find an equation for the line that passes through the two points $(-3, 1)$ and $(7, -3)$. Write your answer in slope–intercept form.

The various forms of the equation of a line that we have discussed are summarized in Table 3 for convenient reference. Note that the standard form includes all the other forms as special cases.

Table 3 Equations of a Line

Standard form	$Ax + By = C$	A and B not both 0
Slope–intercept form	$y = mx + b$	Slope: m; y intercept: b
Point–slope form	$y - y_1 = m(x - x_1)$	Slope: m; point: (x_1, y_1)
Horizontal line	$y = b$	Slope: 0
Vertical line	$x = a$	Slope: undefined

Recognizing Parallel and Perpendicular Lines

EXPLORE-DISCUSS 6

(A) Graph all of the following lines in the same viewing window. Discuss the relationship between these graphs and the slopes of the lines.

$$y = 2x - 5 \qquad y = 2x - 1 \qquad y = 2x + 3$$

(B) Graph each pair of lines in the same *squared* viewing window. Discuss the relationship between each pair of lines and their respective slopes.

$$y = 2x \quad \text{and} \quad y = -0.5x$$

$$y = -3x \quad \text{and} \quad y = \frac{1}{3}x$$

$$y = \frac{4}{5}x \quad \text{and} \quad y = -\frac{5}{4}x$$

From geometry, we know that two vertical lines are parallel and that a horizontal line and a vertical line are perpendicular to each other. But how can we tell when two nonvertical lines are parallel or perpendicular to each other? The key is slope. Slope determines the steepness of a line, so if two lines are parallel (i.e., have the same steepness), they also have the same slope. If two lines are perpendicular, it turns out that the slopes have opposite signs and are reciprocals (see Explore-Discuss 6).

Theorem 4, which we state without proof, explicitly states these relationships.

THEOREM 4 Parallel and Perpendicular Lines

Given two nonvertical lines L_1 and L_2, with slopes m_1 and m_2, respectively, then

$$L_1 \parallel L_2 \quad \text{if and only if } m_1 = m_2$$
$$L_1 \perp L_2 \quad \text{if and only if } m_1 m_2 = -1$$

The symbols $\parallel$ and $\perp$ mean, respectively, "is parallel to" and "is perpendicular to."

In words, two lines are parallel when their slopes are equal, and perpendicular when their slopes are negative reciprocals.

EXAMPLE 7 **Parallel and Perpendicular Lines**

Given the line L with equation $3x - 2y = 5$ and the point P with coordinates $(-3, 5)$, find an equation of a line through P that is

(A) Parallel to L (B) Perpendicular to L

SOLUTIONS

We already know a point on the lines, so all we need is the slope of each.

First we write the equation for L in the slope–intercept form to find the slope of L:

$$\begin{aligned} 3x - 2y &= 5 && \text{Subtract } 3x \text{ from each side.} \\ -2y &= -3x + 5 && \text{Divide both sides by } -2. \\ y &= \tfrac{3}{2}x - \tfrac{5}{2} \end{aligned}$$

The coefficient of x is $\frac{3}{2}$, so this is the slope of L. The slope of a line parallel to L will also be $\frac{3}{2}$, and the slope of a line perpendicular to L will be $-\frac{2}{3}$. We can now find the equations of the two lines in parts A and B using the point–slope form.

(A) Parallel ($m = \frac{3}{2}$):

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 5 &= \tfrac{3}{2}(x + 3) \\ y - 5 &= \tfrac{3}{2}x + \tfrac{9}{2} \\ y &= \tfrac{3}{2}x + \tfrac{19}{2} \end{aligned}$$

(B) Perpendicular ($m = -\frac{2}{3}$):

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 5 &= -\tfrac{2}{3}(x + 3) \\ y - 5 &= -\tfrac{2}{3}x - 2 \\ y &= -\tfrac{2}{3}x + 3 \end{aligned}$$

MATCHED PROBLEM 7

Given the line L with equation $4x + 2y = 3$ and the point P with coordinates $(2, -3)$, find an equation of a line through P that is

(A) Parallel to L (B) Perpendicular to L

Mathematical Modeling: Slope as a Rate of Change

In 2006 in the United States, babies were born at the rate of about 11,600 per day. At this rate, in 1 day 11,600 would be born; in 2 days, 23,200; in 3 days, 34,800; and so forth. In general, the function $b(x) = 11{,}600x$ describes the number of births after x days. Note that this is a linear function with slope 11,600. This illustrates an interesting point: The slope of a linear function tells us the **rate of change** of the function with respect to the independent variable.

This interpretation of slope can be applied to a wide variety of everyday situations. If you're driving at an average speed of 50 miles/hour, the rate of change of your position with respect to time is 50, and $y = 50x$ is the mileage driven in x hours. If you make \$8.00 per hour at a part-time job, the rate of change of money earned with respect to hours worked is 8, and $y = 8x$ describes the amount earned in x hours.

The study of rates of change is pretty simple for linear functions, when the rate of change is constant. For others, computing rates of change is one of the fundamental goals of calculus.

EXAMPLE 8 Rates of Change

The average price in dollars of a gallon of gas in Cincinnati, Ohio, between January 3 and January 18, 2007, can be modeled by the function $P(x) = -0.024x + 2.31$, $0 \le x \le 15$, where x is days after January 3.

(A) At what rate was the price changing during that time period? Was the price going up or down?

(B) What was the price on January 3? On January 18? Do your results from parts A and B agree?

SOLUTIONS

(A) $P(x)$ is a linear function with slope -0.024, so the rate of change is -0.024 dollars per day, or -2.4 cents per day. The negative sign indicates that the price was dropping.

(B) January 3 and January 18 are zero and 15 days after January 3, respectively.

$$P(0) = -0.024(0) + 2.31 = \$2.31 \text{ on January 3}$$
$$P(15) = -0.024(15) + 2.31 = \$1.95 \text{ on January 18}$$

With a decrease of 2.4 cents per day, after 15 days the price would have gone down by $2.4 \cdot 15 = 36$ cents. This matches the calculated difference from January 3 to January 18.

MATCHED PROBLEM 8

The average price in dollars of a gallon of gas in Allentown, Pennsylvania, between January 2 and January 22, 2007, can be modeled by the function $P(x) = -0.007x + 2.39$, $0 \le x \le 20$, where x is days after January 2.

(A) At what rate was the price changing during that time period? Was the price going up or down?

(B) What was the price on January 2? On January 22? Do your results from parts A and B agree?

››› CAUTION ›››

When interpreting the rate of change of a function, don't forget to consider the sign. A negative rate of change always means that the value of the function is decreasing.

EXAMPLE 9 Underwater Pressure

The atmospheric pressure at sea level is 14.7 pounds per square inch. As you descend into the ocean, the pressure increases at a constant rate of about 0.445 pounds per square inch per foot.

(A) Find the pressure p at a depth of d feet.

(B) If a diver's equipment is rated to be safe up to a pressure of 40 pounds per square inch, is it safe to use this equipment at a depth at 60 feet?

SOLUTIONS

(A) The rate of change of pressure is 0.445, so that will be the slope. The equation should look like $p = 0.445d + b$. We know that the pressure at depth zero (the surface) is 14.7, so when $d = 0$, we get

$$p = 0.445(0) + b = b = 14.7$$

The pressure at a depth of d feet is given by

$$p = 0.445d + 14.7$$

(B) The pressure at a depth of 60 feet is given by

$$p = 0.445(60) + 14.7 = 41.4$$

It's not safe to use the equipment at this depth. ◉

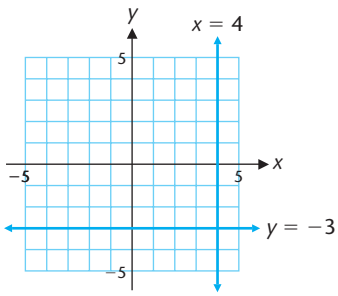
MATCHED PROBLEM 9

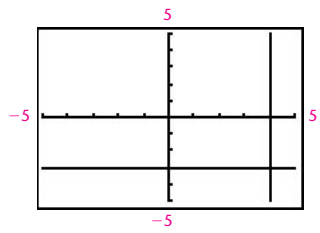
The rate of change of pressure in freshwater is 0.432 pounds per square inch per foot. Repeat Example 9 for a body of freshwater. ◉

ANSWERS TO MATCHED PROBLEMS

1. x intercept: $\frac{15}{4} = 3.75$; y intercept: 5

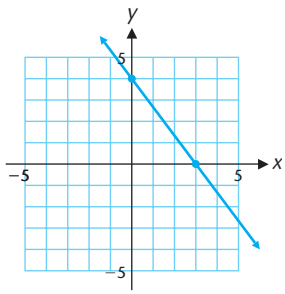
2. (A) The graph of $x = 4$ is a vertical line with x intercept 4. The graph of $y = -3$ is a horizontal line with y intercept -3.

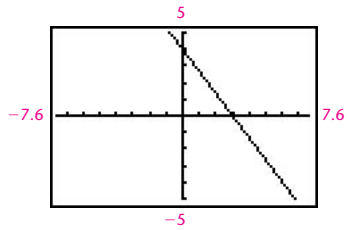




(B) Vertical: $x = -7$; horizontal: $y = 5$

(C)

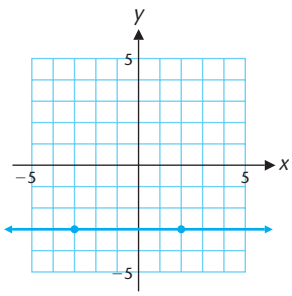


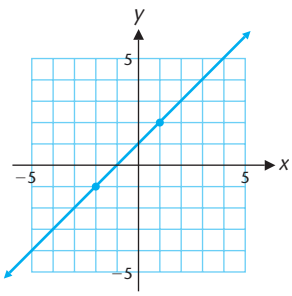


3. (A) Run $= 5$, rise $= 4$, slope $= \frac{4}{5} = 0.8$

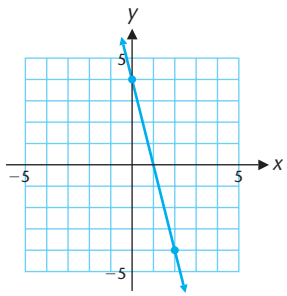
(B) Run $= 3$, rise $= -6$, slope $= \frac{-6}{3} = -2$

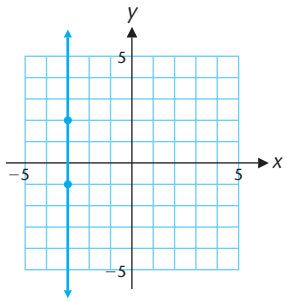
4. (A) $m = 0$ (B) $m = 1$





(C) $m = -4$ (D) m is not defined





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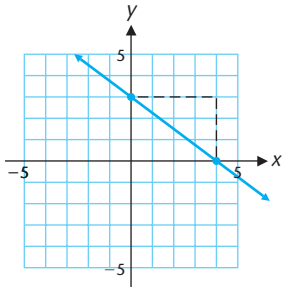
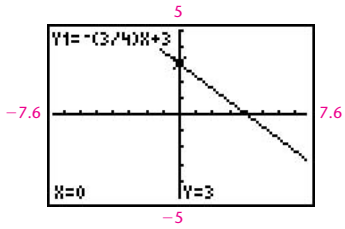




6. (A) $y = -\frac{2}{5}x - \frac{4}{5}$ (B) $y = -\frac{2}{5}x - \frac{1}{5}$

7. (A) $y = -2x + 1$ (B) $y = \frac{1}{2}x - 4$

8. (A) The price was going down by 0.7 cents per day.
(B) The price was \$2.39 on January 2 and \$2.25 on January 22. This matches a decrease of 0.7 cents per day.

9. (A) $p = 0.432d + 14.7$ (B) It is not safe.

2-1 Exercises

1. What is the slope–intercept form of a line? Why is it given that name?

2. What is the point–slope form of a line? Why is it given that name?

3. Explain in your own words what the slope of a line tells us about the graph.

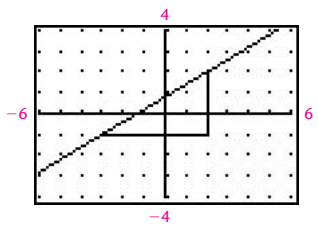
4. If a linear function describes the height in feet of a model rocket in terms of seconds after it was launched, what information would the slope of the line provide? Why?

5. Given a function defined by a formula, how can you tell if it's a linear function?

6. Explain why the graph of a constant function is a horizontal line.

In Problems 7–12, use the graph of each linear function to find the rise, run, and slope. Write the equation of each line in the standard form $Ax + By = C, A \geq 0$. (All the horizontal and vertical line segments have integer lengths.)

7.



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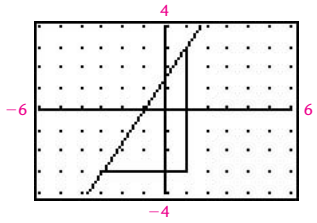

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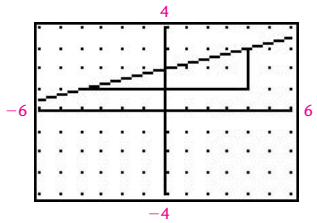

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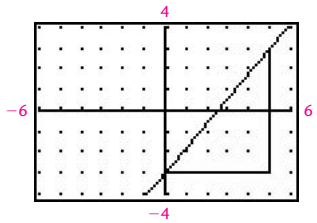

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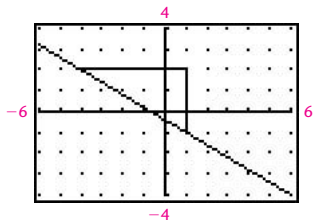

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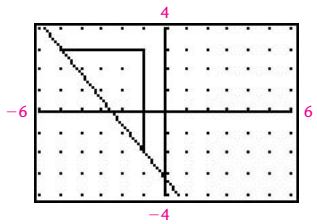
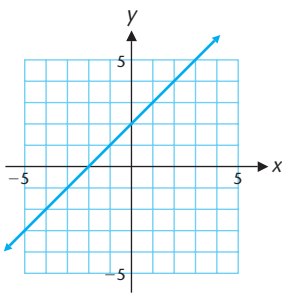


In Problems 13–18, use the graph of each linear function to find the x intercept, y intercept, and slope. Write the slope–intercept form of the equation of each line.

13.



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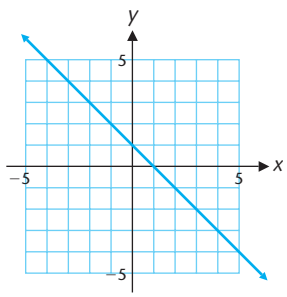

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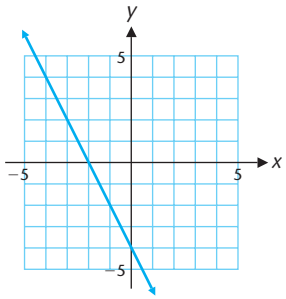

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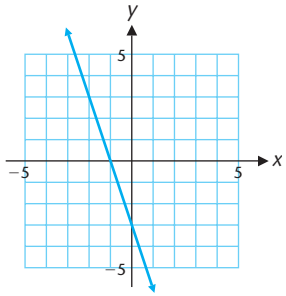

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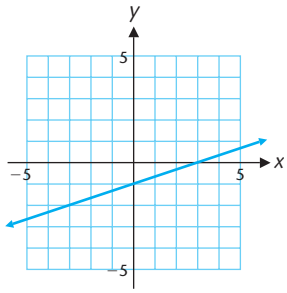

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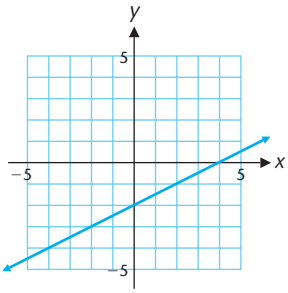


Which equations in Problems 19–28 define linear functions? Justify your answer.

19. $y = 2x^2$

20. $y = 5 - 3x^3$

21. $y = \dfrac{x - 5}{3}$

22. $y = \dfrac{3 - x}{2}$

23. $y = \frac{2}{3}(x - 7) - \frac{1}{2}(3 - x)$

24. $y = -\frac{1}{5}(2 - 3x) + \frac{2}{7}(x + 8)$

25. $y = \frac{1}{4}(2x + 2) + \frac{1}{2}(4 - x)$

26. $y = \frac{4}{3}(2 - x) + \frac{2}{3}(x + 2)$

27. $y = \dfrac{3}{x - 5}$

28. $y = \dfrac{2}{3 - x}$

In Problems 29–40, find the x intercept, y intercept, and slope, if they exist, and graph each equation.

29. $y = -\frac{3}{5}x + 4$

30. $y = -\frac{3}{2}x + 6$

31. $y = -\frac{3}{4}x$

32. $y = \frac{2}{3}x - 3$

33. $2x - 3y = 15$

34. $4x + 3y = 24$

35. $\dfrac{y}{8} - \dfrac{x}{4} = 1$

36. $\dfrac{y}{6} - \dfrac{x}{5} = 1$

37. $x = -3$

38. $y = -2$

39. $y = 3.5$

40. $x = 2.5$

In Problems 41–48, write the equation of the line described.

41. Vertical, goes through $(2, -4)$

42. Vertical, goes through $(-7, 12)$

43. Horizontal, goes through $(2, -4)$

44. Horizontal, goes through $(-7, 12)$

45. Goes through $(-3, 4)$ and $(5, 4)$

46. Goes through $(0, -2)$ and $(4, -2)$

47. Goes through $(4, 6)$ and $(4, -3)$

48. Goes through $(-3, 1)$ and $(-3, 4)$

In Problems 49–52, write the slope–intercept form of the equation of the line with indicated slope and y intercept. Then write the equation in the standard form $Ax + By = C$, where A, B, and C are integers, and $A \geq 0$.

49. Slope $= 1$; y intercept $= 0$

50. Slope $= -1$; y intercept $= 7$

51. Slope $= -\frac{2}{3}$; y intercept $= -4$

52. Slope $= \frac{5}{3}$; y intercept $= 6$

In Problems 53–70, find the equation of the line described. Write your answer in slope–intercept form.

53. Slope -3, goes through $(0, 4)$

54. Slope 2, goes through $(2, 0)$

55. Slope $-\frac{2}{5}$, goes through $(-5, 4)$

56. Slope $\frac{1}{2}$, goes through $(-4, -2)$

57. Goes through $(1, 6)$ and $(5, -2)$

58. Goes through $(-3, 4)$ and $(6, 1)$

59. Goes through $(-4, 8)$ and $(2, 0)$

60. Goes through $(2, -1)$ and $(10, 5)$

61. Has x intercept -4 and y intercept 3

62. Has x intercept -4 and y intercept -5

63. Goes through $(-3, 4)$; parallel to $y = 3x - 5$

64. Goes through $(-4, 0)$; parallel to $y = -2x + 1$

65. Goes through $(2, -3)$; perpendicular to $y = -\frac{1}{3}x$

66. Goes through $(-2, -4)$; perpendicular to $y = \frac{2}{3}x - 5$

67. Goes through $(5, 0)$; parallel to $3x - 2y = 4$

68. Goes through $(3, 5)$; parallel to $3x + 4y = 8$

69. Goes through $(0, -4)$; perpendicular to $x + 3y = 9$

70. Goes through $(-2, 4)$; perpendicular to $4x + 5y = 0$

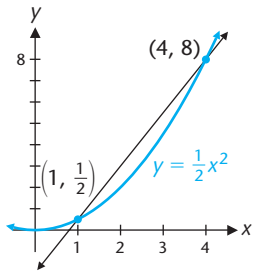
71. Discuss the relationship between the graphs of the lines with equation $y = mx + 2$, where m is any real number.

72. Discuss the relationship between the graphs of the lines with equation $y = -0.5x + b$, where b is any real number.

73. (A) Find the linear function f whose graph passes through the points $(-1, -3)$ and $(7, 2)$.
(B) Find the linear function g whose graph passes through the points $(-3, -1)$ and $(2, 7)$.
(C) Graph both functions and discuss how they are related.

74. (A) Find the linear function f whose graph passes through the points $(-2, -3)$ and $(10, 5)$.
(B) Find the linear function g whose graph passes through the points $(-3, -2)$ and $(5, 10)$.
(C) Graph both functions and discuss how they are related.

*Problems 75–80 are calculus related. A line connecting two points on a graph is called a **secant line.** For the graph of the function $f(x) = \frac{1}{2}x^2$, find the slope of each secant line.*



75. Connecting the points at $x = 1$ and $x = 4$. (See the figure.)

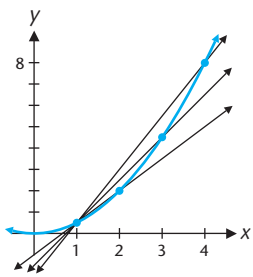
76. Connecting the points at $x = 1$ and $x = 3$.

77. Connecting the points at $x = 1$ and $x = 2$.

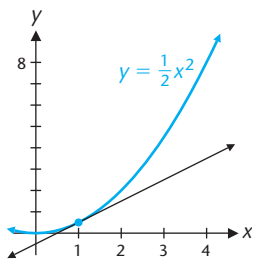
78. Connecting the points at $x = 1$ and $x = \frac{3}{2}$.

79. Connecting the points at $x = 1$ and $x = \frac{5}{4}$.

80. As the right-hand point gets closer and closer to $(1, \frac{1}{2})$ (see Figure A), the secant lines approach a line that intersects the graph only at $(1, \frac{1}{2})$ (see Figure B). This line is called the **tangent line** at $x = 1$. Use your answers from Problems 75–79 to estimate the slope of the tangent line at $x = 1$. (Note that Figure A contains only the secant lines for Problems 75–77.)



Problem 80A



Problem 80B

81. (A) Graph the following equations in a squared viewing window:

$$3x + 2y = 6 \qquad 3x + 2y = 3$$
$$3x + 2y = -6 \qquad 3x + 2y = -3$$

(B) From your observations in part A, describe the family of lines obtained by varying C in $Ax + By = C$ while holding A and B fixed.
(C) Verify your conclusions in part B with a proof.

82. (A) Graph the following two equations in a squared viewing window:

$$3x + 4y = 12 \qquad 4x - 3y = 12$$

(B) Graph the following two equations in a squared viewing window:

$$2x + 3y = 12 \qquad 3x - 2y = 12$$

(C) From your observations in parts A and B, describe the apparent relationship of the graphs of

$$Ax + By = C \text{ and } Bx - Ay = C.$$

(D) Verify your conclusions in part C with a proof.

83. Prove that if a line L has x intercept $(a, 0)$ and y intercept $(0, b)$, then the equation of L can be written in the **intercept form**

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad a, b \neq 0$$

84. For $y = mx + b$, prove that the slope is m by using Definition 2 to find the slope of the line connecting any two points on the graph. [*Hint:* Find the points corresponding to two x values x_1 and x_2.]

APPLICATIONS

85. **POLITICS** The *Washington Post*/ABC News approval rating for the president of the United States between October 2001 and January 2003 can be modeled by the function $A(x) = -2.13x + 91$, $0 \le x \le 15$, where x is months after October 2001, and A is the percentage of those polled that approved of the president's job performance.
(A) At what rate was the approval rating changing during that time period? Was it going up or down?
(B) What was the approval rating in October 2001? In January 2003? Do your results from parts A and B agree?

86. **THE STOCK MARKET** The price in dollars of one share of Apple Computer stock over the span of 108 days from August 15 to December 1, 2006, can be modeled by the function $P(x) = 0.23x + 66.45$, $0 \le x \le 108$, where x is days after August 15.
(A) At what rate was the price changing during that time period? Was it going up or down?
(B) What was the price on August 15? On December 1? Do your results from parts A and B agree?

87. **IMMIGRATION** The number of immigrants (in millions) living in the United States from 1996 to 2004 can be modeled by the function $N(x) = 1.04x + 25.95$, $0 \le x \le 8$, where x is years after 1996.
(A) At what rate was the number of immigrants changing during that time period? Was it going up or down?
(B) How many immigrants were there in 1996? In 2004? Do your results from parts A and B agree?

88. **GLOBAL WARMING** The average global temperature in degrees Celsius from 1965 to 2001 can be modeled by the function $T(x) = 0.0183x + 13.86$, $0 \le x \le 36$, where x is years after 1965.
(A) At what rate was the temperature changing during that time period? Was it going up or down?
(B) What was the average global temperature in 1965? In 2001? Do your results from parts A and B agree?
(Source: www.giss.nasa.gov/data)

89. **METEOROLOGY** A meteorologist tracking a storm finds that it is 145 miles away at 11 P.M. and moving toward her city at the rate of 23 miles per hour. Find a linear function describing the distance d of the storm from the city h hours after 11 P.M. Will the storm arrive by the beginning of rush hour at 6 A.M. the next morning?

90. **GEOLOGY** A geologist is alerted to a seismic disturbance at sea that caused a tsunami headed toward the coast of Japan. At that time, the tsunami is moving at the rate of 235 miles per hour, and is 700 miles away. Find a linear function describing the distance d of the tsunami from Japan h hours later. If it would take $2\frac{1}{2}$ hours to evacuate the coastal communities, will they be able to accomplish this before the tsunami reaches Japan?

91. **PHYSICS** The two temperature scales Fahrenheit (F) and Celsius (C) are linearly related. It is known that water freezes at 32°F or 0°C and boils at 212°F or 100°C.
(A) Find a linear equation that expresses F in terms of C.
(B) If a European house thermostat is set at 20°C, what is the setting in degrees Fahrenheit? If the outside temperature in Milwaukee is 86°F, what is the temperature in degrees Celsius?
(C) What is the slope of the graph of the linear equation found in part A? Interpret verbally.

92. PHYSICS Hooke's law states that the relationship between the stretch s of a spring and the weight w causing the stretch is linear (a principle on which all spring scales are constructed). For a particular spring, a 5-pound weight causes a stretch of 2 inches, whereas with no weight the stretch of the spring is 0.
(A) Find a linear equation that expresses s in terms of w.
(B) What weight will cause a stretch of 3.6 inches?
(C) What is the slope of the graph of the equation? Interpret verbally.

93. BUSINESS–DEPRECIATION A copy machine was purchased by a law firm for $8,000 and is assumed to have a depreciated value of $0 after 5 years. The firm takes straight-line depreciation over the 5-year period.
(A) Find a linear equation that expresses value V in dollars in terms of time t in years.
(B) What is the depreciated value after 3 years?
(C) What is the slope of the graph of the equation found in part A? Interpret verbally.

94. BUSINESS–MARKUP POLICY A clothing store sells a shirt costing $20 for $33 and a jacket costing $60 for $93.
(A) If the markup policy of the store for items costing over $10 is assumed to be linear, write an equation that expresses retail price R in terms of cost C (wholesale price).
(B) What does a store pay for a suit that retails for $240?
(C) What is the slope of the graph of the equation found in part A? Interpret verbally.

95. COST ANALYSIS A plant can manufacture 80 golf clubs per day for a total daily cost of $8,147 and 100 golf clubs per day for a total daily cost of $9,647.
(A) Assuming that the daily cost function is linear, find the total daily cost of producing x golf clubs.
(B) Write a brief verbal interpretation of the slope and y intercept of this cost function.

96. COST ANALYSIS A plant can manufacture 50 tennis rackets per day for a total daily cost of $4,174 and 60 tennis rackets per day for a total daily cost of $4,634.
(A) Assuming that the daily cost function is linear, find the total daily cost of producing x tennis rackets.
(B) Write a brief verbal interpretation of the slope and y intercept of this cost function.

97. MEDICINE Cardiovascular research has shown that above the 210 cholesterol level, each 1% increase in cholesterol level increases coronary risk 2%. For a particular age group, the coronary risk at a 210 cholesterol level is found to be 0.160 and at a level of 231 the risk is found to be 0.192.
(A) Find a linear equation that expresses risk R in terms of cholesterol level C.
(B) What is the risk for a cholesterol level of 260?
(C) What is the slope of the graph of the equation found in part A? Interpret verbally.
Express all calculated quantities to three significant digits.

98. DEMOGRAPHICS The average number of persons per household in the United States has been shrinking steadily for as long as such statistics have been kept and is approximately linear with respect to time. In 1900, there were about 4.76 persons per household and in 1990, about 2.5.
(A) If N represents the average number of persons per household and t represents the number of years since 1900, write a linear equation that expresses N in terms of t.
(B) What is the predicted household size in the year 2015?
Express all calculated quantities to three significant digits.

99. METEOROLOGY In stable air, as the altitude of a weather balloon increases, the temperature drops at a rate of about 5°F for each 1,000-foot rise in altitude.
(A) If the temperature at sea level is 70°F, write a linear equation that expresses the temperature T in terms of altitude A (in thousands of feet above sea level).
(B) What would the temperature be at an altitude of 10,000 feet?
(C) What is the slope of the graph of the equation found in part A? What does the slope describe physically?

100. FLIGHT NAVIGATION An airspeed indicator on some aircraft is affected by the changes in atmospheric pressure at different altitudes. A pilot can estimate the true airspeed by observing the indicated airspeed and adding to it about 2% for every 1,000 feet of altitude.
(A) If a pilot maintains a constant reading of 200 miles per hour on the airspeed indicator as the aircraft climbs from sea level to an altitude of 10,000 feet, write a linear equation that expresses true airspeed T (miles per hour) in terms of altitude A (thousands of feet).
(B) What would be the true airspeed of the aircraft at 6,500 feet?
(C) What is the slope of the graph of the equation found in part A? Interpret verbally.

2-2 Linear Equations and Models

- Solving Linear Equations
- Modeling with Linear Equations
- Modeling Distance-Rate-Time Problems
- Modeling Mixture Problems
- Data Analysis and Linear Regression

In this section, we will discuss methods for solving equations that involve linear functions. Some problems are best solved using algebraic techniques, while others benefit from a graphical approach. Because graphs often give additional insight into relationships, especially in applications, we will usually emphasize graphical techniques over algebraic methods. But mastering both will lead to a far greater understanding of linear equations and how they are used. Some of the problems in this section can only be solved algebraically, and near the end of the section, we are going to introduce an important new tool—linear regression—that will be one of the most useful applications of a graphing calculator.

Solving Linear Equations

We will begin with a quick review of solving a basic linear equation. For a more detailed coverage, see Appendix B, Section B-1.

EXAMPLE 1 Solving an Equation

Solve $5x - 8 = 2x + 1$.

SOLUTION

Algebraic Solution

We will use the familiar properties of equality to transform the given equation into an equivalent equation with an obvious solution.

$5x - 8 = 2x + 1$	Subtract $2x$ from both sides.
$5x - 8 - 2x = 2x + 1 - 2x$	Combine like terms.
$3x - 8 = 1$	Add 8 to both sides.
$3x - 8 + 8 = 1 + 8$	Combine like terms.
$3x = 9$	Divide both sides by 3.
$\frac{3x}{3} = \frac{9}{3}$	Simplify.
$x = 3$	

Graphical Solution

Enter each side of the equation in the equation editor of a graphing calculator (Fig. 1) and use the INTERSECT command (Fig. 2).

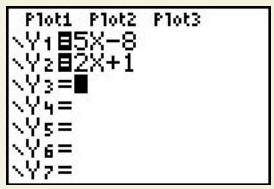
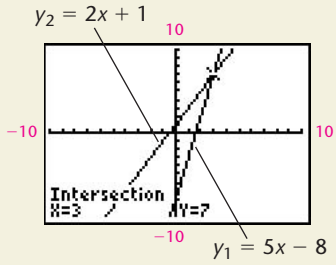


Figure 1

It follows from the properties of equality that $x = 3$ is also the solution set of all the preceding equations in our solution, including the original equation.



› Figure 2

We see that $x = 3$ is the solution to the original equation.

MATCHED PROBLEM 1

Solve $2x + 1 = 4x + 5$.

In the solution of Example 1, notice that we used the informal notation $x = 3$ for the solution set rather than the more formal statement: solution set $= \{3\}$.

››› EXPLORE-DISCUSS 1

An equation that is true for all values of the variable for which both sides of the equation are defined is called an **identity.** An equation that is true for some values of the variable and false for others is called a **conditional equation.** An equation that is false for all permissible values of the variable is called a **contradiction.** Use algebraic and/or graphical techniques to classify each of the following as an identity, a conditional equation, or a contradiction. Solve any conditional equations.

(A) $2(x - 4) = 2x - 12$ **(B)** $2(x - 4) = 3x - 12$

(C) $2(x - 4) = 2x - 8$ **(D)** $\dfrac{2}{x - 1} + 3 = \dfrac{x}{x - 1}$

(E) $\dfrac{1}{x - 1} + 3 = \dfrac{x}{x - 1}$ **(F)** $\dfrac{1}{x - 1} + 1 = \dfrac{x}{x - 1}$

When an equation is an identity, we say that the solution is all real numbers for which the equation is defined. When an equation is a contradiction, we say that it has no solution.

When equations involve fractions, it's always a good idea to begin by removing all of the fractions. This can be accomplished with a two-step procedure: (1) Find the least common denominator (LCD) for all fractions, and (2) multiply both sides of the equation by that LCD.

EXAMPLE 2 Solving an Equation*

Solve $\frac{7}{2x} - 3 = \frac{8}{3} - \frac{15}{x}$.

SOLUTION

Algebraic Solution

The denominators have factors 2, 3, and x, so the LCD is $2 \cdot 3 \cdot x$, or $6x$.

$$\frac{7}{2x} - 3 = \frac{8}{3} - \frac{15}{x}$$ Multiply both sides by $6x$.

$$6x\left(\frac{7}{2x} - 3\right) = 6x\left(\frac{8}{3} - \frac{15}{x}\right)$$ Distribute.

$$6x \cdot \frac{7}{2x} - 6x \cdot 3 = 6x \cdot \frac{8}{3} - 6x \cdot \frac{15}{x}$$ Simplify each fraction. The equation is now free of fractions.

$$21 - 18x = 16x - 90$$ Subtract $16x$ from both sides.

$$21 - 18x - 16x = 16x - 90 - 16x$$ Combine like terms.

$$21 - 34x = -90$$ Subtract 21 from both sides.

$$21 - 34x - 21 = -90 - 21$$ Combine like terms.

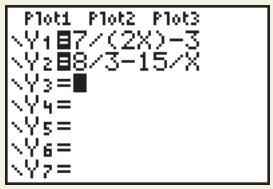
$$-34x = -111$$ Divide both sides by -34.

$$\frac{-34x}{-34} = \frac{-111}{-34}$$ Simplify.

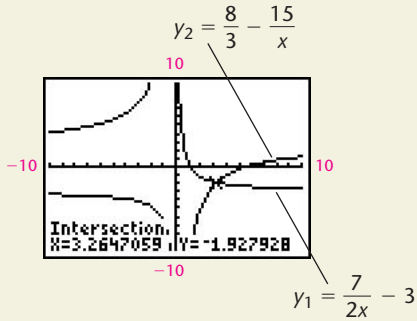
$$x = \frac{111}{34}$$

Graphical Solution

Enter $y_1 = \frac{7}{2x} - 3$ and $y_2 = \frac{8}{3} - \frac{15}{x}$ (Fig. 3) in the equation editor of a graphing calculator. Note the use of parentheses in Figure 3 to be certain that $\frac{7}{2x}$ is evaluated correctly. After looking at various viewing windows to convince ourselves that the graphs only cross once, we use the INTERSECT command (Fig. 4).



› Figure 3



› Figure 4

We see that $x = 3.2647059$ is the solution of the original equation. Note that $\frac{111}{34} \approx 3.2647059$ to seven decimal places.

MATCHED PROBLEM 2

Solve $\frac{7}{3x} + 2 = \frac{1}{x} - \frac{3}{5}$.

*Note that the equation of Example 2 is not a linear equation. However, its solution can be found algebraically by solving a related linear equation. Note also that the functions in the graphical solution are not linear; their graphs are not lines. These same observations apply to Example 3.

››› CAUTION ›››

1. When multiplying both sides of an equation by the LCD, make sure you multiply by *every* term. Students often forget to multiply by terms with no denominator.
2. When entering fractional expressions in a graphing calculator, you should get in the habit of putting parentheses around any numerator or denominator that consists of more than one symbol.

The equation in Example 2 has a factor of x in the denominator, so a value of $x = 0$ has to be excluded. Fortunately, that wasn't an issue, since the solution is $x = \frac{111}{34}$. But it brings up an important point: When solving an equation with a variable denominator, you always need to make sure that any potential solution does not result in a zero denominator.

EXAMPLE 3 Solving an Equation

Solve $\dfrac{x}{x+2} = 4 - \dfrac{2}{x+2}$.

SOLUTION

Algebraic Solution

There is only one denominator, $x + 2$, so it's the LCD.

$$\frac{x}{x+2} = 4 - \frac{2}{x+2}$$ Multiply both sides by $x + 2$.

$$(x+2)\frac{x}{x+2} = (x+2)\left(4 - \frac{2}{x+2}\right)$$ Distribute.

$$(x+2)\frac{x}{x+2} = 4(x+2) - (x+2)\frac{2}{x+2}$$ Simplify each term.

$$x = 4x + 8 - 2$$ Subtract $4x$ from each side.

$$x - 4x = 4x + 8 - 2 - 4x$$ Combine like terms.

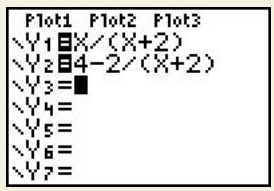
$$-3x = 6$$ Divide both sides by -3.

$$\frac{-3x}{-3} = \frac{6}{-3}$$ Simplify.

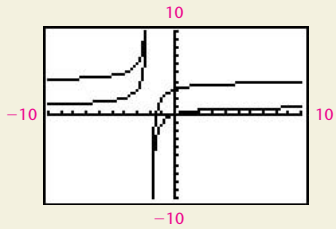
$$x = -2$$

Graphical Solution

Enter $y_1 = \frac{x}{x+2}$ and $y_2 = 4 - \frac{2}{x+2}$ into the equation editor of a graphing calculator. (Fig. 5). The graph in a standard window (Fig. 6) shows that the graphs might intersect near $x = -2$ (Fig. 7). Using the INTERSECT command with a guess near $x = -2$ results in an error screen (Fig. 8). This indicates that the graphs do not intersect, and there is no solution. (This should be repeated for an x value slightly less than -2.)

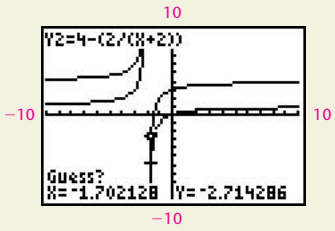


› Figure 5



› Figure 6

It appears that $x = -2$ is the solution. But $x = -2$ makes the original equation undefined, so we conclude that the equation has no solution.

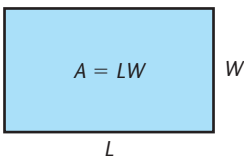


› Figure 7

› Figure 8

MATCHED PROBLEM 3

Solve $\dfrac{2x}{x + 10} = 3 - \dfrac{20}{x + 10}$.



› Figure 9 Area of a rectangle.

In practical applications, we frequently encounter equations involving more than one variable. For example, if L and W are the length and width of a rectangle, respectively, the area of the rectangle is given by (Fig. 9)

$$A = LW$$

Depending on the situation, we may want to solve this equation for L or W. To solve for W, we simply treat A and L as constants and W as the variable. Then the equation $A = LW$ becomes a linear equation with variable W, which can be solved easily by dividing both sides by L:

$$W = \frac{A}{L} \qquad L \neq 0$$

EXAMPLE 4 Solving an Equation with More than One Variable

Solve for P in terms of the other variables: $A = P + Prt$.

SOLUTION

$$A = P + Prt$$ Think of A, r, and t as constants. Factor to isolate P.

$$A = P(1 + rt)$$ Divide both sides by $1 + rt$.

$$P = \frac{A}{1 + rt}$$ Restriction: $1 + rt \neq 0$

MATCHED PROBLEM 4

Solve for r in terms of the other variables: $A = P + Prt$.

› Modeling with Linear Equations

Linear equations can be used to model a wide variety of real-world situations. In the remainder of the section, we'll study a sampling of these applications. To construct models for word problems, we translate verbal statements into mathematical statements. Explore-Discuss 2 will help you review this process.

››› EXPLORE-DISCUSS 2

Translate each of the following sentences involving two numbers into an equation.

(A) The first number is 10 more than the second number.

(B) The first number is 15 less than the second number.

(C) The first number is half the second number.

(D) The first number is three times the second number.

(E) Ten times the first number is 15 more than the second number.

EXAMPLE 5 Cost Analysis

A hot dog vendor pays \$25 per day to rent a pushcart and \$1.25 for the ingredients in one hot dog.

(A) Find the cost of selling x hot dogs in 1 day.
(B) What is the cost of selling 200 hot dogs in 1 day?
(C) If the daily cost is \$355, how many hot dogs were sold that day?

SOLUTIONS

(A) The rental charge of \$25 is the vendor's **fixed cost**—a cost that is accrued every day and does not depend on the number of hot dogs sold. The cost of ingredients does depend on the number sold. The cost of the ingredients for x hot dogs is $\$1.25x$ (\$1.25 times the number of hot dogs sold). This is the vendor's **variable cost**—a cost that depends on the number of hot dogs sold. The total cost for selling x hot dogs is

$$C(x) = 1.25x + 25 \qquad \text{Total cost} = \text{Variable cost} + \text{Fixed cost}$$

(B) The cost of selling 200 hot dogs in 1 day is

$$C(200) = 1.25(200) + 25 = \$275$$

(C) The number of hot dogs that can be sold for \$355 is the solution of the equation

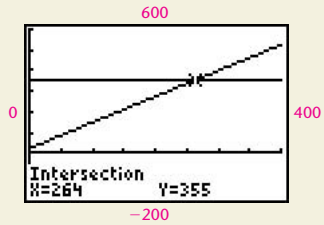
$$1.25x + 25 = 355$$

Algebraic Solution

$$\begin{aligned} 1.25x + 25 &= 355 && \text{Subtract 25 from each side.} \\ 1.25x &= 330 && \text{Divide both sides by 1.25.} \\ x &= \frac{330}{1.25} \\ &= 264 \text{ hot dogs} \end{aligned}$$

Graphical Solution

Entering $y_1 = 1.25x + 25$ and $y_2 = 355$ in a graphing calculator and using the INTERSECT command (Fig. 10) shows that 264 hot dogs can be sold for \$355.



› Figure 10

MATCHED PROBLEM 5

It costs a pretzel vendor \$20 per day to rent a cart and \$0.75 for each pretzel.

(A) Find the cost of selling x pretzels in 1 day.
(B) What is the cost of selling 150 pretzels in 1 day?
(C) If the daily cost is \$275, how many pretzels were sold that day?

In Example 5, the vendor's cost increases at the rate of \$1.25 per hot dog. Thus, the rate of change of the cost function $C(x) = 1.25x + 25$ is the slope $m = 1.25$. This constant rate can also be viewed as the cost of selling one additional hot dog. In economics, this quantity is referred to as the **marginal cost.**

› Modeling Distance-Rate-Time Problems

If you drive for 2 hours and cover 120 miles, what was your average speed? If you answered 60 miles per hour, you already know our next important formula. To get that result, we used the formula

$$\text{Average speed} = \frac{\text{Distance traveled}}{\text{Time passed}}$$

When the speed of an object is constant, we can write this as

$$r = \frac{d}{t} \qquad \text{Rate is distance divided by time.}$$

or, equivalently, $d = rt$. Distance is rate times time.

››› EXPLORE-DISCUSS 3

A bus leaves Milwaukee at 12:00 noon and travels due west on Interstate 94 at a constant rate of 55 miles per hour. A passenger that was left behind leaves Milwaukee in a taxicab at 1:00 P.M. in pursuit of the bus. The taxicab travels at a constant rate of 65 miles per hour. Let t represent time in hours after 12:00 noon.

(A) How far has the bus traveled after t hours?

(B) If $t \geq 1$, how far has the taxicab traveled after t hours?

(C) When will the taxicab catch up with the bus?

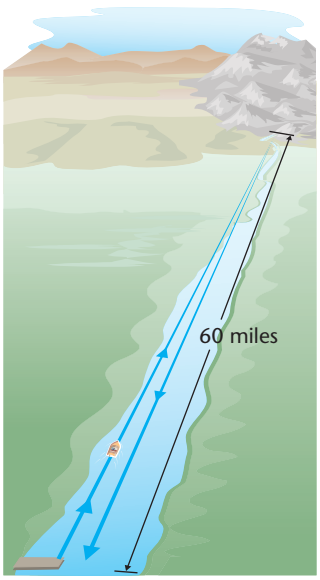
EXAMPLE 6 A Distance-Rate-Time Problem

An excursion boat takes 1.5 times as long to go 60 miles up a river as it does to return. If the boat cruises at 16 miles per hour in still water, what is the rate of the current in the river?

SOLUTION

It's usually helpful in distance-rate-time problems to build a table that helps to organize the information. We were asked to find the speed of the current, so we let x = rate of current (in mph). Then the rate of the boat upstream (against the current) is $16 - x$ mph, and the rate downstream (with the current) is $16 + x$ mph. The distance in each case is 60 miles.

	Distance	Rate	Time
Upstream	60	$16 - x$	
Downstream	60	$16 + x$	



Now we can use the formula $d = rt$ to find an expression for each time. We solve $d = rt$ for t, and get

$$t = \frac{d}{r}$$

Time upstream: $t_u = \dfrac{60}{16 - x} \quad (x \neq 16)$

Time downstream: $t_d = \dfrac{60}{16 + x} \quad (x \neq -16)$

Since the upstream trip took 1.5 times as long,

$$t_u = 1.5t_d$$

$$\frac{60}{16 - x} = 1.5\frac{60}{16 + x}$$

Multiplying out the right side, we get the equation

$$\frac{60}{16 - x} = \frac{90}{16 + x}$$

Algebraic Solution

$$\frac{60}{16 - x} = \frac{90}{16 + x}$$ Multiply both sides by $(16 - x)(16 + x)$.

$$60(16 + x) = 90(16 - x)$$ Distribute.

$$960 + 60x = 1{,}440 - 90x$$ Add $90x$ to each side.

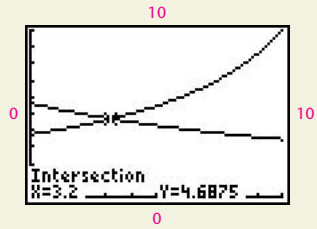
$$960 + 150x = 1{,}440$$ Subtract 960 from each side.

$$150x = 480$$ Divide both sides by 150.

$$x = \frac{480}{150} = 3.2 \text{ mph}$$

Graphical Solution

We first enter $y_1 = \frac{60}{16 - x}$ and $y_2 = \frac{90}{16 + x}$. Since x is a speed and y is a time, we restrict our viewing window to positive values. The INTERSECT command (Fig. 11) shows that $x = 3.2$ mph is the solution.



› Figure 11

MATCHED PROBLEM 6

A jetliner takes 1.2 times as long to fly from Paris to New York (3,600 miles) as to return. If the jet cruises at 550 miles per hour in still air, what is the average rate of the wind blowing in the direction of Paris from New York?

› Modeling Mixture Problems

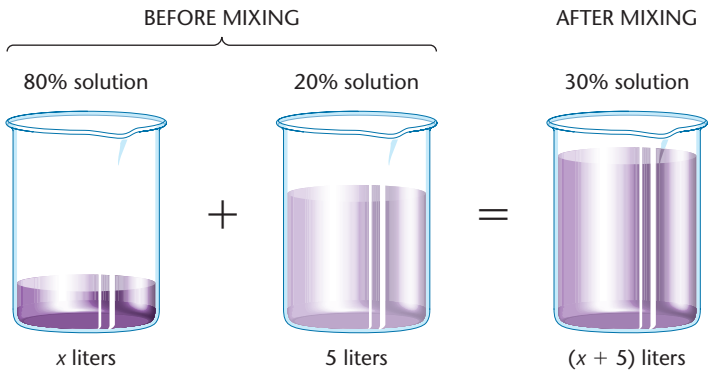
A variety of applications can be classified as mixture problems. Although the problems come from different areas, their mathematical treatment is essentially the same.

EXAMPLE 7 A Mixture Problem

How many liters of a mixture containing 80% alcohol should be added to 5 liters of a 20% solution to yield a 30% solution?

SOLUTION

Let x = amount of 80% solution used. Then $0.8x$ is the amount of alcohol in that solution, and $x + 5$ is the amount of the 30% solution that results.



$$\begin{pmatrix}\text{Amount of}\\ \text{alcohol in}\\ \text{80\% solution}\end{pmatrix} + \begin{pmatrix}\text{Amount of}\\ \text{alcohol in}\\ \text{20\% solution}\end{pmatrix} = \begin{pmatrix}\text{Amount of}\\ \text{alcohol in}\\ \text{30\% mixture}\end{pmatrix}$$

$$0.8x + 0.2(5) = 0.3(x + 5) \quad \text{Multiply.}$$
$$0.8x + 1 = 0.3x + 1.5 \quad \text{Subtract } 0.3x \text{ from both sides.}$$
$$0.5x + 1 = 1.5 \quad \text{Subtract 1 from both sides.}$$
$$0.5x = 0.5 \quad \text{Divide both sides by 0.5.}$$
$$x = 1$$

Add 1 liter of the 80% solution.

CHECK

	Liters of solution	Liters of alcohol	Percent alcohol
First solution	1	0.8(1) = 0.8	80%
Second solution	5	0.2(5) = 1	20%
Mixture	6	1.8	1.8/6 = 0.3, or 30%

MATCHED PROBLEM 7

A chemical storeroom has a 90% acid solution and a 40% acid solution. How many centiliters of the 90% solution should be added to 50 centiliters of the 40% solution to yield a 50% solution?

› Data Analysis and Linear Regression

In most math courses you've taken, there were probably a lot of application problems in which an equation or function was provided that describes some real-world situation. There have been dozens in this book already, and we're only in the second section of Chapter 2! This may have left you wondering where in the world these functions come from.

There is a mathematical technique known as **regression analysis** that is used to find a function that provides a useful model for a set of data points. Graphs of equations are often called **curves** and regression analysis is also referred to as **curve fitting.** In Example 8, we use **linear regression** to construct a mathematical model in the form of a linear function that fits a data set.

EXAMPLE 8 Diamond Prices

Table 1 Round-Cut Diamond Prices

Weight (carats)	Price
0.5	$1,340
0.6	$1,760
0.7	$2,540
0.8	$3,350
0.9	$4,130
1.0	$4,920

Source: www.tradeshop.com

Prices for round-cut diamonds taken from an online trader are given in Table 1.

(A) Use linear regression on a graphing calculator to find a linear model $y = f(x)$ that fits these data, where x is the weight of a diamond (in carats) and y is the associated price of that diamond (in dollars). Round the constants a and b to three significant digits. Compare the model and the data both graphically and numerically.

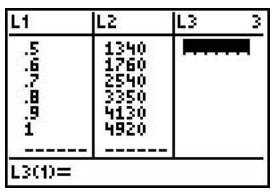
(B) Use the model to estimate the cost of a 0.85 carat diamond and the cost of a 1.2 carat diamond. Round answers to the nearest dollar.

(C) Use the model to estimate the weight of a diamond that sells for $3,000. Round the answer to two significant digits.

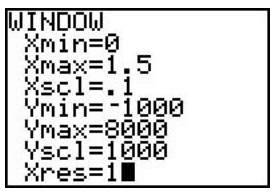
SOLUTIONS

(A) The first step in fitting a curve to a data set is to enter the data in two lists in a graphing calculator, usually by pressing STAT and selecting EDIT (see Fig. 12).* We enter the given values of the independent variable x in L_1 and the corresponding values of the dependent variable y in L_2 (Fig. 13). Next, we select a viewing window that will show all the data (Fig. 14).

› Figure 12



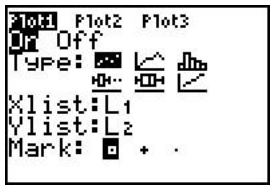
› Figure 13



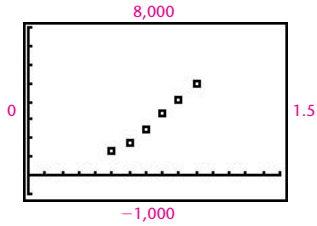
› Figure 14

To check that all the data will be visible in this window, we need to graph the points in the form (x, y), where x is a number in list L_1 and y is the corresponding number in list L_2. This is called a **scatter diagram** or **scatter plot.** On most graphing calculators, a scatter diagram can be drawn by first pressing STAT PLOT and selecting the options displayed in Figure 15 on the next page. Then press GRAPH to display the scatter diagram (Fig. 16).

*Remember, we are using a TI-83 or TI-84 to produce the screen images in this book. Other graphing calculators will produce different images.

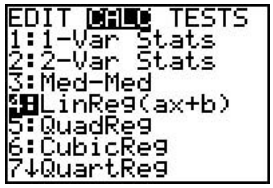


› Figure 15



› Figure 16

Now we are ready to fit a curve to the data graphed in Figure 16. First we find the screen on the graphing calculator that lists the various regression options, usually by pressing STAT and selecting CALC (Fig. 17).



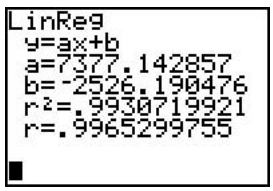
(a)

(b)

› Figure 17

Any of options 4 through C in Figure 17 can be used to fit a curve to a data set. As you progress through this text, you will become familiar with most of the choices in Figure 17. In this example, we are directed to select option 4 (or, equivalently, option 8), linear regression (Fig. 18). Notice that we entered the names of the two lists of data, L_1 and L_2, after the command LinReg($ax + b$) in Figure 18. The order in which we enter these two names is important. The name of the list of independent values must precede the name of the list of dependent values. Press ENTER to obtain the results in Figure 19. The values r^2 and r displayed in Figure 19 are called **diagnostics.*** They provide a measure of how well the regression curve fits the data. Values of r close to 1 or -1 indicate a good fit. Values of r close to 0 indicate a poor fit.

› Figure 18



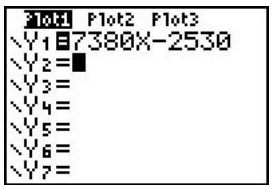
› Figure 19

After rounding a and b in Figure 19 to three significant digits, the linear regression model for these data is

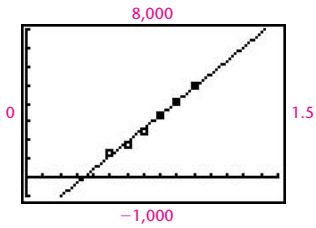
$$y = f(x) = 7{,}380x - 2{,}530$$

*If your graphing calculators doesn't display values of r, try pressing CATALOG, then selecting "diagnostics on."

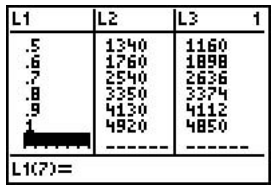
Enter this equation in the equation editor (Fig. 20). The graph of the model and the scatter plot of the data are shown in Figure 21 and a table comparing the data and the corresponding values of the model is shown in Figure 22.*



› Figure 20



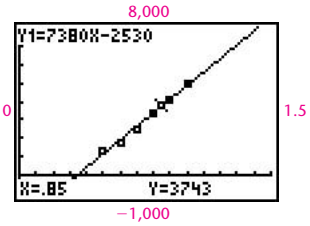
› Figure 21



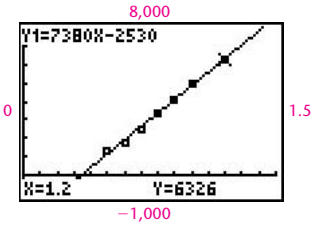
› Figure 22

Examining Figures 21 and 22, we see that the model does seem to provide a reasonable fit for these data.

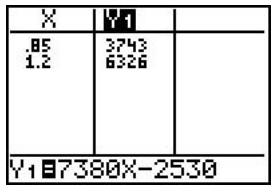
(B) The carat weight is represented by x, so we need to find the dollar values for $x = 0.85$ and $x = 1.2$. Because $x = 0.85$ and $x = 1.2$ are not in Table 1, we use the model to estimate the corresponding prices. From Figure 23 we see that the estimated price of a 0.85-carat diamond is \$3,743. Figure 24 shows that the estimated price of a 1.2-carat diamond is \$6,326. Figure 25 shows how TABLE can be used in place of TRACE to obtain the same results.



› Figure 23

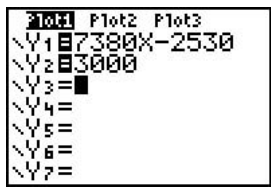


› Figure 24

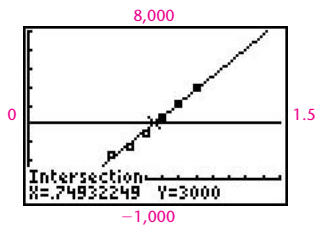


› Figure 25

(C) This time we are given a value of the dependent variable y (\$3,000) and asked to solve for the independent variable (carat weight). To find the weight, we add $y_2 = 3{,}000$ to the equation list (Fig. 26) and use the INTERSECT command (Fig. 27).



› Figure 26



› Figure 27

*On most graphing calculators, the values of the model displayed as L_3 in Figure 22 can be computed in a single operation by entering $y_1(L_1) \rightarrow L_3$ on the home screen.

From Figure 27 we see that $x = 0.75$ (to two significant digits) when $y = 3{,}000$. Thus, a \$3,000 diamond should weigh approximately 0.75 carats.

MATCHED PROBLEM* 8

Prices for emerald-cut diamonds taken from an online trader are given in Table 2. Repeat Example 8 for this data set.

Table 2 Emerald-Cut Diamond Prices

Weight (carats)	Price
0.5	\$1,350
0.6	\$1,740
0.7	\$2,610
0.8	\$3,320
0.9	\$4,150
1.0	\$4,850

Source: www.tradeshop.com

The quantity of a product that consumers are willing to buy during some period depends on its price. Generally, the higher the price, the lower the demand; the lower the price, the greater the demand. Similarly, the quantity of a product that producers are willing to sell during some period also depends on the price. Generally, a producer will be willing to supply more of a product at higher prices and less of a product at lower prices. In Example 9 we use linear regression to analyze supply and demand data and construct linear models.

EXAMPLE 9 **Supply and Demand**

Table 3 contains supply and demand data for broccoli at various price levels. Express all answers in numbers rounded to three significant digits.

(A) Use the data in Table 3 and linear regression to find a linear supply model $p = f(s)$, where s is the supply (in thousand pounds) and p is the corresponding price of broccoli (in cents).

(B) Use the data in Table 3 and linear regression to find a linear demand model $p = g(d)$, where d is the demand (in thousand pounds) and p is the corresponding price of broccoli (in cents).

*Be certain to delete the old values in the lists L_1 and L_2 before you work Matched Problem 8. Select the list title and push CLEAR, then ENTER.

(C) Graph both functions in the same viewing window and discuss possible interpretations of any intersection points.

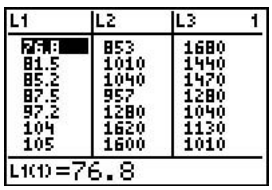
Table 3 Supply and Demand for Broccoli

Price (Cents)	Supply (Thousand lb)	Demand (Thousand lb)
76.8	853	1,680
81.5	1,010	1,440
85.2	1,040	1,470
87.5	957	1,280
97.2	1,280	1,040
104	1,620	1,130
105	1,600	1,010

SOLUTIONS

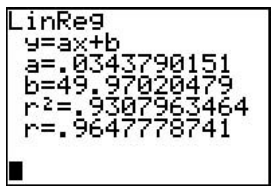
(A) First, we enter the data from Table 3 in the list editor of a graphing utility (Fig. 28). Next we select the linear regression command LinReg($ax + b$) followed by L_2, L_1 (Fig. 29) to make supply the independent variable and price the dependent variable. We also added the variable y_1, which automatically assigns the resulting regression equation into the equation editor as y_1, enabling us to view the graph without having to enter the equation manually. This produces the results shown in Figure 30. Thus, the linear model for the supply function is

$$p = f(s) = 0.0344s + 50.0$$



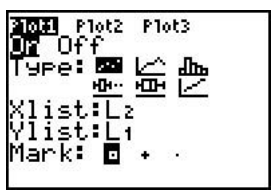
› Figure 28

› Figure 29

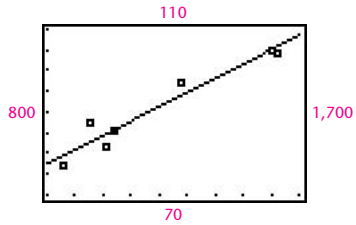


› Figure 30

To graph the supply data we use STAT PLOT, setting Xlist to L_2 and Ylist to L_1 (Fig. 31). Figure 32 shows a graph of the supply data and the supply model.



› Figure 31

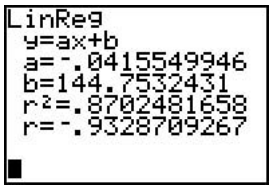


› Figure 32

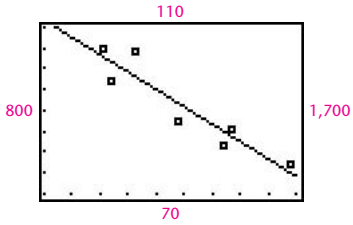
(B) This time we use the command LinReg($ax + b$) followed by L_3, L_1 to make demand the independent variable and price the dependent variable. This produces the results shown in Figure 33. Thus, the linear model for the demand function is

$$p = g(d) = -0.0416d + 145$$

To graph the demand data we use STAT PLOT, setting Xlist to L_3 and Ylist to L_1. Figure 34 shows a graph of the demand data and the demand model.



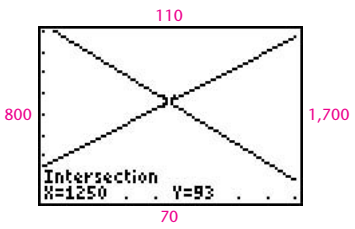
› Figure 33



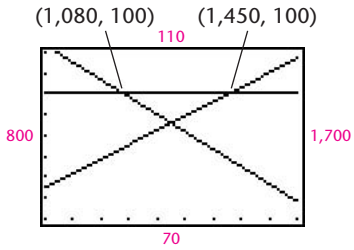
› Figure 34

(C) We graph both models in the same viewing window and use the INTERSECT command to find the intersection point (Fig. 35). The graphs intersect at $p = 93$ and $s = d = 1{,}250$. This point is called the **equilibrium point,** the value of p is called the **equilibrium price,** and the common value of s and d is called the **equilibrium quantity.** To help understand price fluctuations, suppose the current price of broccoli is 100 cents. We add the constant function $p = 100$ to the graph (Fig. 36).

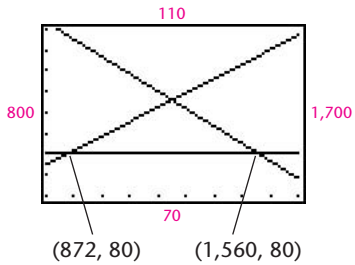
Using the INTERSECT command (details omitted), we find that the constant price line intersects the demand curve at 1,080 thousand pounds and the supply curve at 1,450 thousand pounds. Because the supply at a price level of 100 is greater than the demand, the producers will lower their prices. Suppose the price drops to 80 cents per pound. Changing the constant function to $p = 80$ produces the graph in Figure 37. This time the constant price line intersects the demand curve at 1,560 thousand pounds and the supply curve at 872 thousand pounds. Now supply is less than demand and producers will raise their prices. If the producers set the price at $p = 93$ cents, then, as we saw in Figure 35, the supply and demand are equal.



› Figure 35



› Figure 36



› Figure 37

MATCHED PROBLEM 9

Table 4 contains supply and demand data for cauliflower at various prices. Repeat Example 9 with these data.

Table 4 Supply and Demand for Cauliflower

Price (Cents)	Supply (Thousand lb)	Demand (Thousand lb)
26.5	583	653
27.1	607	629
27.2	596	635
27.4	627	631
27.5	604	638
28.1	661	610
28.6	682	599

ANSWERS TO MATCHED PROBLEMS

1. $x = -2$ **2.** $x = -\frac{20}{39} \approx -0.5128205$ **3.** No solution

4. $r = \frac{A - P}{Pt}$ $Pt \neq 0$ **5.** (A) $C(x) = 0.75x + 20$ (B) \$132.50 (C) 340 pretzels

6. 50 miles per hour **7.** 12.5 centiliters

8. (A) $y = 7{,}270x - 2{,}450$ (B) \$3,730; \$6,270 (C) 0.75 carats

9. (A) $p = 0.0180s + 16.3$ (B) $p = -0.0362d + 50.2$
(C) The price stabilizes at the equilibrium price of 27.6 cents.

2-2 Exercises

1. What exactly does it mean to solve an equation?

2. Explain why the following does not make sense: Solve the equation $P = 2l + 2w$.

3. Explain the difference between a conditional equation, an identity, and a contradiction.

4. Think of a real-world situation that would likely be modeled accurately with a linear function, and one that would not.

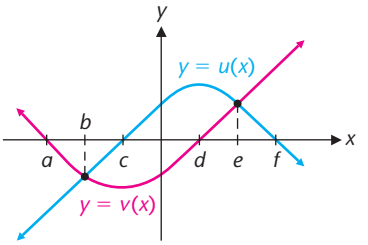
5. Explain how you could decide that an equation is an identity using a graphing calculator.

6. Explain how you could decide that an equation is a contradiction using a graphing calculator.

7. Why is it so important to check your answer when solving equations that contain fractions?

8. What is meant by the term "linear regression"?

Use the graphs of functions u and v in the figure to solve the equations in Problems 9–12. (Assume the graphs continue as indicated beyond the portions shown here.)



9. $u(x) = 0$ **10.** $v(x) = 0$

11. $u(x) = v(x)$ **12.** $u(x) - v(x) = 0$

In Problems 13–18, classify each equation as an identity, a conditional equation, or a contradiction. Solve each conditional equation.

13. $3(x - 2) - 2(x + 1) = x - 8$

14. $4(x - 1) - 2(x + 2) = 2x + 7$

15. $2(x - 1) - 3(2 - x) = 3x - 8$

16. $4(2 - x) + 2(x - 3) = 5x + 2$

17. $5(x + 2) - 3(x - 1) = 2x + 4$

18. $2(x + 1) + 3(2 - x) = 8 - x$

In Problems 19–30, solve the equation.

19. $10x - 7 = 4x - 25$ **20.** $11 + 3y = 5y - 5$

21. $3(x + 2) = 5(x - 6)$ **22.** $5x + 10(x - 2) = 40$

23. $5 + 4(t - 2) = 2(t + 7) + 1$

24. $5w - (7w - 4) - 2 = 5 - (3w + 2)$

25. $\frac{x}{2} - 2 = \frac{x}{5} + \frac{2}{5}$ **26.** $\frac{2x}{3} + \frac{1}{2} = \frac{3x}{2} - 2$

27. $3 + 5(x - 7) = 3x - 2(16 - x)$

28. $8x - 5(2 + x) = 2 + 3(x - 4)$

29. $5 - \frac{2x - 1}{4} = \frac{x + 2}{3}$ **30.** $\frac{x + 3}{4} - \frac{x - 4}{2} = \frac{3}{8}$

In Problems 31–46, solve the equation.

31. $\frac{7}{t} + 4 = \frac{2}{t}$ **32.** $\frac{9}{w} - 3 = \frac{2}{w}$

33. $\frac{1}{m} - \frac{1}{9} = \frac{4}{9} - \frac{2}{3m}$ **34.** $\frac{2}{3x} + \frac{1}{2} = \frac{4}{x} + \frac{4}{3}$

35. $(x - 2)(x + 3) = (x - 4)(x + 5)$

36. $(x - 2)(x - 4) = (x - 1)(x - 5)$

37. $(x + 2)(x - 3) = (x - 4)(x + 5)$

38. $(x - 2)(x + 4) = (x + 3)(x - 5)$

39. $(x - 2)^2 = (x - 1)(x + 2)$

40. $(x + 3)^2 = (x + 2)(x - 4)$

41. $\frac{2x}{x - 3} = 2 + \frac{6}{x - 3}$ **42.** $\frac{2x}{x + 4} = 2 - \frac{8}{x + 4}$

43. $\frac{2x}{x - 3} = 7 + \frac{4}{x - 3}$ **44.** $\frac{2x}{x + 4} = 7 - \frac{6}{x + 4}$

45. $\frac{2x}{x - 3} = 7 + \frac{6}{x - 3}$ **46.** $\frac{2x}{x + 4} = 7 - \frac{8}{x + 4}$

In Problems 47–56, solve for the indicated variable in terms of the other variables.

47. $P = 2l + 2w$ for w (perimeter of a rectangle)

48. $D = L + 2W + 2H$ for H (shipping dimensions)

49. $a_n = a_1 + (n - 1)d$ for d (arithmetic progressions)

50. $F = \frac{9}{5}C + 32$ for C (temperature scale)

51. $\frac{1}{f} = \frac{1}{d_1} + \frac{1}{d_2}$ for f (simple lens formula)

52. $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ for R_1 (electric circuit)

53. $A = 2ab + 2ac + 2bc$ for a (surface area of a rectangular solid)

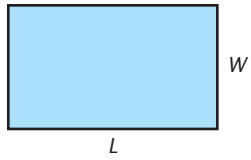
54. $A = 2ab + 2ac + 2bc$ for c

55. $y = \frac{2x - 3}{3x + 5}$ for x **56.** $x = \frac{3y + 2}{y - 3}$ for y

57. Discuss the relationship between the graphs of $y_1 = x$ and $y_2 = \sqrt{x^2}$. What does this tell you about the equation $\sqrt{x^2} = x$?

58. Discuss the relationship between the graphs of $y_1 = |x|$ and $y_2 = \sqrt{x^2}$. What does this tell you about the equation $\sqrt{x^2} = |x|$?

Problems 59–66 refer to a rectangle with width W and length L (see the figure). Write a mathematical expression in terms of W and L for each of the verbal statements in Problems 59–66.



59. The length is twice the width.

60. The width is three times the length.

61. The width is half the length.

62. The length is one-third of the width.

63. The length is three more than the width.

64. The width is five less than the length.

65. The length is four less than the width.

66. The width is ten more than the length.

67. Use linear regression to fit a line to each of the following data sets. How are the graphs of the two functions related? How are the two functions related?

(A)

x	y
−3	−3
1	−1
5	1

(B)

x	y
−3	−3
−1	1
1	5

68. Repeat Problem 67 for the following data sets.

(A)

x	y
−5	−5
−1	0
3	5

(B)

x	y
−5	−5
0	−1
5	3

In Problems 69–72, solve for x.

69. $\dfrac{x-\dfrac{1}{x}}{1+\dfrac{1}{x}}=1$

70. $\dfrac{x-\dfrac{1}{x}}{x+1-\dfrac{2}{x}}=1$

71. $\dfrac{x+1-\dfrac{2}{x}}{1-\dfrac{1}{x}}=x+2$

72. $\dfrac{x-1-\dfrac{2}{x}}{1-\dfrac{2}{x}}=x$

73. Find three consecutive integers whose sum is 84.

74. Find four consecutive integers whose sum is 182.

75. Find four consecutive even integers so that the sum of the first three is 2 more than twice the fourth.

76. Find three consecutive even integers so that the first plus twice the second is twice the third.

77. Find the dimensions of a rectangle if the perimeter is 60 inches and the length is twice the width.

78. Find the dimensions of a rectangle if the perimeter is 60 inches and the length is half the width.

APPLICATIONS

79. **COST ANALYSIS** A doughnut shop has a fixed cost of $124 per day and a variable cost of $0.12 per doughnut. Find the total daily cost of producing x doughnuts. How many doughnuts can be produced for a total daily cost of $250?

80. **COST ANALYSIS** A small company manufactures picnic tables. The weekly fixed cost is $1,200 and the variable cost is $45 per table. Find the total weekly cost of producing x picnic tables. How many picnic tables can be produced for a total weekly cost of $4,800?

81. **SALES COMMISSIONS** One employee of a computer store is paid a base salary of $2,150 a month plus an 8% commission on all sales over $7,000 during the month. How much must the employee sell in 1 month to earn a total of $3,170 for the month?

82. **SALES COMMISSIONS** A second employee of the computer store in Problem 81 is paid a base salary of $1,175 a month plus a 5% commission on all sales during the month.
(A) How much must this employee sell in 1 month to earn a total of $3,170 for the month?
(B) Determine the sales level at which both employees receive the same monthly income. If employees can select either of these payment methods, how would you advise an employee to make this selection?

83. **COMPETITIVE ROWING** A two-woman rowing team can row 1,200 meters with the current in a river in the same amount of time it takes them to row 1,000 meters against that same current. In each case, their average rowing speed without the effect of the current is 3 meters per second. Find the speed of the current.

84. **COMPETITIVE ROWING** The winners of the men's 1,000-meter double sculls event in the 2006 Asian games rowed at an average of 11.3 miles per hour. If this team were to row this speed for a half mile with a current in 80% of the time they were able to row that same distance against the current, what would be the speed of the current?

85. **AERONAUTICS** The cruising speed of an airplane is 150 miles per hour (relative to the ground). You wish to hire the plane for a 3-hour sightseeing trip. You instruct the pilot to fly north as far as he can and still return to the airport at the end of the allotted time.
(A) How far north should the pilot fly if the wind is blowing from the north at 30 miles per hour?
(B) How far north should the pilot fly if there is no wind?

86. **NAVIGATION** Suppose you are at a river resort and rent a motor boat for 5 hours starting at 7 A.M. You are told that the

boat will travel at 8 miles per hour upstream and 12 miles per hour returning. You decide that you would like to go as far up the river as you can and still be back at noon. At what time should you turn back, and how far from the resort will you be at that time?

87. **EARTHQUAKES** An earthquake emits a primary wave and a secondary wave. Near the surface of the Earth the primary wave travels at about 5 miles per second, and the secondary wave travels at about 3 miles per second. From the time lag between the two waves arriving at a given seismic station, it is possible to estimate the distance to the quake. Suppose a station measures a time difference of 12 seconds between the arrival of the two waves. How far is the earthquake from the station? (The *epicenter* can be located by obtaining distance bearings at three or more stations.)

88. **SOUND DETECTION** A ship using sound-sensing devices above and below water recorded a surface explosion 39 seconds sooner on its underwater device than on its above-water device. If sound travels in air at about 1,100 feet per second and in water at about 5,000 feet per second, how far away was the explosion?

89. **CHEMISTRY** How many gallons of distilled water must be mixed with 50 gallons of 30% alcohol solution to obtain a 25% solution?

90. **CHEMISTRY** How many gallons of hydrochloric acid must be added to 12 gallons of a 30% solution to obtain a 40% solution?

91. **CHEMISTRY** A chemist mixes distilled water with a 90% solution of sulfuric acid to produce a 50% solution. If 5 liters of distilled water is used, how much 50% solution is produced?

92. **CHEMISTRY** A fuel oil distributor has 120,000 gallons of fuel with 0.9% sulfur content, which exceeds pollution control standards of 0.8% sulfur content. How many gallons of fuel oil with a 0.3% sulfur content must be added to the 120,000 gallons to obtain fuel oil that complies with the pollution control standards?

93. **EARTH SCIENCE** In 1984, the Soviets led the world in drilling the deepest hole in the Earth's crust—more than 12 kilometers deep. They found that below 3 kilometers the temperature T increased 2.5°C for each additional 100 meters of depth.

(A) If the temperature at 3 kilometers is 30°C and x is the depth of the hole in kilometers, write an equation using x that will give the temperature T in the hole at any depth beyond 3 kilometers.

(B) What would the temperature be at 15 kilometers? (The temperature limit for their drilling equipment was about 300°C.)

(C) At what depth (in kilometers) would the temperature reach 280°C?

94. **AERONAUTICS** Because air is not as dense at high altitudes, planes require a higher ground speed to become airborne. A rule of thumb is 3% more ground speed per 1,000 feet of elevation, assuming no wind and no change in air temperature. (Compute numerical answers to three significant digits.)

(A) Let

V_s = Takeoff ground speed at sea level for a particular plane (in miles per hour)

A = Altitude above sea level (in thousands of feet)

V = Takeoff ground speed at altitude A for the same plane (in miles per hour)

Write a formula relating these three quantities.

(B) What takeoff ground speed would be required at Lake Tahoe airport (6,400 feet), if takeoff ground speed at San Francisco airport (sea level) is 120 miles per hour?

(C) If a landing strip at a Colorado Rockies hunting lodge (8,500 feet) requires a takeoff ground speed of 125 miles per hour, what would be the takeoff ground speed in Los Angeles (sea level)?

(D) If the takeoff ground speed at sea level is 135 miles per hour and the takeoff ground speed at a mountain resort is 155 miles per hour, what is the altitude of the mountain resort in thousands of feet?

DATA ANALYSIS AND LINEAR REGRESSION

In Problems 95–100, use linear regression to construct linear models of the form $y = ax + b$.

95. **COLLEGE TUITION** Find a linear model for the public college tuition data given in Table 5, where x is years after 1999 and y is the average tuition in dollars. Round a and b to the nearest dollar. Use your model to predict the average public college tuition in 2008, and to estimate when the average public college tuition will reach $15,000 per year.

Table 5 Average Annual Tuition at Public and Private Colleges

School Year Ending	Public	Private
1999	$7,107	$19,368
2000	$7,310	$20,186
2001	$7,586	$21,368
2002	$8,022	$22,413
2003	$8,502	$23,340
2004	$9,249	$24,636
2005	$9,877	$26,025

Source: www.infoplease.com

96. **COLLEGE TUITION** Find a linear model for the private college tuition data given in Table 5, where x is years after 1999 and y is the average tuition in dollars. Round a and b to the nearest dollar. Use your model to predict the average private college tuition in 2008, and to estimate when the average private college tuition will reach $35,000 per year.

97. **OLYMPIC GAMES** Find a linear model $y = ax + b$ for the men's 100-meter freestyle data given in Table 6 where x is years since 1968 and y is winning time (in seconds). Do the same for the women's 100-meter freestyle data (round to three decimal places). Do these models indicate that the women will eventually catch up with the men? If so, when? Do you think this will actually occur?

Table 6 Winning Times in Olympic Swimming Events

	100-Meter Freestyle		200-Meter Backstroke	
	Men (seconds)	Women (seconds)	Men (minutes: seconds)	Women (minutes: seconds)
1968	52.20	60.00	2:09.60	2:24.80
1972	51.22	58.59	2:02.82	2:19.19
1976	49.99	55.65	1:59.19	2:13.43
1980	50.40	54.79	2:01.93	2:11.77
1984	49.80	55.92	2:00.23	2:12.38
1988	48.63	54.93	1:59.37	2:09.29
1992	49.02	54.65	1:58.47	2:07.06
1996	48.74	54.50	1.58.54	2:07.83
2000	48.30	53.83	1:56.76	2:08.16
2004	48.17	53.84	1:54.95	2:09.19

Source: www.infoplease.com

98. **OLYMPIC GAMES** Find a linear model $y = ax + b$ for the men's 200-meter backstroke data given in Table 6, where x is years since 1968 and y is winning time (in seconds). Do the same for the women's 200-meter backstroke data (round to three decimal places). Do these models indicate that the women will eventually catch up with the men? If so, when? Do you think this will actually occur?

99. **SUPPLY AND DEMAND** Table 7 contains price–supply data and price–demand data for corn. Find a linear model $y = ax + b$ for the price–supply data where x is price (in dollars) and y is supply (in billions of bushels). Do the same for the price–demand data. Find the equilibrium price for corn.

Table 7 Supply and Demand for U.S. Corn

Price $/bu	Supply (billion bu)	Price $/bu	Demand (billion bu)
2.15	6.29	2.07	9.78
2.29	7.27	2.15	9.35
2.36	7.53	2.22	8.47
2.48	7.93	2.34	8.12
2.47	8.12	2.39	7.76
2.55	8.24	2.47	6.98
2.71	9.23	2.59	5.57

Source: www.usda.gov/nass/pubs/histdata.htm

100. **SUPPLY AND DEMAND** Table 8 contains price–supply data and price–demand data for soybeans. Find a linear model $y = ax + b$ for the price–supply data where x is supply (in billions of bushels) and y is price (in dollars). Do the same for the price–demand data. Find the equilibrium price for soybeans.

Table 8 Supply and Demand for U.S. Soybeans

Price $/bu	Supply (billion bu)	Price $/bu	Demand (billion bu)
5.15	1.55	4.93	2.60
5.79	1.86	5.48	2.40
5.88	1.94	5.71	2.18
6.07	2.08	6.07	2.05
6.15	2.15	6.40	1.95
6.25	2.27	6.66	1.85
6.65	2.53	7.25	1.67

Source: www.usda.gov/nass/pubs/histdata.htm

2-3 Quadratic Functions

- Defining Quadratic Functions
- The Vertex Form of a Quadratic Function
- Completing the Square
- Finding the Equation of a Parabola
- Modeling with Quadratic Functions

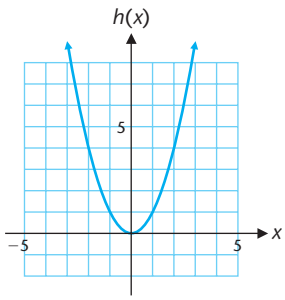


Figure 1 Squaring function $h(x) = x^2$.

The graph of the squaring function $h(x) = x^2$ is shown in Figure 1. Notice that h is an even function; that is, the graph of h is symmetrical with respect to the y axis. Also, the lowest point on the graph is (0, 0). Let's explore the effect of applying a sequence of basic transformations to the graph of h. (A brief review of Section 1-4 would be helpful at this point.)

EXPLORE-DISCUSS 1

Indicate how the graph of each function is related to the graph of $h(x) = x^2$. Discuss the symmetry of the graphs and find the highest or lowest point, whichever exists, on each graph.

(A) $f(x) = (x - 3)^2 - 7 = x^2 - 6x + 2$

(B) $g(x) = 0.5(x + 2)^2 + 3 = 0.5x^2 + 2x + 5$

(C) $m(x) = -(x - 4)^2 + 8 = -x^2 + 8x - 8$

(D) $n(x) = -3(x + 1)^2 - 1 = -3x^2 - 6x - 4$

Defining Quadratic Functions

Graphing the functions in Explore-Discuss 1 produces figures similar in shape to the graph of the squaring function in Figure 1. These figures are called *parabolas*. The functions that produced these parabolas are examples of the important class of *quadratic functions*, which we will now define.

DEFINITION 1 Quadratic Functions

If a, b, and c are real numbers with $a \neq 0$, then the function

$$f(x) = ax^2 + bx + c$$

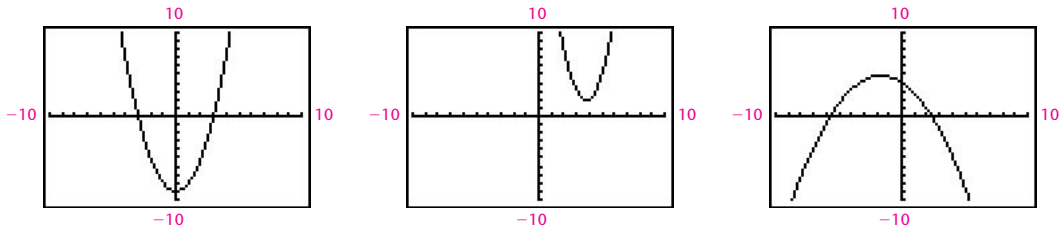
is called a **quadratic function** and its graph is called a **parabola.*** This is known as the **general form** of a quadratic function.

*A more general definition of a parabola that is independent of any coordinate system is given in Section 8-1.

Because the expression $ax^2 + bx + c$ represents a real number no matter what number we substitute for x,

the domain of a quadratic function is the set of all real numbers.

We will discuss methods for determining the range of a quadratic function later in this section. Typical graphs of quadratic functions are illustrated in Figure 2.



› **Figure 2** Graphs of quadratic functions.

› The Vertex Form of a Quadratic Function

We will begin our detailed study of quadratic functions by examining some in a special form, which we will call the **vertex form:***

$$f(x) = a(x - h)^2 + k$$

We'll see where the name comes from in a bit. For now, refer to Explore-Discuss 1. Any function of this form is a transformation of the basic squaring function $g(x) = x^2$, so we can use transformations to analyze the graph.

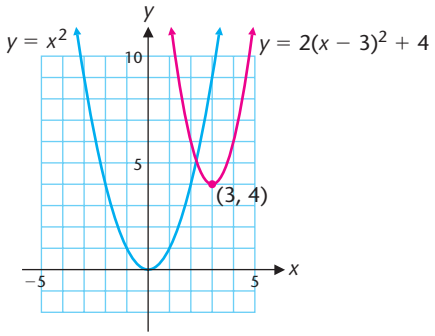
EXAMPLE 1 **The Graph of a Quadratic Function**

Use transformations of $g(x) = x^2$ to graph the function $f(x) = 2(x - 3)^2 + 4$. Use your graph to determine the graphical significance of the constants 2, 3, and 4 in this function.

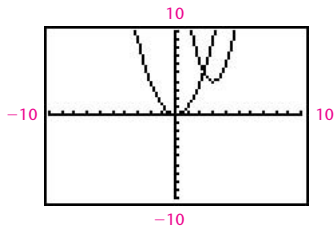
SOLUTION

Multiplying by 2 vertically stretches the graph by a factor of 2. Subtracting 3 inside the square moves the graph 3 units to the right. Adding 4 outside the square moves the graph 4 units up. The graph of f is shown in Figure 3, along with the graph of $g(x) = x^2$.

*In Problem 63 of Exercises 2-3, you will be asked to show that any function of this form fits the definition of quadratic function in Definition 1.



› Figure 3



› Figure 4

The lowest point on the graph of f is (3, 4), so $h = 3$ and $k = 4$ determine the key point where the graph changes direction. The constant $a = 2$ affects the width of the parabola. Our results are checked by graphing f and $g(x) = x^2$ with a graphing calculator (Fig. 4).

MATCHED PROBLEM 1

Use transformations of $g(x) = x^2$ to graph the function $f(x) = -\frac{1}{2}(x - 2)^2 + 5$. Use your graph to determine the significance of the constants $-\frac{1}{2}$, 2, and 5 in this function.

Explore-Discuss 2 will help to clarify the significance of the constants a, h, and k in the form $f(x) = a(x - h)^2 + k$.

››› EXPLORE-DISCUSS 2

Explore the effect of changing the constants a, h, and k on the graph of $f(x) = a(x - h)^2 + k$.

(A) Let $a = 1$ and $h = 5$. Graph function f for $k = -4, 0$, and 3 simultaneously in the same viewing window. Explain the effect of changing k on the graph of f.

(B) Let $a = 1$ and $k = 2$. Graph function f for $h = -4, 0$, and 5 simultaneously in the same viewing window. Explain the effect of changing h on the graph of f.

(C) Let $h = 5$ and $k = -2$. Graph function f for $a = 0.25, 1$, and 3 simultaneously in the same viewing window. Then graph function f for $a = 1, -1$, and -0.25 simultaneously in the same viewing window. Explain the effect of changing a on the graph of f.

(D) Can all quadratic functions of the form $y = ax^2 + bx + c$ be rewritten as $a(x - h)^2 + k$?

Every parabola has a point where the graph reaches a maximum or minimum and changes direction. We will call that point the **vertex** of the parabola. Finding the vertex is key to many of the things we'll do with parabolas. Example 1 and Explore-Discuss 2 demonstrate that *if a quadratic function is in the form* $f(x) = a(x - h)^2 + k$, *then the vertex is the point* (h, k).

Next, notice that the graph of $h(x) = x^2$ is symmetric about the y axis. As a result, the transformation $f(x) = 2(x - 3)^2 + 4$ is symmetric about the vertical line $x = 3$ (which runs through the vertex). We will call this vertical line of symmetry the **axis,** or **axis of symmetry** of a parabola. If the page containing the graph of f is folded along the line $x = 3$, the two halves of the graph would match exactly.

Finally, Explore-Discuss 2 illustrates the significance of the constant a in $f(x) = a(x - h)^2 + k$. If a is positive, the graph has a minimum and opens upward. But if a is negative, the graph will be a vertical reflection of $h(x) = x^2$ and will have a maximum and open downward. The size of a determines the width of the parabola: if $|a| > 1$, the graph is narrower than $h(x) = x^2$, and if $|a| < 1$, it is wider.

These properties of a quadratic function in vertex form are summarized next.

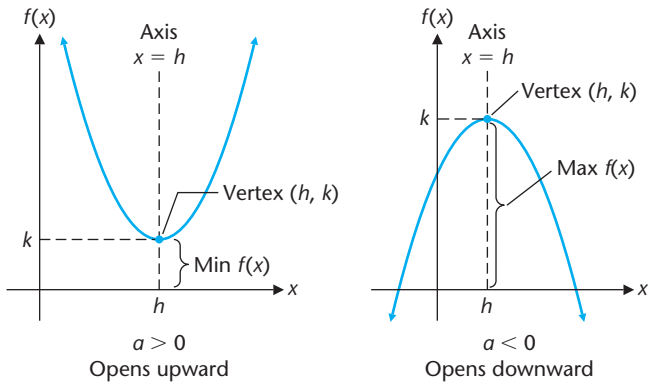
› PROPERTIES OF A QUADRATIC FUNCTION AND ITS GRAPH

Given a quadratic function in vertex form

$$f(x) = a(x - h)^2 + k \qquad a \neq 0$$

we summarize general properties as follows:

1. The graph of f is a parabola:



2. Vertex: (h, k) (parabola rises on one side of the vertex and falls on the other).

3. Axis (of symmetry): $x = h$ (parallel to y axis).

4. $f(h) = k$ is the minimum if $a > 0$ and the maximum if $a < 0$.

5. Domain: all real numbers; range: $(-\infty, k]$ if $a < 0$ or $[k, \infty)$ if $a > 0$.

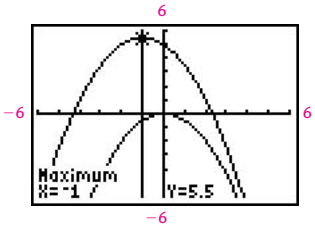
6. The graph of f is the graph of $g(x) = ax^2$ translated horizontally h units and vertically k units.

EXAMPLE 2 Analyzing a Quadratic Function

For the following quadratic function, analyze the graph, and check your results with a graphing calculator:

$$f(x) = -0.5(x + 1)^2 + 5.5$$

SOLUTION



› Figure 5

We can rewrite the function as $f(x) = -0.5[x - (-1)]^2 + 5.5$. Comparing this equation to $y = a(x - h)^2 + k$, we see that $a = -0.5$, $h = -1$, and $k = 5.5$. Therefore, the vertex is $(-1, 5.5)$, the axis of symmetry is $x = -1$, the maximum value is $f(-1) = 5.5$, and the range is $(-\infty, 5.5]$. The function f is increasing on $(-\infty, -1]$ and decreasing on $[-1, \infty)$. The graph of f is the graph of $g(x) = -0.5x^2$ shifted to the left one unit and upward five and one-half units. To check these results, we graph f and g simultaneously in the same viewing window, use the MAXIMUM command to locate the vertex, and add the graph of the axis of symmetry (Fig. 5).

MATCHED PROBLEM 2

For the following quadratic function, analyze the graph, and check your results with a graphing calculator:

$$f(x) = -(x - 1.5)^2 + 1.25$$

››› CAUTION ›››

Be careful with the sign when finding the first coordinate of a vertex. The generic vertex form has $(x - h)^2$ in it, so when we have $(x + 1)^2$, the first coordinate of the vertex is actually *negative* 1.

› Completing the Square

Now that we can recognize the properties of a quadratic function in vertex form, the obvious question is "What if a quadratic function is *not* in vertex form?" More often than not, the quadratic functions we encounter will be in the form $f(x) = ax^2 + bx + c$. The method of *completing the square* can be used to find the vertex form of such a quadratic function. We'll also find this process useful in solving equations later in the chapter.

››› EXPLORE-DISCUSS 3

Replace ? in each of the following with a number that makes the equation valid.

(A) $(x + 1)^2 = x^2 + 2x + ?$ **(B)** $(x + 2)^2 = x^2 + 4x + ?$

(C) $(x + 3)^2 = x^2 + 6x + ?$ **(D)** $(x + 4)^2 = x^2 + 8x + ?$

Replace ? in each of the following with a number that makes the expression a perfect square of the form $(x + h)^2$.

(E) $x^2 + 10x + ?$ **(F)** $x^2 + 12x + ?$

(G) $x^2 + bx + ?$

Given the quadratic expression

$$x^2 + bx$$

what number should be added to this expression to make it a perfect square? To find out, consider the square of the following expression:

$$(x + m)^2 = x^2 + \underbrace{2mx} + \underbrace{m^2}$$

m^2 is the square of one-half the coefficient of x.

We see that the third term on the right side of the equation is the square of one-half the coefficient of x in the second term on the right; that is, m^2 is the square of $\frac{1}{2}(2m)$. This observation leads to the following rule:

› COMPLETING THE SQUARE

To **complete the square** of the quadratic expression

$$x^2 + bx$$ Leading coefficient 1

add the square of one-half the coefficient of x; that is, add

$$\left(\frac{b}{2}\right)^2 \quad \text{or} \quad \frac{b^2}{4}$$

The resulting expression can be factored as a perfect square:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

EXAMPLE 3 **Completing the Square**

Complete the square for each of the following:

(A) $x^2 - 6x$ (B) $x^2 + \frac{4}{5}x$ (C) $x^2 - \frac{3}{4}x$

SOLUTIONS

(A) $x^2 - 6x$

$x^2 - 6x + 9 = (x - 3)^2$ Add $\left[\frac{1}{2}(-6)\right]^2$; that is, 9.

(B) $x^2 + \frac{4}{5}x$

$x^2 + \frac{4}{5}x + \frac{4}{25} = \left(x + \frac{2}{5}\right)^2$ Add $\left(\frac{1}{2} \cdot \frac{4}{5}\right)^2$; that is, $\frac{4}{25}$.

(C) $x^2 - \frac{3}{4}x$

$x^2 - \frac{3}{4}x + \frac{9}{64} = \left(x - \frac{3}{8}\right)^2$ Add $\left(\frac{1}{2} \cdot \frac{-3}{4}\right)^2$; that is, $\frac{9}{64}$.

Note: In each case, the quadratic expression ends up factoring as (x + half the coefficient of the x term).

MATCHED PROBLEM 3

Complete the square for each of the following:

(A) $x^2 - 8x$ (B) $x^2 + \frac{7}{4}x$ (C) $x^2 - \frac{2}{3}x$

It is important to note that the rule for completing the square applies to only quadratic expressions in which the coefficient of x^2 is 1. We'll see how to overcome this limitation later.

› Finding the Equation of a Parabola

EXAMPLE 4 **Finding the Vertex Form of a Parabola**

Find the vertex form of $f(x) = x^2 - 8x + 4$, then write the vertex and axis.

SOLUTION

We will separate $x^2 - 8x$ with parentheses, then use completing the square to factor part of f as a perfect square.

$$\begin{aligned} f(x) &= x^2 - 8x + 4 && \text{Group } x^2 - 8x \text{ together.} \\ &= (x^2 - 8x) + 4 && \text{Find the number needed to complete the square.} \\ &= (x^2 - 8x + ?) + 4 && \left[\tfrac{1}{2}(-8)\right]^2 = 16\text{; add 16, then subtract it at the end.} \\ &= (x^2 - 8x + 16) + 4 - 16 && \text{Factor parentheses; combine like terms.} \\ &= (x - 4)^2 - 12 && f \text{ is now in vertex form.} \end{aligned}$$

The vertex form is $f(x) = (x - 4)^2 - 12$. The vertex is $(4, -12)$ and the axis is $x = 4$.

MATCHED PROBLEM 4

Find the vertex form of $g(x) = x^2 + 10x - 1$, then write the vertex and axis.

When the coefficient of x^2 is not 1, the procedure is just a bit more complicated.

EXAMPLE 5 **Finding the Vertex Form of a Parabola**

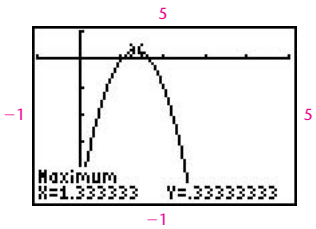
Find the vertex form of $f(x) = -3x^2 + 8x - 5$. Find the vertex and axis, then describe the graph verbally and check your answer with a graphing calculator.

SOLUTION

We need a coefficient of 1 on the x^2, so after grouping the first two terms, we'll factor out -3.

$$\begin{aligned} f(x) &= (-3x^2 + 8x) - 5 && \text{Factor } -3 \text{ out of the first two terms.} \\ &= -3\left(x^2 - \frac{8}{3}x\right) - 5 && \left(\tfrac{1}{2} \cdot \tfrac{-8}{3}\right)^2 = \tfrac{16}{9}\text{; add this number inside the parentheses.} \\ &= -3\left(x^2 - \frac{8}{3}x + \frac{16}{9}\right) - 5 + ? && \text{Because of the } -3 \text{ factor, we actually subtracted } \tfrac{16}{3}\text{, so we also add } \tfrac{16}{3}. \\ &= -3\left(x^2 - \frac{8}{3}x + \frac{16}{9}\right) - 5 + \frac{16}{3} && \text{Factor the parentheses; combine like terms.} \\ &= -3\left(x - \frac{4}{3}\right)^2 + \frac{1}{3} && f \text{ is in vertex form.} \end{aligned}$$

The vertex is $(\frac{4}{3}, \frac{1}{3})$ and the axis is $x = \frac{4}{3}$. Because $a = -3$ is negative, the parabola opens downward and has a maximum value of $\frac{1}{3}$. The function f is increasing on $(-\infty, \frac{4}{3}]$ and decreasing on $[\frac{4}{3}, \infty)$. The range of f is $(-\infty, \frac{1}{3}]$. The graph is shown in Figure 6.



› Figure 6

MATCHED PROBLEM 5

Repeat Example 5 for $g(x) = -2x^2 - 7x + 3$.

››› CAUTION ›››

When completing the square on a quadratic function with $a \neq 1$, the number that you add or subtract at the end will be different from the number you added inside the parentheses.

A key observation based on Examples 4 and 5 will help us to find the vertex of a parabola quickly, without completing the square. In both Example 4 and 5, the first coordinate of the vertex worked out to be $-\frac{b}{2a}$, where the function f was written in the form $f(x) = ax^2 + bx + c$. This provides a simple way to find the vertex of a parabola in that form. (For a proof of this fact, see Problem 64 in Exercises 2-3.)

› FINDING THE VERTEX OF A PARABOLA

When a quadratic function is written in the form $f(x) = ax^2 + bx + c$, the first coordinate of the vertex can be found using the formula

$$x = -\frac{b}{2a}$$

The second coordinate can then be found by evaluating f at the first coordinate.

EXAMPLE 6 Finding the Vertex of a Parabola

Find the vertex of the parabola $f(x) = -5x^2 + 30x - 2$.

SOLUTION

The first coordinate is given by

$$x = -\frac{b}{2a} = -\frac{30}{-10} = 3$$

The second coordinate is $f(3)$:

$$f(3) = -5(3)^2 + 30(3) - 2 = 43$$

The vertex is (3, 43).

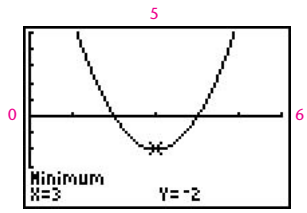
MATCHED PROBLEM 6

Find the vertex of the parabola $g(x) = 2x^2 - 16x + 10$.

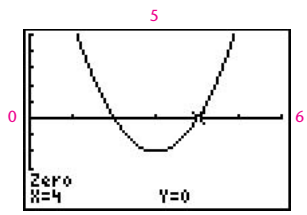
So far we've used the vertex form of a parabola to help us learn all about the graphs of quadratic functions. But it can also be used for something equally as important, especially when it comes to mathematical modeling: finding the equation of a parabola.

EXAMPLE 7 Finding the Equation of a Parabola

Find an equation for the parabola whose graph is shown in Figure 7. Write your answer in the form $y = ax^2 + bx + c$.



(a)



(b)

› Figure 7

SOLUTION

Figure 7a shows that the vertex of the parabola is $(h, k) = (3, -2)$. So the vertex form of the parabola must look like

$$f(x) = a(x - 3)^2 - 2 \tag{1}$$

All that remains is to find a. Figure 7b shows that the point $(4, 0)$ is on the graph of f, so $f(4) = 0$. According to equation (1),

$$\begin{aligned} f(4) &= a(4 - 3)^2 - 2 \\ &= a - 2 \end{aligned}$$

Since 0 and $a - 2$ are both $f(4)$, they must be equal:

$$\begin{aligned} a - 2 &= 0 \\ a &= 2 \end{aligned}$$

The equation for the parabola is

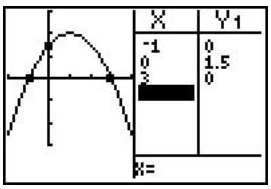
$$\begin{aligned} f(x) &= 2(x - 3)^2 - 2 \\ &= 2(x^2 - 6x + 9) - 2 \\ &= 2x^2 - 12x + 16 \end{aligned}$$

MATCHED PROBLEM 7

Find the equation of the parabola with vertex $(2, 4)$ and y intercept $(0, 2)$. Write your answer in the form $y = ax^2 + bx + c$.

EXAMPLE 8 Finding the Equation of a Parabola

Find an equation for the parabola whose graph is shown in Figure 8. Write your answer in the form $y = ax^2 + bx + c$.



› Figure 8

SOLUTION

Let

$$f(x) = a(x - h)^2 + k$$

Because $f(-1) = f(3)$, the axis of symmetry $x = h$ must contain the midpoint of the interval $[-1, 3]$. That is,

$$h = \frac{-1 + 3}{2} = 1 \qquad \text{and} \qquad f(x) = a(x - 1)^2 + k$$

Now we can use either x intercept to find a relationship between a and k. We choose $f(-1) = 0$.

$$\begin{aligned} f(-1) = a(-1 - 1)^2 + k &= 0 \\ 4a + k &= 0 \\ k &= -4a \end{aligned}$$

Now we can write

$$f(x) = a(x - 1)^2 - 4a$$

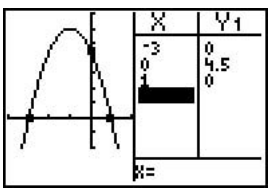
and use the y intercept to find a. (We can't use the other x intercept to find a. Try it to see why.)

$$\begin{aligned} f(0) = a(0 - 1)^2 - 4a &= 1.5 \\ a - 4a &= 1.5 \\ -3a &= 1.5 \\ a &= -0.5 \end{aligned}$$

So the equation is

$$f(x) = -0.5(x - 1)^2 + 2 = -0.5x^2 + x + 1.5$$

MATCHED PROBLEM 8



› Figure 9

Find an equation for the parabola whose graph is shown in Figure 9. Write your answer in the form $y = ax^2 + bx + c$.

Modeling with Quadratic Functions

We now look at several applications that can be modeled using quadratic functions.

EXAMPLE 9 **Maximum Area**

A dairy farm has a barn that is 150 feet long and 75 feet wide. The owner has 240 feet of fencing and wishes to use all of it in the construction of two identical adjacent outdoor pens, with part of the long side of the barn as one side of the pens, and a common fence between the two (Fig. 10). The owner wants the pens to be as large as possible.

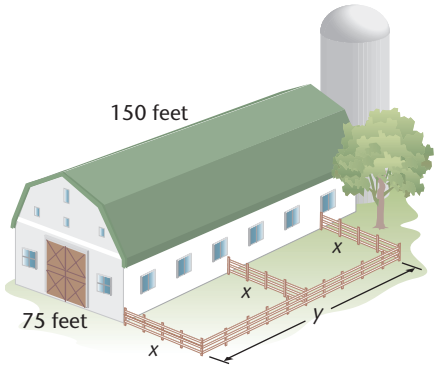


Figure 10

(A) Construct a mathematical model for the combined area of both pens in the form of a function $A(x)$ (see Fig. 10) and state the domain of A.

(B) Find the value of x that produces the maximum combined area.

(C) Find the dimensions and the area of each pen.

SOLUTIONS

(A) The combined area of the two pens is

$$A = xy$$

Adding up the lengths of all four segments of fence, we find that building the pens will require $3x + y$ feet of fencing. We have 240 feet of fence to use, so

$$3x + y = 240$$
$$y = 240 - 3x$$

Because the distances x and y must be nonnegative, x and y must satisfy $x \geq 0$ and $y = 240 - 3x \geq 0$. It follows that $0 \leq x \leq 80$. Substituting for y in the combined area equation, we have the following model for this problem:

$$A(x) = x(240 - 3x) = 240x - 3x^2 \qquad 0 \leq x \leq 80$$

(B) Algebraic Solution

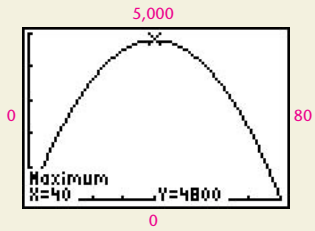
The function $A(x) = 240x - 3x^2$ is a parabola that opens downward, so the maximum value of area will occur at the vertex.

$$x = -\frac{b}{2a} = -\frac{240}{2(-3)} = 40;$$
$$A(40) = 240(40) - 3(40)^2 = 4{,}800$$

A value of $x = 40$ gives a maximum area of 4,800 square feet.

(B) Graphical Solution

Entering $y_1 = 240x - 3x^2$ and using the MAXIMUM command produces the graph in Figure 11. This shows that the maximum combined area of 4,800 square feet occurs at $x = 40$ feet.



› Figure 11

(C) When $x = 40$, $y = 240 - 3(40) = 120$. Each pen is x by $\frac{y}{2}$, or 40 feet by 60 feet. The area of each pen is 40 feet $\times$ 60 feet = 2,400 square feet.

MATCHED PROBLEM 9

Repeat Example 9 with the owner constructing three identical adjacent pens instead of two.

The great sixteenth-century astronomer and physicist Galileo was the first to discover that the distance an object falls is proportional to the square of the time it has been falling. This makes quadratic functions a natural fit for modeling falling objects. Neglecting air resistance, the quadratic function

$$h(t) = h_0 - 16t^2$$

represents the *height of an object* t seconds after it is dropped from an initial height of h_0 feet. The constant -16 is related to the force of gravity and is dependent on the units used. That is, -16 only works for distances measured in feet and time measured in seconds. If the object is thrown either upward or downward, the quadratic model will also have a term involving t. (See Problems 87 and 88 in Exercises 2-3.)

EXAMPLE 10 Projectile Motion

As a publicity stunt, a late-night talk show host drops a pumpkin from a rooftop that is 200 feet high. When will the pumpkin hit the ground? Round your answer to two decimal places.

SOLUTION

Because the initial height is 200 feet, the quadratic model for the height of the pumpkin is

$$h(t) = 200 - 16t^2$$

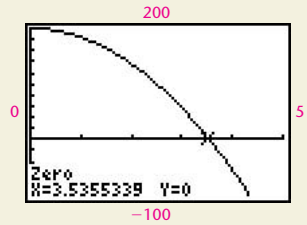
Because $h(t) = 0$ when the pumpkin hits the ground, we must solve this equation for t.

Algebraic Solution

$$\begin{aligned} h(t) = 200 - 16t^2 &= 0 \\ 16t^2 &= 200 \\ t^2 &= \frac{200}{16} = 12.5 \\ t &= \sqrt{12.5} \\ &= 3.54 \text{ seconds} \end{aligned}$$

Graphical Solution

Graphing $y_1 = 200 - 16x^2$ and using the ZERO command (Fig. 12) shows that $h = 0$ at $t = 3.54$ seconds.



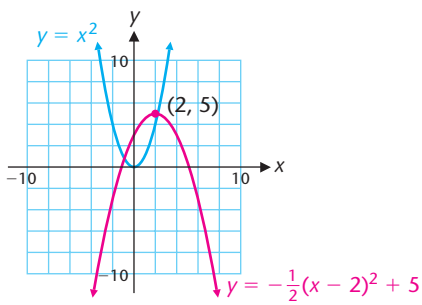
› Figure 12

MATCHED PROBLEM 10

A watermelon is dropped from a rooftop that is 300 feet high. When will the melon hit the ground? Round your answer to two decimal places.

ANSWERS TO MATCHED PROBLEMS

1.



The $-\frac{1}{2}$ makes the graph open downward and vertically shrinks it by a factor of $\frac{1}{2}$, the 2 moves it 2 units right, and the 5 moves it 5 units up.

2. Vertex form: $f(x) = -(x - 1.5)^2 + 1.25$. The vertex is (1.5, 1.25), the axis of symmetry is $x = 1.5$, the maximum value of $f(x)$ is 1.25, and the range of f is $(-\infty, 1.25]$. The function f is increasing on $(-\infty, 1.5]$ and decreasing on $[1.5, \infty)$. The graph of f is the graph of $g(x) = -x^2$ shifted one and a half units to the right and one and a quarter units upward.

3. (A) $x^2 - 8x + 16 = (x - 4)^2$ (B) $x^2 + \frac{7}{4}x + \frac{49}{64} = \left(x + \frac{7}{8}\right)^2$
(C) $x^2 - \frac{2}{3}x + \frac{1}{9} = \left(x - \frac{1}{3}\right)^2$

4. $g(x) = (x + 5)^2 - 26$; vertex is $(-5, -26)$, axis is $x = -5$.

5. $g(x) = -2\left(x + \frac{7}{2}\right)^2 + \frac{55}{2}$; vertex is $(-\frac{7}{2}, \frac{55}{2})$, axis is $x = -\frac{7}{2}$. The parabola opens downward, has a maximum value of $\frac{55}{2}$, and a range of $(-\infty, \frac{55}{2}]$. The function g is increasing on $(-\infty, -\frac{7}{2}]$ and decreasing on $[-\frac{7}{2}, \infty)$.

6. $(4, -22)$ **7.** $f(x) = -0.5x^2 + 2x + 2$ **8.** $f(x) = -1.5x^2 - 3x + 4.5$

9. (A) $A(x) = (240 - 4x)x$, $0 \le x \le 60$ (B) The maximum combined area of 3,600 ft^2 occurs at $x = 30$ feet. (C) Each pen is 30 feet by 40 feet with area 1,200 ft^2. **10.** 4.33 seconds

2-3 Exercises

1. Describe the graph of any quadratic function.

2. How can you tell from a quadratic function whether its graph opens up or down?

3. True or False: every quadratic function has a maximum. Explain.

4. Using transformations, explain why the vertex of $f(x) = a(x - h)^2 + k$ is (h, k).

5. What information does the constant a provide about the graph of a function of the form $f(x) = ax^2 + bx + c$?

6. Explain how to find the maximum or minimum value of a quadratic function.

In Problems 7–12, find the vertex and axis of the parabola, then draw the graph by hand and verify with a graphing calculator.

7. $f(x) = (x + 3)^2 - 4$ **8.** $f(x) = (x + 2)^2 - 2$

9. $f(x) = -\left(x - \frac{3}{2}\right)^2 - 5$ **10.** $f(x) = -\left(x - \frac{11}{2}\right)^2 + 3$

11. $f(x) = 2(x + 10)^2 + 20$ **12.** $f(x) = -\frac{1}{2}(x + 8)^2 + 12$

In Problems 13–18, write a brief verbal description of the relationship between the graph of the indicated function and the graph of $y = x^2$.

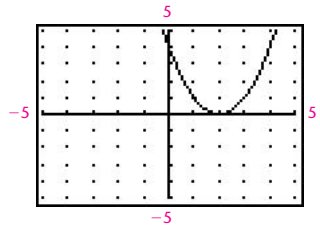
13. $f(x) = (x - 2)^2 + 1$ **14.** $g(x) = -(x + 1)^2 - 2$

15. $h(x) = -(x + 1)^2$ **16.** $k(x) = (x - 2)^2$

17. $m(x) = (x - 2)^2 - 3$ **18.** $n(x) = -(x + 1)^2 + 4$

In Problems 19–24, match each graph with one of the functions in Problems 13–18.

19.



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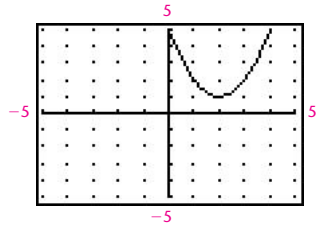

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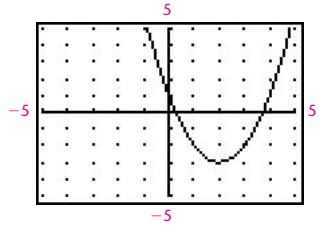

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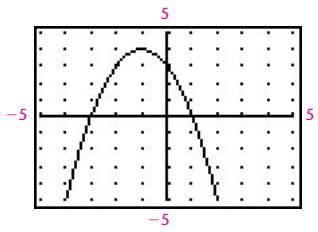

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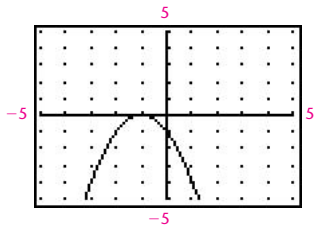

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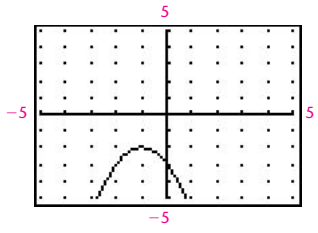


In Problems 25–30, complete the square for each expression.

25. $x^2 + 10x$

26. $x^2 + 8x$

27. $x^2 - 7x$

28. $x^2 - 3x$

29. $x^2 + \frac{7}{5}x$

30. $x^2 + \frac{11}{3}x$

In Problems 31–40, complete the square and find the vertex form of each quadratic function, then write the vertex and the axis.

31. $f(x) = x^2 - 4x + 5$

32. $g(x) = x^2 - 6x + 1$

33. $h(x) = -x^2 - 2x - 3$

34. $k(x) = -x^2 - 10x + 3$

35. $m(x) = 2x^2 - 12x + 22$

36. $n(x) = 3x^2 + 6x - 2$

37. $f(x) = \frac{1}{2}x^2 + 3x - \frac{7}{2}$

38. $g(x) = -\frac{3}{2}x^2 + 9x + \frac{11}{2}$

39. $f(x) = 2x^2 - 24x + 90$

40. $g(x) = 3x^2 + 24x + 30$

In Problems 41–48, use the formula $x = -\frac{b}{2a}$ to find the vertex. Then write a description of the graph using all of the following words: axis, increases, decreases, range, and maximum or minimum. Check your answer with a graphing calculator.

41. $f(x) = x^2 + 8x + 8$

42. $f(x) = x^2 + 10x + 10$

43. $f(x) = -x^2 - 7x + 4$

44. $f(x) = -x^2 - 11x + 1$

45. $f(x) = 4x^2 - 18x + 25$

46. $f(x) = 5x^2 + 30x - 17$

47. $f(x) = -10x^2 + 50x + 12$

48. $f(x) = -8x^2 - 24x + 16$

In problems 49–54, find the equation of a quadratic function whose graph satisfies the given conditions.

49. Vertex: (4, 8); x intercept: 6

50. Vertex: (−2, −12); x intercept: −4

51. Vertex: (−4, 12); y intercept: 4

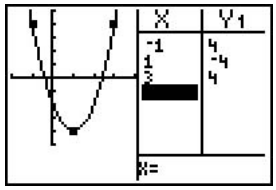
52. Vertex: (5, 8); y intercept: −2

53. Vertex: (−5, −25); additional point on graph: (−2, 20)

54. Vertex: (6, −40); additional point on graph: (3, 50)

In Problems 55–62, use the graph of the parabola to find the equation of the corresponding quadratic function.

55.



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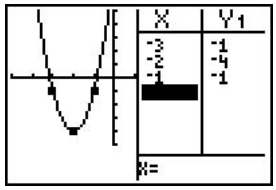

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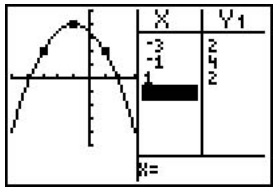

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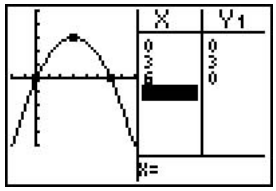

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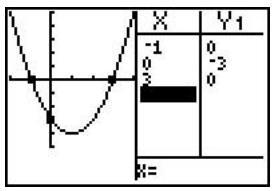

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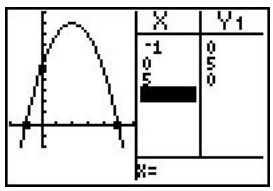

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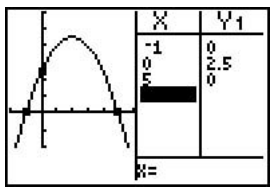

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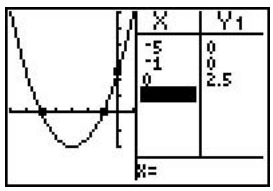


63. For $f(x) = a(x - h)^2 + k$, expand the parentheses and simplify to write in the form $f(x) = ax^2 + bx + c$. This proves that any function in vertex form is a quadratic function as defined in Definition 1.

64. For $f(x) = ax^2 + bx + c$:
(A) Group the first two terms, then factor out a.
(B) Multiply the coefficient of the middle term by $\frac{1}{2}$ and square the result.

(C) Add your result from (B) inside the parentheses, and subtract an appropriate number outside the parentheses so that the function is unchanged.
(D) Factor the parentheses. [*Hint:* The number you squared in (B) is important here!] What is the x coordinate of the vertex for any function of the form $f(x) = ax^2 + bx + c$?

65. Let $g(x) = x^2 + kx + 1$. Graph g for several different values of k and discuss the relationship among these graphs.

66. Confirm your conclusions in Problem 65 by finding the vertex form for g.

67. Let $f(x) = (x - 1)^2 + k$. Discuss the relationship between the values of k and the number of x intercepts for the graph of f. Generalize your comments to any function of the form

$$f(x) = a(x - h)^2 + k, a > 0$$

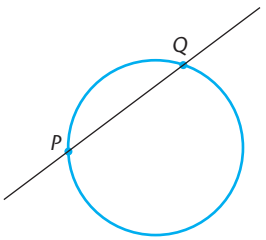
68. Let $f(x) = -(x - 2)^2 + k$. Discuss the relationship between the values of k and the number of x intercepts for the graph of f. Generalize your comments to any function of the form

$$f(x) = a(x - h)^2 + k, a < 0$$

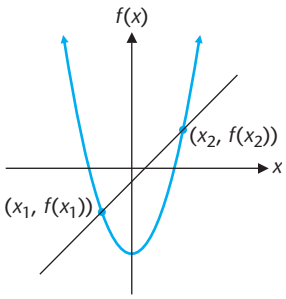
69. Let $f(x) = a(x - h)^2 + k$. Compare the values of $f(h + r)$ and $f(h - r)$ for any real number r. Interpret the results in terms of the graph of f.

70. For the equation $x = y^2 - 2y + 1$:
(A) Plot points corresponding to $y = -2, -1, 0, 1, 2, 3$, and 4. Use these points to sketch the graph of the equation.
(B) Is your graph a function? Explain.
(C) Use your results to describe the graph of any equation of the form $x = ay^2 + by + c$.

Problems 71–74 are calculus related. In geometry, a line that intersects a circle in two distinct points is called a secant line, as shown in figure (a). In calculus, the line through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is called a ***secant line*** *for the graph of the function f, as shown in figure (b).*



Secant line for a circle
(a)



Secant line for the graph of a function
(b)

In Problems 71 and 72, find the equation of the secant line through the indicated points on the graph of f. Graph f and the secant line on the same coordinate system.

71. $f(x) = x^2 - 4; (-1, -3), (3, 5)$

72. $f(x) = 9 - x^2; (-2, 5), (4, -7)$

73. Let $f(x) = x^2 - 3x + 5$. If h is a nonzero real number, then $(2, f(2))$ and $(2 + h, f(2 + h))$ are two distinct points on the graph of f.
(A) Find the slope of the secant line through these two points.
(B) Evaluate the slope of the secant line for $h = 1$, $h = 0.1$, $h = 0.01$, and $h = 0.001$. What value does the slope seem to be approaching?

74. Repeat Problem 73 for $f(x) = x^2 + 2x - 6$.

75. Find the minimum product of two numbers whose difference is 30. Is there a maximum product? Explain.

76. Find the maximum product of two numbers whose sum is 60. Is there a minimum product? Explain.

APPLICATIONS

77. **PROFIT ANALYSIS** A consultant hired by a small manufacturing company informs the company owner that their annual profit can be modeled by the function $P(x) = -1.2x^2 + 62.5x - 491$, where x represents the number of employees and P is profit in thousands of dollars. How many employees should the company have to maximize annual profit? What is the maximum annual profit they can expect in that case?

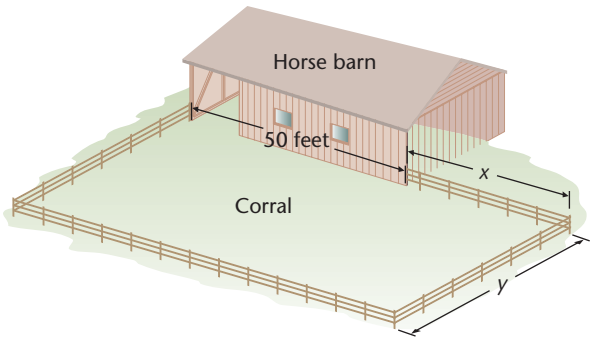
78. **PROFIT ANALYSIS** The annual profits (in thousands of dollars) from 1998 to 2007 for the company in Problem 77 can be modeled by the function $P(t) = 6.8t^2 - 80.5t + 427.3$, $0 \le t \le 9$, where t is years after 1998. How much profit did the company make in their worst year?

79. **MOVIE INDUSTRY REVENUE** The annual U.S. box office revenue in billions of dollars for a span of years beginning in 2000 can be modeled by the function $B(x) = -0.19x^2 + 1.2x + 7.6$, $0 \le x \le 7$, where x is years after 2000.
(A) In what year was box office revenue at its highest in that time span?
(B) Explain why you should not use the exact vertex in answering part A in this problem.

80. **GAS MILEAGE** The speed at which a car is driven can have a big effect on gas mileage. Based on EPA statistics for compact cars, the function $m(x) = -0.025x^2 + 2.45x - 30$, $30 \le x \le 65$, models the average miles per gallon for compact cars in terms of the speed driven x (in miles per hour).
(A) At what speed should the owner of a compact car drive to maximize miles per gallon?

(B) If one compact car has a 14-gallon gas tank, how much farther could you drive it on one tank of gas driving at the speed you found in part A than if you drove at 65 miles per hour?

81. **CONSTRUCTION** A horse breeder wants to construct a corral next to a horse barn that is 50 feet long, using all of the barn as one side of the corral (see the figure). He has 250 feet of fencing available and wants to use all of it.



(A) Express the area $A(x)$ of the corral as a function of x and indicate its domain.
(B) Find the value of x that produces the maximum area.
(C) What are the dimensions of the corral with the maximum area?

82. **CONSTRUCTION** Repeat Problem 81 if the horse breeder has only 140 feet of fencing available for the corral. Does the maximum value of the area function still occur at the vertex? Explain.

83. **FALLING OBJECT** A sandbag is dropped off a high-altitude balloon at an altitude of 10,000 ft. When will the sandbag hit the ground?

84. **FALLING OBJECT** A prankster drops a water balloon off the top of a 144-foot-high building. When will the balloon hit the ground?

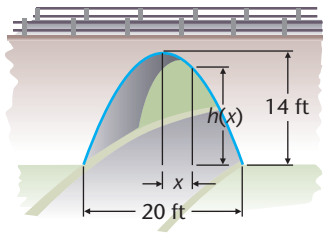
85. **FALLING OBJECT** A cliff diver hits the water 2.5 seconds after diving off the cliff. How high is the cliff?

86. **FALLING OBJECT** A forest ranger drops a coffee cup off a fire watchtower. If the cup hits the ground 1.5 seconds later, how high is the tower?

87. **PROJECTILE FLIGHT** An arrow shot vertically into the air from ground level with a crossbow reaches a maximum height of 484 feet after 5.5 seconds of flight. Let the quadratic function $d(t)$ represent the distance above ground (in feet) t seconds after the arrow is released. (If air resistance is neglected, a quadratic model provides a good approximation for the flight of a projectile.)
(A) Find $d(t)$ and state its domain.
(B) At what times (to two decimal places) will the arrow be 250 feet above the ground?

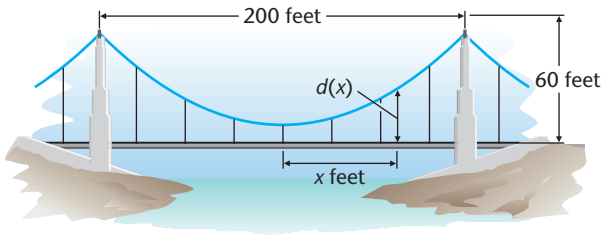
88. **PROJECTILE FLIGHT** Repeat Problem 87 if the arrow reaches a maximum height of 324 feet after 4.5 seconds of flight.

89. **ENGINEERING** The arch of a bridge is in the shape of a parabola 14 feet high at the center and 20 feet wide at the base (see the figure).



(A) Express the height of the arch $h(x)$ in terms of x and state its domain.
(B) Can a truck that is 8 feet wide and 12 feet high pass through the arch?
(C) What is the tallest 8-foot-wide truck that can pass through the arch?
(D) What (to two decimal places) is the widest 12-foot-high truck that can pass through the arch?

90. **ENGINEERING** The roadbed of one section of a suspension bridge is hanging from a large cable suspended between two towers that are 200 feet apart (see the figure). The cable forms a parabola that is 60 feet above the roadbed at the towers and 10 feet above the roadbed at the lowest point.



(A) Express the vertical distance $d(x)$ (in feet) from the roadbed to the suspension cable in terms of x and state the domain of d.
(B) The roadbed is supported by seven equally spaced vertical cables (see the figure). Find the combined total length of these supporting cables.

MODELING AND LINEAR REGRESSION

91. **MAXIMIZING REVENUE** A company that manufactures flashlights has collected the price–demand data in Table 1 on the next page. Round all numbers to three significant digits.
(A) Use the data in Table 1 and linear regression to find a linear price–demand function $p = d(x)$, where x is the number of flashlights (in thousands) that the company can sell at a price of p dollars.
(B) Find the price that maximizes the company's revenue from the sale of flashlights. Recall that revenue = price times quantity sold.

Table 1

Price	Demand
$3.55	45,800
$3.95	40,500
$4.13	37,900
$4.85	34,700
$5.19	30,400
$5.55	28,900
$6.15	25,400

92. MAXIMIZING REVENUE A company that manufactures pencil sharpeners has collected the price–demand data in Table 2. Round all numbers to three significant digits.
(A) Use the data in Table 2 and linear regression to find a linear price–demand function $p = d(x)$, where x is the number of pencil sharpeners (in thousands) that the company can sell at a price of p dollars.
(B) Find the price that maximizes the company's revenue from the sale of pencil sharpeners. Recall that revenue = price times quantity sold.

Table 2

Price	Demand
$4.23	47,800
$4.89	45,600
$5.43	42,700
$5.97	39,600
$6.47	34,700
$7.12	31,600
$7.84	27,800

2-4 Complex Numbers

- Imaginary Numbers
- The Complex Number System
- Complex Numbers and Radicals
- Solving Equations Involving Complex Numbers

The idea of inventing new numbers may sound very odd to you, but it's not as strange as you might think. Consider the simple quadratic equation

$$x^2 - 2 = 0 \tag{1}$$

The solution is a number whose square is 2, so you are probably thinking "the square root of two." But there was a time when square roots had not yet been defined. Twenty-six hundred years ago, early mathematicians would have told you that there was no number whose square is 2. They only knew about rational numbers, and around 500 B.C. a group of mathematicians known as the Pythagoreans found that it is impossible to square a rational number and get 2. For equation (1) to have a solution, a new kind of number had to be invented—an irrational number. The study of irrational numbers didn't get a firm mathematical foundation for over 2,000 years after that!

We now accept $\sqrt{2}$ as a number because we can approximate it very well with technology, but at one time almost everyone would have said that it was just made up, and didn't *really* exist. In this section, we ask this question: Is there any reason to invent any other kinds of numbers?

››› EXPLORE-DISCUSS 1

Graph $g(x) = x^2 - 1$ in a standard viewing window and discuss the relationship between the real zeros of the function and the x intercepts of its graph. Do the same for $f(x) = x^2 + 1$.

Does the simple quadratic equation

$$x^2 + 1 = 0 \tag{2}$$

have a solution? If equation (2) is to have a solution, x^2 has to be negative. But the square of a real number is never negative. Therefore, equation (2) cannot have any real-number solutions. Once again, a new type of number must be invented—a number whose square is negative one. The concept of square roots of negative numbers had been kicked around for a couple of centuries, but in 1748, the Swiss mathematician Euler used the letter i to represent a square root of -1. From this simple beginning, it's possible to build a new system of numbers called the complex number system, which we explore in this section.

› Imaginary Numbers

We begin with imaginary numbers, one of which we know already: i is a square root of -1. The number i is called the **imaginary unit**. Explore-Discuss 2 will help you to become acquainted with this number.

››› EXPLORE-DISCUSS 2

Natural number powers of i take on particularly simple forms:

$$\begin{aligned} i &= i & i^5 &= i^4 \cdot i = (1)i = i \\ i^2 &= -1 & i^6 &= i^4 \cdot i^2 = 1(-1) = -1 \\ i^3 &= i^2 \cdot i = (-1)i = -i & i^7 &= i^4 \cdot i^3 = 1(-i) = -i \\ i^4 &= i^2 \cdot i^2 = (-1)(-1) = 1 & i^8 &= i^4 \cdot i^4 = 1 \cdot 1 = 1 \end{aligned}$$

In general, what are the possible values for i^n, n a natural number? Explain how you could easily evaluate i^n for any natural number n. Then evaluate each of the following:

(A) i^{17} **(B)** i^{24} **(C)** i^{38} **(D)** i^{47}

If your graphing calculator can perform complex arithmetic, use it to check your calculations in parts A–D.

A **pure imaginary number** is defined to be any multiple of the imaginary unit i, that is, any number of the form bi, where b is a real number.

EXAMPLE 1 Finding Powers of Pure Imaginary Numbers

Find the requested power of each pure imaginary number:

(A) $(5i)^2$ (B) $(-3i)^3$ (C) $(2i)^4$

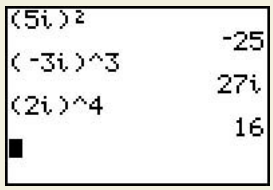
SOLUTIONS

Algebraic Solutions

(A) $(5i)^2 = 5^2 \cdot i^2 = 25(-1) = -25$
(B) $(-3i)^3 = (-3)^3 \cdot i^3 = -27 \cdot (-i) = 27i$
(C) $(2i)^4 = 2^4 \cdot i^4 = 16(1) = 16$

Graphical Solutions

Most graphing calculators can do arithmetic with the imaginary unit i (Fig. 1). On the TI-84, i is entered with the key combination 2^{nd} decimal point.



› Figure 1

MATCHED PROBLEM 1

Find the requested power of each pure imaginary number:

(A) $(-7i)^2$ (B) $(4i)^3$ (C) $(-2i)^4$

As we will see in the next section, to solve many equations that would otherwise not have solutions, we will need to go a bit beyond pure imaginary numbers. With the imaginary unit as starting point, we can build a new system of numbers called the complex number system. The complex numbers evolved over a long period, but, like the real numbers, it was not until the nineteenth century that they were given a firm mathematical foundation. Table 1 gives a brief history of the evolution of complex numbers.

Table 1 Brief History of Complex Numbers

Approximate Date A.D.	Person	Event
50	Heron of Alexandria	First recorded encounter of a square root of a negative number.
850	Mahavira of India	Said that a negative has no square root, because it is not a square.
1545	Cardano of Italy	Solutions to cubic equations involved square roots of negative numbers.
1637	Descartes of France	Introduced the terms *real* and *imaginary*.
1748	Euler of Switzerland	Used i for $\sqrt{-1}$.
1832	Gauss of Germany	Introduced the term *complex number*.

The Complex Number System

We start our development of the complex number system by defining what we mean by a complex number, and several special types of complex numbers.

DEFINITION 1 Complex Number

A **complex number** is a number of the form

$$a + bi \qquad \text{Standard form}$$

where a and b are real numbers and i is the **imaginary unit,** a square root of -1.

For a complex number $a + bi$, a is called the **real part** and bi is called the **imaginary part.**

Some examples of complex numbers are

$$\begin{array}{lll} 3 - 2i & \frac{1}{2} + 5i & 2 - \frac{1}{3}i \\ 0 + 3i & 5 + 0i & 0 + 0i \end{array}$$

Particular kinds of complex numbers are given special names as follows:

DEFINITION 2 Names for Particular Kinds of Complex Numbers

Imaginary Unit:	i	(A square root of -1)
Complex Number:	$a + bi$	a and b real numbers
Imaginary Number:	$a + bi$	$b \neq 0$ (The imaginary part is nonzero.)
Pure Imaginary Number:	$0 + bi = bi$	$b \neq 0$ (The real part is zero.)
Real Number:	$a + 0i = a$	(The imaginary part is zero.)
Zero:	$0 + 0i = 0$	
Conjugate of $a + bi$:	$a - bi$	

EXAMPLE 2 Special Types of Complex Numbers

Given the list of complex numbers:

$$\begin{array}{lll} 3 - 2i & \frac{1}{2} + 5i & 2 - \frac{1}{3}i \\ 0 + 3i = 3i & 5 + 0i = 5 & 0 + 0i = 0 \end{array}$$

(A) List all the imaginary numbers, pure imaginary numbers, real numbers, and zero.

(B) Write the conjugate of each.

SOLUTIONS

(A) Imaginary numbers: $3 - 2i, \frac{1}{2} + 5i, 2 - \frac{1}{3}i, 3i$
Pure imaginary numbers: $0 + 3i = 3i$
Real numbers: $5 + 0i = 5, 0 + 0i = 0$
Zero: $0 + 0i = 0$

(B)

$3 + 2i$	$\frac{1}{2} - 5i$	$2 + \frac{1}{3}i$
$0 - 3i = -3i$	$5 - 0i = 5$	$0 - 0i = 0$

MATCHED PROBLEM 2

Given the list of complex numbers:

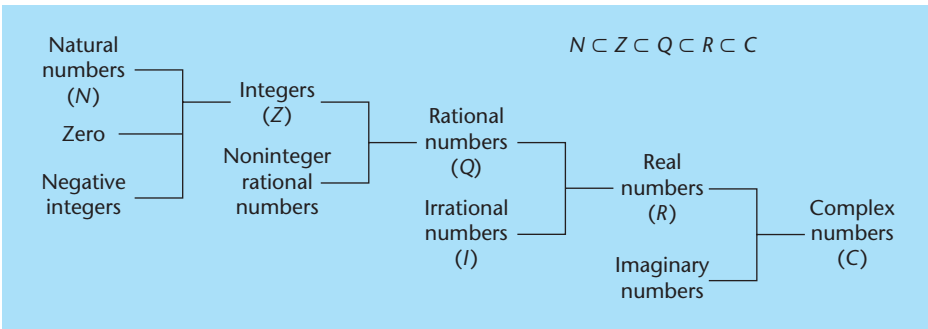
$6 + 7i$	$\sqrt{2} - \frac{1}{3}i$	$0 - i = -i$
$0 + \frac{2}{3}i = \frac{2}{3}i$	$-\sqrt{3} + 0i = -\sqrt{3}$	$0 - 0i = 0$

(A) List all the imaginary numbers, pure imaginary numbers, real numbers, and zero.

(B) Write the conjugate of each.

In Definition 2, notice that we identify a complex number of the form $a + 0i$ with the real number a. This means that every real number is also a complex number; in other words, the set of real numbers is contained in the set of complex numbers, just as the set of rational numbers is contained in the set of real numbers. Any complex number that is not a real number is called an **imaginary number.** If we combine the set of all real numbers with the set of all imaginary numbers, we obtain ***C*, the set of complex numbers.** The relationship of the complex number system to the other number systems we have studied is shown in Figure 2. In each column, adding a new set of numbers to the set on top produces a bigger set containing the one listed before it.

› Figure 2



If we are going to work with this new type of number, it seems reasonable to begin by considering the basic operations of addition, subtraction, multiplication, and division. But first, we should make sure we are clear on what exactly it means for two complex numbers to be equal.

DEFINITION 3 Equality and Basic Operations

1. Two complex numbers are equal exactly when their real and imaginary parts are equal. That is,

$$a + bi = c + di \text{ if and only if } a = c \text{ and } b = d$$

2. To add complex numbers, add their real and imaginary parts separately. That is,

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

3. Multiplication of complex numbers is defined by the following formula:

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Using the definitions of addition and multiplication of complex numbers (Definition 3), it can be shown that basic properties of the real number system* extend to the following basic properties of the complex number system.

BASIC PROPERTIES OF THE COMPLEX NUMBER SYSTEM

1. Addition and multiplication of complex numbers are commutative and associative operations.
2. There is an additive identity and a multiplicative identity for complex numbers.
3. Every complex number has an additive inverse or negative.
4. Every nonzero complex number has a multiplicative inverse or reciprocal.
5. Multiplication distributes over addition.

This is actually really good news: it tells us that we don't have to memorize the formulas for adding and multiplying complex numbers in Definition 3. Instead:

We can treat complex numbers of the form $a + bi$ exactly as we treat algebraic expressions of the form $a + bx$. We just need to remember that in this case, i stands for the imaginary unit; it is not a variable that represents a real number.

The first two arithmetic operations we consider are **addition** and **subtraction.**

*Basic properties of the real number system are discussed in Appendix A, Section A-1.

EXAMPLE 3 Addition and Subtraction of Complex Numbers

Carry out each operation and express the answer in standard form:

(A) $(2 - 3i) + (6 + 2i)$ (B) $(-5 + 4i) + (0 + 0i)$
(C) $(7 - 3i) - (6 + 2i)$ (D) $(-2 + 7i) + (2 - 7i)$

SOLUTIONS

Algebraic Solutions

(A) We could apply the definition of addition directly, but it is easier to use complex number properties.

$$(2 - 3i) + (6 + 2i) = 2 - 3i + 6 + 2i$$ Remove parentheses.

$$= (2 + 6) + (-3 + 2)i$$ Combine like terms.

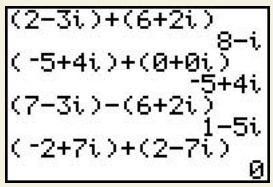
$$= 8 - i$$

(B) $(-5 + 4i) + (0 + 0i) = -5 + 4i + 0 + 0i$
$= -5 + 4i$

(C) $(7 - 3i) - (6 + 2i) = 7 - 3i - 6 - 2i$
$= 1 - 5i$

(D) $(-2 + 7i) + (2 - 7i) = -2 + 7i + 2 - 7i = 0$

Graphical Solutions



› Figure 3

››› CAUTION ›››

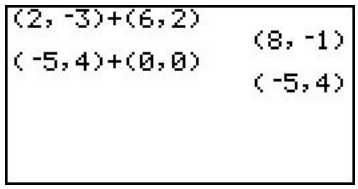
When subtracting complex numbers, the parentheses around the second number are crucial. Make sure that you distribute the negative sign.

MATCHED PROBLEM 3

Carry out each operation and express the answer in standard form:

(A) $(3 + 2i) + (6 - 4i)$ (B) $(0 + 0i) + (7 - 5i)$
(C) $(3 - 5i) - (1 - 3i)$ (D) $(-4 + 9i) + (4 - 9i)$

Not all graphing calculators make use of the $a + bi$ notation for complex numbers. For example, on the TI-86, the complex number $a + bi$ is entered as the ordered pair (a, b). Figure 4 shows the solution to parts A and B of Example 2 on a TI-86.



› **Figure 4** Complex number arithmetic on a TI-86.

Example 2, part B, illustrates the following general result: For any complex number $a + bi$,

$$(a + bi) + (0 + 0i) = (0 + 0i) + (a + bi) = a + bi$$

Thus, $0 + 0i$ is the **additive identity** or **zero** for the complex numbers. We anticipated this result in Definition 2 when we identified the complex number $0 + 0i$ with the real number 0.

Example 2, part D, illustrates a different result: In general, the **additive inverse** or **negative** of $a + bi$ is $-a - bi$ because

$$(a + bi) + (-a - bi) = (-a - bi) + (a + bi) = 0$$

Now we turn our attention to multiplication. Just like addition and subtraction, **multiplication of complex numbers** can be carried out by treating $a + bi$ in the same way we treat the algebraic expression $a + bx$. The key difference is that we replace i^2 with -1 each time it occurs.

EXAMPLE 4 Multiplying Complex Numbers

Carry out each operation and express the answer in standard form:

(A) $(2 - 3i)(6 + 2i)$ (B) $1(3 - 5i)$

(C) $i(1 + i)$ (D) $(3 + 4i)(3 - 4i)$

SOLUTIONS

Algebraic Solutions

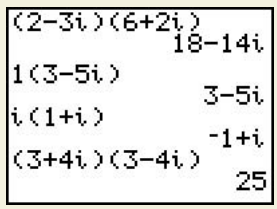
(A) $(2 - 3i)(6 + 2i) = 12 + 4i - 18i - 6i^2$ Combine like terms; replace i^2 with -1.
$= 12 - 14i - 6(-1)$
$= 18 - 14i$

(B) $1(3 - 5i)$ $= 1 \cdot 3 - 1 \cdot 5i$ $= 3 - 5i$

(C) $i(1 + i) = i + i^2 = i - 1 = -1 + i$

(D) $(3 + 4i)(3 - 4i) = 9 - 12i + 12i - 16i^2$ $-16i^2 = -16(-1) = 16$
$= 9 + 16 = 25$

Graphical Solutions



› **Figure 5**

MATCHED PROBLEM 4

Carry out each operation and express the answer in standard form:

(A) $(5 + 2i)(4 - 3i)$
(B) $3(-2 + 6i)$
(C) $i(2 - 3i)$
(D) $(2 + 3i)(2 - 3i)$

Notice that for any complex number $a + bi$,

$$1(a + bi) = (a + bi)1 = a + bi$$

(see Example 4, part B). This tells us that 1 is the **multiplicative identity** for complex numbers, just as it is for real numbers.

Part D of Example 4 illustrates an important property of the conjugate of a complex number.

› THEOREM 1 Product of a Complex Number and Its Conjugate

$$(a + bi)(a - bi) = a^2 + b^2 \quad \text{A real number}$$

You will be asked to prove this theorem in the exercises. Theorem 1 comes in very handy in finding reciprocals and **dividing complex numbers.**

Earlier we stated that every nonzero complex number has a multiplicative inverse or reciprocal. We will denote this as a fraction, just as we do with real numbers. The **reciprocal** of $a + bi$ is

$$\frac{1}{a + bi} \text{ (for } a + bi \neq 0).$$

As before, when dividing we can manipulate $a + bi$ in the same way we manipulate the real binomial form $a + bx$, except we replace i^2 with -1 each time it occurs. The trick to dividing and writing the result in standard form is to multiply numerator and denominator by the conjugate of the denominator.

EXAMPLE 5 Reciprocals and Quotients

Carry out each operation and express the answer in standard form:

(A) $\dfrac{1}{2+3i}$ (B) $\dfrac{7-3i}{1+i}$

SOLUTIONS

Algebraic Solutions

(A) Multiply numerator and denominator by the conjugate of the denominator:

$$\frac{1}{2+3i}=\frac{1}{2+3i}\cdot\frac{2-3i}{2-3i}=\frac{2-3i}{4-9i^2}=\frac{2-3i}{4+9}$$

$$=\frac{2-3i}{13}$$ Write in standard form.

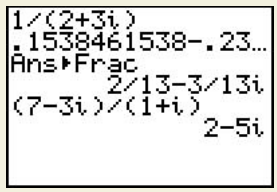
$$=\frac{2}{13}-\frac{3}{13}i$$

(B) $$\frac{7-3i}{1+i}=\frac{7-3i}{1+i}\cdot\frac{1-i}{1-i}=\frac{7-7i-3i+3i^2}{1-i^2}$$

$$=\frac{4-10i}{2}$$ Write in standard form.

$$=2-5i$$

Graphical Solutions



› Figure 6

In Figure 6, note that we used the FRACTION command to convert the decimal form to the fraction form.

MATCHED PROBLEM 5

Carry out each operation and express the answer in standard form:

(A) $\dfrac{1}{4+2i}$ (B) $\dfrac{6+7i}{2-i}$

EXAMPLE 6 Combined Operations

Carry out the indicated operations and write each answer in standard form:

(A) $(3-2i)^2-6(3-2i)+13$ (B) $\dfrac{2-3i}{2i}$

SOLUTIONS

Algebraic Solutions

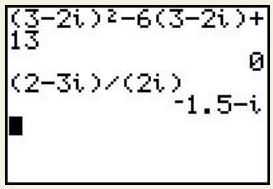
(A) $(3 - 2i)^2 - 6(3 - 2i) + 13$

$$= 9 - 12i + 4i^2 - 18 + 12i + 13$$
$$= 9 - 12i - 4 - 18 + 12i + 13$$
$$= 0$$

(B) If a complex number is divided by a pure imaginary number, we can make the denominator real by multiplying numerator and denominator by i. (We could also multiply by the conjugate of $2i$, which is $-2i$.)

$$\frac{2 - 3i}{2i} \cdot \frac{i}{i} = \frac{2i - 3i^2}{2i^2} = \frac{2i + 3}{-2} = -\frac{3}{2} - i$$

Graphical Solutions



› Figure 7

MATCHED PROBLEM 6

Carry out the indicated operations and write each answer in standard form:

(A) $(3 + 2i)^2 - 6(3 + 2i) + 13$ (B) $\dfrac{4 - i}{3i}$

› Complex Numbers and Radicals

Recall that we say that a is a square root of b if $a^2 = b$. If x is a positive real number, then x has two square roots, the principal square root, denoted by $\sqrt{x}$, and its negative, $-\sqrt{x}$. If x is a negative real number, we would previously have said that x didn't have any square roots. Now we can find two square roots for negative real numbers: They will both be imaginary numbers.

››› EXPLORE-DISCUSS 3

(A) Find the square of each expression.

$(i\sqrt{3})^2 =$ _____ $(5i)^2 =$ _____ $(i\sqrt{11})^2 =$ _____

$(-i\sqrt{3})^2 =$ _____ $(-5i)^2 =$ _____ $(-i\sqrt{11})^2 =$ _____

(B) What are the two square roots of -3? Of -5? Of -11?

› DEFINITION 4 Principal Square Root of a Negative Real Number

The **principal square root of a negative real number,** denoted by $\sqrt{-a}$, where a is positive, is defined by

$$\sqrt{-a} = i\sqrt{a} \qquad \sqrt{-3} = i\sqrt{3} \qquad \sqrt{-9} = i\sqrt{9} = 3i$$

The other square root of $-a$, $a > 0$, is $-\sqrt{-a} = -i\sqrt{a}$.

Note in Definition 4 that we wrote $i\sqrt{a}$ and $i\sqrt{3}$ in place of the standard forms $\sqrt{a}i$ and $\sqrt{3}i$. We follow this convention to avoid confusion over whether the i should or should not be under the radical. (Notice that $\sqrt{3}i$ and $\sqrt{3i}$ look a lot alike, but are not the same number.)

EXAMPLE 7 Complex Numbers and Radicals

Write in standard form:

(A) $\sqrt{-4}$ (B) $4 + \sqrt{-5}$ (C) $\dfrac{-3 - \sqrt{-5}}{2}$ (D) $\dfrac{1}{1 - \sqrt{-9}}$

SOLUTIONS

Algebraic Solutions

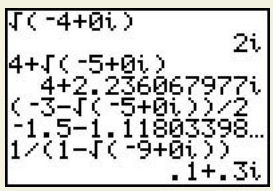
(A) $\sqrt{-4} = i\sqrt{4} = 2i$

(B) $4 + \sqrt{-5} = 4 + i\sqrt{5}$

(C) $$\frac{-3 - \sqrt{-5}}{2} = \frac{-3 - i\sqrt{5}}{2} = -\frac{3}{2} - \frac{\sqrt{5}}{2}i$$

(D) $$\frac{1}{1 - \sqrt{-9}} = \frac{1}{1 - 3i} = \frac{1 \cdot (1 + 3i)}{(1 - 3i) \cdot (1 + 3i)}$$
$$= \frac{1 + 3i}{1 - 9i^2} = \frac{1 + 3i}{10}$$
$$= \frac{1}{10} + \frac{3}{10}i$$

Graphical Solutions



› Figure 8

Note that principal square roots like $\sqrt{-4}$ must be entered as $\sqrt{-4 + 0i}$ to indicate that we want to perform complex arithmetic rather than real arithmetic (see Fig. 8). Press the right arrow key to display the i in the answer to part C.

MATCHED PROBLEM 7

Write in standard form:

(A) $\sqrt{-16}$ (B) $5 + \sqrt{-7}$ (C) $\dfrac{-5 - \sqrt{-2}}{2}$ (D) $\dfrac{1}{3 - \sqrt{-4}}$

››› EXPLORE-DISCUSS 4

From basic algebra, we know that if a and b are positive real numbers, then

$$\sqrt{a}\sqrt{b} = \sqrt{ab} \tag{3}$$

So, we can evaluate expressions like $\sqrt{9}\ \sqrt{4}$ two ways:

$$\sqrt{9}\ \sqrt{4} = \sqrt{(9)(4)} = \sqrt{36} = 6 \quad \text{and} \quad \sqrt{9}\ \sqrt{4} = (3)(2) = 6$$

Evaluate each of the following in two ways. Is equation (3) a valid property to use in all cases?

(A) $\sqrt{9}\ \sqrt{-4}$ **(B)** $\sqrt{-9}\ \sqrt{-4}$ **(C)** $\sqrt{-9}\ \sqrt{-4}$

››› CAUTION ›››

Note that in Example 7, part D, we wrote $1 - \sqrt{-9} = 1 - 3i$ before proceeding with the simplification. This is a necessary step because some of the properties of radicals that are true for real numbers turn out not to be true for complex numbers. In particular, for positive real numbers a and b,

$$\sqrt{a}\sqrt{b} = \sqrt{ab} \quad \text{but} \quad \sqrt{-a}\ \sqrt{-b} \neq \sqrt{(-a)(-b)}$$

(See Explore-Discuss 4.)

To avoid having to worry about this, always convert expressions of the form $\sqrt{-a}$ to the equivalent form in terms of i before performing any operations.

› Solving Equations Involving Complex Numbers

EXAMPLE 8 Solving Equations Involving Complex Numbers

(A) Solve for real numbers x and y:

$$(3x + 2) + (2y - 4)i = -4 + 6i$$

(B) Solve for complex number z:

$$(3 + 2i)z - 3 + 6i = 8 - 4i$$

SOLUTIONS

(A) This equation is really a statement that two complex numbers are equal: $(3x + 2) + (2y - 4)i$, and $-4 + 6i$. In order for these numbers to be equal, the real parts must be the same, and the imaginary parts must be the same as well.

$$\begin{aligned} 3x + 2 &= -4 & \qquad 2y - 4 &= 6 \\ 3x &= -6 & \qquad 2y &= 10 \\ x &= -2 & \qquad y &= 5 \end{aligned}$$

(B) Solve for z, then write the answer in standard form.

$$\begin{aligned} (3 + 2i)z - 3 + 6i &= 8 - 4i && \text{Add 3 and subtract } 6i \text{ from both sides.} \\ (3 + 2i)z &= 11 - 10i && \text{Divide both sides by } 3 + 2i. \\ z &= \frac{11 - 10i}{3 + 2i} && \text{Multiply numerator and denominator by the conjugate of the denominator.} \\ &= \frac{(11 - 10i)(3 - 2i)}{(3 + 2i)(3 - 2i)} && \text{Simplify.} \\ &= \frac{13 - 52i}{13} && \text{Write in standard form.} \\ &= 1 - 4i \end{aligned}$$

A check is left to the reader.

MATCHED PROBLEM 8

(A) Solve for real numbers x and y:

$$(2y - 7) + (3x + 4)i = 1 + i$$

(B) Solve for complex number z:

$$(1 + 3i)z + 4 - 5i = 3 + 2i$$

The truth is that the types of numbers we studied in this section weren't received very well when they were invented. (It could have been worse, though. According to legend, the first mathematician to accept irrational numbers was sentenced to drowning!) In fact, the names given to these numbers are indicative of this resistance to accept them: *complex* and *imaginary*. These are not exactly names that would be given to widely accepted ideas!

In spite of this early resistance, complex numbers have come into widespread use in both pure and applied mathematics. They are used extensively, for example, in electrical engineering, physics, chemistry, statistics, and aeronautical engineering. Our first use of them will be in connection with solutions of second-degree equations in the next section.

ANSWERS TO MATCHED PROBLEMS

1. (A) -49 (B) $-64i$ (C) 16

2. (A) Imaginary numbers: $6 + 7i$, $\sqrt{2} - \frac{1}{3}i$, $0 - i = -i$, $0 + \frac{2}{3}i = \frac{2}{3}i$
Pure imaginary numbers: $0 - i = -i$, $0 + \frac{2}{3}i = \frac{2}{3}i$
Real numbers: $-\sqrt{3} + 0i = -\sqrt{3}$, $0 - 0i = 0$
Zero: $0 - 0i = 0$
(B) $6 - 7i$, $\sqrt{2} + \frac{1}{3}i$, $0 + i = i$, $0 - \frac{2}{3}i = -\frac{2}{3}i$, $-\sqrt{3} - 0i = -\sqrt{3}$, $0 + 0i = 0$

3. (A) $9 - 2i$ (B) $7 - 5i$ (C) $2 - 2i$ (D) 0 **4.** (A) $26 - 7i$ (B) $-6 + 18i$
(C) $3 + 2i$ (D) 13 **5.** (A) $\frac{1}{5} - \frac{1}{10}i$ (B) $1 + 4i$ **6.** (A) 0 (B) $-\frac{1}{3} - \frac{4}{3}i$

7. (A) $4i$ (B) $5 + i\sqrt{7}$ (C) $-\frac{5}{2} - (\sqrt{2}/2)i$ (D) $\frac{3}{13} + \frac{2}{13}i$ **8.** (A) $x = -1$, $y = 4$
(B) $z = 2 + i$

2-4 Exercises

1. Do negative real numbers have square roots? Explain.

2. Arrange the following sets of numbers so that each one contains the one that comes before it in the list: rational numbers, complex numbers, integers, real numbers, natural numbers

3. Is it possible to square an imaginary number and get a real number? Explain.

4. What is the conjugate of a complex number? How do we use conjugates?

5. Which statement is false, and which is true? Justify your response.
(A) Every real number is a complex number
(B) Every complex number is a real number

6. Is it possible to add a real number and an imaginary number? If so, what kind of number is the result?

In Problems 7–12, classify each number into one or more of the following types: imaginary, pure imaginary, real, complex.

7. $2 + 5i$ **8.** $-3 + 0i$ **9.** $0 + 7i$

10. $14 - \frac{1}{2}i$ **11.** $100 + 0i$ **12.** $0 - \pi i$

In Problems 13–46, perform the indicated operations and write each answer in standard form.

13. i^2 **14.** i^3 **15.** $(4i)^2$ **16.** $(11i)^2$

17. $(-i\sqrt{3})^2$ **18.** $(-i\sqrt{17})^2$

19. $(4i)(6i)$ **20.** $(3i)(8i)$

21. $(2 + 4i) + (5 + i)$ **22.** $(3 + i) + (4 + 2i)$

23. $(-2 + 6i) + (7 - 3i)$ **24.** $(6 - 2i) + (8 - 3i)$

25. $(6 + 7i) - (4 + 3i)$ **26.** $(9 + 8i) - (5 + 6i)$

27. $(3 + 5i) - (-2 - 4i)$ **28.** $(8 - 4i) - (11 - 2i)$

29. $(4 - 5i) + 2i$ **30.** $6 + (3 - 4i)$

31. $-3i(2 - 4i)$ **32.** $-2i(5 - 3i)$

33. $(3 + 3i)(2 - 3i)$ **34.** $(-2 - 3i)(3 - 5i)$

35. $(2 - 3i)(7 - 6i)$ **36.** $(3 + 2i)(2 - i)$

37. $(7 + 4i)(7 - 4i)$ **38.** $(5 + 3i)(5 - 3i)$

39. $(4 - 3i)^2$ **40.** $(-2 + 8i)^2$

41. $\dfrac{1}{2 + i}$ **42.** $\dfrac{1}{3 - i}$ **43.** $\dfrac{3 + i}{2 - 3i}$

44. $\dfrac{2 - i}{3 + 2i}$ **45.** $\dfrac{13 + i}{2 - i}$ **46.** $\dfrac{15 - 3i}{2 - 3i}$

In Problems 47–54, evaluate and express results in standard form.

47. $\sqrt{2}\sqrt{8}$ **48.** $\sqrt{3}\sqrt{12}$ **49.** $\sqrt{2}\sqrt{-8}$

50. $\sqrt{-3}\sqrt{12}$ **51.** $\sqrt{-2}\sqrt{8}$ **52.** $\sqrt{3}\sqrt{-12}$

53. $\sqrt{-2}\sqrt{-8}$ **54.** $\sqrt{-3}\sqrt{-12}$

In Problems 55–64, convert imaginary numbers to standard form, perform the indicated operations, and express answers in standard form.

55. $(2 - \sqrt{-4}) + (5 - \sqrt{-9})$

56. $(3 - \sqrt{-4}) + (-8 + \sqrt{-25})$

57. $(9 - \sqrt{-9}) - (12 - \sqrt{-25})$

58. $(-2 - \sqrt{-36}) - (4 + \sqrt{-49})$

59. $(3 - \sqrt{-4})(-2 + \sqrt{-49})$

60. $(2 - \sqrt{-1})(5 + \sqrt{-9})$

61. $\dfrac{5 - \sqrt{-4}}{7}$ **62.** $\dfrac{6 - \sqrt{-64}}{2}$

63. $\dfrac{1}{2 - \sqrt{-9}}$ **64.** $\dfrac{1}{3 - \sqrt{-16}}$

Write Problems 65–70 in standard form.

65. $\dfrac{2}{5i}$ **66.** $\dfrac{1}{3i}$ **67.** $\dfrac{1 + 3i}{2i}$ **68.** $\dfrac{2 - i}{3i}$

69. $(2 - 3i)^2 - 2(2 - 3i) + 9$

70. $(2 - i)^2 + 3(2 - i) - 5$

71. Let $f(x) = x^2 - 2x + 2$.
(A) Show that the conjugate complex numbers $1 + i$ and $1 - i$ are both zeros of f.
(B) Does f have any real zeros? Any x intercepts? Explain.

72. Let $g(x) = -x^2 + 4x - 5$.
(A) Show that the conjugate complex numbers $2 + i$ and $2 - i$ are both zeros of g.
(B) Does g have any real zeros? Any x intercepts? Explain.

73. Simplify: i^{18}, i^{32}, and i^{67}.

74. Simplify: i^{21}, i^{43}, and i^{52}.

In Problems 75–78, solve for x and y.

75. $(2x - 1) + (3y + 2)i = 5 - 4i$

76. $3x + (y - 2)i = (5 - 2x) + (3y - 8)i$

77. $\dfrac{(1 + x) + (y - 2)i}{1 + i} = 2 - i$

78. $\dfrac{(2 + x) + (y + 3)i}{1 - i} = -3 + i$

In Problems 79–82, solve for z. Express answers in standard form.

79. $(2 + i)z + i = 4i$

80. $(3 - i)z + 2 = i$

81. $3iz + (2 - 4i) = (1 + 2i)z - 3i$

82. $(2 - i)z + (1 - 4i) = (-1 + 3i)z + (4 + 2i)$

83. Show that $2 - i$ and $-2 + i$ are square roots of $3 - 4i$.

84. Show that $-3 + 2i$ and $3 - 2i$ are square roots of $5 - 12i$.

85. Explain what is wrong with the following "proof" that $-1 = 1$:

$$-1 = i^2 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$$

86. Explain what is wrong with the following "proof" that $1/i = i$. What is the correct value of $1/i$?

$$\frac{1}{i} = \frac{1}{\sqrt{-1}} = \frac{\sqrt{1}}{\sqrt{-1}} = \sqrt{\frac{1}{-1}} = \sqrt{-1} = i$$

87. Prove that the product of any complex number and its conjugate is a real number by multiplying a generic complex number $a + bi$ by its conjugate.

88. Is the sum of a complex number and its conjugate always a real number? What about the difference? The quotient?

In Problems 89–94, perform the indicated operations, and write each answer in standard form.

89. $(a + bi) + (c + di)$ **90.** $(a + bi) - (c + di)$

91. $(a + bi)(a - bi)$ **92.** $(u - vi)(u + vi)$

93. $(a + bi)(c + di)$ **94.** $\dfrac{a + bi}{c + di}$

95. Show that $i^{4k} = 1$, for any natural number k.

96. Show that $i^{4k-1} = -i$, for any natural number k.

97. Let $S_n = i + i^2 + i^3 + \cdots + i^n$, $n \geq 1$. Describe the possible values of S_n.

98. Let $T_n = i^2 + i^4 + i^6 + \cdots + i^{2n}$, $n \geq 1$. Describe the possible values of T_n.

Supply the reasons in the proofs for the theorems stated in Problems 99 and 100.

99. *Theorem:* The complex numbers are commutative under addition.

Proof: Let $a + bi$ and $c + di$ be two arbitrary complex numbers; then,

Statement		Reason
1. $(a + bi) + (c + di)$	$= (a + c) + (b + d)i$	**1.**
2.	$= (c + a) + (d + b)i$	**2.**
3.	$= (c + di) + (a + bi)$	**3.**

100. *Theorem:* The complex numbers are commutative under multiplication.

Proof: Let $a + bi$ and $c + di$ be two arbitrary complex numbers; then,

Statement		Reason
1. $(a + bi) \cdot (c + di)$	$= (ac - bd) + (ad + bc)i$	**1.**
2.	$= (ca - db) + (da + cb)i$	**2.**
3.	$= (c + di) \cdot (a + bi)$	**3.**

2-5 Quadratic Equations and Models

- Solving by Factoring
- Solving by Completing the Square
- Solving Using the Quadratic Formula
- Mathematical Modeling
- Data Analysis and Regression

Now that we are familiar with quadratic functions, we turn our attention to quadratic equations. A **quadratic equation** is an equation that can be written in the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers, and a is not zero. (This is called the **general form** for a quadratic equation.) In this book, we are mostly interested in functions that have real-number domains and ranges. But we saw in the last section that some equations don't have solutions unless we expand our thinking to consider imaginary numbers.

Back in Chapter 1, we defined a *zero* of a function $f(x)$ as any real number solution, or *root,* of the equation $f(x) = 0$. To fully understand the zeros of a function, or the roots of an equation, we will need to extend these definitions to allow for complex zeros and roots. We already know that real zeros of a function are x intercepts of its graph. But this does not extend to imaginary zeros: they are never x intercepts.

››› EXPLORE-DISCUSS 1

Match the zeros of each function on the left with one of the sets A, B, or C on the right:

Function	Zeros
$f(x) = x^2 - 1$	$A = \{1\}$
$g(x) = x^2 + 1$	$B = \{-1, 1\}$
$h(x) = (x - 1)^2$	$C = \{-i, i\}$

Which of these sets of zeros can be found using graphical approximation techniques? Which cannot? Why?

A graphing calculator can be used to approximate the real roots of an equation, but not the imaginary roots; they simply don't appear on the graph. So in this section, we will focus on algebraic techniques for finding the exact value of the roots of a quadratic equation. In some cases, the roots will be real numbers, and in others they

will be imaginary numbers. In the process, we will derive the *quadratic formula,* another essential tool for our mathematical toolbox.

Solving by Factoring

Throughout our remaining study of solving equations, factoring plays a tremendously important role. Simply put, it's unlikely that you will be successful in the remainder of this course unless you are comfortable with the basic techniques of factoring. We strongly recommend that you review factoring in Appendix A, Section A-5, before moving on.

There is one single reason why factoring is so important in solving equations. It's called the **zero product property.**

EXPLORE-DISCUSS 2

(A) Write down a pair of numbers whose product is zero. Is one of them zero? Can you think of two nonzero numbers whose product is zero?

(B) Choose any number other than zero and call it a. Write down a pair of numbers whose product is a. Is one of them a? Can you think of a pair, neither of which is a, whose product is a?

ZERO PRODUCT PROPERTY

If m and n are complex numbers, then

$$m \cdot n = 0 \quad \text{if and only if} \quad m = 0 \text{ or } n = 0 \text{ (or both)}$$

It is very helpful to think about what this says in words: If the product of two factors is zero, then *at least one of those factors has to be zero*. It's also helpful to observe that *zero is the only number for which this is true*.

EXAMPLE 1 Solving Quadratic Equations by Factoring

Solve by factoring:

(A) $(x - 5)(x + 3) = 0$

(B) $6x^2 - 19x - 7 = 0$

(C) $x^2 - 6x + 5 = -4$

(D) $2x^2 = 3x$

SOLUTIONS

(A) The product of two factors is zero, so by the zero product property, one of the two must be zero. This enables us to write two easier equations to solve.

$$(x - 5)(x + 3) = 0$$
$$x - 5 = 0 \quad \text{or} \quad x + 3 = 0$$
$$x = 5 \qquad\qquad x = -3$$

The solution set is $\{-3, 5\}$.

(B) $6x^2 - 19x - 7 = 0$ Factor the left side.

$(2x - 7)(3x + 1) = 0$ Use the zero product property.

$2x - 7 = 0$ or $3x + 1 = 0$

$x = \frac{7}{2}$ $\qquad x = -\frac{1}{3}$

The solution set is $\{-\frac{1}{3}, \frac{7}{2}\}$.

(C) $x^2 - 6x + 5 = -4$ Add 4 to both sides.

$x^2 - 6x + 9 = 0$ Factor left side.

$(x - 3)(x - 3) = 0$ Use the zero product property.

$x - 3 = 0$

$x = 3$

The solution set is $\{3\}$. The equation has one root, 3. But because it came from two factors, we call 3 a *double root* or a *root of multiplicity 2*.

(D) $2x^2 = 3x$ Subtract $3x$ from both sides.

$2x^2 - 3x = 0$ Factor the left side.

$x(2x - 3) = 0$ Use the zero product property.

$x = 0$ or $2x - 3 = 0$

$x = \frac{3}{2}$

Solution set: $\{0, \frac{3}{2}\}$

MATCHED PROBLEM 1

Solve by factoring:

(A) $(2x + 4)(x - 7) = 0$ (B) $3x^2 + 7x - 20 = 0$

(C) $4x^2 + 12x + 9 = 0$ (D) $4x^2 = 5x$

››› CAUTION ›››

1. One side of an equation must be 0 before the zero product property can be applied. So

$$x^2 - 6x + 5 = -4$$
$$(x - 1)(x - 5) = -4$$

does not imply that $x - 1 = -4$ or $x - 5 = -4$. See Example 1, part C, for the correct solution of this equation.

2. The equations

$$2x^2 = 3x \qquad \text{and} \qquad 2x = 3$$

are not equivalent. The first has solution set $\{0, \frac{3}{2}\}$, but the second has solution set $\{\frac{3}{2}\}$. The root $x = 0$ is lost when each member of the first equation is divided by the variable x. See Example 1, part D, for the correct solution of this equation.

Never divide both sides of an equation by an expression containing the variable for which you are solving. You may be dividing by 0, which of course is not allowed.

REMARK: It is common practice to represent solutions of quadratic equations informally by the last equation (such as $x = 3$) rather than by writing a solution set using set notation (see Example 1). From now on, we will follow this practice unless a particular emphasis is necessary.

› Solving by Completing the Square

Factoring is a very efficient method for solving equations when the factors can be quickly identified. But often, that is not the case.* We next turn to an approach that will be able to solve any quadratic equation. But first, we must become acquainted with a simple method for solving certain quadratic equations.

› SQUARE ROOT PROPERTY

If r is a complex number, s is a real number, and $r^2 = s$, then $r = \pm\sqrt{s}$.

*As we will see in Chapter 3, every quadratic expression *can* be factored, but doing so directly is often extremely difficult.

EXAMPLE 2 **Solving Using the Square Root Property**

Solve using the square root property:

(A) $(2x - 3)^2 = 9$ (B) $(x + 5)^2 = -3$ (C) $\left(x + \frac{1}{2}\right)^2 - \frac{5}{4} = 0$

SOLUTIONS

(A) According to the square root property, if the square of $2x - 3$ is 9, then $2x - 3$ is either $\sqrt{9}$ or $-\sqrt{9}$.

$$\begin{aligned} (2x - 3)^2 &= 9 && \text{Apply the square root property.} \\ 2x - 3 &= \pm 3 && \text{Add 3 to both sides.} \\ 2x &= 3 + 3 \quad \text{or} \quad 2x = 3 - 3 \\ 2x &= 6 \quad \text{or} \quad 2x = 0 \\ x &= 3 \quad\quad\quad\;\; x = 0 \end{aligned}$$

(B) $$\begin{aligned} (x + 5)^2 &= -3 && \text{Apply the square root property.} \\ x + 5 &= \pm\sqrt{-3} && \text{Simplify } \sqrt{-3}. \\ x + 5 &= \pm i\sqrt{3} && \text{Subtract 5 from both sides.} \\ x &= -5 \pm i\sqrt{3} \end{aligned}$$

(C) $$\begin{aligned} \left(x + \frac{1}{2}\right)^2 - \frac{5}{4} &= 0 && \text{Add } \tfrac{5}{4} \text{ to both sides.} \\ \left(x + \frac{1}{2}\right)^2 &= \frac{5}{4} && \text{Apply the square root property.} \\ x + \frac{1}{2} &= \pm\sqrt{\frac{5}{4}} && \text{Subtract } \tfrac{1}{2} \text{ from both sides; simplify } \sqrt{\tfrac{5}{4}}. \\ x &= -\frac{1}{2} \pm \frac{\sqrt{5}}{2} \\ x &= \frac{-1 \pm \sqrt{5}}{2} \end{aligned}$$

MATCHED PROBLEM 2

Solve using the square root property:

(A) $(7 - 5x)^2 = 25$ (B) $(x - 7)^2 = -6$ (C) $\left(\frac{1}{2}x - 4\right)^2 - 10 = 0$

››› CAUTION ›››

1. Before applying the square root property, make sure to isolate the squared expression.
2. The single most common mistake that students make in solving quadratic equations is forgetting the "±" when applying the square root property.

Now we will see how completing the square will enable us to rewrite quadratic equations of the form $ax^2 + bx + c = 0$ into a form that can be solved using the square root property.

EXAMPLE 3 Solving by Completing the Square

Use completing the square and the square root property to solve each of the following:

(A) $x^2 + 6x - 2 = 0$ (B) $2x^2 - 4x + 3 = 0$

SOLUTIONS

(A) We can speed up the process of completing the square by taking advantage of the fact that we are working with a quadratic equation, not a quadratic expression.

$$x^2 + 6x - 2 = 0$$ Isolate the squared and first power terms.

$$x^2 + 6x = 2$$ $\left(\frac{1}{2} \cdot 6\right)^2 = 9$; Add 9 to both sides to complete the square.

$$x^2 + 6x + 9 = 2 + 9$$ Factor the left side.

$$(x + 3)^2 = 11$$ Use the square root property.

$$x + 3 = \pm\sqrt{11}$$ Subtract 3 from both sides.

$$x = -3 \pm \sqrt{11}$$

(B) $2x^2 - 4x + 3 = 0$ Divide both sides by 2 to make 1 the coefficient of x^2.

$$x^2 - 2x + \frac{3}{2} = 0$$ Isolate the squared and first power terms.

$$x^2 - 2x = -\frac{3}{2}$$ $\left(\frac{1}{2} \cdot (-2)\right)^2 = 1$; Add 1 to both sides to complete the square.

$$x^2 - 2x + 1 = -\frac{3}{2} + 1$$ Factor the left side.

$$(x - 1)^2 = -\frac{1}{2}$$ Use the square root property.

$$x - 1 = \pm\sqrt{-\frac{1}{2}}$$ Add 1 to both sides; simplify $\sqrt{-\frac{1}{2}}$.

$$x = 1 \pm i\sqrt{\frac{1}{2}}$$ Rationalize the denominator.

$$= 1 \pm \frac{\sqrt{2}}{2}i$$ Answer in $a + bi$ form.

MATCHED PROBLEM 3

Solve by completing the square:

(A) $x^2 + 8x - 3 = 0$ (B) $3x^2 - 12x + 13 = 0$

››› CAUTION ›››

Do not confuse completing the square in a quadratic function with completing the square in a quadratic equation. For functions, we add and subtract $\frac{b^2}{4}$ on the same side. For equations, we add $\frac{b^2}{4}$ to both sides of the equation.

Function	Equation
$f(x) = x^2 - 6x - 4$	$x^2 - 6x - 4 = 0$
$= x^2 - 6x + 9 - 9 - 4$	$x^2 - 6x + 9 = 4 + 9$
$= (x - 3)^2 - 13$	$(x - 3)^2 = 13$

››› EXPLORE-DISCUSS 3

Graph the quadratic functions associated with the two quadratic equations in Example 3. Approximate the x intercepts of each function and compare with the roots found in Example 3. Which of these equations has roots that cannot be approximated graphically?

› Solving Using the Quadratic Formula

The value of completing the square as a method of solving is that it works for *every* quadratic equation. But it is a bit cumbersome to use on a regular basis. Fortunately, the fact that it works for any quadratic equation leads to a very clever idea: We will try to reproduce the process of completing the square on the general quadratic equation $ax^2 + bx + c = 0$ (recall that $a \neq 0$). If all goes well, we will have solved *every* quadratic equation all at once!

Note that the steps we follow are exactly the same as the steps used in Example 3, part B.

$$ax^2 + bx + c = 0$$

Divide both sides by a to make 1 the coefficient of x^2.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Isolate the squared and first power terms.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$(\frac{1}{2} \cdot \frac{b}{a})^2 = \frac{b^2}{4a^2}$; add $\frac{b^2}{4a^2}$ to both sides.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

Factor the left side as (x + half the coefficient of x)^2; subtract on the right side.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Use the square root property.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Subtract $\frac{b}{2a}$ from both sides.

$$x = \frac{-b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$ Simplify the radical. (See Problem 93 in Exercises 2-5.)

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$ Write as a single fraction.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We were successful in solving, and the result is a formula that provides the solution of any quadratic equation in standard form. This is known as the **quadratic formula.**

› THEOREM 1 Quadratic Formula

If $ax^2 + bx + c = 0, a \neq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula and completing the square are equivalent methods. Either can be used to find the exact value of the roots of any quadratic equation, although in most cases the quadratic formula is easier to use.

EXAMPLE 4 Using the Quadratic Formula

Solve $2x + \frac{3}{2} = x^2$ using the quadratic formula. Leave the answer in simplest radical form.

SOLUTION

$$2x + \tfrac{3}{2} = x^2$$ Multiply both sides by 2 to clear fractions.

$$4x + 3 = 2x^2$$ Write in standard form.

$$2x^2 - 4x - 3 = 0$$ Use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Substitute $a = 2, b = -4, c = -3$.

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$$

$$= \frac{4 \pm \sqrt{40}}{4} = \frac{4 \pm 2\sqrt{10}}{4} = \frac{2 \pm \sqrt{10}}{2}$$

››› CAUTION ›››

There are a number of common mistakes to watch out for when using the quadratic formula.

1. Make sure the equation is written in the form $ax^2 + bx + c = 0$ before deciding on values for a, b, and c.
2. $-4^2 \neq (-4)^2$ $\quad -4^2 = -16$ and $(-4)^2 = 16$
3. $2 + \dfrac{\sqrt{10}}{2} \neq \dfrac{2 + \sqrt{10}}{2}$ $\quad 2 + \dfrac{\sqrt{10}}{2} = \dfrac{4 + \sqrt{10}}{2}$
4. $\dfrac{\cancel{4} \pm 2\sqrt{10}}{\cancel{4}} \neq \pm 2\sqrt{10}$ $\quad \dfrac{4 \pm 2\sqrt{10}}{4} = \dfrac{2(2 \pm \sqrt{10})}{4} = \dfrac{2 \pm \sqrt{10}}{2}$

MATCHED PROBLEM 4

Solve $x^2 - \frac{5}{2} = -3x$ using the quadratic formula. Leave the answer in simplest radical form.

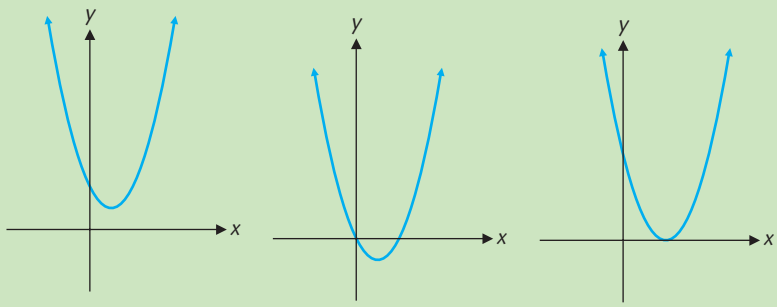
››› EXPLORE-DISCUSS 4

Given the quadratic function $f(x) = ax^2 + bx + c$, let $D = b^2 - 4ac$. How many real zeros does f have if

(A) $D > 0$ **(B)** $D = 0$ **(C)** $D < 0$

In each of these three cases, what type of roots does the quadratic equation $f(x) = 0$ have?

Match each of the three cases with one of the following graphs.



(1) (2) (3)

The quantity $b^2 - 4ac$ in the quadratic formula is called the **discriminant** and gives us information about the roots of the corresponding equation and the zeros of the associated quadratic function. This information is summarized in Table 1.

Table 1 Discriminants, Roots, and Zeros

Discriminant $b^2 - 4ac$	Roots of $ax^2 + bx + c = 0$*	Number of Real Zeros of $f(x) = ax^2 + bx + c$*
Positive	Two distinct real roots	2
0	One real root (a double root)	1
Negative	Two imaginary roots, one the conjugate of the other†	0

*a, b, and c are real numbers with $a \neq 0$.

†See Problem 94 in Exercises 2-5 for a justification.

› Mathematical Modeling

Now we will use our new skills in some applications that involve quadratic equations.

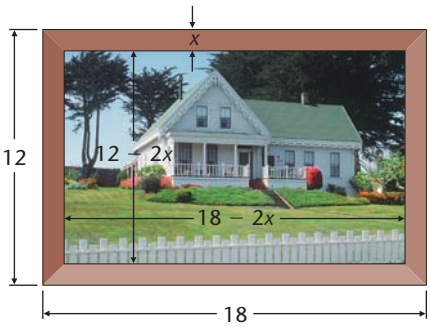
EXAMPLE 5 **Design**

A rectangular picture frame of uniform width has outer dimensions of 12 inches by 18 inches. How wide (to the nearest tenth of an inch) must the frame be to display an area of 140 square inches?

SOLUTION

Constructing the Model

We begin by drawing and labeling a figure:



The width of the frame can't be negative, so x must satisfy $x \geq 0$. The total height is 12 inches, so x has to be less than 6 or there won't be an opening, making the frame kind of pointless. The height and width of the opening are $12 - 2x$ and $18 - 2x$, so the display area is $(12 - 2x)(18 - 2x)$. For a total of 140 square inches, we need

$$(12 - 2x)(18 - 2x) = 140 \qquad 0 \leq x < 6$$

Algebraic Solution

$$(18 - 2x)(12 - 2x) = 140 \quad \text{Multiply parentheses.}$$
$$216 - 36x - 24x + 4x^2 = 140 \quad \text{Write in standard form.}$$
$$4x^2 - 60x + 76 = 0 \quad \text{Divide both sides by 4.}$$
$$x^2 - 15x + 19 = 0 \quad a = 1, b = -15, c = 19$$

$$x = \frac{15 \pm \sqrt{225 - 4(1)(19)}}{2(1)} = \frac{15 \pm \sqrt{149}}{2}$$

The quadratic equation has two solutions (rounded to one decimal place):

$$x = \frac{15 + \sqrt{149}}{2} = 13.6$$

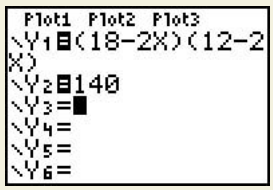
and

$$x = \frac{15 - \sqrt{149}}{2} = 1.4$$

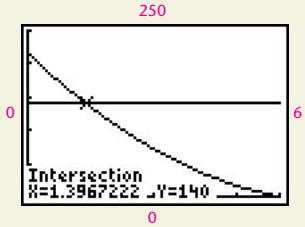
The first must be discarded because x must satisfy $x < 6$. So the width of the frame is 1.4 inches.

Graphical Solution

Entering $y_1 = (18 - 2x)(12 - 2x)$ and $y_2 = 140$ in the equation editor (Fig. 1) and using the INTERSECT command (Fig. 2) shows that the width of the frame is $x = 1.4$ inches.



› Figure 1



› Figure 2

Note: A graphical solution is sufficient in this setting since x is a length, and imaginary solutions wouldn't be of interest.

MATCHED PROBLEM 5

A poster promoting a concert is being designed to fit on 24- by 36-inch pieces of paper. If the printer requires a margin of uniform width around all four sides, how wide (to the nearest tenth of an inch) should the margin be if the artist requires 730 square inches of printable area?

› Data Analysis and Regression

Now that we have added quadratic functions to our mathematical toolbox, we can use this new tool in conjunction with another tool discussed previously—regression analysis. In Example 6, we use both of these tools to investigate the effect of recycling efforts on solid waste disposal.

EXAMPLE 6 Solid Waste Disposal

Franklin Associates, Ltd. of Prairie Village, Kansas, reported the data in Table 2 to the U.S. Environmental Protection Agency.

Table 2 Municipal Solid Waste Disposal

Year	Annual landfill disposal (millions of tons)	Per Person per Day (pounds)
1960	55.5	1.68
1970	88.2	2.37
1980	123.3	2.97
1990	131.6	2.90
1995	118.4	2.50
2000	113.6	2.16
2005	109.1	2.03

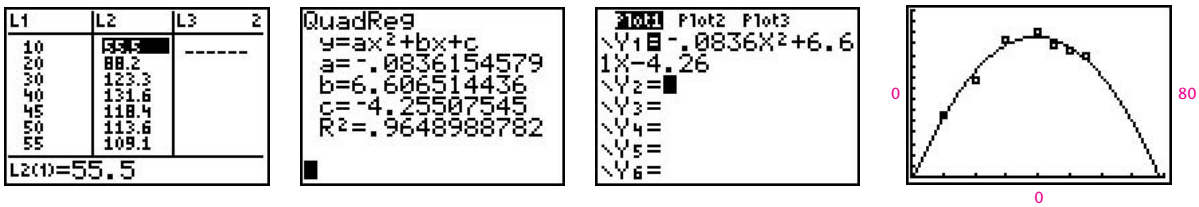
(A) Let x represent time in years with $x = 0$ corresponding to 1950, and let y represent the corresponding annual landfill disposal. Use regression analysis on a graphing calculator to find a quadratic function $y = ax^2 + bx + c$ that models these data. (Round the constants a, b, and c to three significant digits.)

(B) If landfill disposal continues to follow the trend exhibited in Table 2, when (to the nearest year) will the annual landfill disposal return to the 1960 level?

(C) Is it reasonable to expect the annual landfill disposal to follow this trend indefinitely? Explain.

SOLUTIONS

(A) The y values in the annual landfill disposal column increase from 1960 to 1990 but then begin to decrease, so a linear regression model will not fit the data well. But a quadratic model seems like a reasonable alternative. See Section 2-2 for a refresher on entering data and calculating a regression model. The only difference now is that we choose QUADREG from the CALC menu. Figure 3 shows the details of constructing the model on a graphing calculator.



(a) Data (b) Regression equation (c) Regression equation entered in equation editor (d) Graph of data and regression equation

› Figure 3

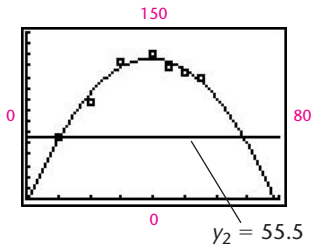
Rounding the constants to three significant digits, a quadratic regression equation for these data is

$$y_1 = -0.0836x^2 + 6.61x - 4.26$$

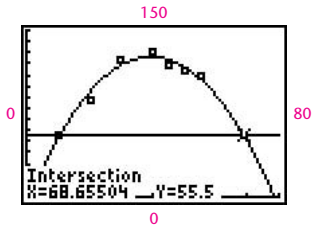
The graph in Figure 3(d) indicates that this is a reasonable model for these data.

(B) To determine when the annual landfill disposal returns to the 1960 level, we add the graph of $y_2 = 55.5$ to the graph (Fig. 4).

The graphs of y_1 and y_2 intersect twice, once at $x = 10$ (1960), and again at a later date. Using the INTERSECT command (Fig. 5) shows that the x coordinate of the second intersection point (to the nearest integer) is 69. Thus, the annual landfill disposal returns to the 1960 level of 55.5 million tons in 2019.



› Figure 4



› Figure 5

(C) The graph of y_1 continues to decrease and reaches 0 somewhere between $x = 78$ and 79 (2028 and 2029). It is highly unlikely that the annual landfall disposal will ever reach 0. As time goes by and more data become available, new models will have to be constructed to better predict future trends.

MATCHED PROBLEM 6

Refer to Table 2.

(A) Let x represent time in years with $x = 0$ corresponding to 1950, and let y represent the corresponding landfill disposal per person per day. Use regression analysis on a graphing calculator to find a quadratic function of the form $y = ax^2 + bx + c$ that models these data. (Round the constants a, b, and c to three significant digits.)

(B) If landfill disposal per person per day continues to follow the trend exhibited in Table 2, when (to the nearest year) will it fall below 1 pound per person per day?

(C) Is it reasonable to expect the landfill disposal per person per day to follow this trend indefinitely? Explain.

Most gasoline engines are more fuel efficient at a midrange speed than at either extremely high or extremely low speeds. Example 7 uses quadratic regression to determine the most economical speed for a speedboat.

EXAMPLE 7 Optimal Speed

Table 3

mph	mpg
4.6	3.07
7.3	3.17
21.0	6.77
29.8	6.62
40.2	2.77
44.6	2.37

Source: www.yamaha-motor.com

Table 3 contains performance data for a speedboat powered by a Yamaha outboard motor. In the work that follows, round all numbers to three significant digits.

(A) Let x be the speed of the boat in miles per hour (mph) and y the associated mileage in miles per gallon (mpg). Use the data in Table 3 to find a quadratic regression function $y = ax^2 + bx + c$ for this boat.

(B) A marina rents this boat for \$20 per hour plus the cost of the gasoline used. If gasoline costs \$2.30 per gallon and you take a 100-mile trip in this boat, construct a mathematical model and use it to answer the following questions:

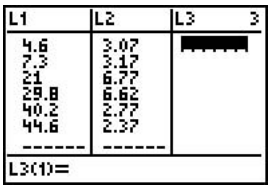
What speed should you travel to minimize the rental charges?

What mileage will the boat get?

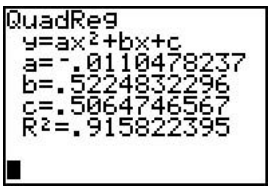
How long is the trip?

How much gasoline will you use?

How much will the trip cost you?



› Figure 6



› Figure 7

SOLUTIONS

(A) Entering the data in the statistics editor (Fig. 6) and selecting the QUADREG option (Fig. 7) produces the following quadratic function relating speed and mileage:

$$y = -0.0110x^2 + 0.522x + 0.506$$

(B) If t is the number of hours the boat is rented and g is the number of gallons of gasoline used, then the cost of the rental (in dollars) is

$$C = 20t + 2.3g \qquad \text{\$20 per hour + \$2.30 per gallon}$$

If x is the speed of the boat and y is the associated mileage, then

$$xt = 100 \qquad \text{rate} \cdot \text{time} = \text{distance} \qquad (1)$$

and

$$yg = 100 \qquad \text{(miles/gallon)(gallons)} = \text{distance} \qquad (2)$$

Solving equation (1) for t, we get

$$t = \frac{100}{x}$$

Next, we solve equation (2) for y, and use the quadratic regression function from part A to represent y.

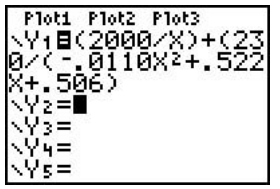
$$g = \frac{100}{y} = \frac{100}{-0.0110x^2 + 0.522x + 0.506}$$

Now that we have expressions for both gallons and hours in terms of x, we return to the cost equation,

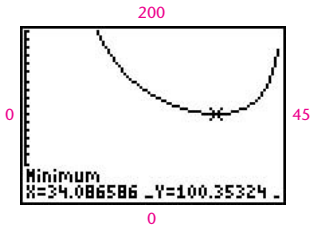
$$\begin{aligned} C &= 20t + 2.3g \\ &= 20\frac{100}{x} + 2.3\frac{100}{-0.0110x^2 + 0.522x + 0.506} \\ &= \frac{2{,}000}{x} + \frac{230}{-0.0110x^2 + 0.522x + 0.506} \end{aligned}$$

This is the mathematical model for the total cost of the trip. Our first objective is to find the speed x that produces the minimum cost. Entering the cost function in the equation editor (Fig. 8) and using the MINIMUM command (Fig. 9), we see that the minimum cost occurs when the boat speed is 34.1 miles per hour. Evaluating the quadratic function in part A at 34.1, we find that the corresponding mileage is $y = 5.52$ miles per gallon. The trip will take $\frac{1000}{34.1} = 2.93$ hours and consume $\frac{1000}{5.52} = 18.1$ gallons of gas. We can see from Figure 9 that the trip will cost \$100. To check this, we can compute the cost directly

$$C = \underset{\text{Rent}}{20(2.93)} + \underset{\text{plus}}{} \underset{\text{Gasoline}}{2.3(18.1)} = \$100 \text{ (to nearest dollar)}$$



› Figure 8



› Figure 9

Table 4

mph	mpg
4.8	2.67
8.9	2.12
18.5	2.89
34.4	3.78
43.8	3.40
48.6	2.63

Source: www.yamaha-motor.com

MATCHED PROBLEM 7

Table 4 contains performance data for a speedboat powered by a Yamaha outboard motor. In the work that follows, round all numbers to three significant digits.

(A) Let x be the speed of the boat in miles per hour (mph) and y the associated mileage in miles per gallon (mpg). Use the data in Table 4 to find a quadratic regression function $y = ax^2 + bx + c$ for this boat.

(B) A marina rents this boat for \$15 per hour plus the cost of the gasoline used. If gasoline costs \$2.50 per gallon and you take a 200-mile trip in this boat, construct a mathematical model and use it to answer the following questions:

What speed should you travel to minimize the rental charges?

What mileage will the boat get?

How long does the trip take?

How much gasoline will you use?

How much will the trip cost you?

ANSWERS TO MATCHED PROBLEMS

1. (A) $\{-2, 7\}$ (B) $\{-4, \frac{5}{3}\}$ (C) $\{-\frac{3}{2}\}$ (a double root) (D) $\{0, \frac{5}{4}\}$

2. (A) $x = \frac{2}{5}, \frac{12}{5}$ (B) $x = 7 \pm i\sqrt{6}$ (C) $x = 8 \pm 2\sqrt{10}$

3. (A) $x = -4 \pm \sqrt{19}$ (B) $x = (6 \pm i\sqrt{3})/3$ or $2 \pm (\sqrt{3}/3)i$

4. $x = (-3 \pm \sqrt{19})/2$ **5.** 1.2 inches

6. (A) $y = -0.00217x^2 + 0.146x + 0.412$ (B) 2013

7. (A) $y = -0.00176x^2 + 0.110x + 1.72$

(B) The boat should travel at 41.6 miles per hour. The mileage is 3.25 miles per gallon. The trip will take 4.81 hours and will consume 61.5 gallons of gasoline. The trip will cost $225.95.

2-5 Exercises

1. Explain what the zero product property says in your own words.

2. Explain what the square root property says in your own words.

3. If you could only use one of factoring, completing the square, and quadratic formula on an important test featuring a variety of quadratic equations, which would you choose, and why?

4. Does every quadratic equation have two solutions? Explain.

5. Explain why the real roots of the quadratic equation $ax^2 + bx + c = 0$ are equivalent to the x intercepts of the function $f(x) = ax^2 + bx + c$.

6. Explain why the imaginary roots of the $ax^2 + bx + c = 0$ do not correspond to x intercepts of the function $f(x) = ax^2 + bx + c$.

In Problems 7–16, solve by factoring.

7. $(x - 8)(2x + 3) = 0$ **8.** $(x + 4)(3x - 10) = 0$

9. $x^2 - 3x - 10 = 0$ **10.** $x^2 + 6x + 8 = 0$

11. $4u^2 = 8u$ **12.** $3A^2 = -12A$

13. $9y^2 = 12y - 4$ **14.** $16x^2 + 8x = -1$

15. $11x = 2x^2 + 12$ **16.** $8 - 10x = 3x^2$

In Problems 17–28, solve by completing the square.

17. $x^2 - 6x - 3 = 0$ **18.** $y^2 - 10y - 3 = 0$

19. $t^2 - 4t + 8 = 0$ **20.** $w^2 - 6w + 25 = 0$

21. $m^2 + 2m + 9 = 0$ **22.** $n^2 + 8n + 34 = 0$

23. $2d^2 + 5d - 25 = 0$ **24.** $2u^2 + 7u + 3 = 0$

25. $2v^2 - 2v + 1 = 0$ **26.** $9x^2 - 12x + 5 = 0$

27. $4y^2 + 3y + 9 = 0$ **28.** $5t^2 + 2t + 5 = 0$

In Problems 29–38, solve using the quadratic formula.

29. $x^2 - 10x - 3 = 0$ **30.** $x^2 - 6x - 3 = 0$

31. $x^2 + 8 = 4x$ **32.** $y^2 + 3 = 2y$

33. $2x^2 + 1 = 4x$ **34.** $2m^2 + 3 = 6m$

35. $5x^2 + 2 = 2x$ **36.** $7x^2 + 6x + 4 = 0$

37. $4(x^2 + 2x) - 4 = 0$ **38.** $-2(5 - x^2) + 2 = 0$

For each equation in Problems 39–44, use the discriminant to determine the number and type of zeros.

39. $2.4x^2 + 6.4x - 4.3 = 0$ **40.** $0.4x^2 - 3.2x + 6.4 = 0$

41. $6.5x^2 - 7.4x + 3.4 = 0$ **42.** $3.4x^2 - 2.5x - 1.5 = 0$

43. $0.3x^2 + 3.6x + 10.8 = 0$ **44.** $1.7x^2 + 2.4x + 1.4 = 0$

For each equation in Problems 45–50, use a graph to determine the number and type of zeros.

45. $0.2x^2 - 3.2x + 12.8 = 0$ **46.** $4.5x^2 + 1.7x - 0.4 = 0$

47. $3.4x^2 - 9.1x - 4.7 = 0$ **48.** $1.3x^2 - 1.5x + 0.8 = 0$

49. $2.4x^2 + 3.7x + 1.5 = 0$ **50.** $0.6x^2 + 6x + 15 = 0$

In Problems 51–60, solve algebraically and confirm with a graphing calculator, if possible.

51. $x^2 - 6x - 3 = 0$ **52.** $y^2 - 10y - 3 = 0$

53. $2y^2 - 6y + 3 = 0$ **54.** $2d^2 - 4d + 1 = 0$

55. $3x^2 - 2x - 2 = 0$ **56.** $3x^2 + 5x - 4 = 0$

57. $12x^2 + 7x = 10$ **58.** $9x^2 + 9x = 4$

59. $x^2 = 3x + 1$ **60.** $x^2 + 2x = 2$

In Problems 61–64, solve for the indicated variable in terms of the other variables. Use positive square roots only.

61. $s = \frac{1}{2}gt^2$ for t **62.** $a^2 + b^2 = c^2$ for a

63. $P = EI - RI^2$ for I **64.** $A = P(1 + r)^2$ for r

In Problems 65–80, solve by any algebraic method and confirm graphically, if possible. Round any approximate solutions to three decimal places.

65. $x^2 - \sqrt{7}x + 2 = 0$ **66.** $x^2 + \sqrt{11}x + 3 = 0$

67. $x^2 - 2\sqrt{3}x + 3 = 0$ **68.** $x^2 - \sqrt{5}x - 5 = 0$

69. $x^2 + \sqrt{3}x - 4 = 0$ **70.** $x^2 + 2\sqrt{5}x + 5 = 0$

71. $1 + \dfrac{9}{x^2} = \dfrac{5}{x}$ **72.** $1 + \dfrac{25}{x^2} = \dfrac{9}{x}$

73. $1 + \dfrac{9}{x^2} = \dfrac{6}{x}$ **74.** $1 + \dfrac{25}{x^2} = \dfrac{10}{x}$

75. $1 + \dfrac{9}{x^2} = \dfrac{7}{x}$ **76.** $1 + \dfrac{25}{x^2} = \dfrac{11}{x}$

77. $3 + \dfrac{5}{x - 4} = \dfrac{7}{x + 4}$ **78.** $5 + \dfrac{6}{x - 2} = \dfrac{4}{x + 2}$

79. $\dfrac{8}{x - 5} = \dfrac{3}{x + 5} - 2$ **80.** $\dfrac{6}{x - 3} = \dfrac{4}{x + 3} - 3$

81. Consider the quadratic equation

$$x^2 + 4x + c = 0$$

where c is a real number. Discuss the relationship between the values of c and the three types of roots listed in Table 1 on page 215.

82. Consider the quadratic equation

$$x^2 - 2x + c = 0$$

where c is a real number. Discuss the relationship between the values of c and the three types of roots listed in Table 1 on page 215.

In Problems 83–86, solve and express your answer in $a + bi$ form.

83. $x^2 + 3ix - 2 = 0$ **84.** $x^2 - 7ix - 10 = 0$

85. $x^2 + 2ix = 3$ **86.** $x^2 = 2ix - 3$

In Problems 87 and 88, find all solutions. [Hint: Factor using special formulas.]

87. $x^3 - 1 = 0$ **88.** $x^4 - 1 = 0$

89. Can a quadratic equation with rational coefficients have one rational root and one irrational root? Explain.

90. Can a quadratic equation with real coefficients have one real root and one imaginary root? Explain.

91. Show that if r_1 and r_2 are the two roots of $ax^2 + bx + c = 0$, then $r_1 r_2 = c/a$.

92. For r_1 and r_2 in Problem 91, show that $r_1 + r_2 = -b/a$.

93. In one stage of the derivation of the quadratic formula, we replaced the expression

$$\pm\sqrt{(b^2 - 4ac)/4a^2}$$

with

$$\pm\sqrt{b^2 - 4ac}/2a$$

What justifies using $2a$ in place of $|2a|$?

Problem 94 addresses the claim in Table 1 on page 215 that imaginary solutions to quadratic equations come in conjugate pairs.

94. If $ax^2 + bx + c = 0$ has two imaginary solutions, then the discriminant $b^2 - 4ac$ is negative.

(A) Write $\sqrt{b^2 - 4ac}$ as i times a root in this case. [*Hint:* Try writing $\sqrt{-3}$ as i times a root, then apply the result to $\sqrt{b^2 - 4ac}$, where $b^2 - 4ac < 0$.]

(B) Rewrite the quadratic formula with your answer to part A in place of $\sqrt{b^2 - 4ac}$.

(C) Split the quadratic formula into two separate solutions, one for the positive root and one for the negative. Then write the right side of each in $a + bi$ form. What do you notice about the two solutions in this case?

95. Find two numbers such that their sum is 21 and their product is 104.

96. Find all numbers with the property that when the number is added to itself the sum is the same as when the number is multiplied by itself.

97. Find two consecutive positive even integers whose product is 168.

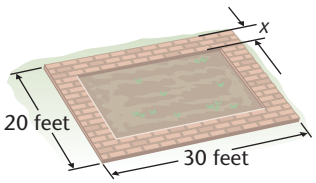
98. Find two consecutive positive integers whose product is 600.

APPLICATIONS

99. **AIR SEARCH** A search plane takes off from an airport at 6:00 A.M. and travels due north at 200 miles per hour. A second plane takes off at 6:30 A.M. and travels due east at 170 miles per hour. The planes carry radios with a maximum range of 500 miles. When (to the nearest minute) will these planes no longer be able to communicate with each other?

100. **NAVIGATION** A speedboat takes 1 hour longer to go 24 miles up a river than to return. If the boat cruises at 10 miles per hour in still water, what is the rate of the current?

101. **CONSTRUCTION** A gardener has a 30 foot by 20 foot rectangular plot of ground. She wants to build a brick walkway of uniform width on the border of the plot (see the figure). If the gardener wants to have 400 square feet of ground left for planting, how wide (to two decimal places) should she build the walkway?

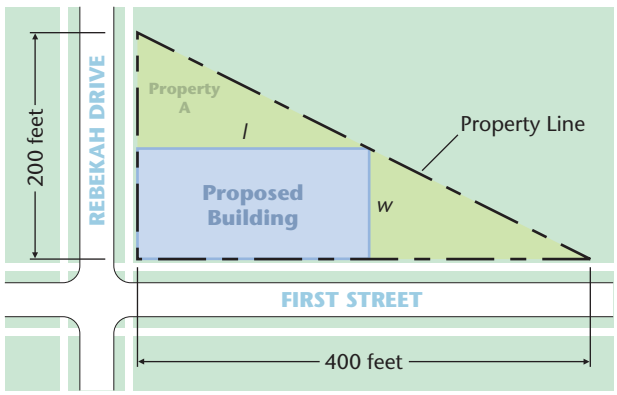


102. **CONSTRUCTION** Refer to Problem 101. The gardener buys enough bricks to build 160 square feet of walkway. Is this sufficient to build the walkway determined in Problem 101? If not, how wide (to two decimal places) can she build the walkway with these bricks?

103. **CONSTRUCTION** A 1,200 square foot rectangular garden is enclosed with 150 feet of fencing. Find the dimensions of the garden to the nearest tenth of a foot.

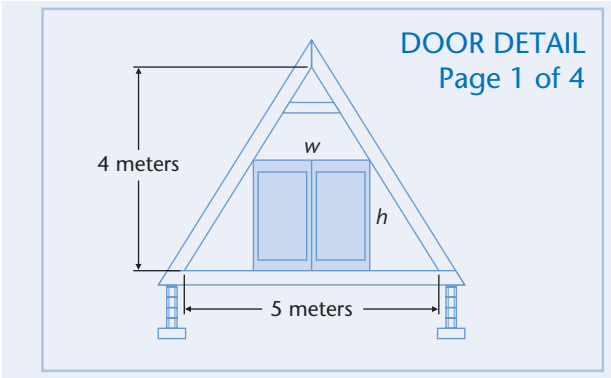
104. **CONSTRUCTION** The intramural fields at a small college will cover a total area of 140,000 square feet, and the administration has budgeted for 1,600 feet of fence to enclose the rectangular field. Find the dimensions of the field.

105. **ARCHITECTURE** A developer wants to erect a rectangular building on a triangular-shaped piece of property that is 200 feet wide and 400 feet long (see the figure).



(A) Express the area $A(w)$ of the footprint of the building as a function of the width w and state the domain of this function. [*Hint:* Use Euclid's theorem* to find a relationship between the length l and width w.]
(B) Building codes require that this building have a footprint of at least 15,000 square feet. What are the widths of the building that will satisfy the building codes?
(C) Can the developer construct a building with a footprint of 25,000 square feet? What is the maximum area of the footprint of a building constructed in this manner?

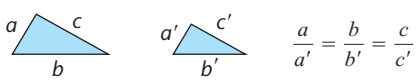
106. **ARCHITECTURE** An architect is designing a small A-frame cottage for a resort area. A cross-section of the cottage is an isosceles triangle with a base of 5 meters and an altitude of 4 meters. The front wall of the cottage must accommodate a sliding door positioned as shown in the figure.



(A) Express the area $A(w)$ of the door as a function of the width w and state the domain of this function. [See the hint for Problem 105.]
(B) A provision of the building code requires that doorways must have an area of at least 4.2 square meters. Find the width of the doorways that satisfy this provision.
(C) A second provision of the building code requires all doorways to be at least 2 meters high. Discuss the effect of this requirement on the answer to part B.

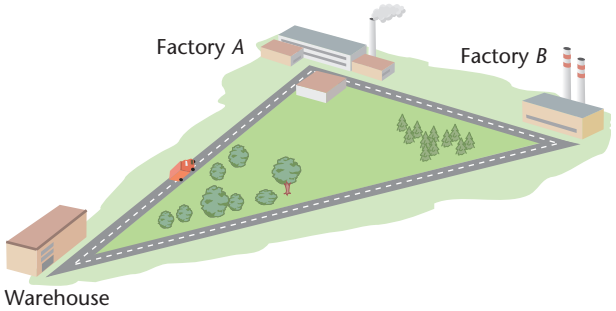
107. **TRANSPORTATION** A delivery truck leaves a warehouse and travels north to factory A. From factory A the truck travels east to factory B and then returns directly to the warehouse (see the figure on the next page). The driver recorded the truck's odometer reading at the warehouse at both the beginning and the end of the trip and also at factory B, but forgot to record it at factory A (see the table on the next page). The driver does recall that it was farther from the warehouse to factory A than it was from factory A to factory B. Because delivery charges are based

**Euclid's theorem:* If two triangles are similar, their corresponding sides are proportional:



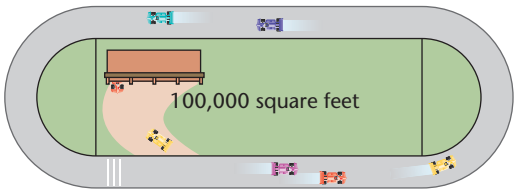
$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

on distance from the warehouse, the driver needs to know how far factory A is from the warehouse. Find this distance.



	Odometer Readings
Warehouse	52846
Factory A	52???
Factory B	52937
Warehouse	53002

108. CONSTRUCTION A $\frac{1}{4}$-mile track for racing stock cars consists of two semicircles connected by parallel straightaways (see the figure). To provide sufficient room for pit crews, emergency vehicles, and spectator parking, the track must enclose an area of 100,000 square feet. Find the length of the straightaways and the diameter of the semicircles to the nearest foot. [*Recall:* The area A and circumference C of a circle of diameter d are given by $A = \pi d^2/4$ and $C = \pi d$.]



DATA ANALYSIS AND QUADRATIC REGRESSION

In Problems 109–116, unless directed otherwise, round all numbers to three significant digits.

109. ALCOHOL CONSUMPTION Table 5 contains data related to the per capita alcohol consumption in the United States from 1960 to 2000.
(A) Let the independent variable x represent years since 1960. Find a quadratic regression model for the per capita beer consumption.
(B) If beer consumption continues to follow the trend exhibited in Table 5, when (to the nearest year) will the consumption return to the 1960 level?
(C) What does your model predict for beer consumption in the year 2005? Use the Internet or a library to compare your predicted results with the actual results.

Table 5 Per Capita Alcohol Consumption (in Gallons)

Year	Beer	Wine
1960	0.99	0.22
1965	1.04	0.24
1970	1.14	0.27
1975	1.26	0.32
1980	1.38	0.34
1985	1.33	0.38
1990	1.34	0.33
1995	1.25	0.29
2000	1.22	0.31

Source: NIAAA

110. ALCOHOL CONSUMPTION Refer to Table 5.
(A) Let the independent variable x represent years since 1960. Find a quadratic regression model for the per capita wine consumption.
(B) If wine consumption continues to follow the trend exhibited in Table 5, when (to the nearest year) will the consumption return to the 1960 level?
(C) What does your model predict for wine consumption in the year 2005? Use the Internet or a library to compare your predicted results with the actual results.

111. CIGARETTE PRODUCTION Table 6 contains data related to the total production and per capita consumption of cigarettes in the United States from 1950 to 2000.
(A) Let the independent variable x represent years since 1950. Find a quadratic regression model for the total cigarette production.

Table 6 Cigarette Consumption

Year	Production (billions)	Per Capita Annual Consumption
1950	370	3,550
1955	396	3,600
1960	484	4,170
1965	529	4,260
1970	537	3,990
1975	607	4,120
1980	632	3,850
1985	594	3,370
1990	525	2,830
1995	487	2,520
2000	430	2,080

Source: CDC

(B) If cigarette production continues to follow the trend exhibited in Table 6, when (to the nearest year) will the production return to the 1950 level?
(C) What does your model predict for cigarette production in the year 2005? Use the Internet or a library to compare your predicted results with the actual results.

112. **CIGARETTE CONSUMPTION** Refer to Table 6.
(A) Let the independent variable x represent years since 1950. Find a quadratic regression model for the per capita cigarette consumption.
(B) If per capita cigarette consumption continues to follow the trend exhibited in Table 6, when (to the nearest year) will the per capita consumption drop to 500 cigarettes?
(C) What does your model predict for per capita cigarette consumption in the year 2005? Use the Internet or a library to compare your predicted results with the actual results.

113. **STOPPING DISTANCE** Table 7 contains data related to the length of the skid marks left by two different automobiles when making emergency stops.
(A) Let x be the speed of the vehicle in miles per hour. Find a quadratic regression model for the braking distance for auto A.
(B) An insurance investigator finds skid marks 200 feet long at the scene of an accident involving auto A. How fast (to the nearest mile per hour) was auto A traveling when it made these skid marks?

Table 7 Skid Marks

Speed (mph)	Length of Skid Marks (in feet)	
	Auto A	Auto B
20	21	29
30	44	53
40	76	86
50	114	124
60	182	193
70	238	263
80	305	332

114. **STOPPING DISTANCE** Refer to Table 7.
(A) Let x be the speed of the vehicle in miles per hour. Find a quadratic regression model for the braking distance for auto B.
(B) An insurance investigator finds skid marks 165 feet long at the scene of an accident involving auto B. How fast (to the nearest mile per hour) was auto B traveling when it made these skid marks?

115. **OPTIMAL SPEED** Table 8 contains performance data for two speedboats powered by Yamaha outboard motors.
(A) Let x be the speed of boat A in miles per hour (mph) and y the associated mileage in miles per gallon (mpg). Use the data in Table 8 to find a quadratic regression function $y = ax^2 + bx + c$ for this boat.
(B) A marina rents this boat for $10 per hour plus the cost of the gasoline used. If gasoline costs $2.30 per gallon and you take a 100-mile trip in this boat, construct a mathematical model and use it to answer the following questions:
What speed should you travel to minimize the rental charges?
What mileage will the boat get?
How long does the trip take?
How much gasoline will you use?
How much will the trip cost you?

Table 8 Performance Data

Boat A		Boat B	
mph	mpg	mph	mpg
5.4	2.84	5.1	1.65
12.3	2.86	9.0	1.45
29.3	4.44	23.9	2.30
41.8	3.80	35.0	2.48
53.1	3.28	44.1	2.19
57.4	2.73	49.1	1.81

Source: www.yamaha-motor.com

116. **OPTIMAL SPEED** Refer to Table 8.
(A) Let x be the speed of boat B in miles per hour (mph) and y the associated mileage in miles per gallon (mpg). Use the data in Table 8 to find a quadratic regression function $y = ax^2 + bx + c$ for this boat.
(B) A marina rents this boat for $15 per hour plus the cost of the gasoline used. If gasoline costs $2.50 per gallon and you take a 200-mile trip in this boat, construct a mathematical model and use it to answer the following questions:
What speed should you travel to minimize the rental charges?
What mileage will the boat get?
How long does the trip take?
How much gasoline will you use?
How much will the trip cost you?

2-6 Additional Equation-Solving Techniques

- Solving Equations Involving Radicals
- Solving Equations of Quadratic Type
- Mathematical Modeling

In this section, we will examine some equations that can be transformed into quadratic equations using various algebraic steps. We will then be able to solve these quadratic equations, and with a little bit of interpretation, use those solutions to solve the original equations.

Solving Equations Involving Radicals

Consider the equation

$$x = \sqrt{x + 2} \tag{1}$$

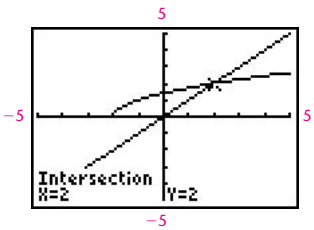


Figure 1 $y_1 = x$, $y_2 = \sqrt{x + 2}$.

Graphing both sides of the equation and using the INTERSECT command on a graphing calculator shows that $x = 2$ is a solution to the equation (Fig. 1). But is it the only solution?

There may be other solutions not visible in this viewing window. Or there may be imaginary solutions (remember, graphical approximation applies only to real solutions). To solve this equation algebraically, we eliminate the radical by squaring both sides of the equation. The result is a quadratic equation.

$$\begin{aligned}
x^2 &= (\sqrt{x + 2})^2 && \text{Simplify right side.} \\
x^2 &= x + 2 && \text{Write in standard form for a quadratic.} \\
x^2 - x - 2 &= 0 && \text{Factor.} \\
(x - 2)(x + 1) &= 0 && \text{Use the zero product property.} \\
x - 2 = 0 \quad &\text{or} \quad x + 1 = 0 \\
x &= 2, -1
\end{aligned} \tag{2}$$

These are the only solutions to the quadratic equation. We have already seen that $x = 2$ is a solution to the original equation. From the graph, it certainly doesn't look like $x = -1$ is a solution. To check, we substitute in equation (1):

$$\begin{aligned}
x &= \sqrt{x + 2} \\
-1 &\stackrel{?}{=} \sqrt{-1 + 2} \\
-1 &\stackrel{?}{=} \sqrt{1} \\
-1 &\neq 1
\end{aligned}$$

It turns out that -1 is not a solution to equation (1). What went wrong?

››› EXPLORE-DISCUSS 1

(A) The three statements below are true. Square both sides of each. Are they still true?

$$5 = \frac{10}{2} \qquad 6 = 2 \cdot 3 \qquad -4 = \frac{-12}{3}$$

(B) The three statements below are false. Square both sides of each. Are they still false? What can you conclude?

$$-4 = 4 \qquad 4 = \frac{6}{2} \qquad -2 = 6 - 4$$

Solving equations is all about deciding when an equation is a true statement. Explore-Discuss 1 demonstrates that squaring both sides can turn a false statement into a true one. That's what happened in equation (1). Substituting in $x = -1$ makes the equation the false statement $-1 = 1$, but squaring both sides made it true.

Then what do we gain by squaring both sides of an equation to solve? Theorem 1 provides a clue.

› THEOREM 1 Power Operation on Equations

If both sides of an equation are raised to the same natural number power, then the solution set of the original equation is a subset of the solution set of the new equation.

Equation	Solution Set
$x = 3$	$\{3\}$
$x^2 = 9$	$\{-3, 3\}$

Theorem 1 indicates that when we square both sides of an equation, the resulting equation might have more solutions than the original. But any solutions of the original will be among the solutions of the new equation.

Referring to equations (1) and (2) on page 226, we know that 2 and -1 are the only solutions to the quadratic equation (2). And we checked that -1 is not a solution to equation (1). Theorem 1 now implies that 2 must be the *only* solution to equation (1).

We call -1 an *extraneous solution*. In general, an **extraneous solution** is a solution introduced during the solution process that does not satisfy the original equation.

When raising both sides of an equation to a power, every solution of the new equation must be checked in the original equation to eliminate extraneous solutions.

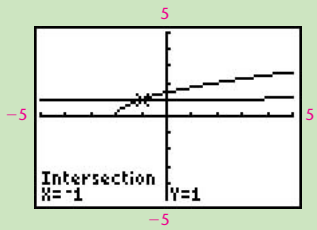
››› EXPLORE-DISCUSS 2

Figure 2 shows that $x = -1$ is a solution of the equation

$$\sqrt{x + 2} = 0.01x + 1.01$$

Are there any other solutions? Find any additional solutions both algebraically and graphically. What are some advantages and disadvantages of each of these solution methods?

› **Figure 2** $y_1 = \sqrt{x + 2}, y_2 = 0.01x + 1.01$.



EXAMPLE 1 Solving Equations Involving Radicals

Solve algebraically $\sqrt{4x^2 + 8x + 7} - x = 1$.

SOLUTION

$$\sqrt{4x^2 + 8x + 7} - x = 1$$ Isolate radical on one side.

$$\sqrt{4x^2 + 8x + 7} = x + 1$$ Square both sides.

$$4x^2 + 8x + 7 = x^2 + 2x + 1$$ Collect like terms.

$$3x^2 + 6x + 6 = 0$$ Divide both sides by 3.

$$x^2 + 2x + 2 = 0$$ Use the quadratic formula.

$$x = \frac{-2 \pm \sqrt{-4}}{2}$$

$$= -1 + i, -1 - i$$

CHECK

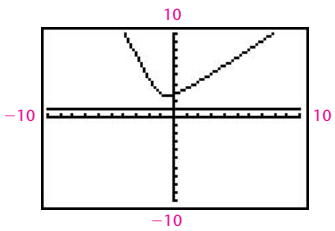
$$x = -1 + i$$
$$\sqrt{4x^2 + 8x + 7} - x = 1$$
$$\sqrt{4(-1+i)^2 + 8(-1+i) + 7} - (-1+i) \stackrel{?}{=} 1 \qquad (-1+i)^2 = 1 - 2i + i^2 = -2i$$
$$\sqrt{-8i - 8 + 8i + 7} + 1 - i \stackrel{?}{=} 1$$
$$\sqrt{-1} + 1 - i \stackrel{?}{=} 1$$
$$1 \stackrel{\checkmark}{=} 1$$

$$x = -1 - i$$
$$\sqrt{4x^2 + 8x + 7} - x = 1$$
$$\sqrt{4(-1-i)^2 + 8(-1-i) + 7} - (-1-i) \stackrel{?}{=} 1 \qquad (-1-i)^2 = 1 + 2i + i^2 = 2i$$
$$\sqrt{8i - 8 - 8i + 7} + 1 + i \stackrel{?}{=} 1$$
$$\sqrt{-1} + 1 + i \stackrel{?}{=} 1$$
$$1 + 2i \neq 1$$

The check shows that $-1 + i$ is a solution to the original equation and $-1 - i$ is extraneous. Thus, the only solution is the imaginary number

$$x = -1 + i$$

Graphing both sides of the equation illustrates that there are no intersection points in a standard viewing window (Fig. 3). The algebraic solution shows that the equation has no real solutions, so there cannot be any intersection points anywhere in the plane.



› **Figure 3**
$y_1 = 1$,
$y_2 = \sqrt{4x^2 + 8x + 7} - x$.

››› CAUTION ›››

1. When solving equations by squaring both sides, it is very important to isolate the radical first.
2. Be sure to square binomials like $x + 1$ by first writing it as $(x + 1)(x + 1)$. Keep in mind that $(x + 1)^2$ is *not* equal to $x^2 + 1^2$.

MATCHED PROBLEM 1

Solve algebraically: $\sqrt{x^2 - 2x - 2} + 2x = 2$.

When an equation has more than one radical, it may be necessary to square both sides more than once. In this case, we begin by isolating one of the radicals.

EXAMPLE 2 Solving Equations Involving Two Radicals

Solve algebraically and graphically: $\sqrt{2x + 3} - \sqrt{x - 2} = 2$.

SOLUTION

Algebraic Solution

$$\sqrt{2x + 3} - \sqrt{x - 2} = 2$$ Isolate one of the radicals.

$$\sqrt{2x + 3} = \sqrt{x - 2} + 2$$ Square both sides.

$$2x + 3 = (\sqrt{x - 2} + 2)(\sqrt{x - 2} + 2)$$ Multiply parentheses.

$$2x + 3 = x - 2 + 4\sqrt{x - 2} + 4$$ Isolate the remaining radical.

$$x + 1 = 4\sqrt{x - 2}$$ Square both sides.

$$x^2 + 2x + 1 = 16(x - 2)$$ Distribute.

$$x^2 + 2x + 1 = 16x - 32$$ Collect like terms.

$$x^2 - 14x + 33 = 0$$ Factor.

$$(x - 3)(x - 11) = 0$$ Use the zero product property.

$$x - 3 = 0 \quad \text{or} \quad x - 11 = 0$$

$$x = 3, 11$$

CHECK

$$x = 3$$
$$\sqrt{2x + 3} - \sqrt{x - 2} = 2$$
$$\sqrt{2(3) + 3} - \sqrt{3 - 2} \stackrel{?}{=} 2$$
$$2 \stackrel{\checkmark}{=} 2$$

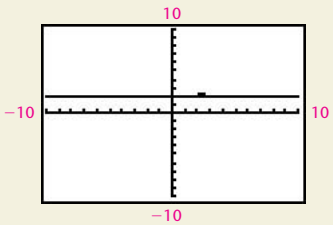
$$x = 11$$
$$\sqrt{2x + 3} - \sqrt{x - 2} = 2$$
$$\sqrt{2(11) + 3} - \sqrt{11 - 2} \stackrel{?}{=} 2$$
$$2 \stackrel{\checkmark}{=} 2$$

Both solutions check. The equation has two solutions.

$$x = 3, 11$$

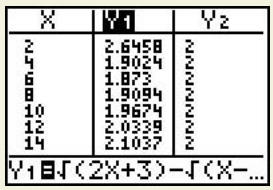
Graphical Solution

Graphing $y_1 = \sqrt{2x + 3} - \sqrt{x - 2}$ and $y_2 = 2$ in a standard viewing window produces a graph that is not very useful (Fig. 4).



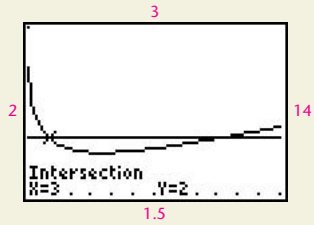
› Figure 4

Examining a table of values (Fig. 5) suggests that choosing Xmin = 2, Xmax = 14, Ymin = 1.5, Ymax = 3 is likely to produce a graph that shows two intersection points.

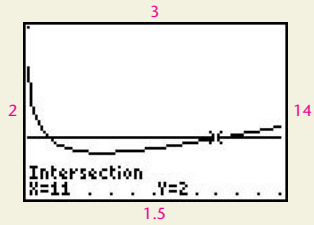


› Figure 5

Using the INTERSECT command, the x coordinates of the intersection points are $x = 3$ (Fig. 6) and $x = 11$ (Fig. 7).



› Figure 6



› Figure 7

MATCHED PROBLEM 2

Solve algebraically and graphically: $\sqrt{2x + 5} + \sqrt{x + 2} = 5$.

How do you choose between algebraic and graphical solution methods? It depends on the type of solutions you want. If you want to find real and complex solutions, you must use algebraic methods, as we did in Example 1. If you are only interested in real solutions, then either method can be used, as in Example 2. To get the most out of this topic, we recommend that you solve each equation algebraically and, when possible, confirm your solutions graphically.

› Solving Equations of Quadratic Type

Quadratic equations in standard form have two terms with the variable; one has power 2, the other power 1. When equations have two variable terms where the larger power is twice the smaller, we can use quadratic solving techniques.

EXAMPLE 3 **Solving an Equation of Quadratic Type**

Solve $x^{2/3} - x^{1/3} - 6 = 0$.

SOLUTIONS

Method I. Direct solution:

Note that the larger power (2/3) is twice the smaller. Using the properties of exponents from basic algebra, we can write $x^{2/3}$ as $(x^{1/3})^2$ and solve by factoring.

$$(x^{1/3})^2 - x^{1/3} - 6 = 0 \qquad \text{Factor left side.}$$
$$(x^{1/3} - 3)(x^{1/3} + 2) = 0 \qquad \text{Use the zero product property.}$$
$$x^{1/3} = 3 \quad \text{or} \quad x^{1/3} = -2 \qquad \text{Cube both sides.}$$
$$(x^{1/3})^3 = 3^3 \qquad (x^{1/3})^3 = (-2)^3$$
$$x = 27 \qquad x = -8$$

The solution is $x = 27, -8$

Method II. Using substitution:

Replace $x^{1/3}$ (the smaller power) with a new variable u. Then the larger power $x^{2/3}$ is u^2. This gives us a quadratic equation with variable u.

$$\begin{aligned} u^2 - u - 6 &= 0 && \text{Factor.} \\ (u - 3)(u + 2) &= 0 && \text{Use the zero product property.} \\ u &= 3, -2 \end{aligned}$$

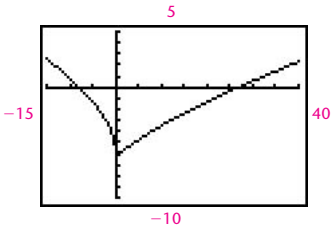
This is not the solution! We still need to find the values of x that correspond to $u = 3$ and $u = -2$.

Replacing u with $x^{1/3}$, we obtain

$$\begin{aligned} x^{1/3} &= 3 \quad \text{or} \quad x^{1/3} = -2 && \text{Cube both sides.} \\ x &= 27 \qquad\quad x = -8 \end{aligned}$$

The solution is $x = 27, -8$.

The graph in Figure 8 confirms these results. [*Note:* In some graphing calculators you may have to enter the left side of the equation in the form $y_1 = (x^2)^{1/3} - x^{1/3} - 6$ rather than $y_1 = x^{2/3} - x^{1/3} - 6$. Try both forms to see what happens.]



› **Figure 8**
$y_1 = x^{2/3} - x^{1/3} - 6$.

MATCHED PROBLEM 3

Solve algebraically using both Method I and Method II and confirm graphically $x^{1/2} - 5x^{1/4} + 6 = 0$.

In general, if an equation that is not quadratic can be transformed to the form

$$au^2 + bu + c = 0$$

where u is an expression in some other variable, then the equation is called an **equation of quadratic type.** Equations of quadratic type often can be solved using quadratic methods.

››› EXPLORE-DISCUSS 3

Which of the following can be transformed into a quadratic equation by making a substitution of the form $u = x^n$? What is the resulting quadratic equation?

(A) $3x^{-4} + 2x^{-2} + 7 = 0$ **(B)** $7x^5 - 3x^2 + 3 = 0$

(C) $2x^5 + 4x^2\sqrt{x} - 6 = 0$ **(D)** $8x^{-2}\sqrt{x} - 5x^{-1}\sqrt{x} - 2 = 0$

In general, if a, b, c, m, and n are nonzero real numbers, when can an equation of the form $ax^m + bx^n + c = 0$ be transformed into an equation of quadratic type?

EXAMPLE 4 Solving Equations of Quadratic Type

Solve algebraically and graphically: $3x^{-2/5} - 6x^{-1/5} + 2 = 0$.

SOLUTION

Algebraic Solution

The larger power $(-2/5)$ is twice the smaller $(-1/5)$. We substitute $u = x^{-1/5}$; then $u^2 = x^{-2/5}$

$$3u^2 - 6u + 2 = 0 \qquad \text{Use the quadratic formula.}$$

$$u = \frac{6 \pm \sqrt{12}}{6}$$

$$u = \frac{3 \pm \sqrt{3}}{3}$$

We now replace u with $x^{-1/5}$.

$$x^{-1/5} = \frac{3 \pm \sqrt{3}}{3} \qquad \text{Raise both sides to the } -5 \text{ power.}$$

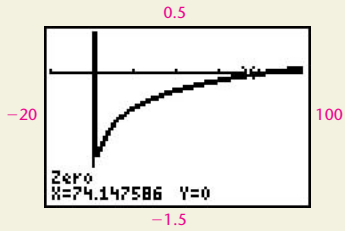
$$x = \left(\frac{3 \pm \sqrt{3}}{3}\right)^{-5} \qquad \text{Apply the negative exponent.}$$

$$x = \left(\frac{3}{3 \pm \sqrt{3}}\right)^{5} \qquad \text{Use a calculator.}$$

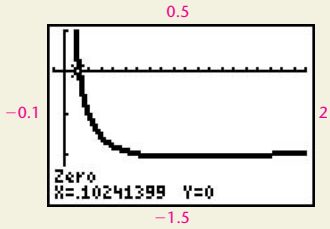
$$\approx 0.102414, 74.147586$$

Graphical Solution

The graph of $y_1 = 3x^{-2/5} - 6x^{-1/5} + 2$ is the thick curve in Figure 9. The graph crosses the x axis near $x = 0$ and again near $x = 75$. The solution near $x = 75$ is easily approximated in this viewing window. The solution near the origin can be approximated in the same viewing window, but it's much easier to see if we change the limits on the x axis (Fig. 10).



› Figure 9



› Figure 10

MATCHED PROBLEM 4

Solve algebraically and graphically: $3x^{-2/5} - x^{-1/5} - 2 = 0$.

EXAMPLE 5 Solving Equations of Quadratic Type

Solve algebraically and confirm graphically, if possible: $x^4 - 3x^2 - 4 = 0$.

SOLUTION

Algebraic Solution

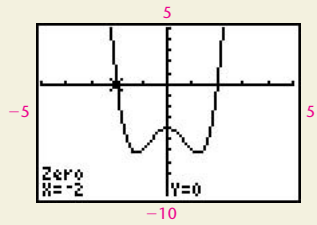
In this case, we will use Method 1 and factor directly.

$$(x^2)^2 - 3x^2 - 4 = 0 \quad \text{Factor.}$$
$$(x^2 - 4)(x^2 + 1) = 0 \quad \text{Use the zero product property.}$$
$$x^2 = 4 \quad \text{or} \quad x^2 = -1$$
$$x = \pm 2 \quad \text{or} \quad x = \pm i$$

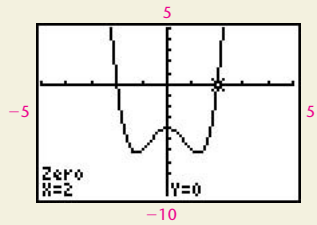
Because we did not raise each side of the equation to a natural number power, we do not have to check for extraneous solutions. (But you should still check the accuracy of the solutions.)

Graphical Confirmation

Figures 11 and 12 show the two real solutions. The imaginary solutions cannot be confirmed graphically.



› Figure 11



› Figure 12

MATCHED PROBLEM 5

Solve algebraically and confirm graphically, if possible: $x^4 + 3x^2 - 4 = 0$.

› Mathematical Modeling

Examples 6 and 7 illustrate the use of radicals in constructing mathematical models.

EXAMPLE 6 Depth of a Well

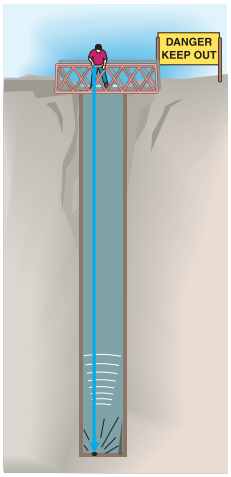
The splash from a stone dropped into a deep well is heard 5 seconds after the stone is released (Fig. 13). If sound travels through air at 1,100 feet per second, how deep is the well? Round answer to the nearest foot.

SOLUTION

Constructing the Model

The time between the instant the stone is released and the instant the splash is heard can be broken down into two parts:

t_1 = Time in seconds stone is falling through the air
t_2 = Time in seconds sound of splash travels back to surface



› Figure 13

From the section before last, the distance traveled by the stone in free fall is $x = 16t_1^2$. Using distance = rate · time, the sound of the splash traveling back up is $x = 1{,}100t_2$. Solving these two equations for t_1 and t_2, we get

$$t_1^2 = \frac{x}{16} \qquad t_2 = \frac{x}{1{,}100}$$

$$t_1 = \frac{\sqrt{x}}{4}$$

t_1 is a time, so we disregard the negative solution.

If we combine t_1 and t_2, we have a model for the total time t between releasing the stone and hearing the splash in terms of the depth of the well x:

$$t = t_1 + t_2 = \frac{\sqrt{x}}{4} + \frac{x}{1{,}100}$$

We are asked to find x when $t = 5$ seconds.

Algebraic Solution

$$\frac{\sqrt{x}}{4} + \frac{x}{1{,}100} = 5$$ Multiply both sides by 1,100.

$$275\sqrt{x} + x = 5{,}500$$ Subtract 5,500 from both sides; rearrange.

$$x + 275x^{1/2} - 5{,}500 = 0$$ Let $u = x^{1/2}$; $u^2 = x$

$$u^2 + 275u - 5{,}500 = 0$$ Use the quadratic formula.

$$u = \frac{-275 \pm \sqrt{275^2 - 4(-5{,}500)}}{2}$$

$$= \frac{-275 \pm \sqrt{97{,}625}}{2}$$

$$= 18.724998 \quad \text{or} \quad -293.724998$$

Because $u = x^{1/2} > 0$, the second solution is discarded.

$$x = u^2$$

$$= 18.724998^2$$ Round to the nearest foot.

$$= 351$$

The well is 351 feet deep.

Graphical Solution

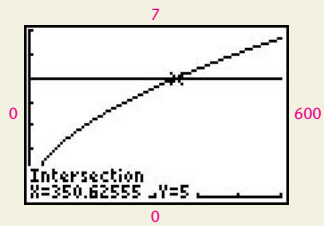
Enter $y_1 = \frac{\sqrt{x}}{4} + \frac{x}{1{,}100}$ and $y_2 = 5$. To determine the window variables, examine a table of values (Fig. 14) with fairly large x values (remember, x is the depth of the well and wells can be thousands of feet deep).

X	Y1	Y2
0	0	5
100	2.5909	5
200	3.7174	5
300	4.6029	5
400	5.3636	5
500	6.0447	5
600	6.6692	5

Y1∎√(X)/4+X/1100

› Figure 14

Now graph y_1 and y_2 and use INTERSECT (Fig. 15).



› Figure 15

From Figure 15, we see that the well is 351 feet deep.

MATCHED PROBLEM 6

The speed of sound through air is actually dependent on the air temperature. A deep well above a hot spring has an air temperature of 115 degrees, which makes sound travel at 1,180 feet per second. If the splash from a stone dropped into this well is heard 10 seconds after the stone is released, how deep is the well? Round to the nearest foot.

EXAMPLE 7 Design

› Figure 16

A window in the shape of a semicircle with radius 20 inches contains a rectangular pane of glass as shown in Figure 16.

(A) Find a mathematical model for the area of the rectangle. Use one-half the length of the base of the rectangle for the independent variable in your model.

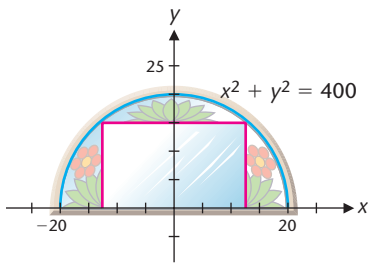
(B) Find the dimensions of the pane if the area of the pane is 320 square inches.

(C) Find the dimensions and the area of the largest possible rectangular pane of glass.

Round all answers to three significant digits.

SOLUTIONS

(A) Place a rectangular coordinate system on the window (Fig. 17). Let x be one-half the base of the rectangle and y be the height of the rectangle.



› Figure 17

Because (x, y) are the coordinates of a point on the circle with radius 20 and center $(0, 0)$, x and y must satisfy the equation of the circle*

$$x^2 + y^2 = 400$$
$$y = \sqrt{400 - x^2} \tag{3}$$

*Circles are reviewed in Appendix B, Section B-3.

The area of the rectangle is

$$A = \text{Base} \cdot \text{Height} = 2xy = 2x\sqrt{400 - x^2}$$

The radius is 20, so x must be less than 20. A model for the area is

$$A(x) = 2x\sqrt{400 - x^2} \qquad 0 < x < 20$$

(B) Solve the equation $A(x) = 320$.

Algebraic Solution

$$2x\sqrt{400 - x^2} = 320$$ Divide both sides by 2.

$$x\sqrt{400 - x^2} = 160$$ Square both sides.

$$x^2(400 - x^2) = 25{,}600$$ Distribute.

$$400x^2 - x^4 = 25{,}600$$ Subtract 25,600 from both sides.

$$400x^2 - x^4 - 25{,}600 = 0$$ Multiply both sides by -1; rearrange.

$$x^4 - 400x^2 + 25{,}600 = 0$$ Write as a quadratic with variable x^2.

$$(x^2)^2 - 400x^2 + 25{,}000 = 0$$

Use the quadratic formula to solve for x^2:

$$x^2 = \frac{400 \pm \sqrt{400^2 - 4 \cdot 25{,}600}}{2}$$

$$= \frac{400 \pm \sqrt{57{,}600}}{2}$$

$$= \frac{400 \pm 240}{2}$$

$$= 80 \quad \text{or} \quad 320$$

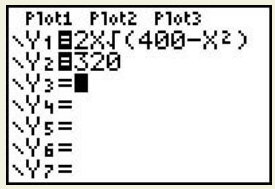
$$x = \sqrt{80} = 8.94 \quad \text{or} \quad x = \sqrt{320} = 17.9$$

A check (which we leave to the reader) shows that neither solution is extraneous.

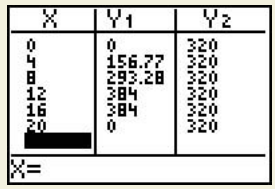
Graphical Solution

To solve the equation $A(x) = 320$, enter both sides in the equation editor of a graphing calculator (Fig. 18).

The values of x must satisfy $0 \le x \le 20$. Examining a table of values over this interval suggests that $0 \le y \le 500$ will produce a usable window (Fig. 19).

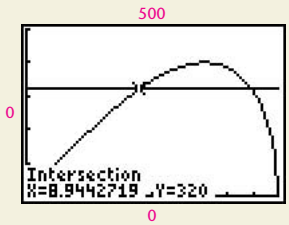


› Figure 18

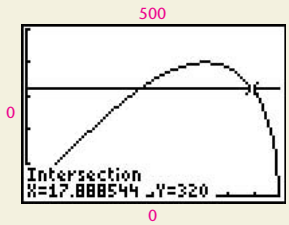


› Figure 19

Graphing y_1 and y_2 and using the INTERSECT command shows that the solutions are $x = 8.94$ (Fig. 20) and $x = 17.9$ (Fig. 21).



› Figure 20



› Figure 21

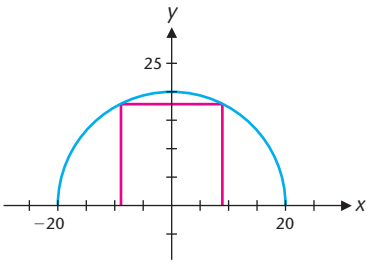
Now that we have determined the solutions to the equation $A(x) = 320$, we use $y = \sqrt{400 - x^2}$ to find the dimensions of the two rectangles:

$$x = \sqrt{80} = 8.94 \quad \text{and} \quad y = \sqrt{400 - 80} = \sqrt{320} = 17.9$$

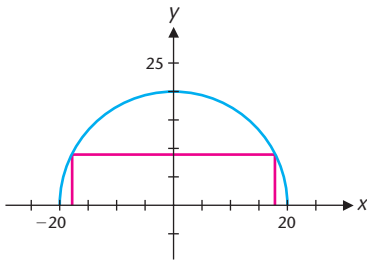
$$x = \sqrt{320} = 17.9 \quad \text{and} \quad y = \sqrt{400 - 320} = \sqrt{80} = 8.94$$

Recalling that x is one-half the base, the dimensions of the rectangles are 17.9 inches wide by 17.9 inches high or 35.8 inches wide by 8.94 inches high. Each solution is illustrated in Figure 22.

(C) Using the MAXIMUM command (Fig. 23), the largest rectangle has an area of 400 square inches when $x = 14.1$ inches. The dimensions of this rectangle are 28.2 inches wide and 14.2 inches high.

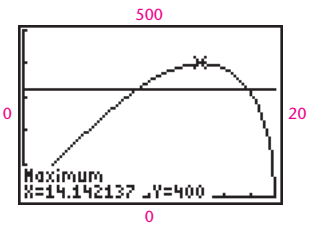


17.9 in. by 17.9 in.



35.8 in. by 8.94 in.

› Figure 22



› Figure 23

MATCHED PROBLEM 7

A window in the shape of a semicircle with radius 25 inches contains a rectangular pane of glass as shown in Figure 16 in Example 7.

(A) Find a mathematical model for the area of the rectangle. Use one-half the length of the base of the rectangle for the independent variable in your model.

(B) Find the dimensions of the pane if the area of the pane is 500 square inches.

(C) Find the dimensions and the area of the largest possible rectangular pane of glass.

Round all answers to three significant digits.

ANSWERS TO MATCHED PROBLEMS

1. $x = 1 - i$ **2.** $x = 2$ **3.** $x = 16, 81$ **4.** $x = 1, -\frac{243}{32}$ **5.** $x = \pm 1, \pm 2i$

6. 1,273 feet **7.** (A) $A(x) = 2x\sqrt{625 - x^2}$, $0 \le x \le 25$

(B) 22.4 inches by 22.4 inches or 44.8 inches by 11.2 inches

(C) 35.4 inches by 17.7 inches, area = 625 square inches

2-6 Exercises

In Problems 1–8, determine the validity of each statement. If a statement is false, explain why.

1. If $x^2 = 5$, then $x = \pm\sqrt{5}$

2. $\sqrt{25} = \pm 5$

3. $(x + 5)^2 = x^2 + 25$

4. $(2x - 1)^2 = 4x^2 - 1$

5. $(\sqrt{x - 1} + 1)^2 = x$

6. $(\sqrt{x - 1})^2 + 1 = x$

7. If $x^3 = 2$, then $x = 8$

8. If $x^{1/3} = 2$, then $x = 8$

In Problems 9–14, transform each equation of quadratic type into a quadratic equation in u and state the substitution used in the transformation. If the equation is not an equation of quadratic type, say so.

9. $2x^{-6} - 4x^{-3} = 0$

10. $\frac{4}{7} - \frac{3}{x} + \frac{6}{x^2} = 0$

11. $3x^3 - 4x + 9 = 0$

12. $7x^{-1} + 3x^{-1/2} + 2 = 0$

13. $\frac{10}{9} + \frac{4}{x^2} - \frac{7}{x^4} = 0$

14. $3x^{3/2} - 5x^{1/2} + 12 = 0$

15. Explain why squaring both sides of an equation sometimes introduces extraneous solutions.

16. Would raising both sides of an equation to the third power ever introduce extraneous solutions? Why or why not?

17. Write an example of a false statement that becomes true when you square both sides. What would every possible example have in common?

18. How can you recognize when an equation is of quadratic type?

In Problems 19–32, solve algebraically and confirm graphically, if possible.

19. $\sqrt{4x - 7} = 5$

20. $\sqrt{4 - x} = 4$

21. $\sqrt{5x + 6} + 6 = 0$

22. $\sqrt{10x + 1} + 8 = 0$

23. $\sqrt[3]{x + 5} = 3$

24. $\sqrt[4]{x - 3} = 2$

25. $\sqrt{x + 5} + 7 = 0$

26. $3 + \sqrt{2x - 1} = 0$

27. $y^4 - 2y^2 - 8 = 0$

28. $x^4 - 7x^2 - 18 = 0$

29. $3x = \sqrt{x^2 - 2}$

30. $x = \sqrt{5x^2 + 9}$

31. $\sqrt{x^2 - 5x} = \sqrt{x - 8}$

32. $\sqrt{2x + 3} = \sqrt{x^2 - 12}$

In Problems 33–56, solve algebraically and confirm graphically, if possible.

33. $\sqrt{5n + 9} = n - 1$

34. $m - 13 = \sqrt{m + 7}$

35. $\sqrt{3x + 4} = 2 + \sqrt{x}$

36. $\sqrt{3w - 2} - \sqrt{w} = 2$

37. $2x^{2/3} + 3x^{1/3} - 2 = 0$

38. $x^{2/3} - 3x^{1/3} - 10 = 0$

39. $(m^2 - m)^2 - 4(m^2 - m) = 12$

40. $(x^2 + 2x)^2 - (x^2 + 2x) = 6$

41. $\sqrt{u - 2} = 2 + \sqrt{2u + 3}$

42. $\sqrt{3t + 4} + \sqrt{t} = -3$

43. $\sqrt{3y - 2} = 3 - \sqrt{3y + 1}$

44. $\sqrt{2x - 1} - \sqrt{x - 4} = 2$

45. $\sqrt{7x - 2} - \sqrt{x + 1} = \sqrt{3}$

46. $\sqrt{3x + 6} - \sqrt{x + 4} = \sqrt{2}$

47. $\sqrt{4x^2 + 12x + 1} - 6x = 9$

48. $6x - \sqrt{4x^2 - 20x + 17} = 15$

49. $3n^{-2} - 11n^{-1} - 20 = 0$

50. $6x^{-2} - 5x^{-1} - 6 = 0$

51. $9y^{-4} - 10y^{-2} + 1 = 0$

52. $4x^{-4} - 17x^{-2} + 4 = 0$

53. $y^{1/2} - 3y^{1/4} + 2 = 0$

54. $4x^{-1} - 9x^{-1/2} + 2 = 0$

55. $(m - 5)^4 + 36 = 13(m - 5)^2$

56. $(x - 3)^4 + 3(x - 3)^2 = 4$

57. Explain why the following "solution" is incorrect:

$$\sqrt{x + 3} + 5 = 12$$
$$x + 3 + 25 = 144$$
$$x = 116$$

58. Explain why the following "solution" is incorrect.

$$\sqrt{x^2 - 16} = 2x + 3$$
$$x - 4 = 2x + 3$$
$$-7 = x$$

In Problems 59–62, solve algebraically and confirm graphically, if possible.

59. $\sqrt{5 - 2x} - \sqrt{x + 6} = \sqrt{x + 3}$

60. $\sqrt{2x + 3} - \sqrt{x - 2} = \sqrt{x + 1}$

61. $2 + 3y^{-4} = 6y^{-2}$

62. $4m^{-2} = 2 + m^{-4}$

In Problems 63–66, solve two ways: by isolating the radical and squaring, and by substitution. Confirm graphically, if possible.

63. $m - 7\sqrt{m} + 12 = 0$

64. $y - 6 + \sqrt{y} = 0$

65. $t - 11\sqrt{t} + 18 = 0$

66. $x = 15 - 2\sqrt{x}$

In Problems 67–70, solve algebraically and graphically. Discuss the advantages and disadvantages of each method.

67. $2\sqrt{x+5} = 0.01x + 2.04$ **68.** $3\sqrt{x-1} = 0.05x + 2.9$

69. $2x^{-2/5} - 5x^{-1/5} + 1 = 0$ **70.** $x^{-2/5} - 3x^{-1/5} + 1 = 0$

APPLICATIONS

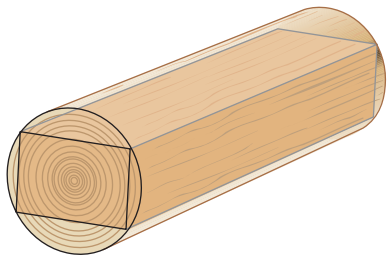
71. **GEOMETRY** The diagonal of a rectangle is 10 inches and the area is 45 square inches. Find the dimensions of the rectangle, correct to one decimal place.

72. **GEOMETRY** The hypotenuse of a right triangle is 12 inches and the area is 24 square inches. Find the dimensions of the triangle, correct to one decimal place.

73. **PHYSICS–WELL DEPTH** If the splash of a stone dropped into a well is heard 14 seconds after the stone is released, how deep (to the nearest foot) is the well? Use 1,100 feet per second for the speed of sound.

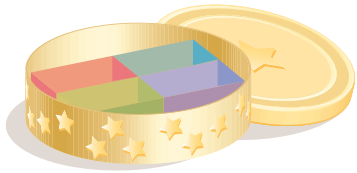
74. **PHYSICS–WELL DEPTH** If the splash of a stone dropped into a well is heard 2 seconds after the stone is released, how deep (to the nearest foot) is the well? Use 1,100 feet per second for the speed of sound.

75. **MANUFACTURING** A lumber mill cuts rectangular beams from circular logs that are 16 inches in diameter (see the figure).



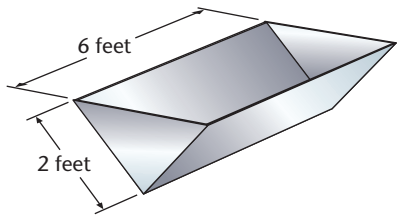
(A) Find a model for the cross-sectional area of the beam. Use the width of the beam as the independent variable.
(B) If the cross-sectional area of the beam is 120 square inches, find the dimensions correct to one decimal place.
(C) Find the dimensions of the beam that has the largest cross-sectional area and find this area. Round answers to one decimal place.

76. **DESIGN** A food-processing company packages an assortment of their products in circular metal tins 12 inches in diameter. Four identically sized rectangular boxes are used to divide the tin into six compartments (see the figure).

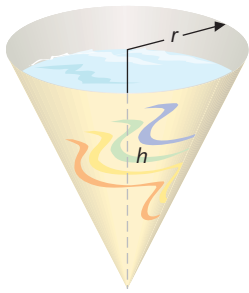


(A) Find a model for the cross-sectional area of one of these boxes. Use the width of the box as the independent variable.
(B) If the cross-sectional area of the box is 15 square inches, find the dimensions correct to one decimal place.
(C) Find the dimensions of the box that has the largest cross-sectional area and find this area. Round answers to one decimal place.

77. **CONSTRUCTION** A water trough is constructed by bending a 4- by 6-foot rectangular sheet of metal down the middle and attaching triangular ends (see the figure). If the volume of the trough is 9 cubic feet, find the width correct to two decimal places.



78. **DESIGN** A paper drinking cup in the shape of a right circular cone is constructed from 125 square centimeters of paper (see the figure). If the height of the cone is 10 centimeters, find the radius correct to two decimal places.



Lateral surface area:

$S = \pi r\sqrt{r^2 + h^2}$

2-7 Solving Inequalities

- Solving Linear Inequalities
- Solving Inequalities Involving Absolute Value
- Solving Quadratic Inequalities
- Mathematical Modeling
- Data Analysis and Regression

Now that we have sharpened our equation-solving skills, we turn our attention to solving various types of inequalities and several applications that involve inequalities.

Solving Linear Inequalities

Any inequality that can be written in one of the four forms in (1) is called a **linear inequality in one variable.**

$$\begin{aligned} mx + b &> 0 \\ mx + b &\geq 0 \\ mx + b &< 0 \\ mx + b &\leq 0 \end{aligned} \qquad \text{Linear inequalities} \qquad (1)$$

As was the case with equations, the **solution set of an inequality** is the set of all values of the variable that make the inequality a true statement. Each element of the solution set is called a **solution.** Two inequalities are said to be **equivalent** if they have the same solution set.

EXPLORE-DISCUSS 1

Associated with the linear equation and inequalities

$$3x - 12 = 0 \qquad 3x - 12 < 0 \qquad 3x - 12 > 0$$

is the linear function

$$f(x) = 3x - 12$$

(A) Graph the function f.

(B) From the graph of f describe verbally the values of x for which

$$f(x) = 0 \qquad f(x) < 0 \qquad f(x) > 0$$

(C) How are the answers to part B related to the solutions of

$$3x - 12 = 0 \qquad 3x - 12 < 0 \qquad 3x - 12 > 0$$

As you discovered in Explore-Discuss 1, solving inequalities graphically is both intuitive and efficient. When an inequality has zero on one side, it is a statement about when the nonzero side is positive or negative. When looking at the graph, this corresponds to x values for which the graph is above or below the x axis.

Our study of solving equations is based on an understanding of the algebraic steps that can be performed to produce an equivalent equation. The same is true for solving inequalities. The necessary facts are summarized in Theorem 1. If you need a refresher on inequalities and interval notation, see Appendix B, Section B-1.

› THEOREM 1 Inequality Properties

An equivalent inequality will result and the **sense (or direction) will remain the same** if each side of the original inequality

- Has the same real number added to or subtracted from it
- Is multiplied or divided by the same positive number

An equivalent inequality will result and the **sense (or direction) will reverse** if each side of the original inequality

- Is multiplied or divided by the same negative number

Note: Multiplication by 0 and division by 0 are not permitted.

Theorem 1 tells us that we can perform essentially the same operations on inequalities that we perform on equations, with the exception that *the sense (or direction) of the inequality reverses if we multiply or divide both sides by a negative number.* Otherwise the sense of the inequality does not change.

EXAMPLE 1 Solving a Linear Inequality

Solve $0.5x + 1 \le 0$.

SOLUTION

Algebraic Solution

$$0.5x + 1 \le 0$$ Subtract 1 from both sides.

$$0.5x + 1 - 1 \le 0 - 1$$

$$0.5x \le -1$$ Divide both sides by 0.5.

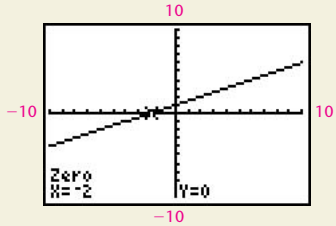
$$\frac{0.5x}{0.5} \le \frac{-1}{0.5}$$

$$x \le -2 \quad \text{or} \quad (-\infty, -2]$$

Graphical Solution

The graph of $f(x) = 0.5x + 1$ is shown in Figure 1. We see from the graph that $f(x)$ is negative to the left of -2 and positive to the right.

The inequality requires that $0.5x + 1$ be zero or negative, so the solution set of $0.5x + 1 \le 0$ is $x \le -2$ or, in interval notation, $(-\infty, -2]$.



› Figure 1

MATCHED PROBLEM 1

Solve $2x - 6 \geq 0$.

Think for a moment about the inequality $-2 < x < 3$. This is another way to describe the interval $(-2, 3)$. In fact, any inequality with three members is a statement about when an expression is between two values. We call these **combined inequalities.** We can solve linear combined inequalities by performing the same operations on all three members.

EXAMPLE 2 Solving a Combined Inequality

Solve $-3 \leq 4 - 7x < 18$.

SOLUTION

Algebraic Solution

To solve algebraically, we perform operations on the combined inequality until we have isolated x in the middle with a coefficient of 1.

$$-3 \leq 4 - 7x < 18 \quad \text{Subtract 4 from each member.}$$

$$-3 - 4 \leq 4 - 7x - 4 < 18 - 4$$

$$-7 \leq -7x < 14 \quad \text{Divide each member by } -7 \text{ and reverse each inequality.}$$

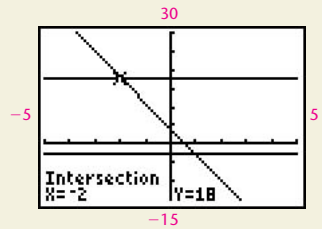
$$\frac{-7}{-7} \geq \frac{-7x}{-7} > \frac{14}{-7}$$

The solution is $1 \geq x > -2$. which can also be written as

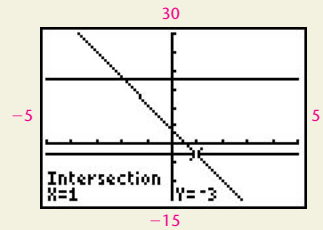
$$-2 < x \leq 1 \quad \text{or} \quad (-2, 1] \qquad (2)$$

Graphical Solution

The inequality is a statement about where $4 - 7x$ is between -3 and 18, so we enter $y_1 = -3$, $y_2 = 4 - 7x$, $y_3 = 18$, and find the intersection points (Fig. 2 and Fig. 3). It is clear from the graph that y_2 is between y_1 and y_3 for x between -2 and 1. Because $y_2 = -3$ at $x = 1$, we include 1 in the solution set, obtaining the same solution as shown in (2).



› Figure 2



› Figure 3

››› CAUTION ›››

When multiplying or dividing both sides of an inequality by a negative number, don't forget to change the direction of each inequality.

MATCHED PROBLEM 2

Solve $-3 < 7 - 2x \leq 7$.

Solving Inequalities Involving Absolute Value

EXPLORE-DISCUSS 2

Recall the definition of the absolute value function (see Section 1-3)

$$f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

(A) Graph the absolute value function $f(x) = |x|$ and the constant function $g(x) = 3$ in the same viewing window.

(B) From the graph in part A, determine the values of x for which:

$$|x| < 3 \qquad |x| = 3 \qquad |x| > 3$$

(C) Find all the points with coordinates $(x, 0)$ that are

Less than three units from the origin

Exactly three units from the origin

More than three units from the origin

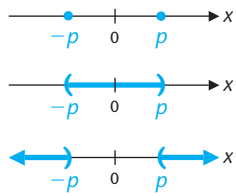
(D) Compare the solutions found in parts B and C.

The absolute value of a number x is simply the distance between x and the origin on a number line.

$|x| = p$ describes the set of all points that are exactly p units from zero. That is, $x = p$ or $x = -p$.

$|x| < p$ describes the set of all points that are less than p units from zero. That is, $-p < x < p$.

$|x| > p$ describes the set of all points that are more than p units from zero. That is, $x < -p$ or $x > p$.



Theorem 2 describes how to apply this to solving inequalities.

› THEOREM 2 Geometric Interpretation of Absolute Value

For $p > 0$

1. $|ax + b| < p$ is equivalent to $-p < ax + b < p$.
2. $|ax + b| = p$ is equivalent to $ax + b = p$ or $ax + b = -p$.
3. $|ax + b| > p$ is equivalent to $ax + b < -p$ or $ax + b > p$.

››› CAUTION ›››

Don't try to memorize Theorem 2. Instead, think about each part as a statement about distance from zero on a number line.

EXAMPLE 3 Solving Inequalities Involving Absolute Value

Solve and write the solution in both inequality and interval notation for $|2x - 1| < 3$.

SOLUTION

Geometric Solution

The solution is the set of points x for which $2x - 1$ is less than three units from the origin. This means that $2x - 1$ must be between -3 and 3.

$$-3 < 2x - 1 < 3 \qquad \text{Add 1 to each member.}$$

$$-3 + 1 < 2x - 1 + 1 < 3 + 1$$

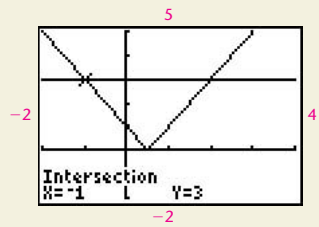
$$-2 < 2x < 4 \qquad \text{Divide each member by 2.}$$

$$\frac{-2}{2} < \frac{2x}{2} < \frac{4}{2}$$

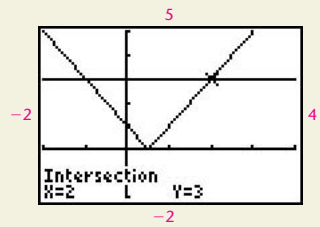
$$-1 < x < 2 \quad \text{or} \quad (-1, 2)$$

Graphical Solution

Enter $y_1 = |2x - 1|$ and $y_2 = 3$ and use INTERSECT to find the intersection points (Fig. 4 and Fig. 5).



› Figure 4



› Figure 5

Examining these graphs, we see that the graph of y_1 is below the graph of y_2 for $-1 < x < 2$.

MATCHED PROBLEM 3

Solve and write the solution in both inequality and interval notation for $|2x + 1| < 5$.

EXAMPLE 4 Solving an Inequality Involving Absolute Value

Solve and express the answer in both inequality and interval notation for $|4x - 5| \geq 2$.

SOLUTION

Geometric Solution

The solution is the set of points x for which $4x - 5$ is two or more units from the origin. This occurs when $4x - 5 \leq -2$ or $4x - 5 \geq 2$:

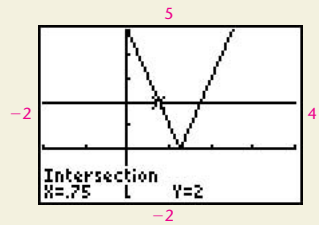
$$\begin{aligned} 4x - 5 &\leq -2 \quad \text{or} \quad 4x - 5 \geq 2 \\ 4x &\leq 3 \quad \text{or} \quad 4x \geq 7 \\ x &\leq \frac{3}{4} = 0.75 \quad \text{or} \quad x \geq \frac{7}{4} = 1.75 \end{aligned}$$

In interval notation, the solution set is

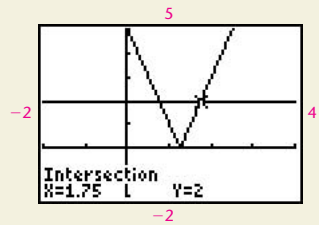
$$(-\infty, 0.75] \cup [1.75, \infty)^* \qquad (3)$$

Graphical Solution

Enter $y_1 = |4x - 5|$ and $y_2 = 2$ and use INTERSECT to find the intersection points (Fig. 6 and Fig. 7).



› Figure 6



› Figure 7

Examining these graphs, we see that if $x \leq 0.75$ or $x \geq 1.75$, then the graph of y_1 is on or above the graph of y_2. This is the same solution given in (3).

MATCHED PROBLEM 4

Solve and express the answer in both inequality and interval notation for $|\frac{2}{3}x + 1| \geq 2$.

*The symbol $\cup$ denotes the union operation for sets. See Appendix B, Section B-1, for a discussion of interval notation and set operations.

Solving Quadratic Inequalities

Solving linear inequalities is very similar to solving linear equations. As we will see, however, the methods we use for solving quadratic equations will not work for quadratic inequalities. This is largely due to the fact that the zero product property does not carry over to inequalities.

EXPLORE-DISCUSS 3

The zero product property tells us that if $a \cdot b = 0$, then one or both of a and b must be zero.

(A) If $a \cdot b > 0$, is it true that $a > 0$ or $b > 0$? Can they both be positive?

(B) If $a \cdot b < 0$, is it true that $a < 0$ or $b < 0$? Can they both be negative?

Consider the following attempt at solving the inequality $(x + 2)(x - 3) > 0$, which is based on the solution of the equation $(x + 2)(x - 3) = 0$.

Equation	**Inequality**
$(x + 2)(x - 3) = 0$	$(x + 2)(x - 3) > 0$
$x + 2 = 0$ or $x - 3 = 0$	$x + 2 > 0$ or $x - 3 > 0$
$x = -2$ or $x = 3$	$x > -2$ or $x > 3$

(C) Graph the alleged solution to the inequality on a number line.

(D) Check this solution by graphing $f(x) = (x + 2)(x - 3)$ on a graphing calculator. Is the alleged solution incorrect? What do you think went wrong?

Now we know that we cannot solve quadratic inequalities just like quadratic equations. But all is not lost. We can use what we know about solving quadratic equations because of the following important fact.

THEOREM 3 The Location Theorem

If $f(x)$ is a continuous function and $f(a)$ and $f(b)$ have opposite signs, then there must be a zero of f somewhere between $x = a$ and $x = b$.*

Why is this helpful? When an inequality has zero on one side, it is a statement about where an expression is positive or negative. Theorem 3 will enable us to identify the x values that make an expression positive or negative by first identifying all of the x values where it can change sign.

*Theorem 3 is actually a special case of a theorem that is very important in calculus called the Intermediate Value Theorem.

EXAMPLE 5 Solving a Quadratic Inequality

Solve the inequality $x^2 - x - 12 > 0$ and write your answer in interval notation.

SOLUTION

Algebraic Solution

Solving the inequality requires finding all x values for which the function $f(x) = x^2 - x - 12$ is positive. This is a continuous function, so Theorem 3 applies.

First, find the zeros of f:

$$x^2 - x - 12 = 0$$
$$(x - 4)(x + 3) = 0$$
$$x = 4, -3$$

These are the only zeros of f, so Theorem 3 implies that f can change sign only at $x = 4$ and $x = -3$. So if f is positive for *any* x values less than -3, it must be positive for *all of them*. Choose a **test number** $x = -5$ (any x value less than -3 will do).

$x = -5$: $(-5)^2 - (-5) - 12 = 18 > 0$ $x^2 - x - 12$ is positive for $x < -3$.

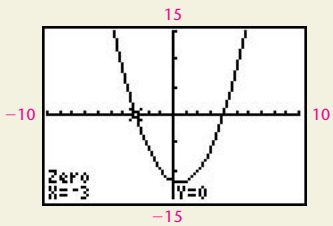
We choose a test number for the intervals $-3 < x < 4$ and $x > 4$, and summarize the information with a table.

Interval	Test Number	Result
$x < -3$	-5	$(-5)^2 - (-5) - 12 = 18$; positive
$-3 < x < 4$	0	$(0)^2 - (0) - 12 = -12$; negative
$x > 4$	10	$(10)^2 - (10) - 12 = 78$; positive

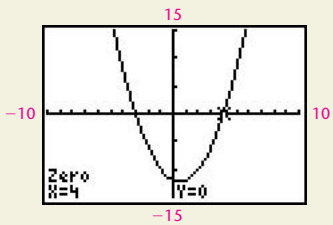
The expression $x^2 - x - 12$ is positive on $x < -3$ or $x > 4$, so the solution to the inequality $x^2 - x - 12 > 0$ in interval notation is $(-\infty, -3) \cup (4, \infty)$.

Graphical Solution

Enter $y_1 = x^2 - x - 12$. Using the ZERO command we see that $y_1 = 0$ at $x = -3$ (Fig. 8) and at $x = 4$ (Fig. 9).



› Figure 8



› Figure 9

The graph of y_1 is above the x axis for $x < -3$ and also for $x > 4$. Thus, the solution to the inequality $y_1 > 0$ is

$$(-\infty, -3) \cup (4, \infty)$$

MATCHED PROBLEM 5

Solve the inequality $x^2 - x - 6 > 0$ and write your answer in interval notation.

Note that both the algebraic and graphical solutions accomplished the same thing. First, we identify the x values where $x^2 - x - 12$ is zero, then decide on which intervals bounded by these x values are positive.

EXAMPLE 6 Solving a Quadratic Inequality

Solve the inequality $2x^2 - 5x + 1 \le 0$. Write your answer in interval notation. Round to two decimal places.

SOLUTION

Algebraic Solution

1. Find the zeros of $f(x) = 2x^2 - 5x + 1$ using the quadratic formula.

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4(2)(1)}}{2(2)}$$
$$= \frac{5 \pm \sqrt{17}}{4} \approx 0.22, 2.28$$

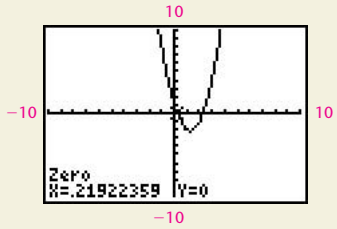
2. Make a table, check test values.

Interval	Test Number	Result
$x < 0.22$	-2	$2(-2)^2 - 5(-2) + 1 = 19$; positive
$0.22 < x < 2.28$	1	$2(1)^2 - 5(1) + 1 = -2$; negative
$x > 2.28$	5	$2(5)^2 - 5(5) + 1 = 26$; positive

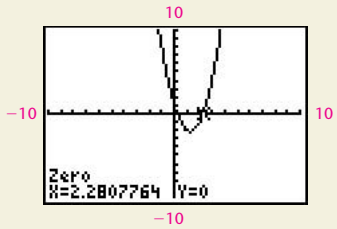
We choose the interval where $2x^2 - 5x + 1$ is negative, and include the values for which it is zero. The solution is $[0.22, 2.28]$ to two decimal places.

Graphical Solution

Enter $y_1 = 2x^2 - 5x + 1$. Using the ZERO command, we see that $y_1 = 0$ at approximately $x = 0.22$ and 2.28 (Figs. 10 and 11).



› Figure 10



› Figure 11

The graph of y_1 is below the x axis for $0.22 < x < 2.28$. We include the zeros in our solution and get $[0.22, 2.28]$.

››› CAUTION ›››

Always think carefully about whether to include the endpoints of intervals when solving quadratic inequalities. If the inequality sign is $\le$ or $\ge$, you will need to include the endpoints.

MATCHED PROBLEM 6

Solve the inequality $-x^2 + 7x - 9 \le 0$. Write your answer in interval notation. Round to two decimal places.

› Mathematical Modeling

EXAMPLE 7 **Projectile Motion**

An artillery shell propelled upward from the ground reaches a maximum height of 576 feet above ground level after 6 seconds. Let the quadratic function $d(t)$ represent the distance above ground level (in feet) t seconds after the shell is released.

(A) Find $d(t)$.

(B) At what times will the shell be more than 320 feet above the ground?

SOLUTIONS

(A) Because the quadratic distance function d has a maximum value of 576 at $t = 6$, the vertex form for $d(t)$ is

$$d(t) = a(t - 6)^2 + 576$$

All that remains is to find a. We use the fact that $d(0) = 0$.

$$\begin{aligned} d(0) = a(-6)^2 + 576 &= 0 \\ 36a &= -576 \\ a &= -16 \end{aligned}$$

The model for the height of this artillery shell is

$$\begin{aligned} d(t) &= -16(t - 6)^2 + 576 \\ &= -16t^2 + 192t \end{aligned}$$

(B) To determine the times when the shell is higher than 320 feet, we solve the inequality

$$d(t) = -16t^2 + 192t > 320$$

Algebraic Solution

$$-16t^2 + 192t > 320$$ Subtract 320 from both sides.

$$-16t^2 + 192t - 320 > 0$$ Divide each side by -16 and reverse the direction of the inequality.

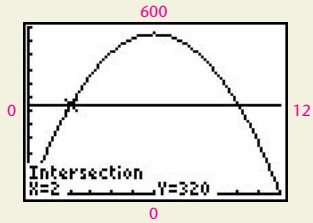
$$\frac{-16t^2}{-16} + \frac{192t}{-16} - \frac{320}{-16} < 0$$

$$t^2 - 12t + 20 < 0$$ Factor.

$$(t - 2)(t - 10) < 0$$

Graphical Solution

Graph $y_1 = -16x^2 + 192x$ and $y_2 = 320$ and find the intersection points (Fig. 12 and Fig. 13).

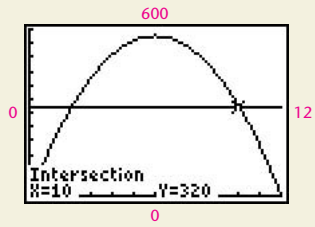


› Figure 12

The zeros of $t^2 - 12t + 20$ are 2 and 10.

Interval	Test Number	Result
$0 < t < 2$	1	$(1)^2 - 12(1) + 20 = 9$; positive
$2 < t < 10$	5	$(5)^2 - 12(5) + 20 = -15$; negative
$t > 10$	12	$(12)^2 - 12(12) + 20 = 20$; positive

The solution to $-16t^2 + 192t > 320$ is $2 < t < 10$, so the shell is above 320 feet between 2 and 10 seconds after it is launched.



› Figure 13

From these graphs we see that the shell will be above 320 feet between 2 and 10 seconds after it is launched.

MATCHED PROBLEM 7

Refer to the shell equation in Example 7. At what times during its flight will the shell be less than 432 feet above the ground?

› Data Analysis and Regression

EXAMPLE 8

Break-Even, Profit, and Loss

A paint manufacturer has weekly fixed costs of \$40,000 and variable costs of \$6.75 per gallon produced. Examining past records produces the price–demand data in Table 1. Round all numbers to three significant digits.

Table 1 Price–Demand Data

Weekly Sales	Price per Gallon
5,610	\$20.50
5,810	\$18.70
5,990	\$17.90
6,180	\$16.20
6,460	\$15.40
6,730	\$13.80
6,940	\$12.90

(A) Use linear regression to find the price–demand equation $p = d(x)$ for the data in Table 1. What is the domain of $d(x)$?

(B) Find the revenue and cost functions as functions of the sales x. What is the domain of each function?

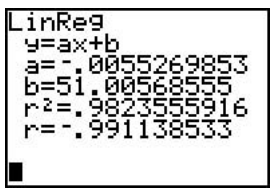
(C) Find the level of sales for which the company will break even. Describe verbally and graphically the sales levels that result in a profit and those that result in a loss.

(D) Find the sales and the price that will produce the maximum profit. Find the maximum profit.

SOLUTIONS

(A) Enter the data in Table 1 and select the LINREG ($ax + b$) option (Fig. 14). After rounding, the price–demand equation is

$$p = d(x) = 51.0 - 0.00553x$$



› Figure 14

Because negative prices don't make sense, x must satisfy

$$\begin{aligned} 51.0 - 0.00553x &\geq 0 && \text{Add } 0.00553x \text{ to both sides.} \\ 51 &\geq 0.00553x && \text{Divide both sides by } 0.00553. \\ x &\leq \frac{51}{0.00553} = 9{,}220 && \text{To three significant digits} \end{aligned}$$

Because sales can't be negative, the domain of the price–demand equation is $0 \leq x \leq 9{,}220$.

(B) The revenue function is

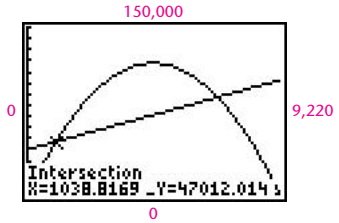
$$\begin{aligned} R(x) &= xp && \text{Revenue is price} \cdot \text{quantity sold.} \\ &= x(51 - 0.00553x) \\ &= 51x - 0.00553x^2 \qquad 0 \leq x \leq 9{,}220 \end{aligned}$$

Note that the domain of R is the same as the domain of d. The cost function is

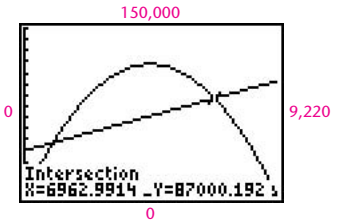
$$C(x) = 40{,}000 + 6.75x \qquad x \geq 0 \qquad \$40{,}000 \text{ fixed costs} + \$6.75 \text{ per gallon}$$

(C) The company will break even when revenue = cost, that is, when $R(x) = C(x)$. An intersection point on the graphs of R and C is often referred to as a **break-even point.** Graphs of both functions and their intersection points are shown in Figures 15 and 16.

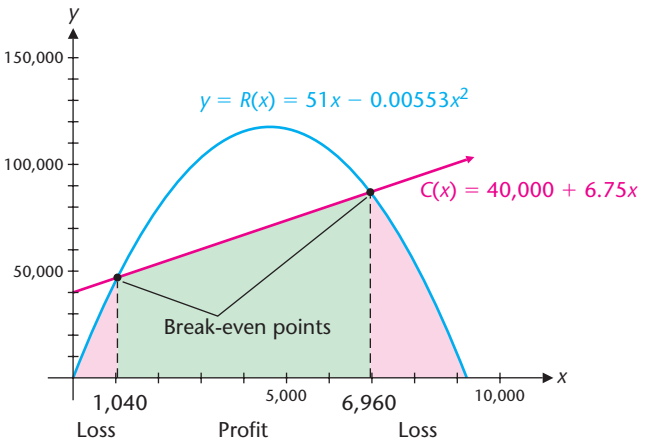
Examining these graphs, we see that the company will break even if they sell 1,040 or 6,960 gallons of paint. If they sell between 1,040 and 6,960 gallons, then revenue is greater than cost and the company will make a profit. If they sell fewer than 1,040 or more than 6,960 gallons, then cost is greater than revenue and the company will lose money. These sales levels are illustrated in Figure 17.



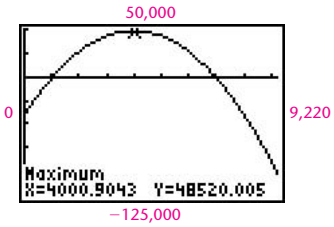
› Figure 15



› Figure 16



› Figure 17



› Figure 18

(D) The profit function for this manufacturer is

$$\begin{aligned} P(x) &= R(x) - C(x) \qquad \text{Profit is revenue − cost.} \\ &= (51x - 0.00553x^2) - (40{,}000 + 6.75x) \\ &= 44.25x - 0.00553x^2 - 40{,}000 \end{aligned}$$

To find the largest profit, enter $y_1 = P(x)$ and use the MAXIMUM command (Fig. 18).

The maximum profit of \$48,500 occurs when 4,000 gallons of paint are sold. The price is

$$\begin{aligned} p = d(4{,}000) &= 51 - 0.00553(4{,}000) \\ &= \$28.90 \end{aligned}$$

MATCHED PROBLEM 8

Table 2 Price–Demand Data

Weekly Sales	Price per Gallon
5,470	\$18.80
5,640	\$17.30
5,910	\$15.90
6,150	\$14.10
6,380	\$13.30
6,530	\$12.40
6,820	\$10.80

A paint manufacturer has weekly fixed costs of \$50,000 and variable costs of \$7.50 per gallon produced. Examining past records produces the price–demand data in Table 2. Round all numbers to three significant digits.

(A) Use linear regression to find the price–demand equation $p = d(x)$ for the data in Table 2. What is the domain of $d(x)$?

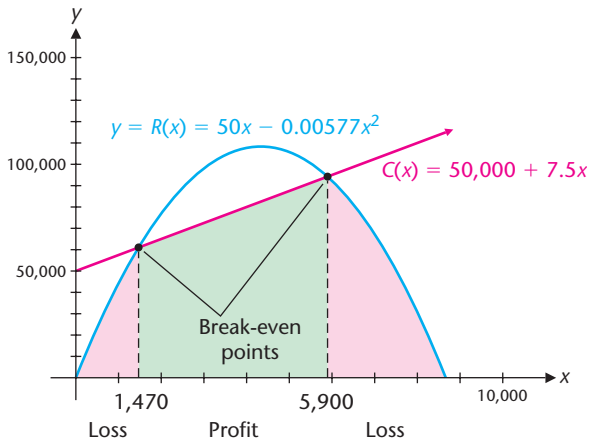
(B) Find the revenue and cost functions as functions of the sales x. What is the domain of each function?

(C) Find the level of sales for which the company will break even. Describe verbally and graphically the sales levels that result in a profit and those that result in a loss.

(D) Find the sales and the price that will produce the maximum profit. Find the maximum profit.

ANSWERS TO MATCHED PROBLEMS

1. $x \geq 3$ or $[3, \infty)$ **2.** $0 \leq x < 5$ or $[0, 5)$ **3.** $-3 < x < 2$ or $(-3, 2)$
4. $x \leq -4.5$ or $x \geq 1.5$; $(-\infty, -4.5] \cup [1.5, \infty)$ **5.** $(-\infty, -2) \cup (3, \infty)$
6. $(-\infty, 1.70] \cup [5.30, \infty)$ **7.** Before 3 seconds and between 9 and 12 seconds after it was launched.
8. (A) $p = d(x) = 50.0 - 0.00577x$, $0 \leq x \leq 8{,}670$
(B) $R(x) = 50x - 0.00577x^2$, $0 \leq x \leq 8{,}670$, $C(x) = 50{,}000 + 7.5x$, $x \geq 0$
(C) The company will break even if they sell 1,470 or 5,900 gallons of paint. If they sell between 1,470 and 5,900 gallons, then revenue is greater than cost and the company will make a profit. If they sell fewer than 1,470 or more than 5,900 gallons, then cost is greater than revenue and the company will lose money.



(D) The company will make a maximum profit of $28,300 when they sell 3,680 gallons at $28.80 per gallon.

2-7 Exercises

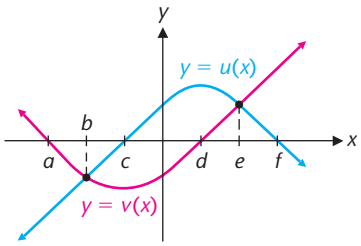
1. Explain in your own words what it means to solve an inequality.

2. What is the main difference between the procedures for solving linear equations and linear inequalities?

3. Why does any inequality of the form $|f(x)| < 0$ have no solution?

4. Explain why it is not true that $a \cdot b > 0$ implies $a > 0$ or $b > 0$.

Use the graphs of functions u and v in the figure to solve the inequalities in Problems 5–12. Express solutions in interval notation.



5. $u(x) > 0$

6. $v(x) \geq 0$

7. $v(x) \geq u(x)$

8. $v(x) < 0$

9. $u(x) \leq 0$

10. $u(x) - v(x) \geq 0$

11. $v(x) - u(x) > 0$

12. $v(x) < u(x)$

In Problems 13–20, write each statement as an absolute value inequality.

13. x is less than five units from 3.

14. w is more than four units from 2.

15. y is more than six units from -1.

16. z is less than eight units from -2.

17. a is no more than five units from 3.

18. c is no less than seven units from -4.

19. d is no less than four units from -2.

20. m is no more than six units from 1.

In Problems 21–24, write each inequality as a verbal statement about distance.

21. $|y| \leq 7$ **22.** $|t| \leq 5$ **23.** $|w| > 7$ **24.** $|r| > 5$

In Problems 25–38, solve and write answers in both interval and inequality notation.

25. $7x - 8 < 4x + 7$

26. $4x + 8 \geq x - 1$

27. $-5t < -10$

28. $-7n \geq 21$

29. $3 - m < 4(m - 3)$ **30.** $2(1 - u) \geq 5u$

31. $(x - 3)(x + 4) < 0$ **32.** $(x + 10)(x - 15) < 0$

33. $x(2x - 7) \geq 0$ **34.** $x(5 - 3x) \geq 0$

35. $|s - 5| < 3$ **36.** $|t - 3| < 4$

37. $|s - 5| > 3$ **38.** $|t - 3| > 4$

In Problems 39–64, solve and write answers in both interval and inequality notation.

39. $-4 < 5t + 6 \leq 21$ **40.** $2 \leq 3m - 7 < 14$

41. $-12 < \frac{3}{4}(2 - x) \leq 24$ **42.** $24 \leq \frac{2}{3}(x - 5) < 36$

43. $\frac{q}{7} - 3 > \frac{q - 4}{3} + 1$ **44.** $\frac{p}{3} - \frac{p - 2}{2} \leq \frac{p}{4} - 4$

45. $x^2 < 10 - 3x$ **46.** $x^2 + x < 12$

47. $x^2 + 21 > 10x$ **48.** $x^2 + 7x + 10 > 0$

49. $x^2 \leq 8x$ **50.** $x^2 \leq 4x$

51. $x^2 + 1 < 2x$ **52.** $x^2 + 25 < 10x$

53. $x^2 - 21 \geq 4x$ **54.** $x^2 + 13x + 40 \leq 0$

55. $x^2 - 5x + 3 > 0$ **56.** $x^2 + 3x - 8 > 0$

57. $x + 7 \geq 2x^2$ **58.** $-10x - 1 \geq 3x^2$

59. $|3x - 7| \leq 4$ **60.** $|5y + 2| \geq 8$

61. $|4 - 2t| > 6$ **62.** $|10 + 4s| < 6$

63. $|0.2u + 1.7| \geq 0.5$ **64.** $|0.5v - 2.5| > 1.6$

In Problems 65–68, replace each question mark with $<$ or $>$ and explain why your choice makes the statement true.

65. If $a - b = 1$, then a ? b.

66. If $u - v = -2$, then u ? v.

67. If $a < 0$, $b < 0$, and $\frac{b}{a} > 1$, then a ? b.

68. If $a > 0$, $b > 0$, and $\frac{b}{a} > 1$, then a ? b.

When studying the concept of limits in calculus, it is very important to specify small distances with inequalities. For Problems 69–72, first write a verbal description of the inequality using distances. Then solve and write your answer in interval notation.

69. $0 < |x - 3| < 0.1$ **70.** $0 < |x - 5| < 0.01$

71. $0 < |x - c| < 2c, c > 0$

72. $0 < |x - 2c| < c, c > 0$

Finding where certain functions are positive and negative plays a very important role in calculus. In Problems 73–76, find the intervals where each function is positive and the intervals where each is negative. Write your answers in interval notation.

73. $f(x) = 2x^2 - 5x - 12$ **74.** $g(x) = -3x^2 - 7x + 10$

75. $h(x) = x^2 + 9x + 2$ **76.** $k(x) = 4 - 5x - 6x^2$

77. Give an example of a quadratic inequality whose solution set is the entire real line.

78. Give an example of a quadratic inequality whose solution set is the empty set.

APPLICATIONS

79. APPROXIMATION The area A of a region is approximately equal to 12.436. The error in this approximation is less than 0.001. Describe the possible values of this area both with an absolute value inequality and with interval notation.

80. APPROXIMATION The volume V of a solid is approximately equal to 6.94. The error in this approximation is less than 0.02. Describe the possible values of this volume both with an absolute value inequality and with interval notation.

81. BREAK-EVEN ANALYSIS An electronics firm is planning to market a new graphing calculator. The fixed costs are \$650,000 and the variable costs are \$47 per calculator. The wholesale price of the calculator will be \$63. For the company to make a profit, revenues must be greater than costs.

(A) How many calculators must be sold for the company to make a profit?

(B) How many calculators must be sold for the company to break even?

(C) Discuss the relationship between the results in parts A and B.

82. BREAK-EVEN ANALYSIS A video game manufacturer is planning to market a handheld version of its game machine. The fixed costs are \$550,000 and the variable costs are \$120 per machine. The wholesale price of the machine will be \$140.

(A) How many game machines must be sold for the company to make a profit?

(B) How many game machines must be sold for the company to break even?

(C) Discuss the relationship between the results in parts A and B.

83. BREAK-EVEN ANALYSIS The electronics firm in Problem 81 finds that rising prices for parts increase the variable costs to \$50.50 per calculator.

(A) Discuss possible strategies the company might use to deal with this increase in costs.

(B) If the company continues to sell the calculators for \$63, how many must they sell now to make a profit?

(C) If the company wants to start making a profit at the same production level as before the cost increase, how much should they increase the wholesale price?

84. **BREAK-EVEN ANALYSIS** The video game manufacturer in Problem 82 finds that unexpected programming problems increase the fixed costs to $660,000.
(A) Discuss possible strategies the company might use to deal with this increase in costs.
(B) If the company continues to sell the game machines for $140, how many must they sell now to make a profit?
(C) If the company wants to start making a profit at the same production level as before the cost increase, how much should they increase the wholesale price?

85. **PROFIT ANALYSIS** A screen printer produces custom silk-screen apparel. The cost $C(x)$ of printing x custom T-shirts and the revenue $R(x)$ from the sale of x T-shirts (both in dollars) are given by

$$C(x) = 200 + 2.25x$$
$$R(x) = 10x - 0.05x^2$$

Determine the production levels x (to the nearest integer) that will result in the printer showing a profit.

86. **PROFIT ANALYSIS** Refer to Problem 85. Determine the production levels x (to the nearest integer) that will result in the printer showing a profit of at least $60.

87. **CELSIUS/FAHRENHEIT** A formula for converting Celsius degrees to Fahrenheit degrees is given by the linear function

$$F = \frac{9}{5}C + 32$$

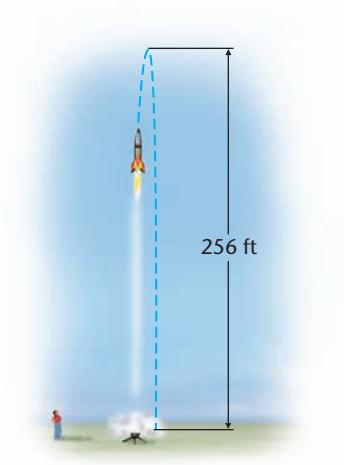
Determine to the nearest degree the Celsius range in temperature that corresponds to the Fahrenheit range of 60°F to 80°F.

88. **CELSIUS/FAHRENHEIT** A formula for converting Fahrenheit degrees to Celsius degrees is given by the linear function

$$C = \frac{5}{9}(F - 32)$$

Determine to the nearest degree the Fahrenheit range in temperature that corresponds to a Celsius range of 20°C to 30°C.

89. **PROJECTILE MOTION** A projectile propelled straight upward from the ground reaches a maximum height of 256 feet above ground level after 4 seconds. Let the quadratic function $d(t)$ represent the distance above ground level (in feet) t seconds after the projectile is released.
(A) Find $d(t)$.
(B) At what times will the projectile be more than 240 feet above the ground? Write and solve an inequality to find the times. Express the answer in inequality notation.



90. **PROJECTILE MOTION** A projectile propelled straight upward from the ground reaches a maximum height of 784 feet above ground level after 7 seconds. Let the quadratic function $d(t)$ represent the distance above ground level (in feet) t seconds after the projectile is released.
(A) Find $d(t)$.
(B) At what times will the projectile be less than 640 feet above the ground? Write and solve an inequality to find the times. Express the answer in inequality notation.

91. **EARTH SCIENCE** Deeper and deeper holes are being bored into the Earth's surface every year in search of energy in the form of oil, gas, or heat. A bore at Windischeschenbach in the North German basin has reached a depth of more than 8 kilometers. The temperature in the bore is 30°C at a depth of 1 kilometer and increases 2.8°C for each additional 100 meters of depth. Find a mathematical model for the temperature T at a depth of x kilometers. At what interval of depths will the temperature be between 150°C and 200°C? Round answers to three decimal places.

92. **EARTH SCIENCE** A bore at Basel, Switzerland, has reached a depth of more than 5 kilometers. The temperature is 35°C at a depth of 1 kilometer and increases 3.6°C for each additional 100 meters of depth. Find a mathematical model for the temperature T at a depth of x kilometers. At what interval of depths will the temperature be between 100°C and 150°C? Round answers to three decimal places.

DATA ANALYSIS AND REGRESSION

Twice each day 70 weather stations in the United States release high-altitude balloons containing instruments that send various data back to the station. Eventually, the balloons burst and the instruments parachute back to Earth to be reclaimed. The air

pressure (in hectopascals), the altitude (in meters), and the temperature (in degrees Celsius) collected on the same day at two midwestern stations are given in Table 3. Round all numbers to three significant digits.*

Table 3 Upper-Air Weather Data

North Platte, NE			Minneapolis, MN		
PRES	HGT	TEMP	PRES	HGT	TEMP
745	2,574	8	756	2,438	2
700	3,087	5	728	2,743	0
627	3,962	−1	648	3,658	−4
559	4,877	−8	555	4,877	−10
551	4,992	−8	500	5,680	−17
476	6,096	−16	400	7,330	−28
404	7,315	−24	367	7,944	−32
387	7,620	−27	300	9,330	−45
300	9,410	−43	250	10,520	−55
259	10,363	−49	241	10,751	−57

Source: NOAA Air Resources Laboratory

93. WEATHER Let x be the altitude of the balloon released from North Platte and let y be the corresponding temperature. Use linear regression to find a linear function $y = ax + b$ that fits these data. For what altitudes will the temperature be between −10°C and −30°C?

94. WEATHER Let x be the altitude of the balloon released from Minneapolis and let y be the corresponding temperature. Use linear regression to find a linear function $y = ax + b$ that fits these data. For what altitudes will the temperature be between −20°C and −40°C?

95. WEATHER Let x be the altitude of the balloon released from North Platte and let y be the corresponding air pressure. Use linear regression to find a linear function $y = ax + b$ that fits these data. For what altitudes will the air pressure be between 350 hectopascals and 650 hectopascals?

96. WEATHER Let x be the altitude of the balloon released from Minneapolis and let y be the corresponding air pressure. Use linear regression to find a linear function $y = ax + b$ that fits these data. For what altitudes will the air pressure be between 350 hectopascals and 650 hectopascals?

*A unit of pressure equivalent to 1 millibar.

97. BREAK-EVEN ANALYSIS Table 4 contains weekly price–demand data for orange juice and grapefruit juice for a fruit juice producer. The producer has weekly fixed costs of $20,000 and variable costs of $0.50 per gallon of orange juice produced.
(A) Use linear regression to find the price–demand equation $p = d(x)$ for the orange juice data in Table 4. What is the domain of $d(x)$?
(B) Find the revenue and cost functions as functions of the sales x. What is the domain of each function?
(C) Find the level of sales for which the company will break even. Describe verbally and graphically the sales levels that result in a profit and those that result in a loss.
(D) Find the sales and the price that will produce the maximum profit. Find the maximum profit.

Table 4 Fruit Juice Production

Orange Juice Demand (gal.)	Price	Grapefruit Juice Demand (gal.)	Price
21,800	$1.95	2,130	$2.32
24,300	$1.81	2,480	$2.21
26,700	$1.43	2,610	$2.07
28,900	$1.37	2,890	$1.87
29,700	$1.28	3,170	$1.81
33,700	$1.14	3,640	$1.68
34,800	$0.96	4,350	$1.56

98. BREAK-EVEN ANALYSIS The juice producer in Problem 97 has weekly fixed costs of $3,000 and variable costs of $0.40 per gallon of grapefruit juice produced.
(A) Use linear regression to find the price–demand equation $p = d(x)$ for the grapefruit juice data in Table 4. What is the domain of $d(x)$?
(B) Find the revenue and cost functions as functions of the sales x. What is the domain of each function?
(C) Find the level of sales for which the company will break even. Describe verbally and graphically the sales levels that result in a profit and those that result in a loss.
(D) Find the sales and the price that will produce the maximum profit. Find the maximum profit.

CHAPTER 2 Review

2-1 Linear Functions

A function f is a **linear function** if $f(x) = mx + b$, $m \neq 0$, where m and b are real numbers. The **domain** is the set of all real numbers and the **range** is the set of all real numbers. If $m = 0$, then f is called a **constant function,** $f(x) = b$, which has the set of all real numbers as its **domain** and the constant b as its **range.** The **standard form** for the equation of a line is $Ax + By = C$, where A, B, and C are real constants, and A and B are not both 0. Every straight line in a Cartesian coordinate system is the graph of an equation of this type. The **slope** of a line is a number that measures how steep a line is. The slope of the line through the points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad x_1 \neq x_2$$

The slope is not defined for a vertical line where $x_1 = x_2$.

Equations of a Line

Standard Form	$Ax + By = C$	A and B not both 0
Slope–Intercept Form	$y = mx + b$	Slope: m; y intercept: b
Point–Slope Form	$y - y_1 = m(x - x_1)$	Slope: m; Point: (x_1, y_1)
Horizontal Line	$y = b$	Slope: 0
Vertical Line	$x = a$	Slope: Undefined

To graph a linear function, it is sufficient to plot two points and connect them with a line. But it is a good idea to plot three points to check for possible errors.

Two nonvertical lines with slopes m_1 and m_2 are **parallel** if and only if $m_1 = m_2$ and **perpendicular** if and only if $m_1 m_2 = -1$.

The slope of a linear function describes the rate at which the output of the function changes as the input changes. In short, slope can be interpreted as a **rate of change.** The y intercept of a linear cost function is called the **fixed cost** and the slope is called the **variable cost.**

2-2 Linear Equations and Models

Solving an equation is the process of finding all values of the variable that make the equation a true statement. An equation that is true for some values of the variable is called a **conditional equation.** An equation that is true for all permissible values of the variable is called an **identity,** and the solution is all real numbers. An equation that is false for all permissible values of the variable is called a **contradiction,** and has no solution.

Linear equations are solved by performing algebraic steps that result in equivalent equations until the result is an equation whose solution is obvious. When an equation has fractions, begin by multiplying both sides by the least common denominator of all the fractions. The formula distance = rate · time is useful in modeling problems that involve motion.

Linear regression is used to fit a curve to a data set. A **scatter diagram** is a graph of a data set. **Diagnostics** indicate how well a curve fits a data set. Supply and demand curves intersect at the **equilibrium point,** which consists of the **equilibrium price** and **equilibrium quantity.**

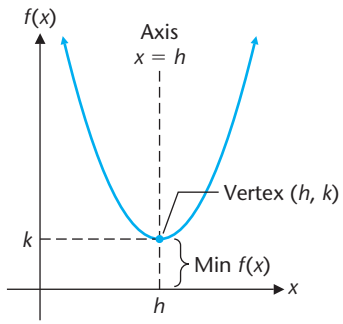
2-3 Quadratic Functions

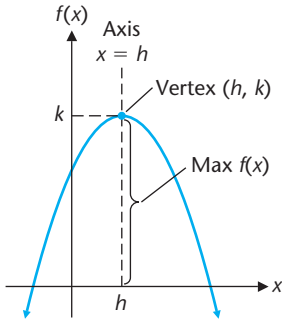
If a, b, and c are real numbers with $a \neq 0$, then the function $f(x) = ax^2 + bx + c$ is a **quadratic function** (in **general form**) and its graph is a **parabola. Completing the square** of the quadratic expression $x^2 + bx$ produces a perfect square:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Completing the square for $f(x) = ax^2 + bx + c$ produces the **vertex form** $f(x) = a(x - h)^2 + k$ and gives the following properties:

1. The graph of f is a parabola:





2. Vertex: (h, k) (Parabola increases on one side of the vertex and decreases on the other.)

3. Axis (of symmetry): $x = h$ (parallel to y axis)

4. $f(h) = k$ is the minimum if $a > 0$ and the maximum if $a < 0$.

5. Domain: All real numbers
Range: $(-\infty, k]$ if $a < 0$ or $[k, \infty)$ if $a > 0$

6. The graph of f is the graph of $g(x) = ax^2$ translated horizontally h units and vertically k units.

The first coordinate of the vertex of a parabola in standard form can be located using the formula $x = -b/2a$. This can then be substituted into the function to find the second coordinate. The vertex form of a parabola can be used to find the equation when the vertex and one other point on the graph are known.

2-4 Complex Numbers

A **complex number** in **standard form** is a number in the form $a + bi$ where a and b are real numbers and i denotes a square root of -1. The number i is known as the **imaginary unit.** For a complex number $a + bi$, a is the **real part** and bi is the **imaginary part.**

If $b \neq 0$ then $a + bi$ is also called an **imaginary number.** If $a = 0$ then $0 + bi = bi$ is also called a **pure imaginary number.** If $b = 0$ then $a + 0i = a$ is a **real number.** The complex **zero** is $0 + 0i = 0$. The **conjugate** of $a + bi$ is $a - bi$. **Equality, addition,** and **multiplication** are defined as follows:

1. $a + bi = c + di$ if and only if $a = c$ and $b = d$

2. $(a + bi) + (c + di) = (a + c) + (b + d)i$

3. $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

Because complex numbers obey the same commutative, associative, and distributive properties as real numbers, most operations with complex numbers are performed by using these properties in the same way that algebraic operations are performed on the expression $a + bx$. Keep in mind that $i^2 = -1$.

The **property of conjugates,**

$$(a + bi)(a - bi) = a^2 + b^2$$

can be used to find **reciprocals** and **quotients.** To divide by a complex number, we multiply the numerator and denominator by the conjugate of the denominator. This enables us to write the result in $a + bi$ form. If $a > 0$, then the **principal square root of the negative real number** $-a$ is $\sqrt{-a} = i\sqrt{a}$.

To solve equations involving complex numbers, set the real and imaginary parts equal to each other and solve.

2-5 Quadratic Equations and Models

A **quadratic equation** is an equation that can be written in the form

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

where x is a variable and a, b, and c are real numbers. This is known as the **general form** for a quadratic equation.

Algebraic methods of solution include:

1. Factoring and using the **zero product property:** $m \cdot n = 0$ if and only if $m = 0$ or $n = 0$ (or both).

2. Completing the square and using the **square root property:** If A is a complex number, C is a real number, and $A^2 = C$, then $A = \pm\sqrt{C}$.

3. Using the **quadratic formula:**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If the **discriminant** $\boldsymbol{b^2 - 4ac}$ is positive, the equation has two distinct **real roots;** if the discriminant is 0, the equation has one real **double root;** and if the discriminant is negative, the equation has two **imaginary roots,** each the conjugate of the other.

2-6 Additional Equation-Solving Techniques

A **radical** can be eliminated from an equation by isolating the radical on one side of the equation and raising both sides of the equation to the same natural number power to produce a new equation. The solution set of the original equation is a subset of the solution set of the new equation. The new equation may have **extraneous solutions** that are not solutions of the original equation. Consequently, **every solution of the new equation must be checked in the original equation to eliminate extraneous solutions.** If an equation contains more than one radical, then the process of isolating a radical and raising both sides to the same natural number power can be repeated until all radicals are eliminated. If a substitution transforms an equation into the form $au^2 + bu + c = 0$, where u is an expression in some other variable, then the equation is an **equation of quadratic type,** which can be solved by quadratic methods.

2-7 Solving Inequalities

Linear inequalities in one variable are expressed using the inequality symbols $<$, $>$, $\leq$, $\geq$. The **solution set of an inequality** is the set of all values of the variable that make the inequality a true statement. Each element of the solution set is called a **solution.** Two inequalities are **equivalent** if they have the same solution set. An equivalent inequality will result and the **sense (or direction) will remain the same** if each side of the original inequality:

- Has the same real number added to or subtracted from it.
- Is multiplied or divided by the same positive number.

An equivalent inequality will result and the **sense (or direction) will reverse** if each side of the original inequality:

- Is multiplied or divided by the same negative number.

Note that multiplication by 0 and division by 0 are not permitted.

A linear **combined inequality** (one that has three members) can be solved by performing the same operations on all three members until the variable is isolated in the center.

The **absolute value function** $|x|$ can also be interpreted as the **distance** between x and the origin. More generally, for $p > 0$:

1. $|ax + b| < p$ is equivalent to $-p < ax + b < p$.
2. $|ax + b| = p$ is equivalent to $ax + b = p$ or $ax + b = -p$.
3. $|ax + b| > p$ is equivalent to $ax + b < -p$ or $ax + b > p$.

Quadratic inequalities can be solved with the following process:

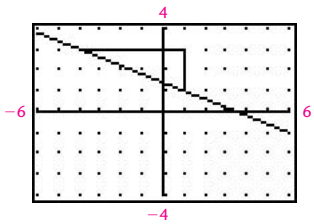
1. Rearrange so that zero is on one side, and a quadratic expression is on the other.
2. Find the roots of that expression.
3. Choose a test number in each interval determined by the roots and substitute into the quadratic expression, noting whether the result is positive or negative.
4. Write the solution based on where the quadratic expression is positive, negative, or zero.

A **break-even point** is an intersection point for the graphs of a cost and a revenue equation.

CHAPTER 2 Review Exercises

Work through all the problems in this chapter review and check answers in the back of the book. Answers to most review problems are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

1. Use the graph of the linear function in the figure to find the rise, run, and slope. Write the equation of the line in the form $Ax + By = C$, where A, B, and C are integers with $A > 0$. (The horizontal and vertical line segments have integer lengths.)



2. Graph $3x + 2y = 9$ and indicate its slope.

3. Write an equation of a line with x intercept 6 and y intercept 4. Write the final answer in the form $Ax + By = C$, where A, B, and C are integers with $A > 0$.

4. Write the slope–intercept form of the equation of the line with slope $-\frac{2}{3}$ and y intercept 2.

5. Write the equations of the vertical and horizontal lines passing through the point $(-3, 4)$. What is the slope of each?

6. Solve algebraically and confirm graphically:

(A) $0.05x + 0.25(30 - x) = 3.3$

(B) $\dfrac{5x}{3} - \dfrac{4 + x}{2} = \dfrac{x - 2}{4} + 1$

In Problems 7 and 8,

(A) *Complete the square and find the vertex form of the function.*

(B) *Write a brief verbal description of the relationship between the graph of the function and the graph of* $y = x^2$.

(C) *Find the x intercepts algebraically and confirm graphically.*

7. $f(x) = -x^2 - 2x + 3$ **8.** $f(x) = x^2 - 3x - 2$

9. Find the vertex of the function $f(x) = -3x^2 + 9x - 4$. Is it a maximum or a minimum?

10. Perform the indicated operations and write the answers in standard form:

(A) $(-3 + 2i) + (6 - 8i)$ (B) $(3 - 3i)(2 + 3i)$

(C) $\dfrac{13 - i}{5 - 3i}$ (D) $(7 + 6i) - (8 + 9i)$

In Problems 11–21, solve algebraically and confirm graphically, if possible.

11. $4 - 3(2x + 7) = 5(4 - x) + 11x - 3$

12. $(2x + 5)^2 = 23$ **13.** $x^2 - 2x + 3 = x^2 + 3x - 7$

14. $2x^2 - 7 = 0$ **15.** $2x^2 = 4x$

16. $2x^2 = 7x - 3$ **17.** $m^2 + m + 1 = 0$

18. $y^2 = \frac{3}{2}(y + 1)$ **19.** $\sqrt{4 + 7x} = 5$

20. $\sqrt{5x - 6} - x = 0$ **21.** $\sqrt{x^2 + 2} = \sqrt{4x + 14}$

In Problems 22–24, solve and express answers in inequality and interval notation.

22. $3(2 - x) - 2 \le 2x - 1$ **23.** $x^2 + x < 20$

24. $x^2 > 4x + 12$

25. Discuss the use of the terms *rising, falling, increasing,* and *decreasing* as they apply to the descriptions of the following:

(A) A line with positive slope

(B) A line with negative slope

(C) A parabola that opens upward

(D) A parabola that opens downward

26. Find an equation of the line through the points $(-4, 3)$ and $(0, -3)$. Write the final answer in the form $Ax + By = C$, where A, B, and C are integers with $A > 0$.

27. Write the slope–intercept form of the equation of the line that passes through the point $(-2, 1)$ and is

(A) parallel to the line $6x + 3y = 5$

(B) perpendicular to the line $6x + 3y = 5$

In Problems 28–30, solve each inequality. Write answers in inequality notation.

28. $|y + 9| < 5$ **29.** $|2x - 8| \ge 3$ **30.** $2x^2 \ge 7x - 1$

In Problems 31 and 32, write each inequality verbally as a statement about distance, then solve. Write answers in both interval and inequality notation.

31. $|y - 5| \le 2$ **32.** $|t + 6| > 9$

For each equation in Problems 33–35, use the discriminant to determine the number and type of zeros and confirm graphically.

33. $0.1x^2 + x + 1.5 = 0$ **34.** $0.1x^2 + x + 2.5 = 0$

35. $0.1x^2 + x + 3.5 = 0$

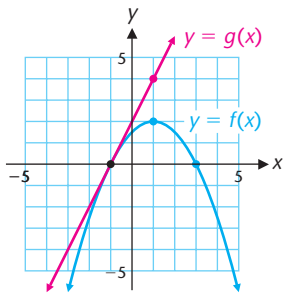
36. Let $f(x) = 0.5x^2 - 4x + 5$.

(A) Sketch the graph of f and label the axis and the vertex.

(B) Where is f increasing? Decreasing? What is the range? (Express answers in interval notation.)

(C) Find the maximum or minimum.

37. Find the equations of the linear function g and the quadratic function f whose graphs are shown in the figure. This line is called the tangent line to the graph of f at the point $(-1, 0)$.



38. Perform the indicated operations and write the final answers in standard form:

(A) $(3 + i)^2 - 2(3 + i) + 3$ (B) i^{27}

39. Convert to $a + bi$ forms, perform the indicated operations, and write the final answers in standard form:

(A) $(2 - \sqrt{-4}) - (3 - \sqrt{-9})$ (B) $\dfrac{2 - \sqrt{-1}}{3 + \sqrt{-4}}$

(C) $\dfrac{4 + \sqrt{-25}}{\sqrt{-4}}$ (D) $\sqrt{-16}\sqrt{-25}$

Solve Problems 40–45 algebraically and confirm graphically, if possible.

40. $(x + \frac{5}{2})^2 = \frac{5}{4}$ **41.** $1 + \dfrac{3}{u^2} = \dfrac{2}{u}$

42. $2x + 3\sqrt{4x^2 - 4x + 9} = 1$

43. $2x^{2/3} - 5x^{1/3} - 12 = 0$

44. $m^4 + 5m^2 - 36 = 0$ **45.** $\sqrt{y - 2} - \sqrt{5y + 1} = -3$

46. Use linear regression to fit a line to each of the following data sets. How are the graphs of the two functions related? How are the two functions related?

(A)

x	y
2	−1
3	1
4	3

(B)

x	y
−1	2
1	3
3	4

47. Can a quadratic function have only imaginary zeros? If not, explain why. If so, give an example and discuss any special relationship between the zeros.

48. If a quadratic function has only imaginary zeros, can the function be graphed? If not, explain why. If so, what is the graph's relationship to the x axis?

49. Consider the quadratic equation

$$x^2 - 6x + c = 0$$

where c is a real number. Discuss the relationship between the values of c and the three types of roots listed in Table 1 in Section 2-5.

Solve Problems 50 and 51 for the indicated variable in terms of the other variables.

50. $P = M - Mdt$ for M (mathematics of finance)

51. $P = EI - RI^2$ for I (electrical engineering)

52. For what values of a and b is the following inequality true?

$$a + b < b - a$$

53. If a and b are negative numbers and $a > b$, then is a/b greater than 1 or less than 1?

54. Solve and graph. Write the answer using interval notation: $0 < |x - 6| < d$

55. Evaluate: $(a + bi)\left(\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i\right)$; $a, b, \neq 0$

56. Are the graphs of $mx - y = b$ and $x + my = b$ parallel, perpendicular, or neither? Justify your answer.

57. Show by substituting for x that $2 + 3i$ is one of the zeros of $f(x) = 2x^2 - 8x + 26$. Without solving an equation, what is the other one?

58. If $5 + 2i$ is one root of $x^2 - bx + c = 0$, find b and c.

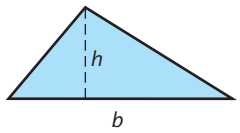
59. Solve $3x^{-2/5} - 4x^{-1/5} + 1 = 0$ algebraically and graphically.

60. Find all solutions of $x^3 + 1 = 0$.

61. Find three consecutive integers whose sum is 144.

62. Find three consecutive even integers so that the first plus twice the second is twice the third.

Problems 63 and 64 refer to a triangle with base b and height h (see the figure). Write a mathematical expression in terms of b and h for each of the verbal statements in Problems 63 and 64.



63. The base is five times the height.

64. The height is one-fourth of the base.

APPLICATONS

65. METEOROLOGY The number of inches of snow on the ground during a blizzard in upstate New York on one February day can be modeled by $S(x) = 3.4x + 11.1$, where x is hours after it started snowing.

(A) At what rate is the snow falling? Include units.

(B) How much snow was on the ground before it started to snow that day?

66. INTERNET GROWTH According to the British newspaper the *Telegraph,* on August 1, 2006, there were 60 million online blogs worldwide, and that number was growing at the rate of 75,000 per day.

(A) Write a linear function describing the number of blogs worldwide in terms of months after August 1, 2006. (Note the difference in units of time.)

(B) When does your function predict that the number of blogs would hit 100 million?

67. TIME SPENT STUDYING As part of a group project for a statistics class, several students at a Midwestern university found that the score students could expect on the final exam in a history class could be predicted by the function $G(x) = -0.23x^2 + 8.54x + 15$, where x is the total number of hours spent studying for the final. If this model is accurate, how many hours should a student in this course study to maximize their score?

68. GEOMETRY The diagonal of a rectangle is 32.5 inches and the area is 375 square inches. Find the dimensions of the rectangle, correct to one decimal place.

69. FALLING OBJECT A worker at the top of a radio tower drops a hammer to the ground. If the hammer hits the ground 3.5 seconds after it is dropped, how high is the tower?

70. **COST ANALYSIS** Cost equations for manufacturing companies are often quadratic—costs are high at very low and very high production levels. The weekly cost $C(x)$ (in dollars) for manufacturing x inexpensive calculators is

$$C(x) = 0.001x^2 - 9.5x + 30{,}000$$

Find the production level(s) (to the nearest integer) that

(A) Produces the minimum weekly cost. What is the minimum weekly cost (to the nearest cent)?

(B) Produces a weekly cost of $12,000.

(C) Produces a weekly cost of $6,000.

71. **BREAK-EVEN ANALYSIS** The manufacturing company in Problem 70 sells its calculators to wholesalers for $3 each. How many calculators (to the nearest integer) must the company sell to break even?

72. **PROFIT ANALYSIS** Refer to Problems 70 and 71. Find the production levels that produce a profit and the production levels that produce a loss.

73. **LINEAR DEPRECIATION** A computer system was purchased by a small company for $12,000 and is assumed to have a depreciated value of $2,000 after 8 years. If the value depreciates linearly from $12,000 to $2,000:

(A) Find the linear equation that relates value V (in dollars) to time t (in years).

(B) What would be the depreciated value of the system after 5 years?

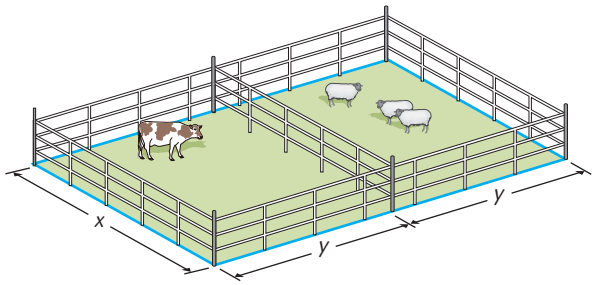
74. **BUSINESS–PRICING** A sporting goods store carries a brand of tennis shorts that costs them $30 per pair and a brand of sunglasses that costs them $20 per pair. They sell the shorts for $48 and the glasses for $32.

(A) If the markup policy of the store for items that cost over $10 is assumed to be linear and is reflected in the pricing of these two items, write an equation that expresses retail price R as a function of cost C.

(B) What should be the retail price of a pair of skis that cost $105?

75. **INCOME** A salesperson receives a base salary of $400 per week and a commission of 10% on all sales over $3,000 during the week. If x represents the salesperson's weekly sales, express the total weekly earnings $E(x)$ as a function of x. Find $E(2{,}000)$ and $E(5{,}000)$.

76. **CONSTRUCTION** A farmer has 120 feet of fencing to be used in the construction of two identical rectangular pens sharing a common side (see the figure).



(A) Express the total area $A(x)$ enclosed by both pens as a function of the width x.

(B) From physical considerations, what is the domain of the function A?

(C) Find the dimensions of the pens that will make the total enclosed area maximum.

77. **SPORTS MEDICINE** The following quotation was found in a sports medicine handout: "The idea is to raise and sustain your heart rate to 70% of its maximum safe rate for your age. One way to determine this is to subtract your age from 220 and multiply by 0.7."

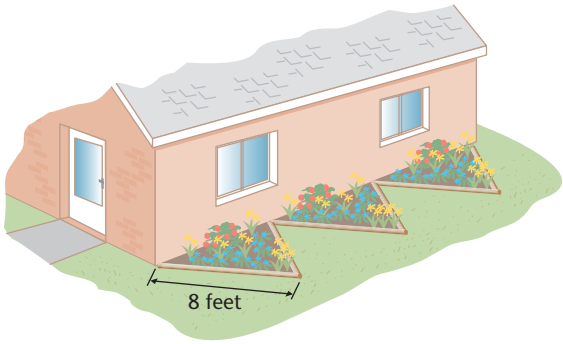
(A) If H is the maximum safe sustained heart rate (in beats per minute) for a person of age A (in years), write a formula relating H and A.

(B) What is the maximum safe sustained heart rate for a 20-year-old?

(C) If the maximum safe sustained heart rate for a person is 126 beats per minute, how old is the person?

78. **DESIGN** The pages of a textbook have uniform margins of 2 centimeters on all four sides (see the figure). If the area of the entire page is 480 square centimeters and the area of the printed portion is 320 square centimeters, find the dimensions of the page.

79. **DESIGN** A landscape designer uses 8-foot timbers to form a pattern of three identical isosceles triangles along the wall of a building (see the figure). If the area of each triangle is 24 square feet, find the base correct to two decimal places.



80. **ARCHITECTURE** An entranceway in the shape of a parabola 12 feet wide and 12 feet high must enclose a rectangular door that is 8.4 feet high. What is the widest doorway (to the nearest tenth of a foot) that can be installed in the entranceway?

DATA ANALYSIS AND REGRESSION

In Problems 81–84, unless directed otherwise, round all numbers to three significant digits.

81. **DRUG USE** The use of marijuana by teenagers declined throughout the 1980s, but began to increase during the 1990s. Table 1 gives the percentage of 12- to 17-year-olds who have ever used marijuana for selected years from 1979 to 2003.

Table 1 Marijuana Use: 12 to 17 Years Old

Year	Ever Used [%]
1979	26.7
1985	20.1
1990	12.7
1994	13.6
1995	16.2
2003	19.6

Source: National Household Survey on Drug Abuse

(A) Find a quadratic regression model for the percentage of 12- to 17-year-olds who have ever used marijuana, using years since 1970 for the independent variable.

(B) Use your model to predict the year during which the percentage of marijuana users will return to the 1979 level.

82. **POLITICAL SCIENCE** Association of economic class and party affiliation did not start with Roosevelt's New Deal; it goes back to the time of Andrew Jackson (1767–1845). Paul Lazarsfeld of Columbia University published an article in the November 1950 issue of *Scientific American* in which he discusses statistical investigations of the relationships between economic class and party affiliation. The data in Table 2 are taken from this article.

Table 2 Political Affiliations in 1836

Ward	Average Assessed Value per Person [in $100]	Democratic Votes [%]
12	1.7	51
3	2.1	49
1	2.3	53
5	2.4	36
2	3.6	65
11	3.7	35
10	4.7	29
4	6.2	40
6	7.1	34
9	7.4	29
8	8.7	20
7	11.9	23

(A) Find a linear regression model for the data in the second and third columns of the table, using the average assessed value as the independent variable.

(B) Use the linear regression model to predict (to two decimal places) the percentage of votes for democrats in a ward with an average assessed value of $300.

83. **SUPPLY AND DEMAND** Table 3 contains price–supply data and price–demand data for a broccoli grower. Find a linear model for the price–supply data where x is supply (in pounds) and y is price (in dollars). Do the same for the price–demand data. Find the equilibrium price for broccoli.

Table 3 Supply and Demand for Broccoli

Price $/lb	Supply (lb)	Demand (lb)
0.71	25,800	41,500
0.77	27,400	38,700
0.84	30,200	36,200
0.91	33,500	32,800
0.96	34,900	29,800
1.01	37,800	27,900
1.08	39,210	25,100

84. BREAK-EVEN ANALYSIS The broccoli grower in Problem 83 has fixed costs of $15,000 and variable costs of $0.20 per pound of broccoli produced.

(A) Find the revenue and cost functions as functions of the sales x. What is the domain of each function?

(B) Find the level of sales for which the company will break even. Describe verbally and graphically the sales levels that result in a profit and those that result in a loss.

(C) Find the sales and the price that will produce the maximum profit. Find the maximum profit.

85. OPTIMAL SPEED Table 4 contains performance data for a speedboat powered by a Yamaha outboard motor.

(A) Let x be the speed of the boat in miles per hour (mph) and y the associated mileage in miles per gallon (mpg). Use the data in Table 4 to find a quadratic regression function $y = ax^2 + bx + c$ for this boat.

(B) A marina rents this boat for $15 per hour plus the cost of the gasoline used. If gasoline costs $2.55 per gallon and you take a 100-mile trip in this boat, construct a mathematical model and use it to answer the following questions:

What speed should you travel to minimize the rental charges?

What mileage will the boat get?

How long does the trip take?

How much gasoline will you use?

How much will the trip cost you?

Table 4 Performance Data

mph	mpg
9.5	1.67
21.1	1.92
28.3	2.16
33.7	1.88
37.9	1.77
42.6	1.49

Source: www.yamaha-motor.com

CHAPTER 2

››› GROUP ACTIVITY Mathematical Modeling in Population Studies

In a study on population growth in California, Tulane University demographer Leon Bouvier recorded the past population totals for every 10 years starting at 1900. Then, using sophisticated demographic techniques, he made high, low, and medium projections to the year 2040. Table 1 shows actual populations up to 1990 and medium projections (to the nearest million) to 2040.

Table 1 California Population 1900–2040

Years after 1900	Date	Population [millions]	
0	1900	2	Actual
10	1910	3	
20	1920	4	
30	1930	5	
40	1940	5	
50	1950	10	
60	1960	15	
70	1970	20	
80	1980	23	
90	1990	30	
100	2000	35	Projected
110	2010	45	
120	2020	53	
130	2030	61	
140	2040	70	

1. Building a Mathematical Model.

(A) Plot the first and last columns in Table 1 up to 1990 (actual populations). Would a linear or a quadratic function be the better model for these data? Why?

(B) Use a graphing utility to compute both a linear and a quadratic regression function to model the data you plotted in part A.

(C) Graph both functions and the data from part A for $0 \leq x \leq 150$.

(D) Based on the graph, which model appears to be more realistic? Explain your reasoning.

2. Using the Mathematical Models for Projections.

For each regression model, answer the following questions.

(A) Calculate projected populations for California at 10-year intervals, starting at 2000 and ending at 2040. Compare your projections with Professor Bouvier's projections, both numerically and graphically.

(B) During what year would each model project that the population will reach 40 million? 50 million?

(C) For what years would each model project the population to be between 34 million and 68 million, inclusive?

(D) Use a library or the Internet to look up the 2000 population of California. Which model predicted it most accurately: linear, quadratic, or Professor Bouvier?

CHAPTERS 1–2

Cumulative Review

Work through all the problems in this cumulative review and check answers in the back of the book. Answers to most review problems are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

1. (A) Plot the points in the table below in a rectangular coordinate system.

(B) Find the smallest viewing window that will contain all of these points. State your answer in terms of the window variables.

(C) Does this set of points define a function? Explain.

x	-3	1	-2	1	3
y	4	-2	4	-4	4

2. Given points $A = (3, 2)$ and $B = (5, 6)$,

(A) Find the slope–intercept form of the equation of the line through A and B.

(B) Find the slope–intercept form of the equation of the line through B and perpendicular to the line through A and B.

(C) Graph the lines from parts B and C on the same coordinate system.

3. Graph $2x - 3y = 6$ and indicate its slope and intercepts.

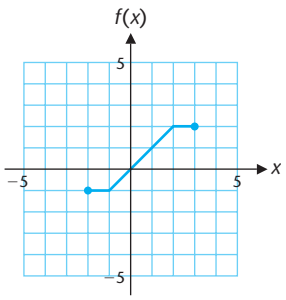
4. For $f(x) = x^2 - 2x + 5$ and $g(x) = 3x - 2$, find

(A) $f(-2) + g(-3)$ (B) $f(1) \cdot g(1)$ (C) $\dfrac{g(0)}{f(0)}$

5. How are the graphs of the following related to the graph of $y = |x|$?

(A) $y = 2|x|$ (B) $y = |x - 2|$ (C) $y = |x| - 2$

Problems 6–8 refer to the function f given by the graph:



6. Find the domain and range of f. Express answers in interval notation.

7. Is f an even function, an odd function, or neither? Explain.

8. Use the graph of f to sketch a graph of the following:

(A) $y = -f(x + 1)$ (B) $y = 2f(x) - 2$

In Problems 9–15, solve algebraically and confirm graphically.

9. $5 - 3(x + 4) = x + 2(11 - 2x)$

10. $(4x + 15)^2 = 9$

11. $\dfrac{7x}{5} - \dfrac{3 + 2x}{2} = \dfrac{x - 10}{3} + 2$

12. $3x^2 = -12x$

13. $4x^2 - 20 = 0$

14. $x^2 - 6x + 2 = 0$

15. $x - \sqrt{12 - x} = 0$

In Problems 16–19, solve and express answers in inequality and interval notation.

16. $2(3 - y) + 4 \le 5 - y$

17. $-3 \le 2 - 3x < 11$

18. $|x - 2| < 7$

19. $x^2 + 3x \ge 10$

20. Let $f(x) = x^2 - 4x - 1$.

(A) Find the vertex form of f.

(B) How is the graph of f related to the graph of $y = x^2$?

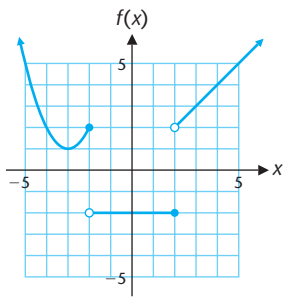
(C) Find the x intercepts algebraically and confirm graphically.

21. Perform the indicated operations and write the answer in standard form:

(A) $(2 - 3i) - (-5 + 7i)$ (B) $(1 + 4i)(3 - 5i)$

(C) $\dfrac{5 + i}{2 + 3i}$

22. Find each of the following for the function f given by the graph shown.



(A) The domain of f

(B) The range of f

(C) $f(-3) + f(-2) + f(2)$

(D) The intervals over which f is increasing.

(E) The x coordinates of any points of discontinuity.

23. Given $f(x) = 1/(x - 2)$ and $g(x) = (x + 3)/x$, find $f \circ g$. What is the domain of $f \circ g$?

24. Find $f^{-1}(x)$ for $f(x) = 2x + 5$.

25. Let $f(x) = \sqrt{x + 4}$

(A) Find $f^{-1}(x)$.

(B) Find the domain and range of f and f^{-1}.

(C) Graph f, f^{-1}, and $y = x$ on the same coordinate system and identify each graph.

26. Which of the following functions is one-to-one?

(A) $f(x) = x^3 + x$ (B) $g(x) = x^3 + x^2$

27. Write the slope–intercept form of the equation of the line passing through the point $(-6, 1)$ that is

(A) parallel to the line $3x + 2y = 12$.

(B) perpendicular to the line $3x + 2y = 12$.

28. Graph $f(x) = x^2 - 2x - 8$. Label the axis of symmetry and the coordinates of the vertex, and find the range, intercepts, and maximum or minimum value of $f(x)$.

In Problems 29 and 30, solve and express answers in inequality and interval notation.

29. $|4x - 9| > 3$ **30.** $\frac{1}{2}x^2 - x - 8 > 0$

31. Perform the indicated operations and write the final answers in standard form.

(A) $(2 - 3i)^2 - (4 - 5i)(2 - 3i) - (2 + 10i)$

(B) $\dfrac{3}{5} + \dfrac{4}{5}i + \dfrac{1}{\dfrac{3}{5} + \dfrac{4}{5}i}$ (C) i^{35}

32. Convert to $a + bi$ form, perform the indicated operations, and write the final answers in standard form.

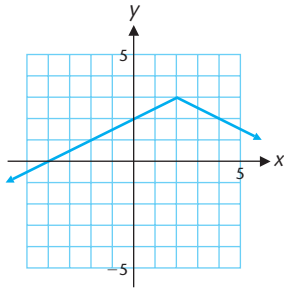
(A) $(5 + 2\sqrt{-9}) - (2 - 3\sqrt{-16})$

(B) $\dfrac{2 + 7\sqrt{-25}}{3 - \sqrt{-1}}$ (C) $\dfrac{12 - \sqrt{-64}}{\sqrt{-4}}$

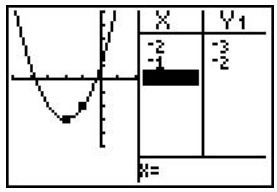
33. Graph, finding the domain, range, and any points of discontinuity.

$$f(x) = \begin{cases} x - 1 & \text{if } x < 0 \\ x^2 + 1 & \text{if } x \geq 0 \end{cases}$$

34. The graph in the figure is the result of applying a sequence of transformations to the graph of $y = |x|$. Describe the transformations verbally and write an equation for the graph in the figure.



35. Find the standard form of the quadratic function whose graph is shown in the figure.



In Problems 36–40, solve algebraically and confirm graphically, if possible.

36. $1 + \dfrac{14}{y^2} = \dfrac{6}{y}$ **37.** $4x^{2/3} - 4x^{1/3} - 3 = 0$

38. $u^4 + u^2 - 12 = 0$ **39.** $\sqrt{8t - 2} - 2\sqrt{t} = 1$

40. $6x = \sqrt{9x^2 - 48}$

41. Consider the quadratic equation

$$x^2 + bx + 1 = 0$$

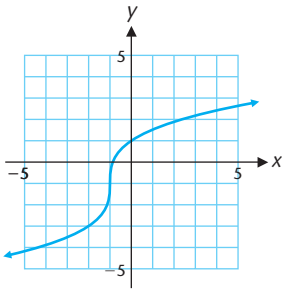
where b is a real number. Discuss the relationship between the values of b and the three types of roots listed in Table 1 in Section 2-5.

42. Give an example of an odd function. Of an even function. Can a function be both even and odd? Explain.

43. Can a quadratic equation with real coefficients have one imaginary root and one real root? One double imaginary root? Explain.

44. If $g(x) = -2x^2 + 3x - 1$, find $\dfrac{g(2 + h) - g(2)}{h}$.

45. The graph is the result of applying one or more transformations to the graph of one of the six basic functions in Figure 1, Section 1-4. Find an equation for the graph.



46. The total surface area of a right circular cylinder with radius r and height h is given by

$$A = 2\pi r(r + h) \qquad r > 0, h > 0$$

(A) Solve for h in terms of the other variables.

(B) Solve for r in terms of the other variables. Why is there only one solution?

47. Given $f(x) = x^2$ and $g(x) = \sqrt{4 - x^2}$, find

(A) Domain of g (B) f/g and its domain

(C) $f \circ g$ and its domain

48. Let $f(x) = x^2 - 2x - 3, x \geq 1$.

(A) Find $f^{-1}(x)$. (B) Find the domain and range of f^{-1}.

(C) Graph f, f^{-1}, and $y = x$ on the same coordinate system.

49. Evaluate $x^2 - x + 2$ for $x = \frac{1}{2} - \frac{i}{2}\sqrt{7}$.

50. For what values of a and b is the inequality $a - b < b - a$ true?

51. Write in standard form: $\frac{a + bi}{a - bi}$; $a, b \neq 0$

In Problems 52–55, solve algebraically and confirm graphically, if possible.

52. $3x^2 = 2\sqrt{2x} - 1$ **53.** $1 + 13x^{-2} + 36x^{-4} = 0$

54. $\sqrt{16x^2 + 48x + 39} - 2x = 3$

55. $3x^{-2/5} - x^{-1/5} - 1 = 0$

56. Show that $5 + i$ and $-5 - i$ are the square roots of $24 + 10i$. Describe how you could find these square roots algebraically.

57. For $f(x) = 0.5x^2 - 3x - 7$, find

(A) $\frac{f(x + h) - f(x)}{h}$ (B) $\frac{f(x) - f(a)}{x - a}$

58. The function f is continuous for all real numbers and its graph passes through the points $(0, 4)$, $(5, -3)$, $(10, 2)$. Discuss the minimum and maximum number of x intercepts for f.

59. Let $f(x) = |x + 2| + |x - 2|$. Find a piecewise definition of f that does not involve the absolute value function. Graph f and find the domain and range.

60. Let $f(x) = 2x - [\![2x]\!]$. Write a piecewise definition for f and sketch the graph of f. Include sufficient intervals to clearly illustrate both the definition and the graph. Find the domain, range, and any points of discontinuity.

61. Find all solutions of $x^3 - 8 = 0$.

APPLICATIONS

62. **CONSTRUCTION** After a water main breakage, the underground level of a building under construction is flooded with 400,000 gallons of water. The construction manager rents three pumps, each of which can remove water at the rate of 4,400 gallons per hour.

(A) Write a linear function that describes the amount of water remaining to be pumped out x hours after the three pumps are started.

(B) Use your function to determine how long it will take for all the water to be pumped out.

(C) Write and solve an inequality that describes the times when the amount of water remaining is between 100,000 and 200,000 gallons.

63. **BREAK-EVEN ANALYSIS** The publisher's fixed costs for the production of a new cookbook are \$41,800. Variable costs are \$4.90 per book. If the book is sold to bookstores for \$9.65, how many must be sold for the publisher to break even?

64. **FINANCE** An investor instructs a broker to purchase a certain stock whenever the price per share p of the stock is within \$10 of \$200. Express this instruction as an absolute value inequality.

65. **PROFIT AND LOSS ANALYSIS** At a price of \$$p$ per unit, the marketing department in a company estimates that the weekly cost C and the weekly revenue R, in thousands of dollars, will be given by the equations

$$C = 88 - 12p \quad \text{Cost equation}$$
$$R = 15p - 2p^2 \quad \text{Revenue equation}$$

Find the prices for which the company has

(A) A profit (B) A loss

66. **DEPRECIATION** Office equipment was purchased for \$20,000 and is assumed to depreciate linearly to a scrap value of \$4,000 after 8 years.

(A) Find a linear function $v = d(t)$ that relates value v in dollars to time t in years.

(B) Find $t = d^{-1}(v)$. What information does $d^{-1}(v)$ provide?

67. **SHIPPING** A ship leaves Port A, sails east to Port B, and then north to Port C, a total distance of 115 miles. The next day the ship sails directly from Port C back to Port A, a distance of 85 miles. Find the distance between Ports A and B and between Ports B and C.

68. **PRICE AND DEMAND** The weekly demand for mouthwash in a chain of drugstores is 1,160 bottles at a price of \$3.79 each. If the price is lowered to \$3.59, the weekly demand increases to 1,340 bottles. Assuming the relationship between the weekly demand x and the price per bottle p is linear, express x as a function of p. How many bottles would the store sell each week if the price were lowered to \$3.29?

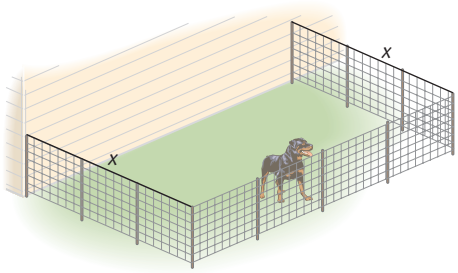
69. **BUSINESS–PRICING** A telephone company begins a new pricing plan that charges customers for local calls as follows: The first 60 calls each month are 6 cents each, the next 90 are 5 cents each, the next 150 are 4 cents each, and any additional calls are 3 cents each. If C is the cost, in dollars, of placing x calls per month, write a piecewise definition of C as a function of x and graph.

70. **CONSTRUCTION** A home owner has 80 feet of chain-link fencing to be used to construct a dog pen adjacent to a house (see the figure).

(A) Express the area $A(x)$ enclosed by the pen as a function of the width x.

(B) From physical considerations, what is the domain of the function A?

(C) Graph A and determine the dimensions of the pen that will make the area maximum.

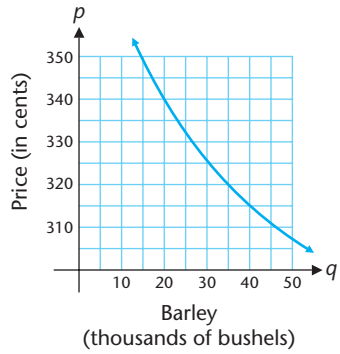


71. **COMPUTER SCIENCE** Let $f(x) = x - 2[\![x/2]\!]$. This function can be used to determine if an integer is odd or even.

(A) Find $f(1)$, $f(2)$, $f(3)$, $f(4)$.

(B) Find $f(n)$ for any integer n. [*Hint:* Consider two cases, $n = 2k$ and $n = 2k + 1$, k is an integer.]

72. **PRICE AND DEMAND** The demand for barley q (in thousands of bushels) and the corresponding price p (in cents) at a midwestern grain exchange are shown in the figure.



(A) What is the demand (to the nearest thousand bushels) when the price is 325 cents per bushel?

(B) Does the demand increase or decrease if the price is increased to 340 cents per bushel? By how much?

(C) Does the demand increase or decrease if the price is decreased to 315 cents per bushel? By how much?

(D) Write a brief description of the relationship between price and demand illustrated by this graph.

(E) Use the graph to estimate the price (to the nearest cent) when the demand is 20, 25, 30, 35, and 40 thousand bushels. Use these data to find a quadratic regression model for the price of barley using the demand as the independent variable.

DATA ANALYSIS AND REGRESSION

In Problems 73–75 round all values to three significant digits, unless directed otherwise.

73. **DEMAND** Egg consumption per capita decreased from a high of about 400 per capita in 1945 to a low of about 230 in 1991, then it began to increase. Table 1 lists the annual per capita consumption of eggs in the United States since 1970.

Table 1 Per Capita Egg Consumption

1970	1975	1980	1985	1990	1995	2000	2005
309	276	271	255	233	234	252	255

Source: Department of Agriculture.

(A) Find a quadratic regression equation $y = f(x)$ for the data in Table 1, where x is the number of years since 1970.

(B) Use the quadratic regression equation to project the year in which the per capita consumption will return to the 1970 level; to the 1945 level.

(C) Write a brief description of egg consumption from 1970 to 2005.

74. **STOPPING DISTANCE** Table 2 contains data related to the length of the skid marks left by an automobile when making an emergency stop.

(A) Let x be the speed of the vehicle in miles per hour. Find a quadratic regression model for the braking distance.

(B) An insurance investigator finds skid marks 220 feet long at the scene of an accident involving this automobile. How fast (to the nearest mile per hour) was the automobile traveling when it made these skid marks?

Table 2 Skid Marks

Speed (mph)	Length of Skid Marks (in feet)
20	24
30	48
40	81
50	118
60	187
70	246
80	312

75. **OPTIMAL SPEED** Table 3 contains performance data for a speedboat powered by a Yamaha outboard motor.

(A) Let x be the speed of the boat in miles per hour (mph) and y the associated mileage in miles per gallon (mpg). Use the data in Table 3 to find a quadratic regression function $y = ax^2 + bx + c$ for this boat.

(B) A marina rents this boat for \$10 per hour plus the cost of the gasoline used. If gasoline costs \$2.60 per gallon and you take a 200-mile trip in this boat, construct a mathematical model and use it to answer the following questions:

What speed should you travel to minimize the rental charges?

What mileage will the boat get?

How long does the trip take?

How much gasoline will you use?

How much will the trip cost you?

Table 3 Performance Data

mph	mpg
4.8	1.41
8.6	1.65
11.9	1.85
26.7	1.78
35.9	1.51
44.5	1.08

Source: www.yamaha-motor.com

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Polynomial and Rational Functions

IN Chapter 2, we were able to model a wide variety of situations with linear and quadratic functions. But both types of functions have relatively simple graphs: At most, the graph can change direction once. What if we want to model more complicated phenomena? We can use a more general class of functions known as *polynomials.* They are relatively easy to work with since they're based on the basic operations of addition, subtraction, multiplication, and division, and their graphs are smooth, continuous curves. Depending on the type of polynomial, the graph can change direction any number of times. This enables us to model data that linear and quadratic functions can't. We can then use polynomials to define *rational functions,* which further expands the number of functions in our library, enabling us to model even more situations.

CHAPTER

3

OUTLINE

3-1 Polynomial Functions and Models

3-2 Polynomial Division

3-3 Real Zeros and Polynomial Inequalities

3-4 Complex Zeros and Rational Zeros of Polynomials

3-5 Rational Functions and Inequalities

3-6 Variation and Modeling

Chapter 3 Review

Chapter 3 Group Activity: Interpolating Polynomials

3-1 Polynomial Functions and Models

- Recognizing Polynomial Functions
- Analyzing Graphs of Polynomials
- Left and Right Behavior of Polynomials
- Mathematical Modeling and Data Analysis

In this section, we will build upon the work we did with linear and quadratic functions in Chapter 2 by defining polynomial functions. After defining this class of functions, we will study their graphs, which show a much wider variety than the graphs we studied in Chapter 2. Finally, we will have a look at how polynomials can be used to model situations that might not be modeled as well with linear or quadratic functions.

Recognizing Polynomial Functions

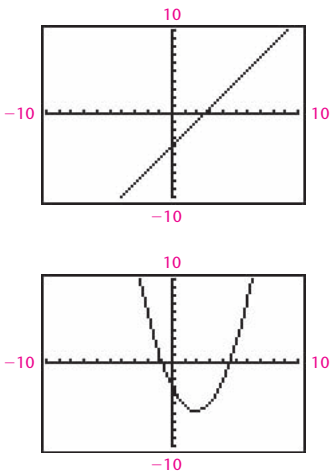


Figure 1 Graphs of linear and quadratic functions.

In Chapter 2 you were introduced to linear and quadratic functions and their graphs (Fig. 1):

$$f(x) = ax + b, \qquad a \neq 0 \qquad \text{Linear function}$$
$$f(x) = ax^2 + bx + c, \qquad a \neq 0 \qquad \text{Quadratic function}$$

In this chapter we will study functions like

$$g(x) = 7x^4 - 5x^3 + 2x^2 + 3x - 1.95$$

Notice that g is the sum of a finite number of terms, each of the form ax^k, where a is a number and k is a nonnegative integer. A function that can be written in this form is called a *polynomial function*. The polynomial function $g(x)$ is said to have *degree* 4 because x^4 is the highest power of x that appears among the terms of $g(x)$. Therefore, linear and quadratic functions are polynomial functions of degrees 1 and 2, respectively. The two functions $h(x) = x^{-1}$ and $k(x) = x^{1/2}$, however, are not polynomial functions (the exponents -1 and $\frac{1}{2}$ are not nonnegative integers).

DEFINITION 1 Polynomial Function

If n is a nonnegative integer, a function that can be written in the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad a_n \neq 0$$

is called a **polynomial function of degree n.** This is called the **general form** for a polynomial. The numbers $a_n, a_{n-1}, \ldots, a_1, a_0$ are called the **coefficients** of $P(x)$.

Note that the order of the terms is not important, but we will usually write the terms in order from largest power to smallest.

EXAMPLE 1 Recognizing Polynomial Functions

Determine whether each function is a polynomial. Write the degree of any polynomials.

(A) $f(x) = 5 - 3x + \frac{1}{2}x^2 - (2 + i)x^4$

(B) $g(x) = 3\sqrt{x} + 7x^5 - 1$

(C) $k(x) = 2x(x - 4)(x + 3)$

SOLUTIONS

(A) The function $f(x)$ in part A is a polynomial since every term is of the form ax^k, where a is a number and k is a nonnegative integer. (You can think of the constant term as $5x^0$.) The degree of f is 4.

(B) The function $g(x)$ in part B is not a polynomial since the term $3\sqrt{x}$ in exponent form is $3x^{1/2}$ and $\frac{1}{2}$ is not an integer.

(C) While the function $k(x)$ in part C is not in the general form for a polynomial, it could be written in that form by multiplying the three factors:

$$\begin{aligned} k(x) = 2x(x - 4)(x + 3) &= (2x^2 - 8x)(x + 3) \\ &= 2x^3 + 6x^2 - 8x^2 - 24x \\ &= 2x^3 - 2x^2 - 24x \end{aligned}$$

So $k(x)$ is a polynomial of degree 3.

MATCHED PROBLEM 1

Determine whether each function is a polynomial. Write the degree of any polynomials.

(A) $y(x) = 4x^2 - \sqrt{5}x^3 + \frac{x^5}{2} - ix^6$

(B) $z(x) = (2x^2 - 3)(x^2 + 4)$

(C) $w(x) = 7x^3 + \frac{5}{x^2} - 8$

Most of the polynomial functions we will study have real-number coefficients, but on occasion we will see complex number coefficients. In some cases, we will restrict our attention to integer or rational number coefficients. Similarly, the domain of a polynomial function can be the set of complex numbers, the set of real numbers, or a subset of either, depending on the situation.

Notice that a constant function like $f(x) = 5$ fits the definition of polynomial with degree zero [it can be written as $f(x) = 5x^0$]. The one exception to this is $f(x) = 0$, which violates the condition that a_n is not zero. This particular function is considered to be a polynomial with no degree.

Analyzing Graphs of Polynomials

We will be working with zeros of polynomials throughout much of this chapter, so a review of the definition of a zero of a function is timely.

DEFINITION 2 Zeros or Roots

A number r is said to be a **zero** or **root** of a function $P(x)$ if $P(r) = 0$.

The zeros of $P(x)$ are the solutions of the equation $P(x) = 0$. So if the coefficients of a polynomial $P(x)$ are real numbers, then the real zeros of $P(x)$ are just the x intercepts of the graph of $P(x)$. For example, the real zeros of the polynomial $P(x) = x^2 - 4$ are 2 and -2, the x intercepts of the graph of $P(x)$ [Fig. 2(a)]. However, a polynomial may have zeros that are not x intercepts. $Q(x) = x^2 + 4$, for example, has zeros $2i$ and $-2i$, but its graph has no x intercepts [Fig. 2(b)].

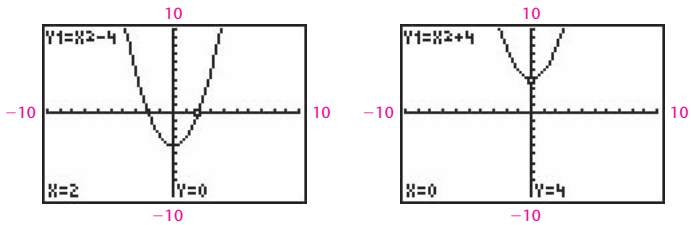


Figure 2 Real zeros are x intercepts.

EXAMPLE 2 Zeros and x Intercepts

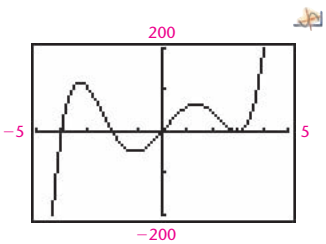


Figure 3

(A) Figure 3 shows the graph of a polynomial function of degree 5. List its real zeros to the nearest integer.

(B) List all zeros of the polynomial function

$$P(x) = (x - 4)(x + 7)^3(x^2 + 9)(x^2 - 2x + 2)$$

Which zeros of $P(x)$ are x intercepts?

SOLUTIONS

(A) The real zeros are the x intercepts: -4, -2, 0, and 3.

(B) Note first that $P(x)$ is a polynomial because it can be written in the form of Definition 1 if multiplied out. The zeros of $P(x)$ are the solutions to the equation $P(x) = 0$. Because a product equals 0 if and only if one of the factors is zero (the zero product property), we can find the zeros by solving each of the following equations. (The last was solved using the quadratic formula with the details omitted.)

$$\begin{array}{llll} x - 4 = 0 & (x + 7)^3 = 0 & x^2 + 9 = 0 & x^2 - 2x + 2 = 0 \\ x = 4 & x = -7 & x = \pm 3i & x = 1 \pm i \end{array}$$

Therefore, the zeros of $P(x)$, are $4, -7, 3i, -3i, 1 + i$, and $1 - i$. Only two of the six zeros are real numbers and thus x intercepts: 4 and -7.

MATCHED PROBLEM 2

(A) Figure 4 shows the graph of a polynomial function of degree 4. List its real zeros to the nearest integer.

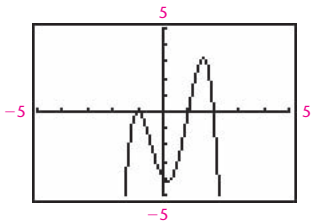


Figure 4

(B) List all zeros of the polynomial function

$$P(x) = (x + 5)(x^2 - 4)(x^2 + 4)(x^2 + 2x + 5)$$

Which zeros of $P(x)$ are x intercepts?

››› CAUTION ›››

When a polynomial is written in factored form like the function in Example 2 part B, you may be tempted to multiply out the parentheses and write it in general form. In this case, that would be a really bad idea! Not only would it be very difficult, it is far easier to find the zeros when a polynomial is in factored form.

A point on a continuous graph where the direction changes from increasing to decreasing, or vice versa, is called a **turning point.** The vertex of a parabola, for example, is a turning point. The graph of a linear function with real coefficients is a line, so it must have exactly one real zero and no turning points. The graph of a quadratic function with real coefficients is a parabola. It must have exactly one turning point, and either two, one, or no real zeros, depending on the location of the vertex.

By definition, the y coordinate of any turning point is either a maximum or minimum. It is possible, however, that $f(c)$ is a local extremum for a continuous function f even though $(c, f(c))$ is not a turning point. See Problems 79 and 80 in Exercises 3-1.

››› EXPLORE-DISCUSS 1

(A) Graph the polynomials

$$f(x) = 0.1x^4 - 0.5x^3 - 0.7x^2 + 4.5x - 3.1$$
$$g(x) = -1.1x^5 + 6.3x^3 - 8.2x - 0.1$$
$$h(x) = 0.01x^8 + x^6 - 7x^4 + 11x^2 - 3$$

in the standard viewing window.

(B) Assuming that all real zeros of $f(x)$, $g(x)$, and $h(x)$ appear in the standard viewing window, how do you think the number of real zeros of a polynomial is related to its degree?

(C) Assuming that all turning points of $f(x)$, $g(x)$, and $h(x)$ appear in the standard viewing window, how do you think the number of turning points of a polynomial is related to its degree?

Explore-Discuss 1 suggests that the graphs of polynomial functions with real coefficients have the properties listed in Theorem 1, which we will accept now without proof. Property 3 is proved in the next section. The other properties are established in calculus.

› THEOREM 1 Properties of Graphs of Polynomial Functions

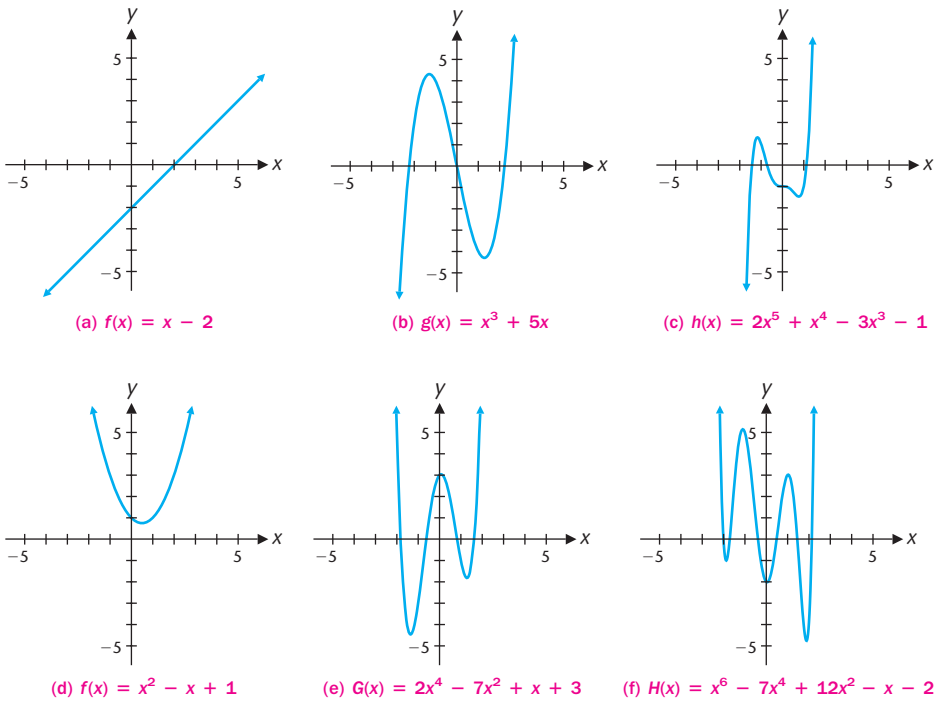
Let $P(x)$ be a polynomial of degree $n > 0$ with real coefficients. Then the graph of $P(x)$:

1. Is continuous for all real numbers
2. Has no sharp corners
3. Has at most n real zeros
4. Has at most $n - 1$ turning points
5. Increases or decreases without bound as x approaches ∞ and as x approaches $-\infty$

Property 5 requires a bit of extra attention. When discussing the behavior of graphs, we will often examine what the graph does "out toward the edges"—that is, for large values of x, both positive and negative. We call this the **left and right behavior** of a graph. We will study left and right behavior in depth later in this section. For now, we notice that when studying the behavior of a graph for large values of x, we use the notation $x \to \infty$ (for increasingly large positive values) and $x \to -\infty$ (for increasingly large negative values).* Keep in mind that ∞ and $-\infty$ are not real numbers, but concepts.

*This notation is used frequently in the study of limits, which is absolutely crucial to calculus.

Figure 5 shows graphs of representative polynomial functions of degrees 1 through 6, illustrating the five properties of Theorem 1.



(a) $f(x) = x - 2$ (b) $g(x) = x^3 + 5x$ (c) $h(x) = 2x^5 + x^4 - 3x^3 - 1$

(d) $f(x) = x^2 - x + 1$ (e) $G(x) = 2x^4 - 7x^2 + x + 3$ (f) $H(x) = x^6 - 7x^4 + 12x^2 - x - 2$

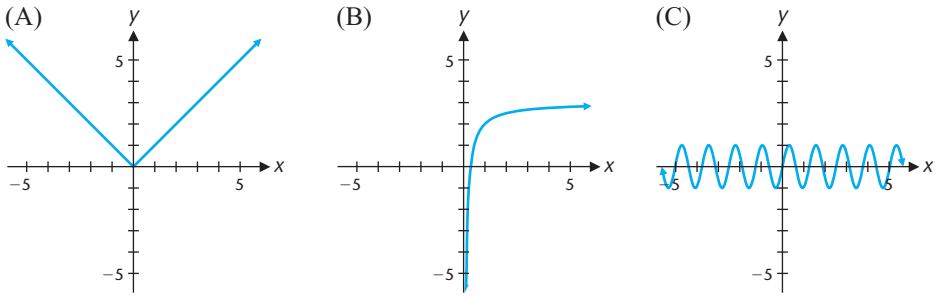
› **Figure 5** Graphs of polynomial functions.

Note that in each case, the number of turning points is at least one less than the degree of the polynomial, but not always exactly one less. Likewise, the number of zeros is never more than the degree, but in some cases is less.

EXAMPLE 3 Properties of Graphs of Polynomials

Explain why each graph is *not* the graph of a polynomial function by listing the properties of Theorem 1 that it *fails* to satisfy.

(A) (B) (C)



SOLUTIONS

(A) The graph has a sharp corner when $x = 0$. Property 2 fails.

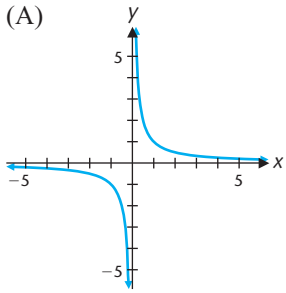
(B) There are no points on the graph with negative x coordinates, so the graph is not continuous for all real numbers (Property 1). Property 5 is also violated. The graph does not increase or decrease without bound as $x \to -\infty$; it does nothing! The graph also levels off at height 3 as $x \to \infty$, which is another violation of Property 5.

(C) There are an infinite number of zeros and an infinite number of turning points, so Properties 3 and 4 fail. Furthermore, the graph is bounded by the horizontal lines $y = \pm 1$ and does not increase or decrease without bound as $x \to \infty$ and $x \to -\infty$. Property 5 fails.

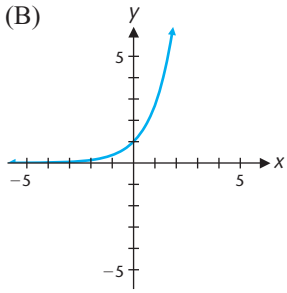
MATCHED PROBLEM 3

Explain why each graph is *not* the graph of a polynomial function by listing the properties of Theorem 1 that it *fails* to satisfy.

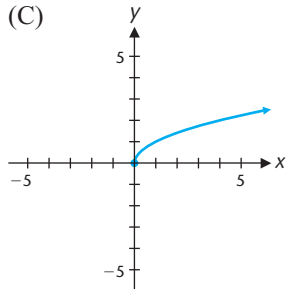
(A)



(B)



(C)



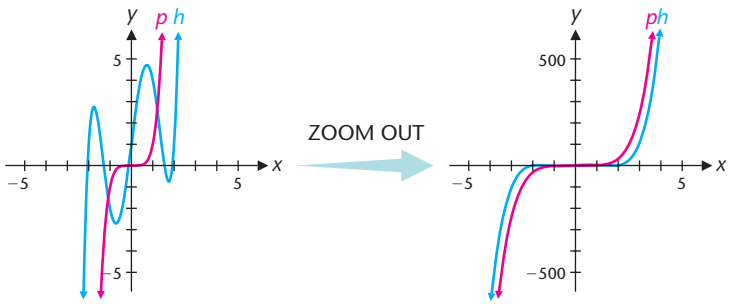
Left and Right Behavior of Polynomials

EXPLORE-DISCUSS 2

If n is a positive integer, then $y_1 = x^n$, $y_2 = x^{n-1}$, and $y_3 = x^n + x^{n-1}$ are all polynomial functions. Is the shape of y_3 more similar to the shape of y_1 or to the shape of y_2? Obtain evidence for your answer by graphing all three functions for several values of n.

Explore-Discuss 2 suggests that the shape of the graph of a polynomial function with real coefficients is similar to the shape of the graph of the **leading term,** that is, the term of highest degree. The coefficient of the leading term is called the **leading**

coefficient. Figure 6 compares the graph of the polynomial $h(x) = x^5 - 6x^3 + 8x + 1$ with the graph of its leading term $p(x) = x^5$. The graphs are dissimilar near the origin, but as we zoom out, the shapes of the two graphs become quite similar.



› **Figure 6** $p(x) = x^5$, $h(x) = x^5 - 6x^3 + 8x + 1$.

››› EXPLORE-DISCUSS 3

Consider the polynomial $h(x) = x^5 - 6x^3 + 8x + 1$. Make a table of values for $h(x)$, using $x = 0, 2, 4, 6, 8$, and 10. Make a separate column for each individual term of h, and put the output of the entire function in the last column. What do you notice about the relative sizes of each term as x gets larger?

Explore-Discuss 3 illustrates why the graphs of $h(x) = x^5 - 6x^3 + 8x + 1$ and $p(x) = x^5$ are very different near the origin, but very similar as x gets larger. The leading term in the polynomial dominates all other terms combined. Because the graph of $p(x)$ increases without bound as $x \to \infty$, the same is true of the graph of $h(x)$. And because the graph of $p(x)$ decreases without bound as $x \to -\infty$, the same is true of the graph of $h(x)$.

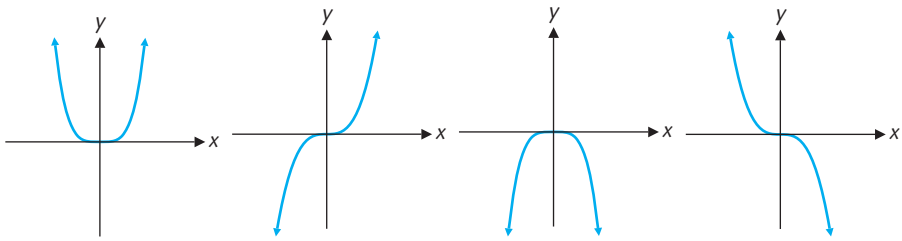
Because of this domination by the highest power, *the left and right behavior of a polynomial function with real coefficients is determined by the left and right behavior of its leading term.* Based on this principle, we can make Property 5 of Theorem 1 much more specific.

When the graph of a function $p(x)$ increases without bound, we can informally describe this as the value of $p(x)$ approaching ∞, and use the symbol $p(x) \to \infty$. When the graph decreases without bound, we write $p(x) \to -\infty$. Using these symbols to describe $p(x) = x^5$ in Figure 6, we would write $p(x) \to \infty$ as $x \to \infty$ and $p(x) \to -\infty$ as $x \to -\infty$. We will use this notation to describe the left and right behavior of polynomial functions.

› THEOREM 2 Left and Right Behavior of Polynomial Functions

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial function with real coefficients, $a_n \neq 0$, $n > 0$. If P has:

1. Positive leading coefficient, even degree: $P(x) \to \infty$ as $x \to \infty$ and $P(x) \to \infty$ as $x \to -\infty$ (like the graph of $y = x^2$).
2. Positive leading coefficient, odd degree: $P(x) \to \infty$ as $x \to \infty$ and $P(x) \to -\infty$ as $x \to -\infty$ (like the graph of $y = x^3$).
3. Negative leading coefficient, even degree: $P(x) \to -\infty$ as $x \to \infty$ and $P(x) \to -\infty$ as $x \to -\infty$ (like the graph of $y = -x^2$).
4. Negative leading coefficient, odd degree: $P(x) \to -\infty$ as $x \to \infty$ and $P(x) \to \infty$ as $x \to -\infty$ (like the graph of $y = -x^3$).



Case 1 Case 2 Case 3 Case 4

The key to remembering the results of Theorem 2 is not trying to memorize each rule, but rather being familiar with the graphs of $y = x^2$, $-x^2$, x^3, and $-x^3$.

EXAMPLE 4 Left and Right Behavior of Polynomials

Determine the left and right behavior of each polynomial. Use the notation from Theorem 2.

(A) $P(x) = 3 - x^2 + 4x^3 - x^4 - 2x^6$

(B) $Q(x) = 4x^5 + 8x^3 + 5x - 1$

SOLUTIONS

(A) The degree of $P(x)$ is 6 (even) and the coefficient a_6 is -2 (negative), so the left and right behavior is the same as that of $-x^6$, and also $-x^2$ (Case 3 of Theorem 2): $P(x) \to -\infty$ as $x \to \infty$ and $P(x) \to -\infty$ as $x \to -\infty$.

(B) The degree of $Q(x)$ is 5 (odd) and the coefficient a_5 is 4 (positive), so the left and right behavior is the same as that of x^5, and also x^3 (Case 2 of Theorem 2): $P(x) \to \infty$ as $x \to \infty$ and $P(x) \to -\infty$ as $x \to -\infty$.

MATCHED PROBLEM 4

Determine the left and right behavior of each polynomial.

(A) $P(x) = 4x^9 - 3x^{11} + 5$
(B) $Q(x) = 1 - 2x^{50} + x^{100}$

››› EXPLORE-DISCUSS 4

(A) What is the least number of turning points that a polynomial function of degree 5, with real coefficients, can have? The greatest number? Explain.

(B) What is the least number of x intercepts that a polynomial function of degree 5, with real coefficients, can have? The greatest number? Explain.

(C) What is the least number of turning points that a polynomial function of degree 6, with real coefficients, can have? The greatest number? Explain.

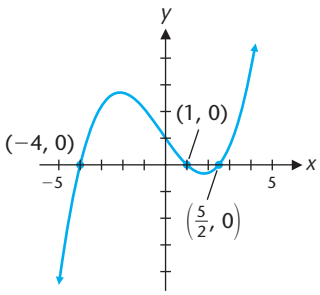
(D) What is the least number of x intercepts that a polynomial function of degree 6, with real coefficients, can have? The greatest number? Explain.

EXAMPLE 5 Sketching the Graph of a Polynomial

Sketch the graph of a polynomial with odd degree, a positive leading coefficient, and zeros -4, 1, and $\frac{5}{2}$.

SOLUTION

The zeros will all be x intercepts of the graph, so we plot those points first (Fig. 7). A polynomial with odd degree and a positive leading coefficient increases without bound to the right and decreases without bound to the left, giving us the graph in Figure 7. Note that we were given no information about the heights of any points not on the x axis, so we did not include a scale on the y axis.



› Figure 7

MATCHED PROBLEM 5

Sketch the graph of a polynomial with even degree, positive leading coefficient, and zeros -3, 0, 1, and 4.

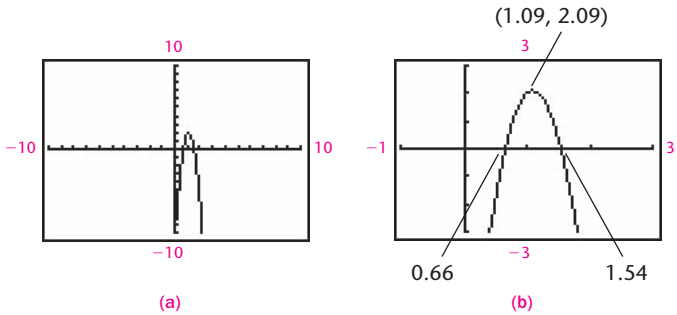
EXAMPLE 6 Analyzing the Graph of a Polynomial

Approximate to two decimal places the zeros and local extrema for

$$P(x) = x^3 - 14x^2 + 27x - 12$$

SOLUTION

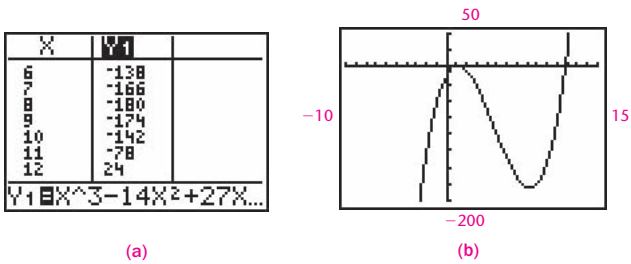
Examining the graph of P in a standard viewing window [Fig. 8(a)], we see two zeros, and a local maximum near $x = 1$. Zooming in shows these points more clearly [Fig. 8(b)]. Using the ZERO and MAXIMUM commands (details omitted), we find that $P(x) = 0$ for $x \approx 0.66$ and $x \approx 1.54$, and that $P(1.09) \approx 2.09$ is a local maximum value.



› **Figure 8** $P(x) = x^3 - 14x^2 + 27x - 12$.

But have we found all the zeros and local extrema? The graph in Figure 8(a) seems to indicate that $P(x) \to -\infty$ as $x \to \infty$ and $P(x) \to -\infty$ as $x \to -\infty$. But $P(x)$ has an odd degree and a positive leading coefficient, so in fact, it must be true that $P(x) \to \infty$ as $x \to \infty$. This tells us that $P(x)$ has to change direction somewhere outside the standard viewing window. In other words, there must be a local minimum and another zero that are not visible in the viewing window. Examining a table of values [Fig. 9(a)], it looks like there is a turning point near $x = 8$, and a zero near $x = 12$. Adjusting the window variables produces the graph in Figure 9(b). Using the ZERO and MAXIMUM commands again, we find that $P(x) = 0$ for $x \approx 11.80$ and that $P(8.24) \approx -180.61$ is a local minimum.

Now are we finished? Because a third-degree polynomial can have at most three zeros and two turning points, we have found all the zeros and local extrema for this polynomial.



› **Figure 9** $P(x) = x^3 - 14x^2 + 27x - 12$.

MATCHED PROBLEM 6

Approximate to two decimal places the zeros and the coordinates of the local extrema for

$$P(x) = -x^3 - 14x^2 - 15x + 5$$

Mathematical Modeling and Data Analysis

In Chapter 2 we saw that regression techniques can be used to construct a linear or quadratic model for a set of data. This was useful in modeling a wide variety of situations. But it's unreasonable to assume that every set of data can be modeled well with either a line or a parabola. Fortunately, most graphing calculators have the ability to use a variety of functions other than linear and quadratic for modeling data. We will discuss polynomial regression models in this section and other types of regression models in later sections.

EXAMPLE 7 Estimating the Weight of Fish

Using the length of a fish to estimate its weight is of interest to both scientists and sport anglers. The data in Table 1 give the average weight of North American sturgeon for certain lengths. Use these data and regression techniques to find a cubic polynomial model that can be used to estimate the weight of a sturgeon for any length. Estimate (to the nearest ounce) the weights of sturgeon of lengths 45, 46, 47, 48, 49, and 50 inches, respectively.

Table 1 Sturgeon

Length (in.) x	Weight (oz.) y
18	13
22	26
26	46
30	75
34	115
38	166
44	282
52	492
60	796

Source: www.thefishernet.com

SOLUTION

We begin by entering the data in Table 1 and examining a scatter plot of the data [Fig. 10(a)]. This makes it clear that linear regression would be a poor choice. And, in fact, we would not expect a linear relationship between length and weight. Instead, because weight is associated with volume, which involves three dimensions, it is more likely that the weight would be related to the cube of the length. We use a cubic (third-degree) regression polynomial to model these data [Fig. 10(b)]. Figure 10(c) adds the graph of the polynomial model to the graph of the data. The graph in Figure 10(c) shows that this cubic polynomial does provide a good fit for the data. (We will have more to say about the choice of functions and the accuracy of the fit provided by regression analysis later in the text.) Figure 10(d) shows the estimated weights for the requested lengths.

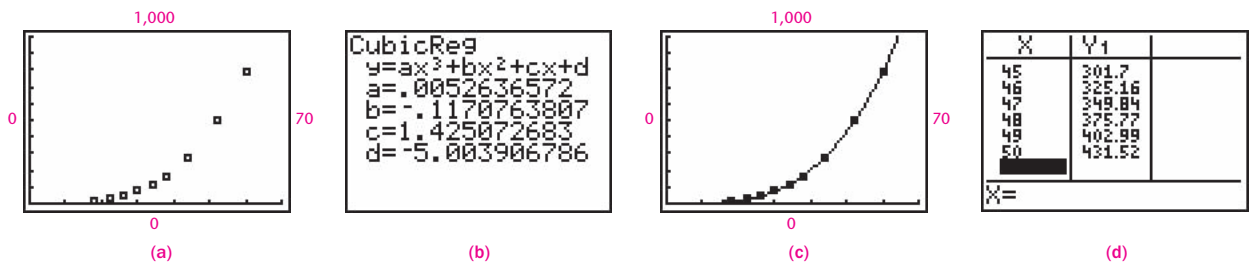


Figure 10

MATCHED PROBLEM 7

Find a quadratic regression model for the data in Table 1 and compare it with the cubic regression model found in Example 7. Which model appears to provide a better fit for these data? Use numerical and/or graphical comparisons to support your choice.

EXAMPLE 8 Hydroelectric Power

Table 2

Year	U.S. Consumption of Hydroelectric Power (quadrillion BTU)
1983	3.90
1985	3.40
1987	3.12
1989	2.99
1991	3.14
1993	3.13
1995	3.48
1997	3.88
1999	3.47
2001	2.38
2003	2.53
2005	2.61

Source: U.S. Department of Energy

The data in Table 2 give the annual consumption of hydroelectric power (in quadrillion BTU) in the United States for selected years since 1983. Use regression techniques to find an appropriate polynomial model for the data. Discuss how well the model is likely to predict annual hydroelectric power consumption in the second decade of the twenty-first century.

SOLUTION

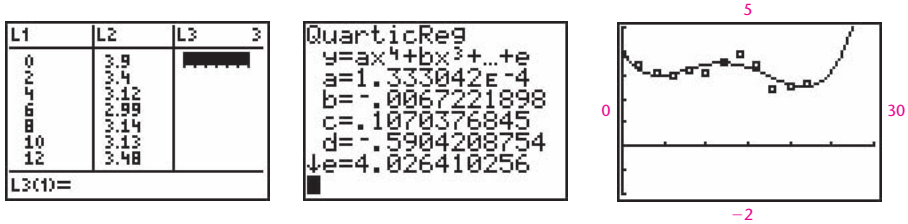
From Table 2 it appears that a polynomial model of the data would have three turning points—near 1989, 1997, and 2001. Because a polynomial with three turning points must have degree at least 4, we use quartic (fourth-degree) regression to find the polynomial of the form

$$y = ax^4 + bx^3 + cx^2 + dx + e$$

that best fits the data. Using $x = 0$ to represent the year 1983, we enter the data [Fig. 11(a)] and find the quartic regression model

$$y = 0.00013x^4 - 0.0067x^3 + 0.107x^2 - 0.59x + 4.03$$

[Fig. 11(b)]. We then plot both the data points and the model [Fig. 11(c)]. The model is not likely to be a good predictor of consumption of hydroelectric power in the second decade of the twenty-first century. In fact, it predicts a sudden dramatic increase to unprecedented levels after 2008. This brings up an important point: A model that fits a set of data points well is not automatically a good model for predicting future trends.



(a) (b) (c)

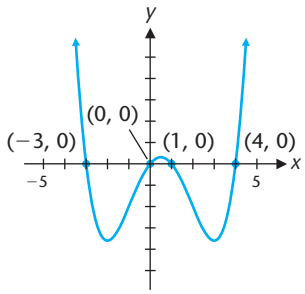
› Figure 11

MATCHED PROBLEM 8

Find the cubic regression model for the data of Table 2. Discuss which model is a better fit of the data—the cubic model or the quartic model of Example 8.

ANSWERS TO MATCHED PROBLEMS

1. (A) Polynomial, degree 6 (B) Polynomial, degree 4 (C) Not a polynomial
2. (A) $-1, 1, 2$ (B) The zeros are $-5, -2, 2, 2i, -2i, -1 + 2i$, and $-1 - 2i$; the x intercepts are $-5, -2$, and 2. **3.** (A) Properties 1 and 5 (B) Property 5 (C) Properties 1 and 5 **4.** (A) $P(x) \to -\infty$ as $x \to \infty$ and $P(x) \to \infty$ as $x \to -\infty$. (B) $P(x) \to \infty$ as $x \to \infty$ and $P(x) \to \infty$ as $x \to -\infty$.
5.



6. Zeros: $-12.80, -1.47, 0.27$; local maximum: $P(-0.57) \approx 9.19$; local minimum: $P(-8.76) \approx -265.71$ **7.** The quadratic regression model is $y = 0.49x^2 - 20.48x + 234.57$. The cubic regression model provides a better model for these data, especially for $18 \le x \le 26$. **8.** The cubic regression model is $y = -0.00086x^3 + 0.026x^2 - 0.23x + 3.79$. The quartic regression model is a better fit.

3-1 Exercises

1. Explain what a polynomial is in your own words.

2. What is the connection between the zeros of a polynomial and its factors? Explain.

3. Explain in your own words what is meant by the statement $P(x) \to \infty$ as $x \to \infty$.

4. Explain in your own words what is meant by the statement $P(x) \to -\infty$ as $x \to -\infty$.

In Problems 5–10, decide if the statement is true or false, then explain your choice.

5. Every quadratic function is a polynomial.

6. There exists a polynomial with degree $\frac{5}{2}$.

7. The coefficients of a polynomial must be real numbers.

8. A polynomial with only imaginary zeros has no x intercepts.

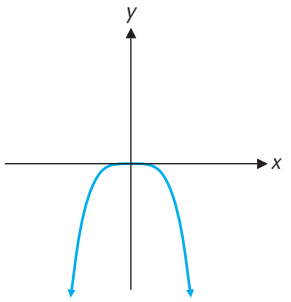
9. A polynomial with no x intercepts has no zeros.

10. The graph of a polynomial can level off at a finite height as x increases without bound.

In Problems 11–14, a is a positive real number. Match each function with one of graphs (a)–(d).

11. $f(x) = ax^3$ **12.** $g(x) = -ax^4$

13. $h(x) = ax^6$ **14.** $k(x) = -ax^5$



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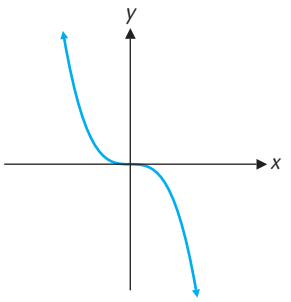

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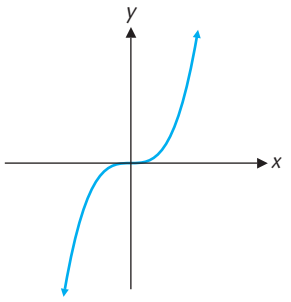

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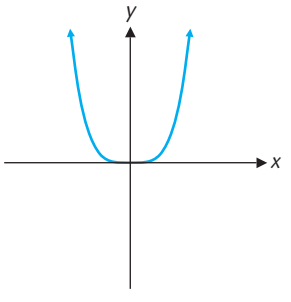
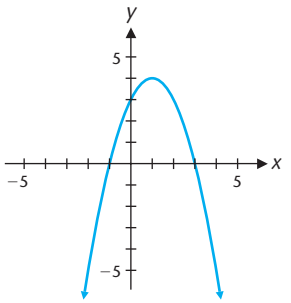


(d)

In Problems 15–18, list the real zeros and turning points, and state the left and right behavior, of the polynomial function $P(x)$ that has the indicated graph.

15.



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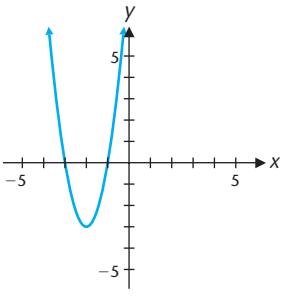

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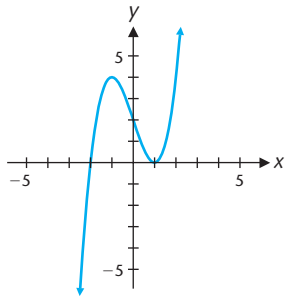

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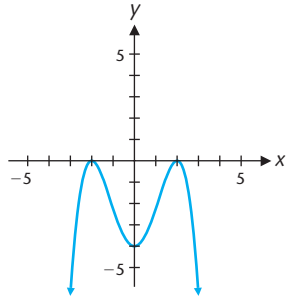
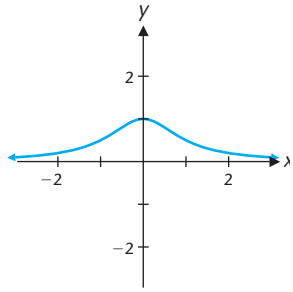


In Problems 19–22, explain why each graph is not the graph of a polynomial function.

19.



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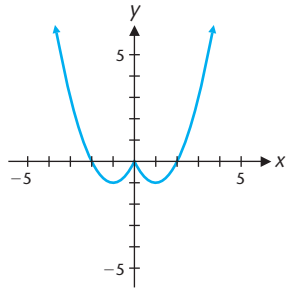

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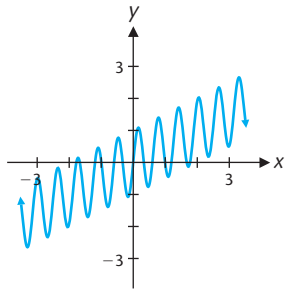

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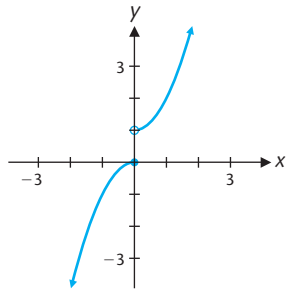


In Problems 23–30, determine whether the function is a polynomial. If it is, state the degree.

23. $f(x) = 4x + 7x^2 - \sqrt{8}x^3$

24. $g(x) = 2 - \sqrt{5}x^2 + x^5$

25. $h(x) = 4x^2 + 3x - 5 + \dfrac{2}{x}$

26. $k(x) = \dfrac{1}{3}x^3 + x - \dfrac{7}{x^2}$

27. $f(x) = 4\sqrt{x} + 7$ **28.** $g(x) = 8x - \sqrt[3]{x}$

29. $h(x) = 4(2x - 7)(x + 1)(3 + i - x)$

30. $k(x) = 2i(4 - x)(5 - x)(6 - x)$

In Problems 31–34, list all zeros of each polynomial function, and specify those zeros that are x intercepts.

31. $P(x) = x(x^2 - 9)(x^2 + 4)$ **32.** $P(x) = (x^2 - 4)(x^4 - 1)$

33. $P(x) = (x + 5)(x^2 + 9)(x^2 + 16)$

34. $P(x) = (x^2 - 5x + 6)(x^2 - 5x + 7)$

For each polynomial function in Problems 35–40:

(A) State the left and right behavior, the maximum number of x intercepts, and the maximum number of local extrema.

(B) Write the intervals where P is increasing, and intervals where P is decreasing.

(C) Approximate (to two decimal places) the x intercepts and the local extrema.

35. $P(x) = x^3 - 5x^2 + 2x + 6$

36. $P(x) = x^3 + 2x^2 - 5x - 3$

37. $P(x) = -x^3 + 4x^2 + x + 5$

38. $P(x) = -x^3 - 3x^2 + 4x - 4$

39. $P(x) = x^4 + x^3 - 5x^2 - 3x + 12$

40. $P(x) = -x^4 + 6x^2 - 3x - 16$

In Problems 41–46, sketch the graph of a polynomial fitting each description.

41. Zeros $x = -4, 3$; even degree, positive leading coefficient

42. Zeros $x = 1, 5$; even degree, negative leading coefficient

43. Zeros $x = -10, 0, 10$; odd degree, negative leading coefficient

44. Zeros $x = -7, 1, 11$; odd degree, positive leading coefficient

45. Zeros $x = -\frac{5}{2}, 0, 3, \frac{11}{2}$; even degree, negative leading coefficient

46. Zeros $x = -3, -\frac{1}{2}, \frac{1}{2}, 2$; even degree, positive leading coefficient

In Problems 47–54, find a polynomial of least degree with integer coefficients that has the given zeros. Write your answer in both factored form and general form.

47. $x = 0, 2, -3$ **48.** $x = 0, -4, 1$

49. $x = -\frac{1}{2}, 4, 0$ **50.** $x = \frac{5}{2}, -1, 0$

51. $x = 2, 3, 4$ **52.** $x = -2, -3, -4$

53. $x = 0, i, -i$ **54.** $x = 0, 2i, -2i$

In Problems 55–58, either give an example of a polynomial with real coefficients that satisfies the given conditions or explain why such a polynomial cannot exist.

55. $P(x)$ is a third-degree polynomial with one x intercept.

56. $P(x)$ is a fourth-degree polynomial with no x intercepts.

57. $P(x)$ is a third-degree polynomial with no x intercepts.

58. $P(x)$ is a fourth-degree polynomial with no turning points.

In calculus, the limit concept is very important. We have used the symbols $x \to \infty$, $x \to -\infty$, $P(x) \to \infty$, and $P(x) \to -\infty$ to describe right and left behavior of polynomials. In calculus, the right behavior of a function $f(x)$ is symbolized as $\lim_{x \to \infty} f(x)$ and read "the limit as x approaches infinity of $f(x)$." Similarly, the left behavior is symbolized as $\lim_{x \to -\infty} f(x)$. In Problems 59–66, use our study of left and right behavior to find each limit.

59. $\lim_{x \to \infty} x^2$ **60.** $\lim_{x \to -\infty} x^2$

61. $\lim_{x \to -\infty} (x^3 - x^2 + x)$ **62.** $\lim_{x \to \infty} (x^3 - x^2 + x)$

63. $\lim_{x \to \infty} (4 - x - 3x^4)$ **64.** $\lim_{x \to -\infty} (4 - x - 3x^4)$

65. $\lim_{x \to -\infty} (-x^4 + x^3 + 5x^2)$ **66.** $\lim_{x \to \infty} (-x^4 + x^3 + 5x^2)$

In Problems 67–74, approximate (to two decimal places) the x intercepts and the local extrema.

67. $P(x) = 40 + 50x - 9x^2 - x^3$

68. $P(x) = 40 + 70x + 18x^2 + x^3$

69. $P(x) = 0.04x^3 - 10x + 5$

70. $P(x) = -0.01x^3 + 2.8x - 3$

71. $P(x) = 0.1x^4 + 0.3x^3 - 23x^2 - 23x + 90$

72. $P(x) = 0.1x^4 + 0.2x^3 - 19x^2 + 17x + 100$

73. $P(x) = x^4 - 24x^3 + 167x^2 - 275x + 131$

74. $P(x) = x^4 + 20x^3 + 118x^2 + 178x + 79$

75. (A) What is the least number of turning points that a polynomial function of degree 4, with real coefficients, can have? The greatest number? Explain and give examples.

(B) What is the least number of x intercepts that a polynomial function of degree 4, with real coefficients, can have? The greatest number? Explain and give examples.

76. (A) What is the least number of turning points that a polynomial function of degree 3, with real coefficients, can have? The greatest number? Explain and give examples.

(B) What is the least number of x intercepts that a polynomial function of degree 3, with real coefficients, can have? The greatest number? Explain and give examples.

77. Is every polynomial of even degree an even function? Explain.

78. Is every polynomial of odd degree an odd function? Explain.

79. Let $f(x) = \begin{cases} x^2 & \text{if } |x| \geq 2 \\ 4 & \text{if } |x| < 2 \end{cases}$

(A) Graph f and observe that f is continuous.
(B) Find all numbers c such that $f(c)$ is a local extremum although $(c, f(c))$ is not a turning point.

80. Explain why the y coordinate of any turning point on the graph of a continuous function is a local extremum.

81. Is it possible for a third-degree polynomial to have real zeros $x = 0$ and $x = 2$ and no other real zeros? Explain, illustrating with graphs if necessary.

82. Is it possible for a fourth-degree polynomial to have real zeros $x = -5$, 1, and 4 and no other real zeros? Explain, illustrating with graphs if necessary.

APPLICATIONS

83. REVENUE The price–demand equation for 8,000-BTU window air conditioners is given by

$$p = 0.0004x^2 - x + 569 \qquad 0 \leq x \leq 800$$

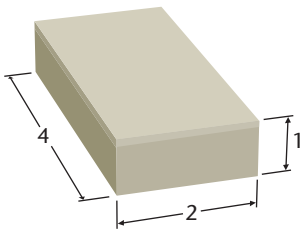
where x is the number of air conditioners that can be sold at a price of p dollars each.
(A) Find the revenue function.
(B) Find the number of air conditioners that must be sold to maximize the revenue, the corresponding price to the nearest dollar, and the maximum revenue to the nearest dollar.

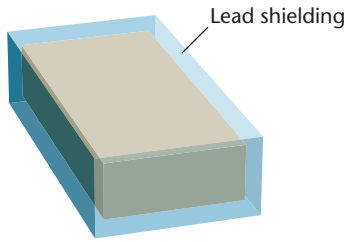
84. PROFIT Refer to Problem 83. The cost of manufacturing 8,000-BTU window air conditioners is given by

$$C(x) = 10{,}000 + 90x$$

where $C(x)$ is the total cost in dollars of producing x air conditioners.
(A) Find the profit function.
(B) Find the number of air conditioners that must be sold to maximize the profit, the corresponding price to the nearest dollar, and the maximum profit to the nearest dollar.

85. CONSTRUCTION A rectangular container measuring 1 foot by 2 feet by 4 feet is covered with a layer of lead shielding of uniform thickness (see the figure below and at the top of the next column).
(A) Find the volume of lead shielding V as a function of the thickness x (in feet) of the shielding.
(B) Find the thickness of the lead shielding to three decimal places if the volume of the shielding is 3 cubic feet.





86. MANUFACTURING A rectangular storage container measuring 2 feet by 2 feet by 3 feet is coated with a protective coating of plastic of uniform thickness.
(A) Find the volume of plastic V as a function of the thickness x (in feet) of the coating.
(B) Find the thickness of the plastic coating to four decimal places if the volume of the shielding is 0.1 cubic feet.

MODELING AND DATA ANALYSIS

87. HEALTH CARE Table 3 shows the total national expenditures (in billion dollars) and the per capita expenditures (in dollars) for selected years since 1960.

Table 3 National Health Expenditures

Year	Total Expenditures (billion $)	Per Capita Expenditures ($)
1960	27.5	143
1970	74.9	348
1980	253.9	1,067
1990	714.0	2,737
1995	1,016.5	3,686
2000	1,353.3	4,790
2005	1,987.7	6,697

Source: U.S. Dept. of Health and Human Services

(A) Let x represent the number of years since 1960 and find a cubic regression polynomial for the total national expenditures. Round coefficients to three significant digits.
(B) Use the polynomial model from part A to estimate the total national expenditures (to the nearest tenth of a billion) for 2010.
(C) Find a quadratic regression model for the total national expenditures. Which model predicts more rapid long-term increase?

88. HEALTH CARE Refer to Table 3.
(A) Let x represent the number of years since 1960 and find a cubic regression polynomial for the per capita expenditures. Round coefficients to three significant digits.
(B) Use the polynomial model from part A to estimate the per capita expenditures (to the nearest dollar) for 2010.

(C) Find a quadratic regression model for the per capita expenditures. Which model predicts more rapid long-term increase?

89. MARRIAGE Table 4 shows the marriage and divorce rates per 1,000 population for selected years since 1950.

Table 4 Marriages and Divorces (per 1,000 Population)

Year	Marriages	Divorces
1950	11.1	2.6
1960	8.5	2.2
1970	10.6	3.5
1980	10.6	5.2
1990	9.8	4.7
2000	8.5	4.2
2005	7.5	3.6

Source: U.S. Census Bureau

(A) Let x represent the number of years since 1950 and find a cubic and a quartic regression polynomial for the marriage rate. Round coefficients to three significant digits.
(B) Use the polynomial models from part A to estimate the marriage rate (to one decimal place) for 2008.
(C) Which model fits the actual data points better, cubic or quartic? Which provides a more realistic prediction for the marriage rate in 2008? Why?
(D) Based on the data in Table 4, why would a quadratic function do a poor job of modeling the data?

90. DIVORCE Refer to Table 4.
(A) Let x represent the number of years since 1950 and find a cubic and a quartic regression polynomial for the divorce rate. Round coefficients to three significant digits.
(B) Use the polynomial models from part A to estimate the divorce rate (to one decimal place) for 2008.
(C) Which model fits the actual data points better, cubic or quartic? Which provides a more realistic prediction for the divorce rate in 2008? Why?
(D) Based on the data in Table 4, why would a quadratic function do a poor job of modeling the data?

3-2 Polynomial Division

- Dividing Polynomials Using Long Division
- Dividing Polynomials Using Synthetic Division
- The Remainder and Factor Theorems

In the last section, we introduced the main properties of graphs of polynomials with real coefficients and saw the usefulness of these properties in analyzing graphs. But to understand why, for example, a polynomial function of degree n can have at most n real zeros, we will need to look at polynomials from an algebraic perspective. The first steps in doing so take place in this section. Two of the key algebraic procedures in studying polynomials from this perspective are division and factorization.

Dividing Polynomials Using Long Division

One method of dividing polynomials is long division, a process similar to long division of numbers in arithmetic. The terminology is very important in being able to

understand our description of the process, so we begin with a simple division problem to illustrate the necessary vocabulary.

$$\underset{\text{Divisor}}{\overset{\text{Dividend}}{\frac{11}{2}}} = \underset{\text{Quotient}}{5} + \underset{\text{Remainder}}{\frac{1}{2}}$$

The process of polynomial long division is illustrated in Example 1.

EXAMPLE 1 Dividing Polynomials Using Long Division

Divide $P(x) = 2x^3 + 5x^2 - 3x + 1$ by $x + 2$.

SOLUTION

We begin by setting up the division:

$$\underset{\text{Divisor}}{x + 2}\overset{\text{Dividend}}{\overline{)2x^3 + 5x^2 - 3x + 1}}$$

The first term of the quotient is found by dividing the first term of the dividend by the first term of the divisor; in this case,

$$\frac{2x^3}{x} = 2x^2$$

We write this result above the division sign:

$$\begin{array}{r} 2x^2 \\ x + 2\overline{)2x^3 + 5x^2 - 3x + 1} \end{array}$$

Now we multiply $2x^2$ by the divisor, writing the result under the first two terms of the dividend. Then we subtract these two pairs of terms.

$$\begin{array}{rl} 2x^2 & \\ x + 2\overline{)2x^3 + 5x^2 - 3x + 1} & \\ \underline{-(2x^3 + 4x^2)} & \\ 0x^3 + x^2 & \text{Subtract.} \end{array} \qquad 2x^2(x + 2) = 2x^3 + 4x^2$$

Next, we bring down the next term of the dividend and repeat the above process.

$$\begin{array}{rl} 2x^2 + x - 5 & \\ x + 2\overline{)2x^3 + 5x^2 - 3x + 1} & \\ \underline{-(2x^3 + 4x^2)} & \\ x^2 - 3x & \frac{x^2}{x} = x;\ x(x + 2) = x^2 + 2x \\ \underline{-(x^2 + 2x)} & \text{Subtract; bring down last term of dividend.} \\ -5x + 1 & -\frac{5x}{x} = -5;\ -5(x + 2) = -5x - 10 \\ \underline{-(-5x - 10)} & \text{Subtract.} \\ 11 & \end{array}$$

The quotient is $2x^2 + x - 5$ and the remainder is 11, so

$$\frac{2x^3 + 5x^2 - 3x + 1}{x + 2} = 2x^2 + x - 5 + \frac{11}{x + 2}$$

Check: You can always check division using multiplication.

$$\begin{aligned}(x + 2)\left(2x^2 + x - 5 + \frac{11}{x + 2}\right) &= (x + 2)(2x^2 + x - 5) + 11 \\ &= 2x^3 + x^2 - 5x + 4x^2 + 2x - 10 + 11 \\ &= 2x^3 + 5x^2 - 3x + 1\end{aligned}$$

MATCHED PROBLEM 1

Divide $P(x) = x^3 - 6x^2 + 8x - 2$ by $x + 3$.

››› CAUTION ›››

When subtracting during the long-division process, make sure that you subtract every term. It may help to include parentheses and think of it as distributing the negative sign.

The procedure illustrated in Example 1 is called the **division algorithm.** The concluding equation of Example 1 (before the check) may be multiplied by the divisor $x + 2$ to give the following form:

$$\underset{\text{Dividend}}{2x^3 + 5x^2 - 3x + 1} = \underset{\text{Divisor}}{(x + 2)} \cdot \underset{\text{Quotient}}{(2x^2 + x - 5)} + \underset{\text{Remainder}}{11}$$

This last equation is an *identity:* it is true for all replacements of x by real or complex numbers including $x = -2$. Theorem 1, which we state without proof, gives the general result of applying the division algorithm.

› THEOREM 1 The Division Algorithm

For any two polynomials $P(x)$ and $D(x)$ with $D(x) \neq 0$, there are unique polynomials $Q(x)$ and $R(x)$ so that

$$P(x) = D(x)\,Q(x) + R(x)$$

and the degree of $R(x)$ is less than the degree of $D(x)$. The polynomial $P(x)$ is called the **dividend,** $D(x)$ is the **divisor,** $Q(x)$ is the **quotient,** and $R(x)$ is the **remainder.** Note that $R(x)$ may be zero.

In essence, Theorem 1 tells us that any two polynomials can be divided (as long as the divisor is not zero), and it also tells us what the result will look like.

When dividing polynomials that have "missing" powers of x, like $x^2 - 2$ (which lacks a first-power term), the missing term must be restored with a zero coefficient for the process to work correctly. This is illustrated in Example 2.

EXAMPLE 2 Dividing Polynomials Using Long Division

Divide $x^4 - 5x^2 + 2x + 1$ by $x^2 - 2$.

SOLUTION

We set up the division, putting in zero coefficients for the missing terms, and proceed as in Example 1.

$$\begin{array}{r} x^2 - 3 \\ x^2 + 0x - 2\overline{)\,x^4 + 0x^3 - 5x^2 + 2x + 1} \\ \underline{-(x^4 + 0x^3 - 2x^2)} \\ -3x^2 + 2x + 1 \\ \underline{-(-3x^2 + 0x + 6)} \\ 2x - 5 \end{array}$$

$\frac{x^4}{x^2} = x^2$; $x^2(x^2 + 0x - 2) = x^4 + 0x^3 - 2x^2$

Subtract; bring down next two terms of dividend.

$-\frac{3x^2}{x^2} = -3$; $-3(x^2 + 0x - 2) = -3x^2 + 0x + 6$

Subtract.

Result: $\dfrac{x^4 - 5x^2 + 2x + 1}{x^2 - 2} = x^2 - 3 + \dfrac{2x - 5}{x^2 - 2}$

Note that we had to bring down the next two terms after the first subtraction since there was only one nonzero term remaining, and the divisor has three terms.

MATCHED PROBLEM 2

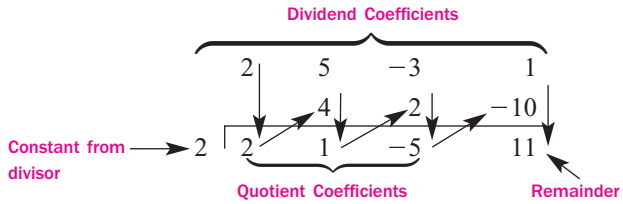
Divide $3x^5 - x^3 + 5x + 6$ by $x^2 + 3$.

Dividing Polynomials Using Synthetic Division

Let's take another look at the long division from Example 1. It turns out that it can be carried out by a shortcut called **synthetic division.** The coefficients that represent the essential elements of the long-division process are indicated in color.

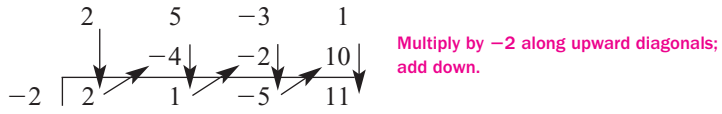
$$\begin{array}{r} 2x^2 + 1x - 5 \\ x + 2\overline{)\,2x^3 + 5x^2 - 3x + 1} \\ \underline{-(2x^3 + 4x^2)} \\ 1x^2 - 3x \\ \underline{-(x^2 + 2x)} \\ -5x + 1 \\ \underline{-(-5x - 10)} \\ 11 \end{array}$$

The coefficients printed in color can be arranged more conveniently as follows:



This probably looks confusing at first, but the arrows indicate how the middle and bottom rows can be found without having to do the long division. The first coefficient from the dividend is simply brought down to be the first digit of the quotient. This is then multiplied by the 2 from the divisor, resulting in 4 placed in the middle row. As in long division, we then subtract downward to get 1. This process is then repeated: Multiply 1 by 2 (the constant term from the divisor), to get the next digit in the middle row, then subtract down. When all the rows are filled in, the bottom row represents the coefficients of the quotient and remainder.

There is a way to make the process of synthetic division a bit quicker and less prone to arithmetic mistakes. We will change the $+2$ from the divisor to a -2. This enables us to add down, rather than subtract.



The quotient is $2x^2 + x - 5$, and the remainder is 11.

Clearly, this is much simpler than long division. But there is a catch: Synthetic division only works when the divisor is of the form $x - r$. If the divisor has any other form, long division is the only choice.

We will now summarize the steps in performing synthetic division.

› KEY STEPS IN THE SYNTHETIC DIVISION PROCESS

To divide the polynomial $P(x)$ by $x - r$:

1. Write the coefficients of $P(x)$ in order of descending powers of x. Write 0 as the coefficient for any missing powers. Draw a horizontal line below, leaving space for the middle row. Draw a short vertical line downward from the left edge of the horizontal line, and write r to the left of it.
2. Bring down the first coefficient from the dividend and write it in the third row, below the horizontal line. Then multiply that coefficient by r, and put the result in the middle row below the second coefficient of the dividend. Add the two numbers in this second column, writing the result in the bottom row.
3. Repeat this "multiply, then add down" process until all columns have been filled in.
4. The last number to the right in the third row of numbers is the remainder. The other numbers in the third row are the coefficients of the quotient, which is of degree 1 less than $P(x)$.

EXAMPLE 3 Synthetic Division

Use synthetic division to divide $P(x) = 4x^5 - 30x^3 - 50x - 2$ by $x + 3$. Find the quotient and remainder. Write the conclusion in the form $P(x) = (x - r)Q(x) + R$.

SOLUTION

Because $x + 3 = x - (-3)$, we have $r = -3$, and

$$\begin{array}{r|cccccc} & 4 & 0 & -30 & 0 & -50 & -2 \\ & & -12 & 36 & -18 & 54 & -12 \\ \hline -3 & 4 & -12 & 6 & -18 & 4 & -14 \end{array}$$

The quotient is $4x^4 - 12x^3 + 6x^2 - 18x + 4$ with a remainder of -14. This means that $4x^5 - 30x^3 - 50x - 2 = (x + 3)(4x^4 - 12x^3 + 6x^2 - 18x + 4) - 14$.

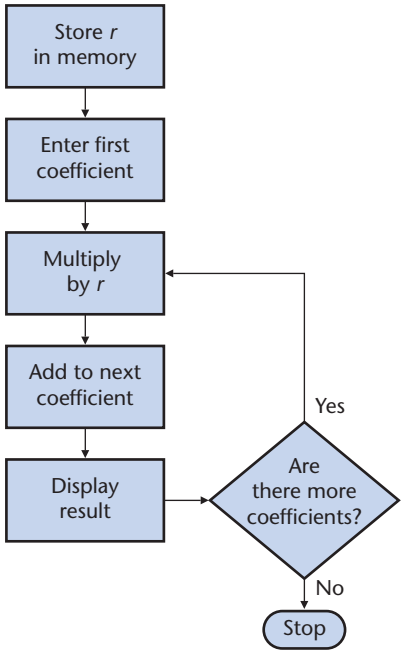
MATCHED PROBLEM 3

Repeat Example 3 with $P(x) = 3x^4 - 11x^3 - 18x + 8$ and divisor $x - 4$.

››› CAUTION ›››

1. Remember that synthetic division only works if the divisor is linear with leading coefficient 1.
2. Don't forget to write the divisor in the form $x - r$ before setting up the division. In Example 3, the divisor $x + 3$ had to be written as $x - (-3)$ to see that $r = -3$.

A calculator is a convenient tool for performing synthetic division. Any type of calculator can be used, although one with a memory will save some keystrokes. The flowchart in Figure 1 shows the repetitive steps in the synthetic division process, and Figure 2 illustrates the results of applying this process to Example 3 on a graphing calculator.



› **Figure 1** Synthetic division.

```
-3→R                    -3
4                        4
Ans*R+0                 -12
Ans*R+(-30)               6
Ans*R+0                 -18
Ans*R+(-50)               4
Ans*R+(-2)              -14
```

› **Figure 2**

› The Remainder and Factor Theorems

››› EXPLORE-DISCUSS 1

Let $P(x) = x^3 - 3x^2 - 2x + 8$.

(A) Find $P(-2)$, $P(1)$, and $P(3)$.

(B) Use synthetic division to find the remainder when $P(x)$ is divided by

$$x + 2, x - 1, \text{ and } x - 3.$$

What conclusion does a comparison of the results in parts A and B suggest?

Explore-Discuss 1 suggests that when a polynomial $P(x)$ is divided by $x - r$, the remainder is equal to $P(r)$, the value of the polynomial $P(x)$ at $x = r$. In Problem 81 in Exercises 3-2, you will be asked to complete a proof of this fact, which we call the *remainder theorem*.

› THEOREM 2 The Remainder Theorem

If R is the remainder after dividing the polynomial $P(x)$ by $x - r$, then

$$P(r) = R$$

EXAMPLE 4 **Two Methods for Evaluating Polynomials**

If $P(x) = 4x^4 + 10x^3 + 19x + 5$, find $P(-3)$ two different ways:

(A) Use the remainder theorem and synthetic division.
(B) Evaluate $P(-3)$ directly.

SOLUTIONS

(A) Use synthetic division to divide $P(x)$ by $x - (-3)$.

$$\begin{array}{r|rrrrr} & 4 & 10 & 0 & 19 & 5 \\ & & -12 & 6 & -18 & -3 \\ \hline -3 & 4 & -2 & 6 & 1 & 2 = R = P(-3) \end{array}$$

(B) $P(-3) = 4(-3)^4 + 10(-3)^3 + 19(-3) + 5$
$= 324 - 270 - 57 + 5 = 2$

MATCHED PROBLEM 4

Repeat Example 4 for $P(x) = 3x^4 - 16x^2 - 3x + 7$ and $x = -2$.

You might think the remainder theorem is not a very effective tool for evaluating polynomials. But let's consider the number of operations performed in parts A and B of Example 4. Synthetic division requires only four multiplications and four additions to find $P(-3)$, whereas the direct evaluation requires ten multiplications and four additions. [Note that evaluating $4(-3)^4$ actually requires five multiplications.] The difference becomes even larger as the degree of the polynomial increases. Computer programs that involve numerous polynomial evaluations often use synthetic division because of its efficiency. We will find synthetic division and the remainder theorem to be useful tools later in this chapter.

One useful consequence of the remainder theorem is that if synthetic division of a polynomial $P(x)$ by $x - r$ results in a zero remainder, then r is actually a zero of $P(x)$. This result can be applied to factoring polynomials.

Suppose that $x - r$ is a factor of a polynomial $P(x)$. This means that $P(x)$ can be written as $P(x) = (x - r)\,Q(x)$, where $Q(x)$ is some other polynomial with degree one less than $P(x)$. If we are trying to find the zeros of $P(x)$, we set up the equation

$$P(x) = 0 \qquad \text{or} \qquad (x - r)Q(x) = 0$$

The zero product property tells us that $x = r$ is a zero of $P(x)$. Conclusion: If $x - r$ is a factor of $P(x)$, then $x = r$ is a zero.

The remainder theorem can be used to show that the reverse is also true. If $x = r$ is a zero, then division by $x - r$ results in a remainder of zero. The division algorithm then gives us

$$P(x) = (x - r)Q(x) + 0$$

In other words, $x - r$ is a factor of $P(x)$. We have proved the *factor theorem*.

› THEOREM 3 The Factor Theorem

If r is a zero of the polynomial $P(x)$, then $x - r$ is a factor of $P(x)$. Conversely, if $x - r$ is a factor of $P(x)$, then r is a zero of $P(x)$.

The factor theorem shows that there is a perfect correlation between the zeros of a polynomial and its linear factors. Find all the linear factors, and you know all the zeros. Find all the zeros, and you know all the linear factors.

EXAMPLE 5 Factors of Polynomials

Use the factor theorem to show that $x + 1$ is a factor of $P(x) = x^{25} + 1$ but is not a factor of $Q(x) = x^{25} - 1$.

SOLUTION

Because

$$P(-1) = (-1)^{25} + 1 = -1 + 1 = 0$$

$x - (-1) = x + 1$ is a factor of $x^{25} + 1$. On the other hand,

$$Q(-1) = (-1)^{25} - 1 = -1 - 1 = -2$$

so -1 is not a zero, and $x + 1$ is not a factor of $x^{25} - 1$.

MATCHED PROBLEM 5

Use the factor theorem to show that $x - i$ is a factor of $P(x) = x^8 - 1$ but is not a factor of $Q(x) = x^8 + 1$.

One consequence of the factor theorem is Theorem 4, which will be proved in Problem 82 in Exercises 3-2.

› THEOREM 4 Zeros of Polynomials

A polynomial of degree n has at most n zeros.

Theorem 4 implies that the graph of a polynomial of degree n with real coefficients has at most n real zeros (Property 3 of Theorem 1 in the last section). The polynomial

$$H(x) = x^6 - 7x^4 + 12x^2 - x - 2$$

for example, has degree 6 and the maximum number, namely six, of real zeros [see Fig. 5(f) on page 279]. Of course polynomials of degree 6 may have fewer than six real zeros. In fact, $p(x) = x^6 + 1$ has no real zeros. However, it can be shown that the polynomial $p(x) = x^6 + 1$ has exactly six complex zeros.

The remainder and factor theorems can work together to help find the zeros of a polynomial. Example 6 illustrates this process, which we will use repeatedly in the section after next.

EXAMPLE 6 Finding the Zeros of a Polynomial

(A) Use synthetic division to show that $x = 3$ is a zero of $P(x) = x^3 - 8x^2 + 9x + 18$.
(B) Use the result of part A to find any other zeros.

SOLUTIONS

(A) We divide by $x - 3$ using synthetic division.

$$\begin{array}{r|rrrr} & 1 & -8 & 9 & 18 \\ & & 3 & -15 & -18 \\ \hline 3 & 1 & -5 & -6 & 0 \end{array}$$

The zero remainder confirms that $x = 3$ is a zero.

(B) The division in part A tells us that

$$\frac{x^3 - 8x^2 + 9x + 18}{x - 3} = x^2 - 5x - 6$$

or, applying the division algorithm,

$$P(x) = x^3 - 8x^2 + 9x + 18 = (x - 3)(x^2 - 5x - 6)$$

To find the remaining zeros, we need to solve $P(x) = 0$.

$$\begin{array}{lll} P(x) = (x - 3)(x^2 - 5x - 6) = 0 & & \\ x - 3 = 0 \quad \text{or} \quad x^2 - 5x - 6 = 0 & & \\ x = 3 \qquad\quad (x - 6)(x + 1) = 0 & & \\ \qquad\quad x - 6 = 0 \quad \text{or} \quad x + 1 = 0 & & \\ \qquad\quad x = 6 \qquad\quad x = -1 & & \end{array}$$

The remaining zeros are $x = 6$ and $x = -1$.

MATCHED PROBLEM 6

(A) Use synthetic division to show that $x = -2$ is a zero of $P(x) = x^3 + 2x^2 - 9x - 18$.
(B) Use the result of part A to find any other zeros.

ANSWERS TO MATCHED PROBLEMS

1. $x^2 - 9x + 35 - \dfrac{107}{x+3}$ **2.** $3x^3 - 10x + \dfrac{35x+6}{x^2+3}$ **3.** $3x^4 - 11x^3 - 18x + 8 = (x-4)(3x^3 + x^2 + 4x - 2)$ **4.** $P(-2) = -3$ for both parts, as it should **5.** $P(i) = 0$, so $x - i$ is a factor of $x^8 - 1$; $Q(i) - 2$, so $x - i$ is not a factor of $x^8 + 1$
6. (B) $x = -3, 3$

3-2 Exercises

1. Write this division problem in the form of the division algorithm, then label the dividend, divisor, quotient, and remainder.

$$\frac{x^2 - 5x + 12}{x - 3} = x - 2 + \frac{6}{x - 3}$$

2. Describe the type of division problem for which synthetic division can be applied.

3. What is wrong with the following setup for dividing $x^4 + 3x^2 - 8x + 1$ by $x - 3$ using synthetic division?

$$\begin{array}{r|llll} 3 & 1 & 3 & -8 & 1 \\ \hline \end{array}$$

4. What is wrong with the following setup for dividing $2x^3 + 5x^2 - x - 3$ by $x + 6$?

$$\begin{array}{r|llll} 6 & 2 & 5 & -1 & -3 \\ \hline \end{array}$$

5. Describe what the remainder theorem says in your own words.

6. What is the relationship between the factors of a polynomial and its zeros?

In Problems 7–14, divide, using algebraic long division.

7. $(4m^2 - 1) \div (2m - 1)$ **8.** $(y^2 - 9) \div (y + 3)$

9. $(6 - 6x + 8x^2) \div (2x + 1)$

10. $(11x - 2 + 12x^2) \div (3x + 2)$

11. $(x^3 - 1) \div (x - 1)$ **12.** $(a^3 + 27) \div (a + 3)$

13. $(3y - y^2 + 2y^3 - 1) \div (y + 2)$

14. $(3 + x^3 - x) \div (x - 3)$

In Problems 15–20, divide using synthetic division.

15. $(x^2 + 3x - 7) \div (x - 2)$

16. $(x^2 + 3x - 3) \div (x - 3)$

17. $(4x^2 + 10x - 9) \div (x + 3)$

18. $(2x^2 + 7x - 5) \div (x + 4)$

19. $(2x^3 - 3x + 1) \div (x - 2)$

20. $(x^3 + 2x^2 - 3x - 4) \div (x + 2)$

In Problems 21–26, evaluate the polynomial two ways: by substituting in the given value of x, and by using synthetic division.

21. Find $P(-2)$ for $P(x) = 3x^2 - x - 10$.

22. Find $P(-3)$ for $P(x) = 4x^2 + 10x - 8$.

23. Find $P(2)$ for $P(x) = 2x^3 - 5x^2 + 7x - 7$.

24. Find $P(5)$ for $P(x) = 2x^3 - 12x^2 - x + 30$.

25. Find $P(-4)$ for $P(x) = x^4 - 10x^2 + 25x - 2$.

26. Find $P(-7)$ for $P(x) = x^4 + 5x^3 - 13x^2 - 30$.

In Problems 27–30, determine whether the second polynomial is a factor of the first polynomial without dividing or using synthetic division. [Hint: Evaluate directly and use the factor theorem.]

27. $x^{18} - 1; x - 1$

28. $x^{18} - 1; x + 1$

29. $3x^3 - 7x^2 - 8x + 2; x + 1$

30. $3x^4 - 2x^3 + 5x - 6; x - 1$

In Problems 31–38, divide using long division.

31. $(-6x^3 + 5x^2 - 5x + 12) \div (-3x + 4)$

32. $(4x^3 - 7x^2 - 14x - 3) \div (4x + 1)$

33. $(2x^3 - 5x^2 - 8x + 1) \div (2x - 1)$

34. $(3x^3 + 8x^2 + 16x - 5) \div (3x + 2)$

35. $(x^3 - 4x^2 - 7x + 1) \div (x^2 - 2)$

36. $(x^4 - x^2 - 3) \div (x^2 + 4)$

37. $(3x^4 - 2x^2 - 3x + 5) \div (x^2 + x + 2)$

38. $(-2x^4 + x^3 + 5x + 2) \div (x^2 + 2x + 3)$

In Problems 39–54, divide, using synthetic division. As coefficients get more involved, a calculator should prove helpful. Do not round off—all quantities are exact.

39. $(3x^4 - x - 4) \div (x + 1)$

40. $(5x^4 - 2x^2 - 3) \div (x - 1)$

41. $(x^5 + 1) \div (x + 1)$ **42.** $(x^4 - 16) \div (x - 2)$

43. $(3x^4 + 2x^3 - 4x - 1) \div (x + 3)$

44. $(x^4 - 3x^3 - 5x^2 + 6x - 3) \div (x - 4)$

45. $(2x^6 - 13x^5 + 75x^3 + 2x^2 - 50) \div (x - 5)$

46. $(4x^6 + 20x^5 - 24x^4 - 3x^2 - 13x + 30) \div (x + 6)$

47. $(4x^4 + 2x^3 - 6x^2 - 5x + 1) \div (x + \frac{1}{2})$

48. $(2x^3 - 5x^2 + 6x + 3) \div (x - \frac{1}{2})$

49. $(4x^3 + 4x^2 - 7x - 6) \div (x + \frac{3}{2})$

50. $(3x^3 - x^2 + x + 2) \div (x + \frac{2}{3})$

51. $(3x^4 - 2x^3 + 2x^2 - 3x + 1) \div (x - 0.4)$

52. $(4x^4 - 3x^3 + 5x^2 + 7x - 6) \div (x - 0.7)$

53. $(3x^5 + 2x^4 + 5x^3 - 7x - 3) \div (x + 0.8)$

54. $(7x^5 - x^4 + 3x^3 - 2x^2 - 5) \div (x + 0.9)$

In Problems 55–60, evaluate the polynomial two ways: by substituting in the given value of x, and by using synthetic division.

55. Find $P(\frac{5}{2})$ for $P(x) = 4x^3 - 12x^2 - 7x + 10$.

56. Find $P(\frac{1}{3})$ for $P(x) = 3x^3 - 13x^2 + 10x + 3$.

57. Find $P(\frac{1}{2})$ for $P(x) = 3x^3 + 5x^2 - x + 2$.

58. Find $P(\frac{1}{2})$ for $P(x) = -5x^3 - 2x^2 + 3x + 4$.

59. Find $P(2i)$ for $P(x) = x^3 + 1$.

60. Find $P(-i)$ for $P(x) = 2x^3 - 8$.

In Problems 61–68, use synthetic division to show that the given x value is a zero of the polynomial. Then find all other zeros.

61. $P(x) = x^3 - 7x + 6; \quad x = 1$

62. $P(x) = x^3 - 2x^2 - 5x + 6; \quad x = 1$

63. $P(x) = 3x^3 - 8x^2 - 5x + 6; \quad x = 3$

64. $P(x) = 2x^3 - 8x^2 + 2x + 12; \quad x = 3$

65. $P(x) = x^3 - 5x^2 + 4x - 20; \quad x = 5$

66. $P(x) = x^3 - 4x^2 + 9x - 36; \quad x = 4$

67. $P(x) = x^3 - 2x^2 - 3x + 10; \quad x = -2$

68. $P(x) = x^3 - x^2 + 8x + 10; \quad x = -1$

In Problems 69 and 70, divide, using synthetic division. Do not use a calculator.

69. $(x^3 - 3x^2 + x - 3) \div (x - i)$

70. $(x^3 - 2x^2 + x - 2) \div (x + i)$

71. Let $P(x) = x^2 + 2ix - 10$. Find
(A) $P(2 - i)$ (B) $P(5 - 5i)$
(C) $P(3 - i)$ (D) $P(-3 - i)$

72. Let $P(x) = x^2 - 4ix - 13$. Find
(A) $P(5 + 6i)$ (B) $P(1 + 2i)$
(C) $P(3 + 2i)$ (D) $P(-3 + 2i)$

73. Use synthetic division to find the value of k so that $x = 2$ is a zero of $P(x) = x^3 - 3x^2 + 4x + k$.

74. Use synthetic division to find the value of k so that $x = -2$ is a zero of $P(x) = 2x^3 + 5x^2 - x + k$.

75. Use synthetic division to find the value of k so that $x = -3$ is a zero of $P(x) = -4x^3 - 6x^2 + kx - 6$.

76. Use synthetic division to find the value of k so that $x = 4$ is a zero of $P(x) = 3x^3 - 14x^2 + kx + 64$.

77. (A) Divide $P(x) = a_2x^2 + a_1x + a_0$ by $x - r$, using both synthetic division and the long-division process, and compare the coefficients of the quotient and the remainder produced by each method.
(B) Expand the expression representing the remainder. What do you observe?

78. Repeat Problem 77 for

$$P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

79. Polynomials also can be evaluated using a "nested factoring" scheme. For example, the polynomial $P(x) = 2x^4 - 3x^3 + 2x^2 - 5x + 7$ can be written in a nested factored form as follows:

$$\begin{aligned} P(x) &= 2x^4 - 3x^3 + 2x^2 - 5x + 7 \\ &= (2x - 3)x^3 + 2x^2 - 5x + 7 \\ &= [(2x - 3)x + 2]x^2 - 5x + 7 \\ &= \{[(2x - 3)x + 2]x - 5\}x + 7 \end{aligned}$$

Use the nested factored form to find $P(-2)$ and $P(1.7)$. [*Hint:* To evaluate $P(-2)$, store -2 in your calculator's memory and proceed from left to right recalling -2 as needed.]

80. Let $P(x) = 3x^4 + x^3 - 10x^2 + 5x - 2$. Find $P(-2)$ and $P(1.3)$ using the nested factoring scheme presented in Problem 79.

81. Prove the remainder theorem:

(A) Write the result of the division algorithm if a polynomial $P(x)$ is divided by $x - r$.

(B) Evaluate both sides of the result for $x = r$. What can you conclude?

82. In this problem we will prove Theorem 4. Write the reason that justifies each step. Let $P(x)$ be a polynomial of degree n, and suppose that P has n distinct zeros $r_1, r_2, \ldots, r_n$.

Step 1: We can write $P(x)$ as $P(x) = (x - r_1)Q_1(x)$, where the degree of $Q_1(x)$ is $n - 1$.

Step 2: Since r_2 is a zero of P, $Q_1(r_2) = 0$.

Step 3: $Q_1(x) = (x - r_2)Q_2(x)$, where the degree of $Q_2(x)$ is $n - 2$.

Step 4: We can now write $P(x)$ as $P(x) = (x - r_1)(x - r_2)Q_2(x)$

Step 5: Continuing, we will get $P(x) = (x - r_1)(x - r_2) \cdots (x - r_n)Q_n(x)$, where the degree of $Q_n(x)$ is zero.

Step 6: $Q_n(x)$ has no zeros, so the only zeros of P are $r_1, r_2, \ldots, r_n$.

83. If you are given one zero for a cubic polynomial, is it *always* possible to find two more real zeros? Explain.

84. If you are given one zero for a cubic polynomial, is it *always* possible to find two more complex zeros? Explain.

3-3 Real Zeros and Polynomial Inequalities

- Finding Upper and Lower Bounds for Real Zeros
- Using the Bisection Method to Locate Zeros
- Approximating Real Zeros at Turning Points
- Solving Polynomial Inequalities
- Mathematical Modeling

We have seen that the real zeros of a polynomial $P(x)$ with real coefficients are just the x intercepts of the graph of $P(x)$. So an obvious strategy for finding the real zeros consists of two steps:

1. Graph $P(x)$ on a graphing calculator.
2. Use the ZERO command to approximate each x intercept.

In this section, we develop two important tools for carrying out this strategy: the *upper and lower bound theorem,* which determines appropriate window variables for step 1, and a simple approximation technique called the *bisection method* that underpins step 2. We also investigate some potential difficulties when the strategy is applied to

polynomials that have a zero at a turning point, and we apply the strategy to solve polynomial inequalities.

In this section we will restrict our attention to the real zeros of polynomials with real coefficients.

Finding Upper and Lower Bounds for Real Zeros

In this section, we will make use of the fact that a polynomial of degree n has at most n zeros. When relying on graphs to find the zeros of a function, we always have to wonder whether the viewing window is showing all of the zeros, or if there may be more that we can't see. When the function is a polynomial, this may not be an issue. For example, if the calculator's viewing window is showing three x intercepts of a cubic polynomial, then we know for a fact that the zero command will find all of the zeros.

EXAMPLE 1 Approximating Real Zeros

Approximate the zeros of $P(x) = x^3 - 6x^2 + 9x - 3$ to three decimal places.

SOLUTION

A graph of $P(x)$ in the standard viewing window shows three x intercepts (Fig. 1). We can find each of them by applying the ZERO command: rounded to three decimal places they are 0.468, 1.653, and 3.879. Because a polynomial of degree 3 can have at most three zeros, we can be sure that we have found all of the zeros of $P(x)$.

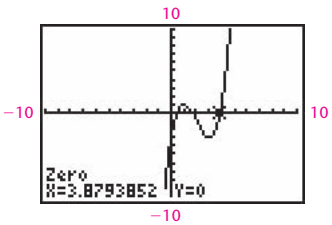


Figure 1 A zero of $P(x) = x^3 - 6x^2 + 9x - 3$.

MATCHED PROBLEM 1

Approximate the zeros of $P(x) = 21 + 10x - 3x^2 - x^3$ to three decimal places.

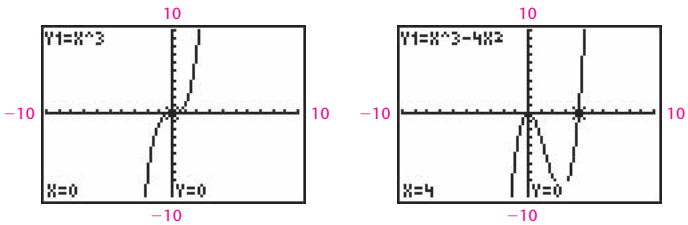


Figure 2 Cubic polynomials having one or two zeros.

Unfortunately, there is a problem with this approach. A polynomial of degree 3 has at most three real zeros, but it may have exactly one or exactly two (Fig. 2). If we can't find a viewing window that displays more than one zero for a particular cubic polynomial, how long do we have to search before we can decide whether the polynomial has one, two, or three real zeros?

››› EXPLORE-DISCUSS 1

For $P(x) = x^3 - 2x^2 + 2x + 3$,

(A) Use synthetic division to divide $P(x)$ by $x - 3$; note the sign of all coefficients in the quotient row.

(B) Evaluate $P(x)$ for five different values of x greater than 3. What do all of the results have in common?

(C) Write the result of the division in the form $P(x) = (x - 3)Q(x) + R$. Explain why the right side of the equation will always be positive for any x values greater than 3.

Explore-Discuss 1 hints at a result that will enable us to know for sure when a given viewing window contains all of the x intercepts for a polynomial. The result is known as the *upper and lower bound theorem*. This theorem indicates how to find two numbers, a **lower bound** that is less than or equal to all real zeros of the polynomial, and an **upper bound** that is greater than or equal to all real zeros of the polynomial. All of the real zeros are guaranteed to lie between the lower bound and the upper bound. So if we graph the polynomial with these two numbers as our least and greatest x values, we'll know for sure that the graph will display all of the real zeros.

› THEOREM 1 Upper and Lower Bound Theorem

Let $P(x)$ be a polynomial of degree $n > 0$ with real coefficients, $a_n > 0$:

1. Upper bound: A number $r > 0$ is an upper bound for the real zeros of $P(x)$ if, when $P(x)$ is divided by $x - r$ using synthetic division, all numbers in the quotient row, including the remainder, are nonnegative.
2. Lower bound: A number $r < 0$ is a lower bound for the real zeros of $P(x)$ if, when $P(x)$ is divided by $x - r$ using synthetic division, all numbers in the quotient row, including the remainder, alternate in sign.

[*Note:* In the lower-bound test, if 0 appears in one or more places in the quotient row, including the remainder, the sign in front of it can be considered either positive or negative, but not both. For example, the numbers 1, 0, 1 can be considered to alternate in sign, whereas 1, 0, -1 cannot.]

Note: The upper and lower bound theorem works only if the leading coefficient of a polynomial is positive. See Problems 71 and 72 in Exercises 3-3 for a look at what to do if the leading coefficient is negative, and Problems 85 and 86 for a sketch of a proof of Theorem 1.

EXAMPLE 2 Bounding Real Zeros

Let $P(x) = x^4 - 2x^3 - 10x^2 + 40x - 90$. Use the upper and lower bound theorem to find the smallest positive integer and the largest negative integer that are upper and lower bounds, respectively, for the real zeros of $P(x)$.

SOLUTION

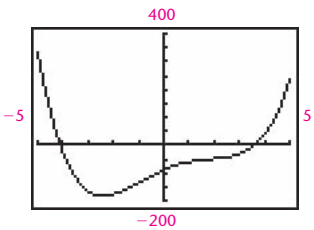
The idea is to perform synthetic division repeatedly until we find the patterns from Theorem 1. We begin with $r = 1$.

	1	−2	−10	40	−90
		1	−1	−11	29
1	1	−1	−11	29	−61

The resulting coefficients are not all positive, so $x = 1$ is not an upper bound for the zeros of P. We should next try $r = 2, 3, \ldots$ until we find all positive coefficients. We should then repeat the process using $r = -1, -2, -3, \ldots$ until the quotient row alternates in sign. But it quickly becomes apparent that it will be cumbersome to keep doing synthetic divisions individually, so instead we will make a *synthetic division table*. We will write only the result of each division, doing the arithmetic that usually generates the middle row mentally.

		1	−2	−10	40	−90	
	1	1	−1	−11	29	−61	
	2	1	0	−10	20	−50	
	3	1	1	−7	19	−33	
	4	1	2	−2	32	38	
UB	5	1	3	5	65	235	← This quotient row is nonnegative, so 5 is an upper bound (UB).
	−1	1	−3	−7	47	−137	
	−2	1	−4	−2	44	−178	
	−3	1	−5	5	25	−165	
	−4	1	−6	14	−16	−26	
LB	−5	1	−7	25	−85	335	← This quotient row alternates in sign, so −5 is a lower bound (LB).

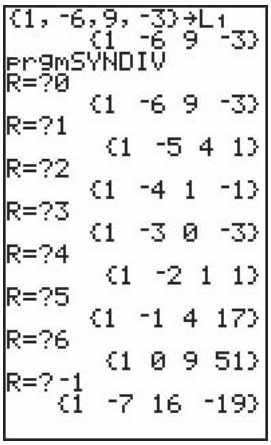
The graph of $P(x) = x^4 - 2x^3 - 10x^2 + 40x - 90$ for $-5 \le x \le 5$ is shown in Figure 3. Since Theorem 1 guarantees that all the real zeros of $P(x)$ are between -5 and 5, we can be certain that the graph does not change direction and cross the x axis somewhere outside the viewing window in Figure 3.



› **Figure 3** $P(x) = x^4 - 2x^3 - 10x^2 + 40x - 90$.

MATCHED PROBLEM 2

Let $P(x) = x^4 - 5x^3 - x^2 + 40x - 70$. Use the upper and lower bound theorem to find the smallest positive integer and the largest negative integer that are upper and lower bounds, respectively, for the real zeros of $P(x)$.



› **Figure 4** Synthetic division on a graphing calculator.

Theorem 1 requires performing synthetic division repeatedly until the desired pattern occurs in the quotient row. This is not terribly difficult, but it can be tedious. SYNDIV* is a program for some TI-model graphing calculators that makes this process routine. Figure 4 shows the results of dividing the polynomial $P(x) = x^3 - 6x^2 + 9x - 3$ from Example 1 by $x - r$ for integer values of r from -1 through 6 using SYNDIV. (When $r = 2$, for example, the quotient is $x^2 - 4x + 1$ and the remainder is -1.)

From Figure 4 we see that the positive number 6 is an upper bound for the real zeros of $P(x)$ because all numbers in the quotient row, including the remainder, are nonnegative. Furthermore, the negative number -1 is a lower bound for the real zeros of $P(x)$ because all numbers in the quotient row, including the remainder, alternate in sign. We can then conclude that all zeros of $P(x)$ are between -1 and 6.

One of the biggest issues in using a graphing calculator is determining the correct viewing window. The upper and lower bound theorem provides some very helpful guidance in dealing with this issue for polynomial functions. Example 3 illustrates how the upper and lower bound theorem and the ZERO command on a graphing calculator work together to analyze the graph of a polynomial.

EXAMPLE 3 Approximating Real Zeros

Let $P(x) = x^3 - 30x^2 + 275x - 720$.

(A) Use the upper and lower bound theorem to find the smallest positive integer multiple of 10 and the largest negative integer multiple of 10 that are upper and lower bounds, respectively, for the real zeros of $P(x)$.

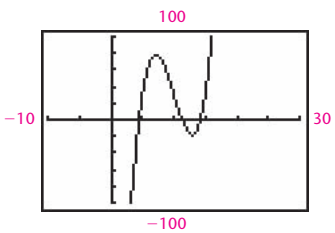
(B) Approximate all real zeros of $P(x)$ to two decimal places.

SOLUTIONS

(A) We construct a synthetic division table to search for bounds for the zeros of $P(x)$. The size of the coefficients in $P(x)$ suggests that we can speed up this search by choosing larger increments between test values.

		1	−30	275	−720
	10	1	−20	75	30
	20	1	−10	75	780
UB	30	1	0	275	7,530
LB	−10	1	−40	675	−7,470

Because all real zeros of $P(x) = x^3 - 30x^2 + 275x - 720$ must be between -10 and 30, we should choose Xmin $= -10$ and Xmax $= 30$.



› **Figure 5** $P(x) = x^3 - 30x^2 + 275x - 720$.

(B) Graphing $P(x)$ for $-10 \le x \le 30$ (Fig. 5) shows that $P(x)$ has three zeros. The approximate values of these zeros (details omitted) are 4.48, 11.28, and 14.23.

*Programs for TI-83, TI-84, and TI-86 graphing calculators can be found at the website for this book (see Preface).

MATCHED PROBLEM 3

Let $P(x) = x^3 - 25x^2 + 170x - 170$.

(A) Use the upper and lower bound theorem to find the smallest positive integer multiple of 10 and the largest negative integer multiple of 10 that are upper and lower bounds, respectively, for the real zeros of $P(x)$.

(B) Approximate all real zeros of $P(x)$ to two decimal places.

››› CAUTION ›››

The upper and lower bound theorem does not tell us the exact location of any zeros, nor how many to expect. It simply gives a range of values that must include all of the real zeros.

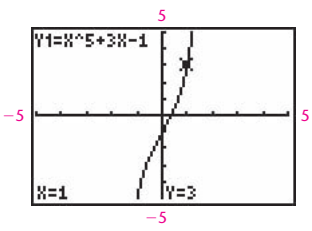
› Using the Bisection Method to Locate Zeros

We begin this topic with a review of the location theorem from Section 2-7. Since it is relevant to two topics in this section, we will restate it as Theorem 2.

› THEOREM 2 The Location Theorem

If $f(x)$ is a continuous function on some interval containing a and b, and $f(a)$ and $f(b)$ have opposite signs, then there must be a zero of f somewhere between $x = a$ and $x = b$.

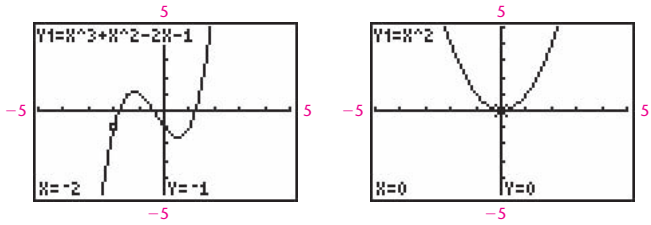
The graph of every polynomial is continuous for all x, so the location theorem can be applied. Consider the polynomial $P(x) = x^5 + 3x - 1$, shown in Figure 6. Note that P is negative at $x = 0$ [$P(0) = -1$] and positive at $x = 1$ [$P(1) = 3$]. According to the location theorem, the graph of P has to cross the x axis at least once between $x = 0$ and $x = 1$.



› Figure 6 $P(x) = x^5 + 3x - 1$.

On the other hand, the conclusion of the location theorem doesn't tell us that there is only one zero between a and b; for example, if $g(x) = x^3 + x^2 - 2x - 1$, then $g(-2)$ and $g(2)$ have opposite signs, and there are actually three real zeros between $x = -2$ and $x = 2$ [Fig. 7(a)].

What if we change the direction of the implication in Theorem 2? If $f(x)$ has a zero between $x = a$ and $x = b$, must $f(a)$ and $f(b)$ have opposite signs? (This is known as the *converse* of Theorem 2.) In fact, this is not true: $h(x) = x^2$ has a zero at $x = 0$, but does not change sign [Fig. 7(b)].



› **Figure 7** Polynomials may or may not change sign at a zero.

››› EXPLORE-DISCUSS 2

When synthetic division is used to divide a polynomial $P(x)$ by $x - 3$ the remainder is -33. When the same polynomial is divided by $x - 4$ the remainder is 38. Must $P(x)$ have a zero between 3 and 4? Explain.

Explore-Discuss 3 will provide an introduction to a repeated systematic application of the location theorem called the *bisection method*. This method forms the basis for the method that many graphing calculators use to find zeros.

››› EXPLORE-DISCUSS 3

Let $P(x) = x^5 + 3x - 1$. Because $P(0)$ is negative and $P(1)$ is positive, the location theorem implies that $P(x)$ must have at least one zero in the interval $(0, 1)$.

(A) Is $P(0.5)$ positive or negative? Does the location theorem guarantee a zero of $P(x)$ in the interval $(0, 0.5)$ or in $(0.5, 1)$?

(B) Let m be the midpoint of the interval from part A that contains a zero of $P(x)$. Is $P(m)$ positive or negative? What does this tell you about the location of the zero?

(C) Explain how this process could be used repeatedly to approximate a zero to any desired accuracy.

(D) Check your answers to parts A and B by using the ZERO command on a graphing calculator.

The **bisection method** is a systematic application of the procedure suggested in Explore-Discuss 3: Let $P(x)$ be a polynomial with real coefficients. If $P(x)$ has opposite signs at the endpoints of an interval (a, b), then by the location theorem $P(x)$ has a zero in (a, b). Bisect this interval (that is, find the midpoint $m = \frac{a+b}{2}$), and find $P(m)$. If $P(m) = 0$, you've found the zero. If not, $P(m)$ has to be either positive or negative, and, in turn, it has to be opposite in sign with either $P(a)$ or $P(b)$. The location theorem then implies that there is a zero in either (a, m) or (m, b)—whichever has opposite signs at the endpoints.

We repeat this bisection procedure (producing a set of intervals, each contained in and half the length of the previous interval, and each containing the zero) until we either find the zero exactly, or the two endpoints agree to the desired accuracy. Example 4 illustrates the procedure, and clarifies when the procedure is finished.

EXAMPLE 4 The Bisection Method

The polynomial $P(x) = x^4 - 2x^3 - 10x^2 + 40x - 90$ from Example 2 has a zero between 3 and 4. Use the bisection method to approximate it to one-decimal-place accuracy.

SOLUTION

We organize the results of our calculations in Table 1.

Table 1 Bisection Approximation

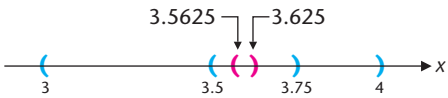
Sign Change Interval (a, b)	Midpoint m	Sign of P			
		$P(a)$	$P(m)$	$P(b)$	
(3, 4)	3.5	−	−	+	Choose (3.5, 4)
(3.5, 4)	3.75	−	+	+	Choose (3.5, 3.75)
(3.5, 3.75)	3.625	−	+	+	Choose (3.5, 3.625)
(3.5, 3.625)	3.5625	−	−	+	Choose (3.5625, 3.625)
(3.5625, 3.625)	We stop here	−		+	

Because the sign of $P(x)$ changes at the endpoints of the interval (3.5625, 3.625), we conclude that a real zero lies in this interval. But notice that each endpoint of that interval rounds to 3.6. So any number in between will round to 3.6 as well, and we can conclude that the zero is 3.6 to one-decimal-place accuracy.

MATCHED PROBLEM 4

The polynomial $P(x) = x^4 - 2x^3 - 10x^2 + 40x - 90$ from Example 2 has a zero between -5 and -4. Use the bisection method to approximate it to one-decimal-place accuracy.

Figure 8 illustrates the nested intervals produced by the bisection method in Table 1. Match each step in Table 1 with an interval in Figure 8. Note how each interval that contains a zero gets smaller and smaller and is contained in the preceding interval that contained the zero. In short, the intervals keep "closing in" on the zero.



› **Figure 8** Nested intervals produced by the bisection method in Table 1.

If we had wanted two-decimal-place accuracy, we would have continued the process in Table 1 until the endpoints of a sign change interval rounded to the same two-decimal-place number.

› Approximating Real Zeros at Turning Points

The bisection method for approximating zeros fails if a polynomial has a turning point at a zero, because the polynomial does not change sign at such a zero. Most graphing calculators use methods that are more sophisticated than the bisection method. Nevertheless, it is not unusual to get an error message when using the ZERO command to approximate a zero that is also a turning point. In this case, we can use the MAXIMUM or MINIMUM command, as appropriate, to approximate the turning point, and thus the zero.

EXAMPLE 5 **Approximating Zeros at Turning Points**

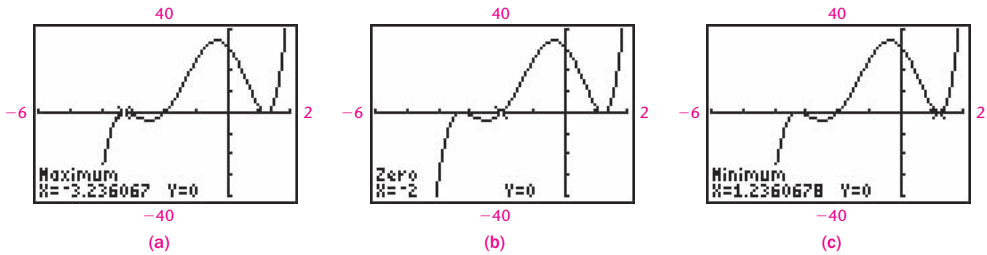
Let $P(x) = x^5 + 6x^4 + 4x^3 - 24x^2 - 16x + 32$. Use the upper and lower bound theorem to find the smallest positive integer and the largest negative integer that are upper and lower bounds, respectively, for the real zeros of $P(x)$. Approximate the zeros to two decimal places, using MAXIMUM or MINIMUM commands to approximate any zeros at turning points.

SOLUTION

The relevant rows of a synthetic division table show that 2 is an upper bound and -6 is a lower bound:

	1	6	4	−24	−16	32	
1	1	7	11	−13	−29	3	
2	1	8	20	16	16	64	← All positive coefficients
−5	1	1	−1	−19	79	−363	
−6	1	0	4	−48	272	−1600	← Coefficients alternate signs

Examining the graph of $P(x)$ on the next page, we find three zeros: the zero -3.24, found using the MAXIMUM command [Fig. 9(a)]; the zero -2, found using the ZERO command [Fig. 9(b)]; and the zero 1.24, found using the MINIMUM command [Fig. 9(c)].



› **Figure 9** Zeros of $P(x) = x^5 + 6x^4 + 4x^3 - 24x^2 - 16x + 32$.

MATCHED PROBLEM 5

Let $P(x) = x^5 - 6x^4 + 40x^2 - 12x - 72$. Use the upper and lower bound theorem to find the smallest positive integer and the largest negative integer that are upper and lower bounds, respectively, for the real zeros of $P(x)$. Approximate the zeros to two decimal places, using MAXIMUM or MINIMUM commands to approximate any zeros at turning points.

››› EXPLORE-DISCUSS 4

(A) Graph the polynomial $P(x) = x^3 - 8.1x^2 + 16.4x$ in a standard viewing window. How many zeros are apparent?

(B) Try to find the zero near $x = 4$ using the MINIMUM command. What does the result indicate?

(C) How many zeros actually exist in the standard viewing window?

(D) Repeat for $P(x) = x^3 - 8.1x^2 + 16.4x + 0.1$

The point of Explore-Discuss 4 is that you have to be careful about zeros at turning points. When it looks like a maximum or minimum occurs at height zero, it's possible that it may be a bit above or below the x axis. When this happens, there might actually be two zeros there, or none at all.

› Solving Polynomial Inequalities

In Section 2-7, we developed a procedure for solving quadratic inequalities. It was based on two things: finding the zeros of a quadratic function, and applying the location theorem. Since we now know that the location theorem applies to polynomials, we can use the same "test number" procedure to solve polynomial inequalities. Of course, we can still rely on graphical methods as well, but once again an understanding of both methods will lead to a better overall grasp of the concepts.

EXAMPLE 6 Solving Polynomial Inequalities

Solve the inequality $x^3 - x^2 - 6x > 0$.

SOLUTION

Algebraic Solution

Let $P(x) = x^3 - x^2 - 6x$. Then $P(x)$ can be factored:

$$x^3 - x^2 - 6x = x(x^2 - x - 6) = x(x + 2)(x - 3)$$

The zeros are $x = 0, -2$, and 3. This divides the x axis into four intervals, which we put in the following table. We then choose a test number in each interval and record the sign of the result.

Interval	Test number	Result
$(-\infty, -2)$	-3	$-3(-3 + 2)(-3 - 3) = -18$; negative
$(-2, 0)$	-1	$-1(-1 + 2)(-1 - 3) = 4$; positive
$(0, 3)$	1	$1(1 + 2)(1 - 3) = -6$; negative
$(3, \infty)$	4	$9(4 + 2)(4 - 3) = 54$; positive

Notice that it's simpler to substitute test numbers into the factored form of $P(x)$ than the original form.

The solution of the inequality is $(-2, 0) \cup (3, \infty)$.

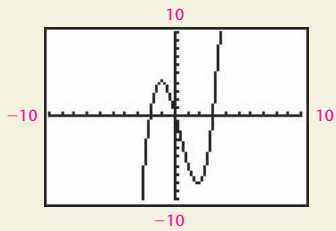
Graphical Solution

We graph P [Fig. 10] and use the ZERO command to find that -2, 0, and 3 are the zeros of P (details omitted). They partition the x axis into four intervals

$$(-\infty, -2), (-2, 0), (0, 3), \text{ and } (3, \infty)$$

By inspecting the graph of P we see that P is above the x axis on the intervals $(-2, 0)$ and $(3, \infty)$. Thus, the solution set of the inequality is

$$(-2, 0) \cup (3, \infty)$$



› Figure 10 $P(x) = x^3 - x^2 - 6x$.

MATCHED PROBLEM 6

Solve the inequality $x^3 + 2x^2 - 8x < 0$.

››› EXPLORE-DISCUSS 5

For the function $P(x) = x^3 - x^2 - 6x$ from Example 6, let $f(x) = \frac{P(x)}{|P(x)|}$.

(A) Graph f in a standard viewing window.

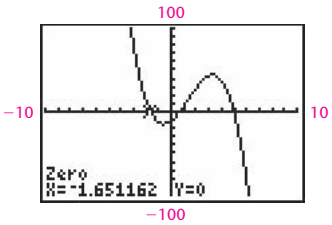
(B) Write the intervals where f takes on value 1, and the intervals where f takes on value -1. Compare the results to the solution of Example 6.

(C) Use the graph of f to write solutions for the following inequalities:

$$x^3 - x^2 - 6x < 0$$
$$x^3 - x^2 - 6x \geq 0$$
$$x^3 - x^2 - 6x \leq 0$$

(D) Discuss this procedure for solving inequalities graphically. Do you prefer this method over the graphical solution in Example 6?

EXAMPLE 7 Solving Polynomial Inequalities



› Figure 11

Solve $3x^2 + 12x - 4 \geq 2x^3 - 5x^2 + 7$ accurate to three decimal places.

SOLUTION

Subtracting the right side gives the equivalent inequality

$$P(x) = -2x^3 + 8x^2 + 12x - 11 \geq 0$$

The zeros of $P(x)$, to three decimal places, are found to be -1.651, 0.669, and 4.983 (Fig. 11) using the graph and the ZERO command.

The graph of P is above the x axis on the intervals $(-\infty, -1.651)$ and $(0.669, 4.983)$. The solution set of the inequality is thus

$$(-\infty, -1.651] \cup [0.669, 4.983]$$

The square brackets indicate that the endpoints of the intervals—the zeros of the polynomial—also satisfy the inequality.

In Example 7, because we needed to find the zeros graphically, it naturally made sense to solve the inequality graphically.

MATCHED PROBLEM 7

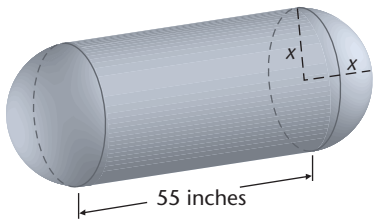
Solve $5x^3 - 13x < 4x^2 + 10x - 5$ accurate to three decimal places.

› Mathematical Modeling

EXAMPLE 8 Construction

An oil tank is in the shape of a right circular cylinder with a hemisphere (half sphere) at each end (Fig. 12). The cylinder is 55 inches long. Let x denote the common radius of the hemispheres and the cylinder.

› Figure 12



(A) Find a polynomial model for the volume of the tank.

(B) If the volume of the tank is required by a distributor to be $11{,}000\pi$ cubic inches (about 20 cubic feet), find x. Round to one decimal place.

SOLUTIONS

(A) The tank is made from a cylinder and a sphere (the two hemispheres). The volume of a cylinder is given by $V = \pi r^2 h$, where r is radius and h is height, so the volume of the cylindrical part of the tank has volume $\pi x^2(55)$. The volume of a sphere is $V = \frac{4}{3}\pi r^3$, which in this case is $V = \frac{4}{3}\pi x^3$. So the total volume can be modeled by

$$\begin{pmatrix}\text{Volume}\\ \text{of}\\ \text{tank}\end{pmatrix} = \begin{pmatrix}\text{Volume}\\ \text{of two}\\ \text{hemispheres}\end{pmatrix} + \begin{pmatrix}\text{Volume}\\ \text{of}\\ \text{cylinder}\end{pmatrix}$$

$$V = \frac{4}{3}\pi x^3 + 55\pi x^2$$

(B) We substitute $11{,}000\pi$ for V and solve for x.

$$\begin{aligned} 11{,}000\pi &= \tfrac{4}{3}\pi x^3 + 55\pi x^2 && \text{Multiply by 3 and divide by } \pi.\\ 33{,}000 &= 4x^3 + 165x^2 && \text{Subtract 33,000 from both sides.}\\ 0 &= 4x^3 + 165x^2 - 33{,}000 \end{aligned}$$

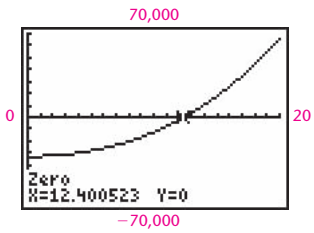
The desired value of x must be a positive zero of

$$P(x) = 4x^3 + 165x^2 - 33{,}000$$

Because the coefficients of $P(x)$ are large, we use large increments in a synthetic division table:

		4	165	0	−33,000
	10	4	205	2,050	−12,500
UB	20	4	245	4,900	65,000

We know x can't be negative, so no lower bound is needed. Graphing $y = P(x)$ for $0 \le x \le 20$ (Fig. 13), we see that $x = 12.4$ inches (to one decimal place).



› **Figure 13**
$P(x) = 4x^3 + 165x^2 - 33{,}000$.

MATCHED PROBLEM 8

Repeat Example 8 if the length of the cylinder is 80 inches, and the volume of the tank is $44{,}000\pi$ cubic inches.

ANSWERS TO MATCHED PROBLEMS

1. −4.190, −1.721, 2.912 **2.** Lower bound: −3; upper bound: 6
3. (A) Lower bound: −10; upper bound: 30 (B) Real zeros: 1.20, 11.46, 12.34
4. $x = -4.1$ **5.** Lower bound: −2; upper bound: 6; −1.65, 2, 3.65
6. $(-\infty, -4) \cup (0, 2)$ **7.** $(-\infty, -1.899) \cup (0.212, 2.488)$
8. (A) $V = \frac{4}{3}\pi x^3 + 80\pi x^2$ (B) 20.3 inches

3-3 Exercises

In Problems 1–6, use the upper and lower bound theorem to decide if the given positive number is an upper bound for the real zeros of P(x), and if the given negative number is a lower bound.

1. $P(x) = 2x^3 - 4x^2 - 18x + 1; x = 4; x = -3$

2. $P(x) = 3x^3 - 2x^2 - 10x + 2; x = 2; x = -2$

3. $P(x) = x^3 + 4x^2 - 3x - 10; x = 2; x = -4$

4. $P(x) = x^3 - 3x^2 + 4x + 10; x = 3; x = -1$

5. $P(x) = x^4 + 2x^3 - 3x^2 + 4x - 5; x = 1; x = -3$

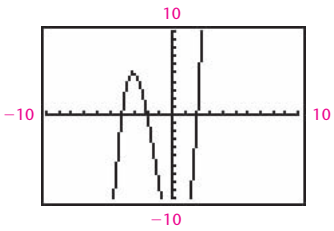
6. $P(x) = x^4 - 4x^3 + 8x^2 + 2x + 1; x = 5; x = -2$

In Problems 7–10, approximate the real zeros of each polynomial to three decimal places.

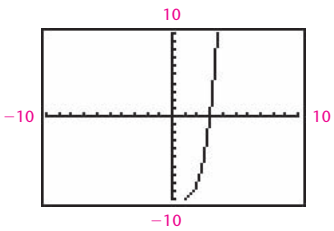
7. $P(x) = x^2 + 5x - 2$ **8.** $P(x) = 3x^2 - 7x + 1$

9. $P(x) = 2x^3 - 5x + 2$ **10.** $P(x) = x^3 - 4x^2 - 8x + 3$

11. The graph of $f(x) = x^3 + 4x^2 - 4x - 16$ is shown here in a standard viewing window. Can you be sure that all of the real zeros can be found using this window? Explain.



12. The graph of $g(x) = x^3 - 3x^2 + 4x - 12$ is shown here in a standard viewing window. Can you be sure that all of the real zeros can be found using this window? Explain.



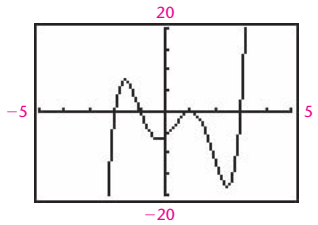
13. Suppose that synthetic division of a polynomial P by $x - 4$ results in all positive numbers in the quotient row. Is $x = 10$ an upper bound for the real zeros of P? Explain.

14. Suppose that synthetic division of a polynomial P by $x + 5$ results in a quotient row with alternating signs. Is $x = -10$ a lower bound for the real zeros of P? Explain.

15. State the location theorem in your own words.

16. Describe the connection between the location theorem and the bisection method.

In Problems 17–20, use the graph of P(x) to write the solution set for each inequality.



17. $P(x) \geq 0$ **18.** $P(x) < 0$

19. $P(x) > 0$ **20.** $P(x) \leq 0$

In Problems 21–24, solve each polynomial inequality to three decimal places (note the connection with Problems 7–10).

21. $x^2 + 5x - 2 > 0$ **22.** $3x^2 - 7x + 1 \geq 0$

23. $2x^3 - 5x + 2 \leq 0$ **24.** $x^3 - 4x^2 - 8x + 3 < 0$

Use the upper and lower bound theorem to find the smallest positive integer and largest negative integer that are upper and lower bounds, respectively, for the real zeros of each of the polynomials given in Problems 25–30.

25. $P(x) = x^3 - 3x + 1$

26. $P(x) = x^3 - 4x^2 + 4$

27. $P(x) = x^4 - 3x^3 + 4x^2 + 2x - 9$

28. $P(x) = x^4 - 4x^3 + 6x^2 - 4x - 7$

29. $P(x) = x^5 - 3x^3 + 3x^2 + 2x - 2$

30. $P(x) = x^5 - 3x^4 + 3x^2 + 2x - 1$

In Problems 31–38,

(A) Use the upper and lower bound theorem to find the smallest positive integer and largest negative integer that are upper and lower bounds, respectively, for the real zeros of P(x).

(B) Approximate the real zeros of each polynomial to two decimal places.

31. $P(x) = x^3 - 2x^2 + 3x - 8$

32. $P(x) = x^3 + 3x^2 + 4x + 5$

33. $P(x) = x^4 + x^3 - 5x^2 + 7x - 22$

34. $P(x) = x^4 - x^3 - 8x^2 - 12x - 25$

35. $P(x) = x^5 - 3x^3 - 4x + 4$

36. $P(x) = x^5 - x^4 - 2x^2 - 4x - 5$

37. $P(x) = x^5 + x^4 + 3x^3 + x^2 + 2x - 5$

38. $P(x) = x^5 - 2x^4 - 6x^2 - 9x + 10$

In Problems 39–46,
(A) Use the location theorem to explain why the polynomial function has a zero in the indicated interval.
(B) Determine the number of additional intervals required by the bisection method to obtain a one-decimal-place approximation to the zero and state the approximate value of the zero.

39. $P(x) = x^3 - 2x^2 - 5x + 4; (3, 4)$

40. $P(x) = x^3 + x^2 - 4x - 1; (1, 2)$

41. $P(x) = x^3 - 2x^2 - x + 5; (-2, -1)$

42. $P(x) = x^3 - 3x^2 - x - 2; (3, 4)$

43. $P(x) = x^4 - 2x^3 - 7x^2 + 9x + 7; (3, 4)$

44. $P(x) = x^4 - x^3 - 9x^2 + 9x + 4; (2, 3)$

45. $P(x) = x^4 - x^3 - 4x^2 + 4x + 3; (-1, 0)$

46. $P(x) = x^4 - 3x^3 - x^2 + 3x + 3; (2, 3)$

Problems 47–50 refer to the polynomial

$$P(x) = (x - 1)^2(x - 2)(x - 3)^4$$

47. Can the zero at $x = 1$ be approximated by the bisection method? Explain.

48. Can the zero at $x = 2$ be approximated by the bisection method? Explain.

49. Can the zero at $x = 3$ be approximated by the bisection method? Explain.

50. Which of the zeros can be approximated using the MAXIMUM command? Using the MINIMUM command? Using the ZERO command?

In Problems 51–56, approximate the zeros of each polynomial function to two decimal places, using MAXIMUM or MINIMUM commands to approximate any zeros at turning points.

51. $P(x) = x^4 - 4x^3 - 10x^2 + 28x + 49$

52. $P(x) = x^4 + 4x^3 - 4x^2 - 16x + 16$

53. $P(x) = x^5 - 6x^4 + 4x^3 + 24x^2 - 16x - 32$

54. $P(x) = x^5 - 6x^4 + 2x^3 + 28x^2 - 15x + 2$

55. $P(x) = x^5 - 6x^4 + 11x^3 - 4x^2 - 3.75x - 0.5$

56. $P(x) = x^5 + 12x^4 + 47x^3 + 56x^2 - 15.75x + 1$

In Problems 57–70, solve each polynomial inequality. Approximate to three decimal places if necessary.

57. $(2x - 5)(x + 3)(x + 7) > 0$

58. $(x + 4)(3x - 9)(x - 1) < 0$

59. $x^2 > 2$

60. $2x^2 \le 38$

61. $x^3 \le 9x$

62. $25x > x^3$

63. $x^3 + 5x^2 \le -6x$

64. $7x^2 > 8x - x^3$

65. $x^2 + 7x - 3 \le x^3 + x + 4$

66. $x^4 + 1 > 3x^2$

67. $x^4 < 8x^3 - 17x^2 + 9x - 2$

68. $x^3 + 5x \ge 2x^3 - 4x^2 + 6$

69. $(x^2 + 2x - 2)^2 \ge 2$

70. $5 + 2x < (x^2 - 4)^2$

71. The statement of the upper and lower bound theorem requires that the leading coefficient of a polynomial be positive. What if the leading coefficient is negative?
(A) Graph $P(x) = -x^3 - x^2 + 6x$ in a standard viewing window. How many real zeros do you see? Are these all of the real zeros? How can you tell?
(B) Based on the graph, is $x = 3$ an upper bound for the real zeros?
(C) Use synthetic division to divide $P(x)$ by $x - 3$. What do you notice about the quotient row? What can you conclude about upper bounds for polynomials with negative leading coefficients?

72. Refer to Problem 71.
(A) Based on the graph, is $x = -4$ a lower bound for the real zeros?
(B) Use synthetic division to divide $P(x)$ by $x + 4$. What do you notice about the quotient row? What can you conclude about lower bounds for polynomials with negative leading coefficients?

In Problems 73–82,
(A) Use the upper and lower bound theorem to find the smallest positive integer multiple of 10 and largest negative integer multiple of 10 that are upper and lower bounds, respectively, for the real zeros of each polynomial.
(B) Approximate the real zeros of each polynomial to two decimal places.

73. $P(x) = x^3 - 24x^2 - 25x + 10$

74. $P(x) = x^3 - 37x^2 + 70x - 20$

75. $P(x) = x^4 + 12x^3 - 900x^2 + 5{,}000$

76. $P(x) = x^4 - 12x^3 - 425x^2 + 7{,}000$

77. $P(x) = x^4 - 100x^2 - 1{,}000x - 5{,}000$

78. $P(x) = x^4 - 5x^3 - 50x^2 - 500x + 7{,}000$

79. $P(x) = 4x^4 - 40x^3 - 1{,}475x^2 + 7{,}875x - 10{,}000$

80. $P(x) = 9x^4 + 120x^3 - 3{,}083x^2 - 25{,}674x - 48{,}400$

81. $P(x) = 0.01x^5 - 0.1x^4 - 12x^3 + 9{,}000$

82. $P(x) = 0.1x^5 + 0.7x^4 - 18.775x^3 - 340x^2 - 1{,}645x - 2{,}450$

83. When synthetic division is used to divide a polynomial $P(x)$ by $x + 4$ the remainder is 10. When the same polynomial is divided by $x + 5$ the remainder is -8. Must $P(x)$ have a zero between -5 and -4? Explain.

84. When synthetic division is used to divide a polynomial $Q(x)$ by $x + 4$ the remainder is 10. When the same polynomial is divided by $x + 5$ the remainder is 8. Could $Q(x)$ have a zero between -5 and -4? Explain.

85. In this problem, we sketch a proof of the upper bound theorem. We suppose that a polynomial $P(x)$ is divided by $x - r$ $(r > 0)$ using synthetic division, and all numbers in the quotient row are nonnegative.

(A) Write the result of the division algorithm for this division. What can we say about the coefficients of $Q(x)$ and the remainder R?

(B) Suppose that x is a number greater than r. What is the sign of $x - r$? Why can we say that the sign of $Q(x)$ must be positive?

(C) Is it possible for $P(x)$ to be zero? Why not?

(D) How does this prove the upper bound theorem?

86. In this problem, we sketch a proof of one case of the lower bound theorem. Suppose that a polynomial $P(x)$ is divided by $x - r (r < 0)$ using synthetic division, and the quotient row alternates in sign.

(A) If we use the division algorithm to write $P(x) = Q(x)(x - r) + R$, where R is positive, what is the sign of the constant term of $Q(x)$? What is the sign of all even-powered terms? What about all odd-powered terms? (Remember: alternating signs!)

(B) Suppose that x is a negative number less than r. What will be the sign of the result when substituting x into all of the even-powered terms? What about all of the odd-powered terms?

(C) Is it possible for $P(x)$ to be zero? Why not?

(D) How does this prove the lower bound theorem?

APPLICATIONS

87. **PROFIT** A small manufacturing company produces toasters for a national chain of discount stores. An economic consultant estimates that the company's weekly profit can be modeled by the function $P(x) = -0.000\,068x^3 + 0.0041x^2 + 15.18x - 1{,}225$, where P is profit in dollars and x is number of toasters produced. The company can produce at most 500 toasters per week.

(A) How many toasters does the company need to produce in order to break even (that is, have profit zero)?

(B) How many toasters should the company produce to make a profit of at least \$1,200 per week?

88. **PROFIT** An independent home builder's annual profit in thousands of dollars can be modeled by the polynomial $P(x) = 5.152x^3 - 143.0x^2 + 1{,}102x - 1{,}673$ where x is the number of houses built in a year. His company can build at most 13 houses in a year.

(A) How many houses must he build to break even (that is, have profit zero)?

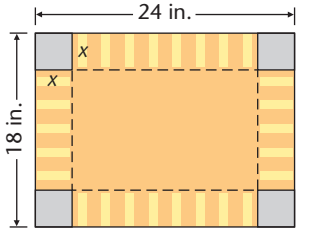
(B) How many houses should he build to have a profit of at least \$400,000?

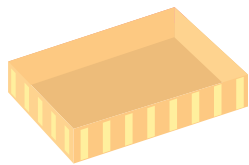
Express the solutions to Problems 89–94 as the roots of a polynomial equation of the form $P(x) = 0$ and approximate these solutions to three decimal places.

89. **GEOMETRY** Find all points on the graph of $y = x^2$ that are one unit away from the point (1, 2). [*Hint:* Use the distance-between-two-points formula from Appendix B, Section B-3.]

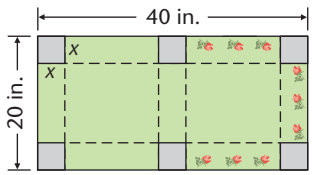
90. **GEOMETRY** Find all points on the graph of $y = x^2$ that are one unit away from the point (2, 1).

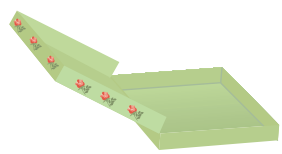
91. **MANUFACTURING** A box is to be made out of a piece of cardboard that measures 18 by 24 inches. Squares, x inches on a side, will be cut from each corner, and then the ends and sides will be folded up (see the figure). Find the value of x that would result in a box with a volume of 600 cubic inches.



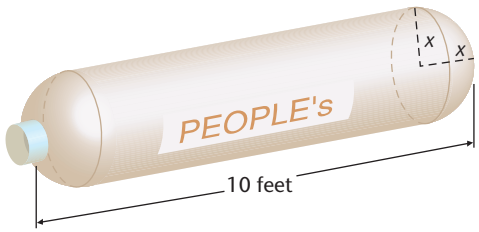


92. **MANUFACTURING** A box with a hinged lid is to be made out of a piece of cardboard that measures 20 by 40 inches. Six squares, x inches on a side, will be cut from each corner and the middle, and then the ends and sides will be folded up to form the box and its lid (see the figure). Find the value of x that would result in a box with a volume of 500 cubic inches.

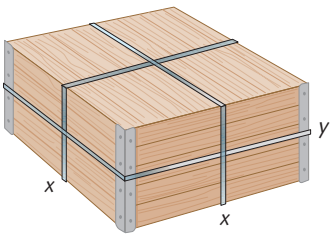




93. CONSTRUCTION A propane gas tank is in the shape of a right circular cylinder with a hemisphere at each end (see the figure). If the overall length of the tank is 10 feet and the volume is 20π cubic feet, find the common radius of the hemispheres and the cylinder.



94. SHIPPING A shipping box is reinforced with steel bands in all three directions (see the figure). A total of 20.5 feet of steel tape is to be used, with 6 inches of waste because of a 2-inch overlap in each direction. If the box has a square base and a volume of 2 cubic feet, find its dimensions.



MATHEMATICAL MODELING AND REGRESSION ANALYSIS

Problems 95–98 refer to Table 2, which shows homeownership rates for households of various races and ethnicities from 1996 to 2005. For each problem, use $x = 0$ for 1996, and round all coefficients to 3 significant digits.

Table 2 Homeownership Rates by Race and Ethnicity

	Hispanic	Asian	White
1996	42.8%	50.8%	69.1%
1998	44.7%	52.6%	70.0%
2000	46.3%	52.8%	71.1%
2002	48.2%	54.7%	71.8%
2003	46.7%	56.3%	72.1%
2004	48.1%	59.8%	72.8%
2005	49.5%	60.1%	72.7%

Source: U.S. Census Bureau

95. Find a cubic regression model for the homeownership rates for Hispanic households. Then set up and solve an inequality that predicts years in which the homeownership for Hispanic households will exceed 60%.

96. Find a cubic regression model for the homeownership rates for Asian households. Then set up and solve an inequality that predicts the years in which the homeownership for Asian households will exceed 70%.

97. Refer to Problem 95. Find a linear regression model for the homeownership rates for white households. Then set up and solve an inequality that predicts the years in which the homeownership for Hispanic households will exceed the percentage for white households.

98. Refer to Problems 95 and 96. Set up and solve an inequality that predicts the years in which the homeownership for Hispanic households will exceed the percentage for Asian households.

Problems 99 and 100 refer to Table 3, which provides the percentage of males and females in the United States who have completed four or more years of college from 1970 to 2005. In each problem, use $x = 0$ for 1970, and round coefficients to three significant digits.

Table 3 Percentage of People in the United States Who Have Completed 4 or More Years of College

Year	Males	Females
1970	14.1	8.2
1980	20.9	13.6
1990	24.4	18.4
1995	26.0	20.2
2000	27.8	23.6
2005	28.9	26.5

Source: www.infoplease.com

99. Find a cubic regression model for the percentage of males having completed 4 or more years of college. Then set up and solve an inequality that predicts the first year in which this percentage will exceed 40%.

100. Refer to Problem 99. Find a cubic regression model for the percentage of females having completed 4 or more years of college. Then set up and solve an inequality that predicts the years in which this percentage will exceed the percentage for males.

3-4 Complex Zeros and Rational Zeros of Polynomials

- The Fundamental Theorem of Algebra
- Finding Factors of Polynomials with Real Coefficients
- Graphs of Polynomials with Real Coefficients
- Finding Rational Zeros

In this section, we will attempt to answer a question that may have occurred to you earlier in this chapter when looking for zeros of polynomials: How do you know exactly how many zeros to look for?

The graph of the polynomial function $P(x) = x^2 + 4$ does not cross the x axis, so $P(x)$ has no real zeros. It does, however, have complex zeros, $2i$ and $-2i$. This tells us that $P(x)$ can be factored as $x^2 + 4 = (x - 2i)(x + 2i)$. The *fundamental theorem of algebra* guarantees that *every* nonconstant polynomial with real or complex coefficients has at least one complex zero. We will use it to show that any such polynomial can be factored completely as a product of linear factors. In this section, we will study the fundamental theorem of algebra and its implications. This will help us improve our understanding of the graphs of polynomials with real coefficients. Finally, we will consider a problem that has led to important advances in mathematics and its applications: When can the zeros of a polynomial be found *exactly*?

The Fundamental Theorem of Algebra

The fundamental theorem of algebra was proved by Karl Friedrich Gauss (1777–1855), one of the greatest mathematicians of all time, in his doctoral thesis. A proof of the theorem is beyond the scope of this book, so we will state and use it without proof.

THEOREM 1 The Fundamental Theorem of Algebra

Every polynomial of degree $n > 0$ with complex coefficients has a complex zero.

CAUTION

Remember that real numbers are complex numbers, too! The fundamental theorem does not say that every polynomial has a complex zero of the form $a + bi$, where b is not zero. It also does not require that any of the coefficients are of that form. It simply guarantees that every nonconstant polynomial has at least one zero, which may or may not be real.

The fundamental theorem of algebra certainly has an important-sounding name for a pretty simple fact. While it may not seem terribly important that every polynomial has a zero, the main value of the fundamental theorem in this course is that it can be used to develop other results that tell us a lot more about polynomials.

If $P(x)$ is any polynomial of degree $n > 0$, then by Theorem 1 it has a zero, which we'll call r_1. So the factor theorem tells us that $x - r_1$ is a factor of $P(x)$. In other words,

$$P(x) = (x - r_1)Q(x)$$

where $Q(x)$ has degree $n - 1$. Now $Q(x)$ is a polynomial also, so by Theorem 1, it has a zero r_2 (which may or may not be equal to r_1). So

$$Q(x) = (x - r_2)Q_2(x) \qquad \text{and}$$
$$P(x) = (x - r_1)(x - r_2)Q_2(x)$$

where $Q_2(x)$ has degree $n - 2$. Do you see where this is headed? If we continue this process n times (recall that n is the degree of P), we develop Theorem 2.

› THEOREM 2 The *n* Linear Factors Theorem

Every polynomial of degree $n > 0$ with complex coefficients can be factored completely as a product of n linear factors.

Theorem 2 guarantees n linear factors for a polynomial of degree n, but it doesn't guarantee n *distinct* linear factors. For example, the polynomial

$$P(x) = (x - 5)^3(x + 1)^2(x - 6i)(x + 2 + 3i) \tag{1}$$

can also be written as

$$P(x) = (x - 5)(x - 5)(x - 5)(x + 1)(x + 1)(x - 6i)(x + 2 + 3i)$$

In this form, we see that there are, in fact, seven linear factors of this seventh-degree polynomial. But because some factors are repeated, there are only four distinct zeros: 5, -1, $6i$, and $-2 - 3i$. Because the factor $x - 5$ appears to the power 3, we say that the corresponding zero $x = 5$ has **multiplicity** 3. Similarly, -1 has multiplicity 2, and $6i$ and $-2 - 3i$ have multiplicity 1. A zero of multiplicity 2 is called a **double zero.** *Note that the sum of multiplicities is always equal to the degree of the polynomial:* For $P(x)$ in equation (1), $3 + 2 + 1 + 1 = 7$. This fact answers our question about the number of zeros for a polynomial:

Every polynomial of degree *n* has *n* zeros, if you count a zero of multiplicity *m* as *m* zeros.

EXAMPLE 1 **Multiplicities of Zeros**

Find all zeros and write their multiplicities:

(A) $P(x) = (x + 2)^7(x - 4)^8(x^2 + 1)$ (B) $Q(x) = (x + 1)^3(x^2 - 1)(x + 1 - i)$

SOLUTIONS

(A) Note that $x^2 + 1 = 0$ has the solutions i and $-i$. The zeros of $P(x)$ are -2 (multiplicity 7), 4 (multiplicity 8), i and $-i$ (each multiplicity 1).

(B) Note that $x^2 - 1 = (x - 1)(x + 1)$, so $Q(x)$ can be written as $Q(x) = (x + 1)^3(x + 1)(x - 1)(x + 1 - i)$, and $x + 1$ appears four times as a factor of $Q(x)$. The zeros of $Q(x)$ are -1 (multiplicity 4), 1 (multiplicity 1), and $-1 + i$ (multiplicity 1). [The factor $x + 1 - i$ can be written as $x - (-1 + i)$.]

MATCHED PROBLEM 1

Find all zeros and write their multiplicities:

(A) $P(x) = (x - 5)^3(x + 3)^2(x^2 + 16)$ (B) $Q(x) = (x^2 - 25)^3(x + 5)(x - i)$

Finding Factors of Polynomials with Real Coefficients

We saw in Section 2-5 that if a quadratic function has an imaginary zero $a + bi$, then its conjugate $a - bi$ is a zero as well. (See Problem 94 in Exercises 2-5.) It turns out that this is actually true for any polynomial of positive degree with real coefficients.

THEOREM 3 Imaginary Zeros of Polynomials with Real Coefficients

Imaginary zeros of polynomials with real coefficients, if they exist, occur in conjugate pairs: If $a + bi$ is a zero, then its conjugate $a - bi$ is a zero as well.

EXPLORE-DISCUSS 1

(A) Suppose that a polynomial has an imaginary zero $a + bi$, and the only other zero is a real number c. Then $x - (a + bi)$ and $x - c$ are factors of P. Multiply these two factors, then simplify. What type of coefficients must P have in this case?

(B) What if $a - bi$ is also a zero? Multiply $x - (a + bi)$ and $x - (a - bi)$, then simplify. Are all the coefficients real now?

Explore-Discuss 1 gives a rough indication of why imaginary zeros of a polynomial with real coefficients must occur in pairs. If they didn't, when multiplying all the factors out, you would always get at least one imaginary coefficient.

But Explore-Discuss 1 actually does much more: It provides a road map for factoring any polynomial with real coefficients. We already know that any such polynomial of degree n can be factored into n linear factors (Theorem 2). If any zeros are imaginary, some of the factors will have imaginary coefficients. But now we know that these factors will occur in pairs, and when we multiply those pairs, the result is a quadratic factor with real coefficients. This result is summarized in Theorem 4.

› THEOREM 4 The Linear and Quadratic Factors Theorem

If $P(x)$ is a polynomial of degree $n > 0$ with real coefficients, then $P(x)$ can be factored as a product of linear factors (with real coefficients) and quadratic factors (with real coefficients and imaginary zeros).

EXAMPLE 2 **Factors of Polynomials**

Given that $x = -1$ is a zero of $P(x) = x^3 + x^2 + 4x + 4$, factor $P(x)$ in two ways:

(A) As a product of linear factors (with real coefficients) and quadratic factors (with real coefficients and imaginary zeros);

(B) As a product of linear factors with complex coefficients.

SOLUTIONS

We first need to find all of the zeros of $P(x)$. (You may want to refer back to Example 6 in Section 3-2.)

First, divide by $x + 1$ using synthetic division:

$$\begin{array}{r|cccc} & 1 & 1 & 4 & 4 \\ & & -1 & 0 & -4 \\ \hline -1 & 1 & 0 & 4 & 0 \end{array}$$

Now we know that $P(x) = (x + 1)(x^2 + 4)$. Next, we find the zeros of the remaining factor:

$$\begin{aligned} x^2 + 4 &= 0 \\ x^2 &= -4 \\ x &= 2i, -2i \end{aligned}$$

(A) We've already solved part A: $P(x) = (x + 1)(x^2 + 4)$.

(B) The zeros of $P(x)$ are -1, $2i$, and $-2i$, so $P(x) = (x + 1)(x - 2i)(x + 2i)$. ◉

MATCHED PROBLEM 2

Factor $P(x) = x^5 - x^4 - x + 1$ in two ways:

(A) As a product of linear factors (with real coefficients) and quadratic factors (with real coefficients and imaginary zeros);

(B) As a product of linear factors with complex coefficients.

Example 2 may have made you wonder how we would know to start with $x = -1$. We'll address this important question at the end of this section.

Graphs of Polynomials with Real Coefficients

The factorization described in Theorem 4 gives additional information about the graphs of polynomial functions with real coefficients. For certain polynomials the factorization of Theorem 4 will involve only linear factors; for others, only quadratic factors. Of course, if only quadratic factors are present, then the degree of the polynomial $P(x)$ must be a multiple of 2; that is, even. On the other hand, a polynomial $P(x)$ of odd degree with real coefficients must have at least one linear factor with real coefficients. This proves Theorem 5.

THEOREM 5 Real Zeros and Polynomials of Odd Degree

Every polynomial of odd degree with real coefficients has at least one real zero, and consequently at least one x intercept.

EXPLORE-DISCUSS 2

The graph of the polynomial $P(x) = x(x - 1)^2(x + 1)^4(x - 2)^3$ is shown in Figure 1. Find the real zeros of $P(x)$ and their multiplicities. How can a real zero of even multiplicity be distinguished from a real zero of odd multiplicity using only the graph?

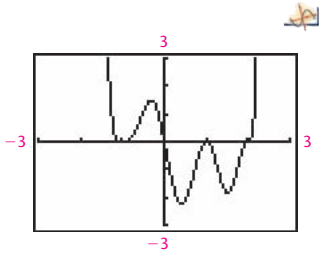


Figure 1 Graph of $P(x) = x(x - 1)^2(x + 1)^4(x - 2)^3$.

For polynomials with real coefficients, as suggested by Explore-Discuss 2, you can easily distinguish real zeros of even multiplicity from those of odd multiplicity using only the graph. Theorem 6, which we state without proof, tells us how to do that.

THEOREM 6 Zeros of Even or Odd Multiplicity

Let $P(x)$ be a polynomial with real coefficients:

1. If r is a real zero of $P(x)$ of even multiplicity, then $P(x)$ has a turning point at r and does not change sign at r. (The graph just touches the x axis, then changes direction.)
2. If r is a real zero of $P(x)$ of odd multiplicity, then $P(x)$ does not have a turning point at r and changes sign at r. (The graph continues through to the opposite side of the x axis.)

EXAMPLE 3 Multiplicities from Graphs

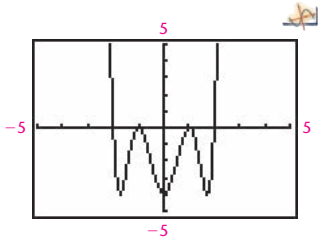


Figure 2

Figure 2 shows the graph of a polynomial function of degree 6. Find the real zeros and their multiplicities.

SOLUTION

The numbers -2, -1, 1, and 2 are real zeros (x intercepts). The graph has turning points at $x = \pm 1$ but not at $x = \pm 2$. Therefore, by Theorem 6, the zeros -1 and 1 have even multiplicity, and -2 and 2 have odd multiplicity. Because the sum of the multiplicities must equal 6 (the degree), the zeros -1 and 1 each have multiplicity 2, and the zeros -2 and 2 each have multiplicity 1. Any multiplicities greater than 1 or 2 would result in a degree greater than 6.

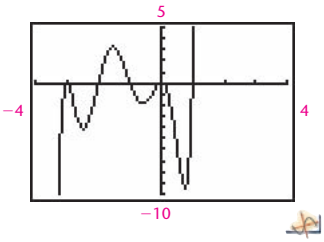


Figure 3

MATCHED PROBLEM 3

Figure 3 shows the graph of a polynomial function of degree 7. Find the real zeros and their multiplicities.

Finding Rational Zeros

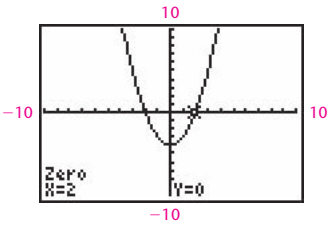


Figure 4 $P(x) = x^2 - (4 + 10^{-9})$.

From a graphical perspective, finding a zero of a polynomial means finding a good approximation to an actual zero. A graphing calculator, for example, might give 2 as a zero of $P(x) = x^2 - (4 + 10^{-9})$ even though $P(2)$ is equal to -10^{-9}, not 0 (Fig. 4).

It is natural, however, to want to find zeros *exactly*. Although this is impossible in general, we will adopt an algebraic strategy to find exact zeros in a special case: when a polynomial with rational coefficients has at least some zeros that are rational numbers. We will find a graphing calculator to be very helpful in carrying out our algebraic strategy.

››› EXPLORE-DISCUSS 3

(A) Use a graphing calculator to find the zeros of $P_1(x) = \frac{1}{9}x^3 - \frac{1}{9}x^2 - x + 1$.

(B) Use a graphing calculator to find the zeros of $P_2(x) = x^3 - x^2 - 9x + 9$.

(C) Explain why the two equations $\frac{1}{9}x^3 - \frac{1}{9}x^2 - x + 1 = 0$ and $x^3 - x^2 - 9x + 9 = 0$ have the same solution.

Using the idea behind Explore-Discuss 3, we can reduce our work in finding zeros of polynomials with rational coefficients. If we multiply a function $P(x)$ by the least common denominator of any fractions in $P(x)$, we will get a new polynomial with integer coefficients, and the exact same zeros as $P(x)$. (The new polynomial will have a different graph, different maximums and minimums, etc., but it will have the same zeros.)

The point is that we can study zeros of polynomials with rational coefficients by studying just the polynomials with integer coefficients.

We will introduce a method for identifying potential rational zeros by examining the following quadratic polynomial whose zeros can be found easily by factoring:

$$P(x) = 6x^2 - 13x - 5 = (2x - 5)(3x + 1)$$

$$\text{Zeros of } P(x): \quad \frac{5}{2} \quad \text{and} \quad -\frac{1}{3} = \frac{-1}{3}$$

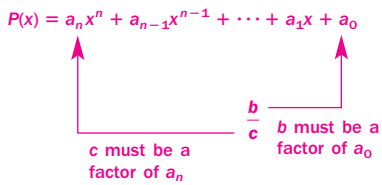
Notice that the numerators, 5 and -1, of the zeros are both integer factors of -5, the constant term in $P(x)$. The denominators 2 and 3 of the zeros are both integer factors of 6, the leading coefficient of $P(x)$. This is not a coincidence. These observations are generalized in Theorem 7.

› THEOREM 7 The Rational Zero Theorem

If the rational number b/c, in lowest terms, is a zero of the polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad a_n \neq 0$$

with integer coefficients, then b must be an integer factor of a_0 (the constant term) and c must be an integer factor of a_n (the leading coefficient).



The rational zero theorem is a bit hard to understand until it is illustrated with an example.

EXAMPLE 4 **Identifying Possible Rational Zeros**

Make a list of all possible rational zeros of $P(x) = 4x^3 - x^2 + 2x + 6$.

SOLUTION

1. Make a list of all factors of the constant term, 6: $\pm 1, \pm 2, \pm 3, \pm 6$
2. Make a list of all possible factors of the leading coefficient, 4: $\pm 1, \pm 2, \pm 4$
3. Write all possible fractions with numerator from the first list and denominator from the second:

$$\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{2}{1}, \pm\frac{2}{2}, \pm\frac{2}{4}, \pm\frac{3}{1}, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm\frac{6}{1}, \pm\frac{6}{2}, \pm\frac{6}{4}$$

4. Reduce and eliminate all repeats from the list:

$$\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm 2, \pm 3, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm 6$$

MATCHED PROBLEM 4

Make a list of all possible rational zeros of $P(x) = 3x^4 - 7x^2 + 12$.

››› EXPLORE-DISCUSS 4

Let $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, where a_3, a_2, a_1, and a_0 are integers.

1. If $P(2) = 0$, there is one coefficient that must be an even integer. Identify this coefficient and explain why it must be even.
2. If $P(\frac{1}{2}) = 0$, there is one coefficient that must be an even integer. Identify this coefficient and explain why it must be even.
3. If $a_3 = a_0 = 1, P(-1) \neq 0$, and $P(1) \neq 0$, does $P(x)$ have any rational zeros? Support your conclusion with verbal arguments and/or examples.

››› CAUTION ›››

The rational zero theorem does *not* say that any of the rational numbers it lists actually are zeros! It's entirely possible that none of them are. But if there are any rational zeros for a given polynomial, they will be on the list. In short, the theorem identifies *potential* zeros, which can then be checked.

Once we have used the rational zero theorem to build a list of potential rational zeros, we can find any that actually are zeros using a process of elimination illustrated in Example 5.

EXAMPLE 5 **Finding Rational Zeros**

Find all the rational zeros for $P(x) = 2x^3 + 9x^2 + 7x - 6$.

SOLUTION

We begin by making lists, as in Example 4.

Factors of constant term, -6: $\pm 1, \pm 2, \pm 3, \pm 6$
Factors of leading coefficient, 2: $\pm 1, \pm 2$

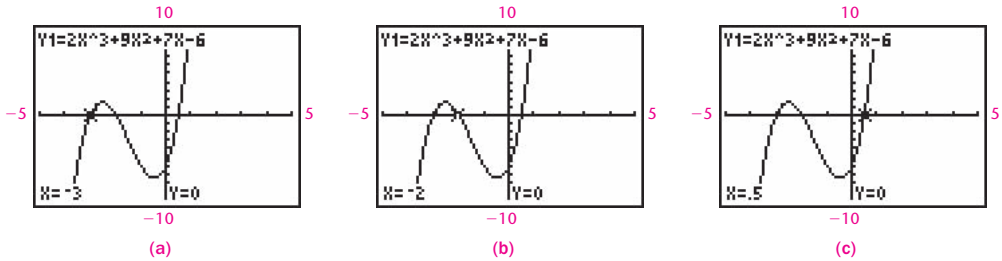
All possible fractions with numerator from the first list and denominator from the second:

$$\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{2}{1}, \pm\frac{2}{2}, \pm\frac{3}{1}, \pm\frac{3}{2}, \pm\frac{6}{1}, \pm\frac{6}{2}$$

Reduce and eliminate repeats to find a list of potential rational zeros:

$$\pm 1, \pm\frac{1}{2}, \pm 2, \pm 3, \pm\frac{3}{2}, \pm 6 \tag{2}$$

If $P(x)$ has any rational zeros, they must be in list (2). We could test each number r in this list by evaluating $P(r)$, either directly or using synthetic division. However, exploring the graph of $y = P(x)$ first will usually indicate which numbers in the list are the most likely candidates for zeros. Examining a graph of $P(x)$, we see that there are zeros near -3, near -2, and between 0 and 1. So the most likely candidates from our list are -3, -2, and $\frac{1}{2}$. We next use a graphing calculator to evaluate $P(x)$ for these three values (Fig. 5).



› Figure 5

It turns out that -3, -2, and $\frac{1}{2}$ are rational zeros of $P(x)$. Because a third-degree polynomial can have at most three zeros, we have found all the rational zeros. There is no need to test the remaining candidates in list (2).

MATCHED PROBLEM 5

Find all rational zeros for $P(x) = 2x^3 + x^2 - 11x - 10$.

As we saw in the solution of Example 5, rational zeros can be located by simply evaluating the polynomial. However, if we want to find multiple zeros, imaginary zeros, or exact values of irrational zeros, we need to consider *reduced polynomials*. If r is a zero of a polynomial $P(x)$, then we know that we can write

$$P(x) = (x - r)Q(x)$$

where $Q(x)$ is a polynomial of degree one less than the degree of $P(x)$. We will call the quotient polynomial $Q(x)$ the **reduced polynomial** for $P(x)$. In Example 5, we could have checked if $x = -3$ is a zero of $P(x)$ using synthetic division.

$$\begin{array}{r|rrrr} & 2 & 9 & 7 & -6 \\ & & -6 & -9 & 6 \\ \hline -3 & 2 & 3 & -2 & 0 \end{array}$$

This would not only show that $x = -3$ is, in fact, a zero, but also that $P(x) = (x + 3)(2x^2 + 3x - 2)$.

Because the reduced polynomial $Q(x) = 2x^2 + 3x - 2$ is a quadratic, we can find its zeros by factoring or the quadratic formula. Thus,

$$P(x) = (x + 3)(2x^2 + 3x - 2) = (x + 3)(x + 2)(2x - 1)$$

and we see that the zeros of $P(x)$ are -3, -2, and $\frac{1}{2}$, as before.

The advantage to this second approach is that if some of the zeros were irrational, we couldn't have found them exactly using a graphing calculator. Worse still, if any were imaginary, we couldn't have found them at all.

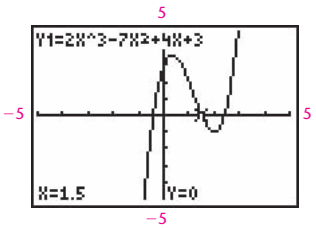
EXAMPLE 6 Finding Rational and Irrational Zeros

Find all zeros exactly for $P(x) = 2x^3 - 7x^2 + 4x + 3$.

SOLUTION

First, list the possible rational zeros as in Example 4:

$$\pm 1, \pm 3, \pm\tfrac{1}{2}, \pm\tfrac{3}{2}$$



› Figure 6

Examining the graph of $y = P(x)$ (Fig. 6), we see that there is a zero between -1 and 0, another between 1 and 2, and a third between 2 and 3. We test the only likely candidates on our list, $-\frac{1}{2}$ and $\frac{3}{2}$, using synthetic division:

$$\begin{array}{r|cccc} & 2 & -7 & 4 & 3 \\ & & -1 & 4 & -4 \\ \hline -\frac{1}{2} & 2 & -8 & 8 & -1 \end{array} \qquad \begin{array}{r|cccc} & 2 & -7 & 4 & 3 \\ & & 3 & -6 & -3 \\ \hline \frac{3}{2} & 2 & -4 & -2 & 0 \end{array}$$

So $-\frac{1}{2}$ is not a zero, but that's okay; $\frac{3}{2}$ is one zero, and the reduced polynomial is $2x^2 - 4x - 2$. Because the reduced polynomial is quadratic, we can use the quadratic formula to find the exact values of the remaining zeros:

$$\begin{aligned} 2x^2 - 4x - 2 &= 0 && \text{Divide both sides by 2.} \\ x^2 - 2x - 1 &= 0 && \text{Use the quadratic formula with } a = 1, b = -2, c = -1. \\ x &= \frac{2 \pm \sqrt{4 - 4(1)(-1)}}{2} \\ &= \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2} \end{aligned}$$

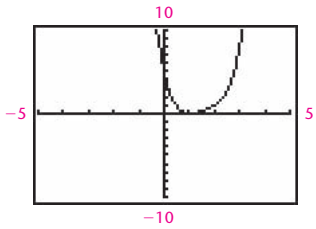
The exact zeros of $P(x)$ are $\frac{3}{2}$ and $1 \pm \sqrt{2}$.

MATCHED PROBLEM 6

Find all zeros exactly for $P(x) = 3x^3 - 10x^2 + 5x + 4$.

EXAMPLE 7 Finding Rational and Imaginary Zeros

Find all zeros exactly for $P(x) = x^4 - 6x^3 + 14x^2 - 14x + 5$.



› Figure 7

SOLUTION

Using the rational zero theorem, the only possible rational zeros are ± 1 and ± 5. The graph of $P(x)$ indicates that $x = 1$ is the only likely candidate (Fig. 7).

$$\begin{array}{r|rrrrr} & 1 & -6 & 14 & -14 & 5 \\ & & 1 & -5 & 9 & -5 \\ \hline 1 & 1 & -5 & 9 & -5 & 0 \end{array}$$

The reduced polynomial is $x^3 - 5x^2 + 9x - 5$.

So $x = 1$ is a zero. Note that it looks like there is a turning point at $x = 1$, so the zero may have even multiplicity. To check, we try synthetic division again, this time with the reduced polynomial.

$$\begin{array}{r|rrrr} & 1 & -5 & 9 & -5 \\ & & 1 & -4 & 5 \\ \hline 1 & 1 & -4 & 5 & 0 \end{array}$$

The new reduced polynomial is $x^2 - 4x + 5$.

This tells us that $x = 1$ is a zero of multiplicity 2, and that $P(x) = (x - 1)^2(x^2 - 4x + 5)$.

We use the quadratic formula to find the zeros of $x^2 - 4x + 5$:

$$x^2 - 4x + 5 = 0 \quad \text{Use the quadratic formula with } a = 1, b = -4, c = 5.$$

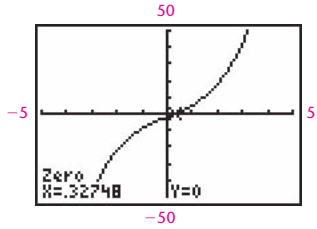
$$x = \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2}$$

$$= \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$$

The exact zeros of $P(x)$ are 1 (multiplicity 2), $2 - i$, and $2 + i$.

MATCHED PROBLEM 7

Find all zeros exactly for $P(x) = x^4 + 4x^3 + 10x^2 + 12x + 5$.



› Figure 8 $P(x) = x^3 + 6x - 2$.

REMARK: We were successful in finding all the zeros of the polynomials in Examples 6 and 7 because we could find sufficient rational zeros to reduce the original polynomial to a quadratic. This is not always possible. For example, the polynomial

$$P(x) = x^3 + 6x - 2$$

has no rational zeros, but does have an irrational zero at $x \approx 0.32748$ (Fig. 8). The other two zeros are imaginary. The techniques we have developed will not find the exact value of these roots.

››› EXPLORE-DISCUSS 5

There is a technique for finding the exact zeros of cubic polynomials, usually referred to as Cardano's formula.* This formula shows that the exact value of the irrational zero of $P(x) = x^3 + 6x - 2$ (see Fig. 8) is

$$x = \sqrt[3]{4} - \sqrt[3]{2}$$

(A) Verify that this is correct by expanding and simplifying

$$P(\sqrt[3]{4} - \sqrt[3]{2})$$

(B) Cardano's formula also shows that the two imaginary zeros are

$$-\tfrac{1}{2}(\sqrt[3]{4} - \sqrt[3]{2}) \pm \tfrac{1}{2}i\sqrt{3}(\sqrt[3]{4} - \sqrt[3]{2})$$

If you like algebraic manipulation, you can also verify that these are correct. (But you'd have to like it an awful lot.)

(C) Find a reference for Cardano's formula in a library or on the Internet. Use this formula to find the exact value of the irrational zero of

$$P(x) = x^3 + 9x - 6$$

Check your answer by comparing it with the approximate value obtained on a graphing calculator.

ANSWERS TO MATCHED PROBLEMS

1. (A) 5 (multiplicity 3), -3 (multiplicity 2), $4i$ and $-4i$ (each multiplicity 1)
(B) -5 (multiplicity 4), 5 (multiplicity 3), i (multiplicity 1)
2. (A) $(x + 1)(x - 1)^2(x^2 + 1)$ (B) $(x + 1)(x - 1)^2(x + i)(x - i)$
3. -3 (multiplicity 2), -2 (multiplicity 1), -1 (multiplicity 1), 0 (multiplicity 2), 1 (multiplicity 1) **4.** $\pm 1, \pm\frac{1}{3}, \pm 2, \pm\frac{2}{3}, \pm 3, \pm 4, \pm\frac{4}{3}, \pm 6, \pm 12$ **5.** $-2, -1, \frac{5}{2}$
6. $\frac{4}{3}, 1 - \sqrt{2}, 1 + \sqrt{2}$ **7.** -1 (multiplicity 2), $-1 - 2i, -1 + 2i$

*Girolamo Cardano (1501–1576), an Italian mathematician and physician, was the first to publish a formula for the solution to cubic equations of the form $x^3 + ax + b = 0$ and the first to realize that this technique could be used to solve other cubic equations. Having predicted that he would live to the age of 75, Cardano committed suicide in 1576.

3-4 Exercises

Write the zeros of each polynomial in Problems 1–8, and indicate the multiplicity of each if more than 1. What is the degree of each polynomial?

1. $P(x) = (x + 8)^3(x - 6)^2$ **2.** $P(x) = (x - 5)(x + 7)^2$

3. $P(x) = 3(x + 4)^3(x - 3)^2(x + 1)$

4. $P(x) = 5(x - 2)^3(x + 3)^2(x - 1)$

5. $P(x) = (x^2 + 4)^3(x^2 - 4)^5(x + 2i)$

6. $P(x) = (x^2 + 7x + 10)^2(x^2 + 6x + 10)^3$

7. $P(x) = (x^3 - 9x)(x^2 + 9)(x + 9)^2$

8. $P(x) = (x^3 - 3x^2 + 3x - 1)(x^2 - 1)(x - i)$

9. Explain in your own words what the fundamental theorem of algebra says. Does it guarantee that every polynomial has a nonreal zero?

10. Do you agree or disagree with the following statement: Every polynomial of degree n has n zeros? Explain.

11. Explain the connection between the zeros of a polynomial, and its linear factors.

12. What is the least degree that a polynomial with zeros 3, i, and $4 + i$ can have if its coefficients are real? Why?

13. True or false: The rational zero test indicates that every polynomial has some rational zeros. Explain.

14. True or false: Every real zero of a polynomial will appear on the list of numbers provided by the rational zero test. Explain.

In Problems 15–20, find a polynomial P(x) of lowest degree, with leading coefficient 1, that has the indicated set of zeros. Leave the answer in a factored form. Indicate the degree of the polynomial.

15. 3 (multiplicity 2) and -4

16. -2 (multiplicity 3) and 1 (multiplicity 2)

17. -7 (multiplicity 3), $-3 + \sqrt{2}$, $-3 - \sqrt{2}$

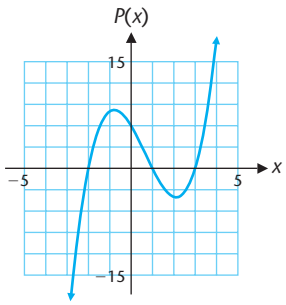
18. $\frac{1}{3}$ (multiplicity 2), $5 + \sqrt{7}$, $5 - \sqrt{7}$

19. $(2 - 3i)$, $(2 + 3i)$, -4 (multiplicity 2)

20. $i\sqrt{3}$ (multiplicity 2), $-i\sqrt{3}$ (multiplicity 2), and 4 (multiplicity 3)

In Problems 21–26, find a polynomial of lowest degree, with leading coefficient 1, that has the indicated graph. Assume all zeros are integers. Leave the answer in a factored form. Indicate the degree of each polynomial.

21.



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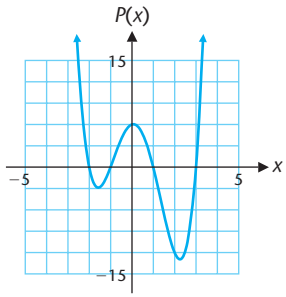

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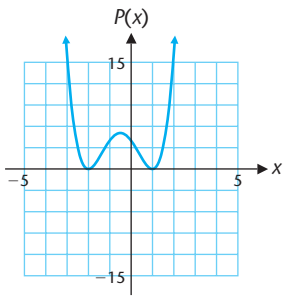

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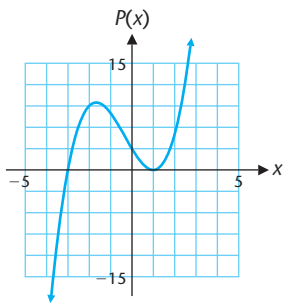

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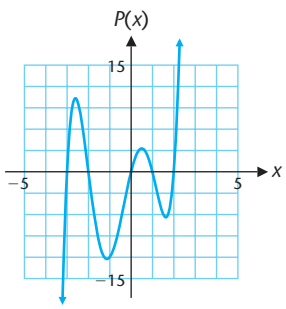

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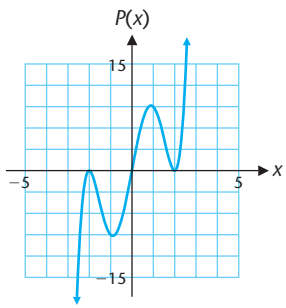


For each polynomial in Problems 27–32, use the rational zero theorem to list all possible rational zeros.

27. $P(x) = x^3 - 2x^2 - 5x + 6$

28. $P(x) = x^3 + 3x^2 - 6x - 8$

29. $P(x) = 3x^3 - 11x^2 + 8x + 4$

30. $P(x) = 2x^3 + x^2 - 4x - 3$

31. $P(x) = 12x^3 - 16x^2 - 5x + 3$

32. $P(x) = 2x^3 - 9x^2 + 14x - 5$

In Problems 33–36, factor each polynomial in two ways:
(A) As a product of linear factors (with real coefficients) and quadratic factors (with real coefficients and imaginary zeros)
(B) As a product of linear factors with complex coefficients

33. $P(x) = x^4 + 5x^2 + 4$ **34.** $P(x) = x^4 + 18x^2 + 81$

35. $P(x) = x^3 - x^2 + 25x - 25$

36. $P(x) = x^5 + x^4 - x - 1$

In Problems 37–42, write P(x) as a product of linear factors.

37. $P(x) = x^3 + 9x^2 + 24x + 16$; -1 is a zero

38. $P(x) = x^3 - 4x^2 - 3x + 18$; 3 is a double zero

39. $P(x) = x^4 - 1$; 1 and -1 are zeros

40. $P(x) = x^4 + 2x^2 + 1$; i is a double zero

41. $P(x) = 2x^3 - 17x^2 + 90x - 41$; $\frac{1}{2}$ is a zero

42. $P(x) = 3x^3 - 10x^2 + 31x + 26$; $-\frac{2}{3}$ is a zero

In Problems 43–52, find all zeros exactly (rational, irrational, and imaginary) for each polynomial.

43. $P(x) = x^3 - 19x + 30$ **44.** $P(x) = x^3 - 7x^2 + 36$

45. $P(x) = x^4 - \frac{21}{10}x^3 + \frac{2}{5}x$

46. $P(x) = x^4 + \frac{7}{6}x^3 - \frac{7}{3}x^2 - \frac{5}{2}x$

47. $P(x) = x^4 - 5x^3 + \frac{15}{2}x^2 - 2x - 2$

48. $P(x) = x^4 - \frac{13}{4}x^2 - \frac{5}{2}x - \frac{1}{4}$ **49.** $P(x) = x^4 + 11x^2 + 30$

50. $P(x) = x^4 + 5x^2 + 6$

51. $P(x) = 3x^5 - 5x^4 - 8x^3 + 16x^2 + 21x + 5$

52. $P(x) = 2x^5 - 3x^4 - 6x^3 + 23x^2 - 26x + 10$

In Problems 53–62, find all zeros exactly (rational, irrational, and imaginary) for each polynomial equation.

53. $2x^3 - 5x^2 + 1 = 0$ **54.** $2x^3 - 10x^2 + 12x - 4 = 0$

55. $x^4 + 4x^3 - x^2 - 20x - 20 = 0$

56. $x^4 - 4x^2 - 4x - 1 = 0$

57. $x^4 - 2x^3 - 5x^2 + 8x + 4 = 0$

58. $x^4 - 2x^2 - 16x - 15 = 0$

59. $x^4 + 10x^2 + 9 = 0$ **60.** $x^4 + 29x^2 + 100 = 0$

61. $2x^5 - 3x^4 - 2x + 3 = 0$

62. $2x^5 + x^4 - 6x^3 - 3x^2 - 8x - 4 = 0$

In Problems 63–68, write each polynomial as a product of linear factors.

63. $P(x) = 6x^3 + 13x^2 - 4$

64. $P(x) = 6x^3 - 17x^2 - 4x + 3$

65. $P(x) = x^3 + 2x^2 - 9x - 4$

66. $P(x) = x^3 - 8x^2 + 17x - 4$

67. $P(x) = 4x^4 - 4x^3 - 9x^2 + x + 2$

68. $P(x) = 2x^4 + 3x^3 - 4x^2 - 3x + 2$

In Problems 69–74, multiply.

69. $[x - (4 - 5i)][x - (4 + 5i)]$

70. $[x - (2 - 3i)][x - (2 + 3i)]$

71. $[x - (3 + 4i)][x - (3 - 4i)]$

72. $[x - (5 + 2i)][x - (5 - 2i)]$

73. $[x - (a + bi)][x - (a - bi)]$ **74.** $(x - bi)(x + bi)$

75. Given that $2 + i$ is a zero of $P(x) = x^3 - 7x^2 + 17x - 15$, find all other zeros by
(A) Writing the conjugate zero.
(B) Writing the two linear factors corresponding to these zeros.
(C) Multiplying these factors to obtain a quadratic factor.
(D) Using long division to divide $P(x)$ by this quadratic factor.

76. See Problem 75. Find all other zeros of $P(x)$ by
(A) Dividing out $2 + i$ using synthetic division.
(B) Dividing out the conjugate zero by synthetic division.
Do you prefer the method of Problem 75 or Problem 76?

In Problems 77–82, find all other zeros of P(x), given the indicated zero. (See Problems 75 and 76.)

77. $P(x) = x^3 - 5x^2 + 4x + 10$; $3 - i$ is one zero

78. $P(x) = x^3 + x^2 - 4x + 6$; $1 + i$ is one zero

79. $P(x) = x^3 - 3x^2 + 25x - 75$; $-5i$ is one zero

80. $P(x) = x^3 + 2x^2 + 16x + 32$; $4i$ is one zero

81. $P(x) = x^4 - 4x^3 + 3x^2 + 8x - 10$; $2 + i$ is one zero

82. $P(x) = x^4 - 2x^3 + 7x^2 - 18x - 18$; $-3i$ is one zero

Prove that each of the real numbers in Problems 83–86 is not rational by writing an appropriate polynomial and making use of the rational zero theorem.

83. $\sqrt{6}$ **84.** $\sqrt{12}$ **85.** $\sqrt[3]{5}$ **86.** $\sqrt[5]{8}$

In Problems 87–90, determine the number of real zeros of each polynomial $P(x)$, and explain why $P(x)$ has no rational zeros.

87. $P(x) = x^4 - 5x^2 - 6$ **88.** $P(x) = 3x^4 + x^2 + 12$

89. $P(x) = x^3 - 3x - 1$ **90.** $P(x) = x^3 - 5x + 3$

In Problems 91–96, find all zeros (rational, irrational, and imaginary) exactly.

91. $P(x) = 3x^3 - 37x^2 + 84x - 24$

92. $P(x) = 2x^3 - 9x^2 - 2x + 30$

93. $P(x) = 4x^4 + 4x^3 + 49x^2 + 64x - 240$

94. $P(x) = 6x^4 + 35x^3 + 2x^2 - 233x - 360$

95. $P(x) = 4x^4 - 44x^3 + 145x^2 - 192x + 90$

96. $P(x) = x^5 - 6x^4 + 6x^3 + 28x^2 - 72x + 48$

97. The solutions to the equation $x^3 - 1 = 0$ are all the cube roots of 1.
(A) How many cube roots of 1 are there?
(B) 1 is obviously a cube root of 1; find all others.

98. The solutions to the equation $x^3 - 8 = 0$ are all the cube roots of 8.
(A) How many cube roots of 8 are there?
(B) 2 is obviously a cube root of 8; find all others.

99. If P is a polynomial function with real coefficients of degree n, with n odd, then what is the maximum number of times the graph of $y = P(x)$ can cross the x axis? What is the minimum number of times?

100. Answer the questions in Problem 99 for n even.

101. Given $P(x) = x^2 + 2ix - 5$ with $2 - i$ a zero, show that $2 + i$ is not a zero of $P(x)$. Does this contradict Theorem 3? Explain.

102. If $P(x)$ and $Q(x)$ are two polynomials of degree n, and if $P(x) = Q(x)$ for more than n values of x, then how are $P(x)$ and $Q(x)$ related?

103. Theorem 5 asserts that every polynomial of odd degree with real coefficients has at least one real zero. Use what you learned about right and left behavior of polynomials in Section 3-1 to justify this fact.

104. Use the rational zero theorem to prove that a polynomial with a nonzero constant term cannot have $x = 0$ as one of its zeros. Then confirm this fact using direct substitution of $x = 0$ into a generic polynomial with nonzero constant term.

105. If the constant term of a polynomial is zero, the rational zero theorem fails because the only potential zero on the list will be zero. How can you overcome this? [*Hint:* What do all of the nonzero terms have in common in this case?]

106. In this problem we will sketch a proof of the rational zero theorem. Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ and suppose that p/q is a rational number in lowest terms that is a zero of $P(x)$.
(A) Substitute p/q in for x and set the result equal to zero. Show that the resulting equation can be written as $a_n p^n + a_{n-1}p^{n-1}q + a_{n-2}p^{n-2}q^2 + \cdots a_1 pq^{n-1} = -a_0 q^n$.
(B) Notice that p appears in every term on the left, so is a factor of the left side. Explain why p cannot be a factor of q^n. [*Hint:* What do we know about p/q?]
(C) Since p is a factor of the left side, and *not* a factor of q^n, what is the relationship between p and a_0? How does this prove half of the rational zero theorem?
(D) Rearrange the equation from part A so that the only term on the left is the one with no power of q. Then adapt parts B and C to prove that q must be a factor of a_n.

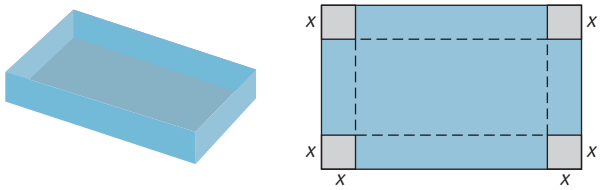
APPLICATIONS

Find all rational solutions exactly, and find irrational solutions to two decimal places.

107. STORAGE A rectangular storage unit has dimensions 1 by 2 by 3 feet. If each dimension is increased by the same amount, how much should this amount be to create a new storage unit with volume 10 times the old?

108. CONSTRUCTION A rectangular box has dimensions 1 by 1 by 2 feet. If each dimension is increased by the same amount, how much should this amount be to create a new box with volume six times the old?

109. PACKAGING An open box is to be made from a rectangular piece of cardboard that measures 8 by 5 inches, by cutting out squares of the same size from each corner and bending up the sides (see the figure). If the volume of the box is to be 14 cubic inches, how large a square should be cut from each corner? [*Hint:* Determine the domain of x from physical considerations before starting.]



110. FABRICATION An open metal chemical tank is to be made from a rectangular piece of stainless steel that measures 10 by 8 feet, by cutting out squares of the same size from each corner and bending up the sides (see the figure for Problem 109). If the volume of the tank is to be 48 cubic feet, how large a square should be cut from each corner?

3-5 Rational Functions and Inequalities

- Rational Functions and Properties of Their Graphs
- Finding Vertical and Horizontal Asymptotes
- Drawing the Graph of a Rational Function
- Solving Rational Inequalities

In this section, we will apply our knowledge of the graphs and zeros of polynomial functions to study *rational functions,* which are functions that are quotients of polynomials. Now more than ever, it's important to understand the features of these graphs before attempting to use a graphing calculator to draw them. Often, the output that appears on the screen will require a lot of interpretation, and unless you have a good idea of what to expect, you may struggle with this interpretation. Our goal will be to produce hand sketches (with help from a graphing calculator) that clearly show all of the important features of the graph.

Rational Functions and Properties of Their Graphs

The number $\frac{7}{13}$ is called a *rational number* because it is a quotient (or ratio) of integers. The function

$$f(x) = \frac{x + 1}{x^2 - x - 6}$$

is called a *rational function* because it is a quotient of polynomials.

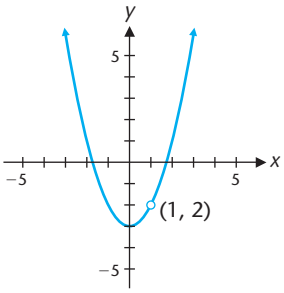
DEFINITION 1 Rational Function

A function f is a **rational function** if it can be written in the form

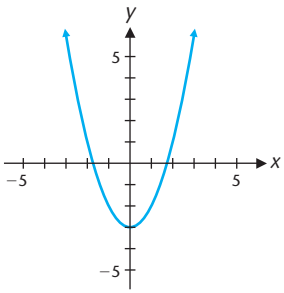
$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials.

When working with rational functions, we will assume that the coefficients of $p(x)$ and $q(x)$ are real numbers, and that the domain of f is the set of all real numbers x such that $q(x) \neq 0$.



(a) $p(x) = \dfrac{(x-1)(x^2-3)}{x-1}$



(b) $q(x) = x^2 - 3$

› Figure 1

Consider the rational function

$$p(x) = \frac{(x-1)(x^2-3)}{x-1}$$

and the polynomial $q(x) = x^2 - 3$. Are they actually the same function? If you answered yes, you're almost right. They're not identical because $p(1)$ is undefined, while $q(1) = -2$. For every other x value, they are equal. Their graphs are identical except at $x = 1$, where the graph of $p(x)$ has a hole (Fig. 1).

In general, when a real number c is a zero of both $p(x)$ and $q(x)$, then $x - c$ is a factor of both, and we can write their quotient as

$$f(x) = \frac{p(x)}{q(x)} = \frac{(x-c)p_r(x)}{(x-c)q_r(x)}$$

$p_r(x)$ and $q_r(x)$ are polynomials with degree one less than p and q, respectively.*

Then the graph of $f_r(x) = p_r(x)/q_r(x)$, is identical to the graph of $f(x) = p(x)/q(x)$, except possibly for a hole at $x = c$.

Later in this section we will discuss how to handle the minor complication caused by common real zeros of $p(x)$ and $q(x)$. But to avoid that complication now, **unless stated to the contrary, we will assume that for any rational function *f* we consider, *p*(*x*) and *q*(*x*) have no real zero in common.**

For any rational function, the zeros of the numerator and denominator each have significance. A rational function is undefined for any x value that is a zero of its denominator, so those values are not in the domain. If the degree of the denominator is n, there are at most n such values. The real zeros of the numerator, on the other hand, are the zeros (and x intercepts) of a rational function. (This is because a fraction is zero exactly when its numerator is zero.) If the degree of the numerator is m, there are at most m x intercepts.

EXAMPLE 1 Domain and *x* Intercepts

Find the domain and x intercepts for $f(x) = \dfrac{2x^2 - 2x - 4}{x^2 - 9}$.

SOLUTION

Since we need the zeros of the numerator and denominator, it's a good idea to begin by factoring both.

$$f(x) = \frac{2x^2 - 2x - 4}{x^2 - 9} = \frac{2(x-2)(x+1)}{(x-3)(x+3)}$$

Because the denominator is zero for $x = 3$ and $x = -3$, the domain of f is

$$x \neq \pm 3 \qquad \text{or} \qquad (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$$

*We referred to p_r and q_r as reduced polynomials in the last section.

Because the numerator is zero for $x = 2$ and $x = -1$, the zeros of f, and thus the x intercepts of f, are -1 and 2.

MATCHED PROBLEM 1

Find the domain and x intercepts for $f(x) = \dfrac{3x^2 - 12}{x^2 + 2x - 3}$.

The graph of the rational function

$$f(x) = \frac{x^2 - 1.44}{x^3 - x}$$

is shown in Figure 2.

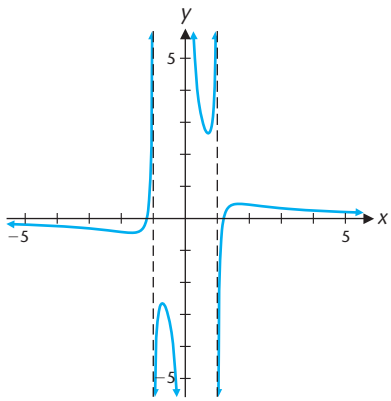


Figure 2 $f(x) = \dfrac{x^2 - 1.44}{x^3 - x}$.

The domain of f consists of all real numbers except $x = -1$, $x = 0$, and $x = 1$ (the zeros of the denominator $x^3 - x$). The dotted vertical lines at $x = \pm 1$ indicate that those values of x are excluded from the domain. (Zero is excluded as well, but a dotted vertical line at $x = 0$ would coincide with the y axis and is omitted.) The graph is discontinuous at $x = -1$, $x = 0$, and $x = 1$, but is continuous elsewhere and has no sharp corners. The zeros of f are the zeros of the numerator $x^2 - 1.44$, which are $x = -1.2$ and $x = 1.2$. The graph of f has four turning points. Its left and right behavior is the same as that of the function $g(x) = \frac{1}{x}$, which we learned in Chapter 1. (The graph is close to the x axis for very large positive and negative values of x.) The graph of f illustrates the general properties of rational functions that are listed in Theorem 1. We have already justified Property 3; the other properties are established in calculus.

THEOREM 1 Properties of Rational Functions

Let $f(x) = p(x)/q(x)$ be a rational function where $p(x)$ and $q(x)$ are polynomials of degrees m and n, respectively. Then the graph of $f(x)$:

1. Is continuous with the exception of at most n real numbers (the zeros of the denominator)
2. Has no sharp corners
3. Has at most m real zeros
4. Has at most $m + n - 1$ turning points
5. Has the same left and right behavior as the quotient of the highest-power terms of $p(x)$ and $q(x)$

Figure 3 shows graphs of several rational functions, illustrating the properties of Theorem 1.

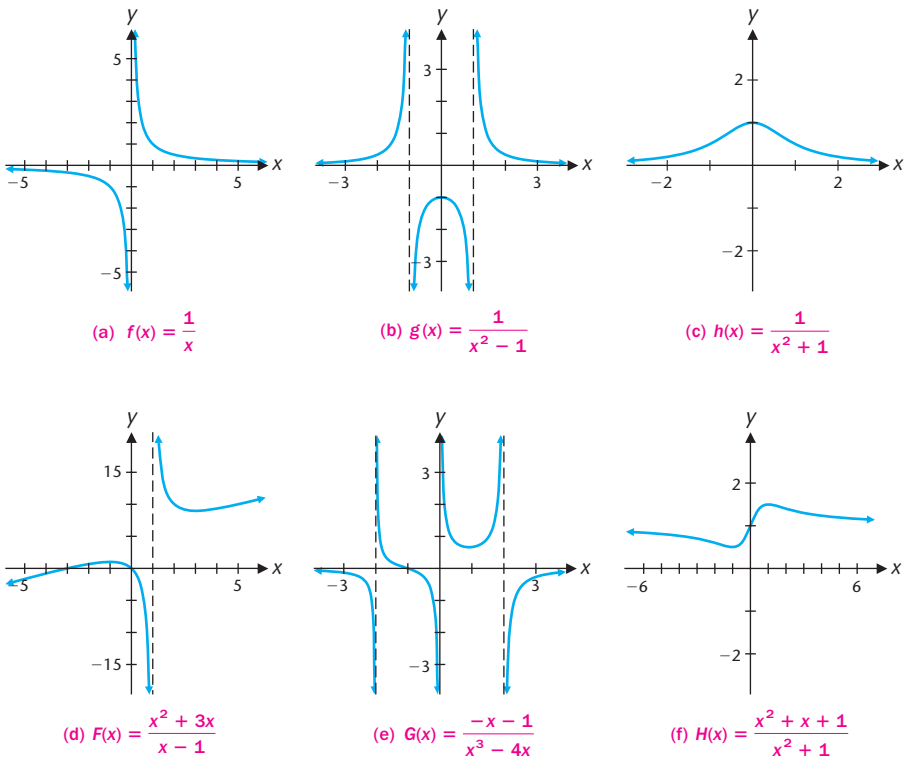
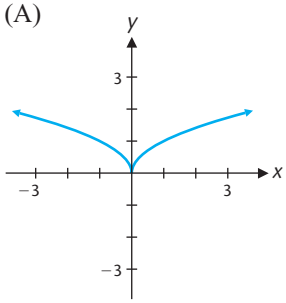


Figure 3 Graphs of rational functions.

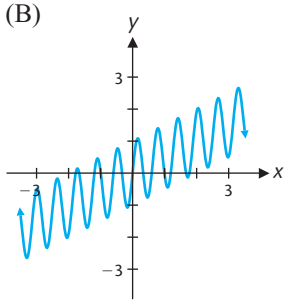
EXAMPLE 2 Properties of Graphs of Rational Functions

Use Theorem 1 to explain why each graph is not the graph of a rational function.

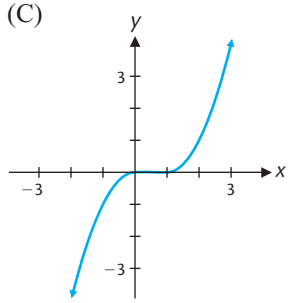
(A)



(B)



(C)



SOLUTIONS

(A) The graph has a sharp corner when $x = 0$, so Property 2 is not satisfied.

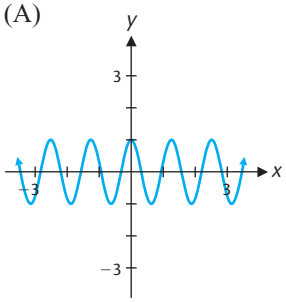
(B) The graph has an infinite number of turning points, so Property 4 is not satisfied.

(C) The graph has an infinite number of zeros (all values of x between 0 and 1, inclusive, are zeros), so Property 3 is not satisfied.

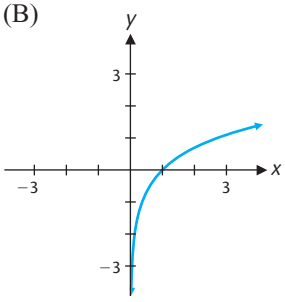
MATCHED PROBLEM 2

Use Theorem 1 to explain why each graph is not the graph of a rational function.

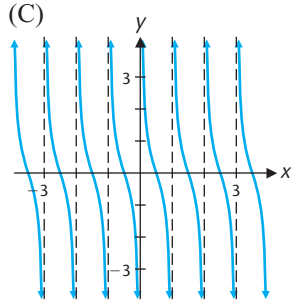
(A)



(B)



(C)



› Finding Vertical and Horizontal Asymptotes

All of the graphs in Figure 3 exhibit similar behavior near their points of discontinuity. This behavior can be described using the concept of *vertical asymptote*.

››› EXPLORE-DISCUSS 1

(A) Complete Table 1.

Table 1 Behavior of $\frac{5}{x}$ as x Approaches Zero from the Right

x	1	0.1	0.01	0.001	0.0001	0.000 01
$5/x$						

Describe the outputs as the x values approach zero from the right.

(B) Complete Table 2.

Table 2 Behavior of $\frac{5}{x}$ as x Approaches Zero from the Left

x	-1	-0.1	-0.01	-0.001	-0.0001	$-0.000\,01$
$5/x$						

Describe the outputs as the x values approach zero from the left.

(C) Use parts A and B and a graph of $y = 5/x$ to discuss how the output of $y = 5/x$ for x values close to zero affects the graph near $x = 0$.

In Explore-Discuss 1, we see why most of the graphs in Figure 3 exhibit the behavior they do near the dotted vertical lines. As x approaches a zero of the denominator, the outputs of the function get larger and larger (either positive or negative) and approach ∞ or $-\infty$. For the function $f(x) = 5/x$, we would say that as x approaches zero from the right, $5/x$ approaches ∞, and write this as

$$\frac{5}{x} \to \infty \quad \text{as} \quad x \to 0^+$$

As x approaches zero from the left, $5/x$ approaches $-\infty$; we write

$$\frac{5}{x} \to -\infty \quad \text{as} \quad x \to 0^-$$

In this case, we say that the vertical line $x = 0$ (the y axis) is a *vertical asymptote* for the graph of $f(x)$.

››› EXPLORE-DISCUSS 2

Construct tables similar to Tables 1 and 2 for $g(x) = 1/x^2$ and discuss the behavior of the graph of $g(x)$ near $x = 0$.

› DEFINITION 2 Vertical Asymptote

The vertical line $x = a$ is a **vertical asymptote** for the graph of $y = f(x)$ if

$$f(x) \to \infty \quad \text{or} \quad f(x) \to -\infty \quad \text{as} \quad x \to a^+ \quad \text{or as} \quad x \to a^-$$

(that is, if $f(x)$ either increases or decreases without bound as x approaches a from the right or from the left).

Informally, a vertical asymptote is a vertical line that the graph approaches but never intersects. To find the vertical asymptotes of a rational function, you simply need to find the zeros of the denominator. For example, the denominator of

$$f(x) = \frac{x^2 - 1.44}{x^3 - x} = \frac{x^2 - 1.44}{x(x - 1)(x + 1)}$$

has three zeros (0, 1, and -1), so the graph of $f(x)$ has three vertical asymptotes, $x = 0$, $x = 1$, and $x = -1$ (see Fig. 2 on page 338).

› THEOREM 2 Vertical Asymptotes of Rational Functions

Let $f(x) = p(x)/q(x)$ be a rational function. If a is a zero of $q(x)$, then the line $x = a$ is a vertical asymptote of the graph of f.*

We next turn our attention to the left and right behavior of rational functions. For many (but not all) rational functions, it can be described using the concept of *horizontal asymptotes*.

*Recall that we are assuming that $p(x)$ and $q(x)$ have no real zeros in common. Theorem 2 may not be valid without this assumption.

››› EXPLORE-DISCUSS 3

Refer back to your graph of $y = 5/x$ in part C of Explore-Discuss 1.

(A) Complete Table 3.

Table 3 Behavior of $\frac{5}{x}$ as $x \to \infty$

x	1	10	100	1,000	10,000	100,000
$5/x$						

Describe the outputs as the x values approach ∞.

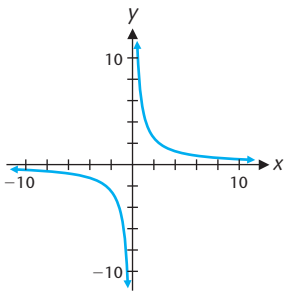
(B) Complete Table 4.

Table 4 Behavior of $\frac{5}{x}$ as $x \to -\infty$

x	-1	-10	-100	$-1{,}000$	$-10{,}000$	$-100{,}000$
$5/x$						

Describe the outputs as the x values approach $-\infty$.

(C) Use parts A and B and a graph of $y = 5/x$ to discuss how the output of $y = 5/x$ for large values of x affects the graph.



› Figure 4

Note that the left and right behavior of the graph of $f(x) = 5/x$ shows the graph approaching the x axis in both directions (Fig. 4).

In symbols, we would write

$$\frac{5}{x} \to 0 \quad \text{as} \quad x \to \infty \quad \text{and as} \quad x \to -\infty$$

In this case, we would say that the line $y = 0$ (the x axis) is a *horizontal asymptote* of the graph of $f(x)$.

››› EXPLORE-DISCUSS 4

Construct tables similar to Tables 3 and 4 for each of the following functions, and discuss the behavior of each as $x \to \infty$ and as $x \to -\infty$:

(A) $f(x) = \dfrac{3x}{x^2 + 1}$ **(B)** $g(x) = \dfrac{3x^2}{x^2 + 1}$ **(C)** $h(x) = \dfrac{3x^3}{x^2 + 1}$

› DEFINITION 3 Horizontal Asymptote

The horizontal line $y = b$ is a **horizontal asymptote** for the graph of $f(x)$ if

$$f(x) \to b \quad \text{as} \quad x \to -\infty \quad \text{or as} \quad x \to \infty$$

(that is, if $f(x)$ approaches b as x increases without bound or as x decreases without bound).

Informally, a horizontal asymptote is a horizontal line that the graph gets infinitely close to as x increases or decreases without bound.

While the mathematical behavior of a function near horizontal and vertical asymptotes is quite different, they are very similar graphically—they are lines that a graph gets infinitely close to.

A rational function $f(x) = p(x)/q(x)$ has the same left and right behavior as the quotient of the leading terms of $p(x)$ and $q(x)$ (Property 5 of Theorem 1). Consequently, **a rational function has at most one horizontal asymptote.** Moreover, we can determine easily whether a rational function has a horizontal asymptote, and if it does, find its equation. Theorem 3 gives the details.

› THEOREM 3 Horizontal Asymptotes of Rational Functions

Consider the rational function

$$f(x) = \frac{a_m x^m + \cdots + a_1 x + a_0}{b_n x^n + \cdots + b_1 x + b_0}$$

where $a_m \neq 0$, $b_n \neq 0$. Note that m is the degree of the numerator, and n is the degree of the denominator.

1. If $m < n$, the line $y = 0$ (the x axis) is a horizontal asymptote.
2. If $m = n$, the line $y = a_m/b_n$ is a horizontal asymptote.
3. If $m > n$, there is no horizontal asymptote.

In 1 and 2, the graph of f approaches the horizontal asymptote both as $x \to \infty$ and as $x \to -\infty$.

EXAMPLE 3 Finding Vertical and Horizontal Asymptotes for a Rational Function

Find all vertical and horizontal asymptotes for

$$\text{(A) } f(x) = \frac{2x^2 - 2x - 4}{x^2 - 9} \qquad \text{(B) } g(x) = \frac{3x}{x^3 + 5x^2} \qquad \text{(C) } k(x) = \frac{x^3 + 2x - 100}{x^2 + 10}$$

SOLUTIONS

(A) Because the denominator can be factored as $(x + 3)(x - 3)$, its zeros are 3 and -3, and the graph of $f(x)$ has vertical asymptotes at $x = 3$ and $x = -3$. Because the numerator and denominator have the same degree, according to Theorem 3, part 2, the line

$$y = \frac{a_2}{b_2} = \frac{2}{1} = 2^* \qquad a_2 = 2, b_2 = 1$$

is a horizontal asymptote (Theorem 3, part 2).

(B) The denominator factors as $x^3 + 5x^2 = x^2(x + 5)$, so its zeros are 0 and -5, and the graph of $g(x)$ has vertical asymptotes at $x = 0$ and $x = -5$. The degree of the denominator is greater than that of the numerator, so the line $y = 0$ is a horizontal asymptote (Theorem 3, part 1).

(C) The denominator has no real zeros, so there are no vertical asymptotes. The degree of the numerator is greater than that of the denominator, so there is no horizontal asymptote (Theorem 3, part 3).

MATCHED PROBLEM 3

Find all vertical and horizontal asymptotes for

(A) $f(x) = \dfrac{3x^2 - 12}{x^2 + 2x - 3}$ (B) $g(x) = \dfrac{x^2 - 10}{3x^4 - 3x^2}$ (C) $k(x) = \dfrac{8x^4}{x^2 + x + 5}$

››› CAUTION ›››

The asymptotes of a graph are *lines,* not numbers. You should always write the equation of an asymptote in the form $x = a$ (vertical) or $y = a$ (horizontal).

› Drawing the Graph of a Rational Function

We will now use the techniques for locating asymptotes, along with other graphing aids discussed in the text, to graph several rational functions. First, we outline a step-by-step approach to the problem of graphing rational functions.

*The dashed "think boxes" are used to enclose steps that may be performed mentally.

› ANALYZING AND SKETCHING THE GRAPH OF A RATIONAL FUNCTION: $f(x) = p(x)/q(x)$

Step 1. *Intercepts.* Find the real solutions of the equation $p(x) = 0$ and use these solutions to plot any x intercepts of the graph of f. Evaluate $f(0)$, if it exists, and plot the y intercept.

Step 2. *Vertical Asymptotes.* Find the real solutions of the equation $q(x) = 0$ and use these solutions to determine the domain of f, the points of discontinuity, and the vertical asymptotes. Sketch any vertical asymptotes as dashed lines.

Step 3. *Horizontal Asymptotes.* Determine whether there is a horizontal asymptote and if so, sketch it as a dashed line.

Step 4. *Complete the Sketch.* Use the information determined in steps 1–3, and a graphing calculator, to sketch the graph by hand.

EXAMPLE 4 Graphing a Rational Function

Graph $f(x) = \dfrac{2x}{x - 3}$.

SOLUTION

$$f(x) = \frac{2x}{x - 3}$$

Step 1. *Intercepts.* Find real zeros of $2x$ and find $f(0)$:

$$\begin{aligned} 2x &= 0 \\ x &= 0 \quad \text{x intercept} \\ f(0) &= 0 \quad \text{y intercept} \end{aligned}$$

The graph crosses the coordinate axes only at the origin. Plot this intercept, as shown in Figure 5.

Step 2. *Vertical Asymptotes.* Find real zeros of $x - 3$:

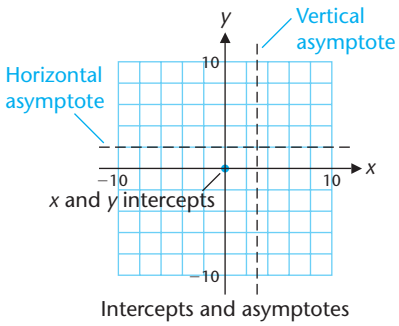
$$\begin{aligned} x - 3 &= 0 \\ x &= 3 \end{aligned}$$

The graph has a vertical asymptote at $x = 3$. The domain of f is $x \neq 3$, and f is discontinuous at $x = 3$. Sketch this asymptote, as shown in Figure 5.

Step 3. *Horizontal Asymptote.* The numerator and denominator have the same degree, so we can use part 2 of Theorem 3. The leading coefficients of the numerator and denominator are 2 and 1, respectively, so

$$y = \frac{2}{1} = 2$$

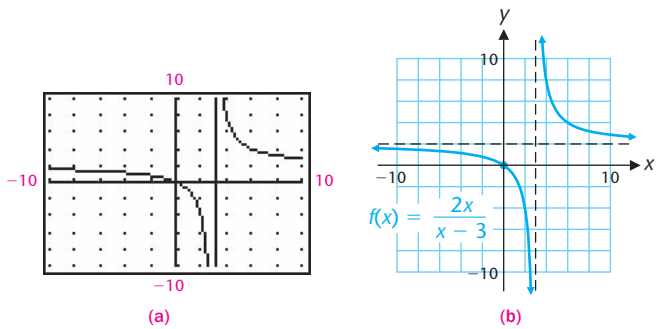
is a horizontal asymptote, as shown in Figure 5.



› Figure 5

Step 4. Based on the information we've obtained, it looks like a standard viewing window is likely to show all of the key features of the graph.

Using the graphing calculator graph in Figure 6(a), we obtain the graph in Figure 6(b). Notice that the graph is a smooth continuous curve over the interval $(-\infty, 3)$ and over the interval $(3, \infty)$. As expected, there is a break in the graph at $x = 3$, and the graph levels off at height 2.



› Figure 6

MATCHED PROBLEM 4

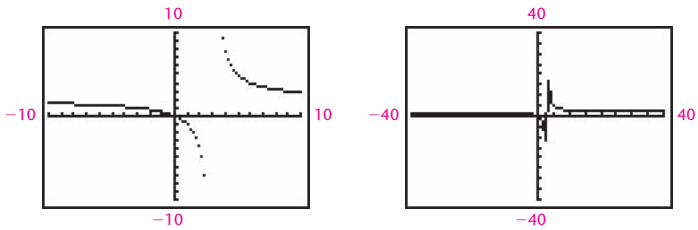
Proceed as in Example 4 and sketch the graph of $f(x) = \dfrac{3x}{x + 2}$.

››› CAUTION ›››

When drawing the graph of a rational function, do not draw asymptotes as part of the graph! See the following remarks.

REMARK: When $f(x) = 2x/(x - 3)$ (see Example 4) is graphed on a graphing calculator [Fig. 6(a)], it appears that the graphing calculator has also drawn the vertical asymptote at $x = 3$, but this is not the case. Many graphing calculators, when set in *connected mode,* calculate points on a graph and connect these points with line segments. The last point plotted to the left of the asymptote and the first plotted to the right of the asymptote will usually have very large y coordinates. If these y coordinates have opposite signs, then the graphing calculator may connect the two points with a nearly vertical line segment, which gives the appearance of an asymptote. If you wish, you can set the calculator in *dot mode* to plot the points without the connecting line segments [Fig. 7(a)].

Depending on the scale, a graph may even appear to be continuous at a vertical asymptote [Fig. 7(b)]. That's why it is important to always locate the vertical asymptotes as we did in step 2 *before* turning to the graphing calculator to complete the sketch. If you already know there's a vertical asymptote at a certain x value, you'll be a lot less likely to misinterpret the calculator's display.



(a) Dot mode (b) Connected mode

› **Figure 7** Graphing calculator graphs of $f(x) = \dfrac{2x}{x - 3}$.

EXAMPLE 5 Graphing a Rational Function

Graph $f(x) = \dfrac{x^2 - 6x + 9}{x^2 - 9x - 10}$.

SOLUTION

$$f(x) = \frac{x^2 - 6x + 9}{x^2 - 9x - 10} = \frac{(x - 3)^2}{(x - 10)(x + 1)}$$ Factor numerator and denominator.

Step 1. *Intercepts*

$$(x - 3)^2 = 0$$ Find zeros of numerator.

$$x = 3$$ x intercept

$$f(0) = \frac{0 - 0 + 9}{0 - 0 - 10} = -\frac{9}{10}$$ Evaluate $f(0)$ to find y intercept.

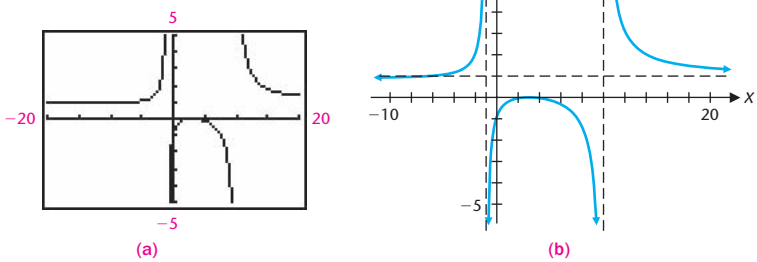
Step 2. *Vertical Asymptotes*

$$(x - 10)(x + 1) = 0 \qquad \text{Find zeros of denominator.}$$
$$x = 10 \qquad x = -1$$

The vertical asymptotes are $x = -1$ and $x = 10$. The domain is $x \neq -1$ or 10, and f is discontinuous at $x = -1$ and $x = 10$.

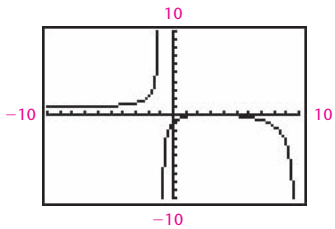
Step 3. *Horizontal Asymptote*. The numerator and denominator have the same degree, so $y = \frac{1}{1} = 1$ is a horizontal asymptote.

Step 4. *Complete the Sketch*. The vertical asymptote at $x = 10$ suggests that we should look at x values of at least 20; the small y intercept suggests that smaller y values may be appropriate. The resulting graphing calculator graph in Figure 8(a) leads to the graph in Figure 8(b).



› Figure 8

Note: If we had used a standard viewing window, we would have totally missed the right portion of the graph (Fig. 9). This is exactly why it's so important to find the asymptotes and intercepts before using the calculator to draw the graph.



› Figure 9

MATCHED PROBLEM 5

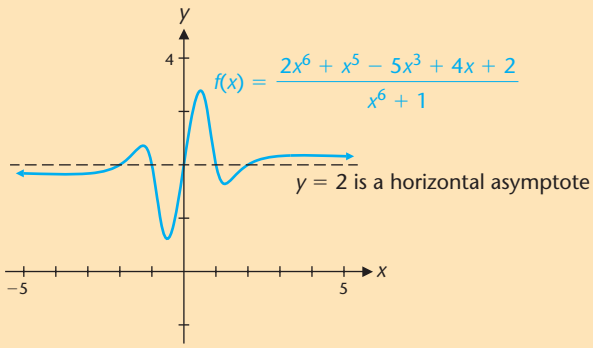
Graph $f(x) = \dfrac{x^2}{x^2 - 11x + 18}$.

››› CAUTION ›››

The graph of a function cannot cross a vertical asymptote, but the same statement is not true for horizontal asymptotes. The rational function

$$f(x) = \frac{2x^6 + x^5 - 5x^3 + 4x + 2}{x^6 + 1}$$

has the line $y = 2$ as a horizontal asymptote. The graph of f in Figure 10 clearly shows that **the graph of a function can cross a horizontal asymptote.**



› **Figure 10** Multiple intersections of a graph and a horizontal asymptote.

Remember, horizontal asymptotes are all about what the graph looks like for large values of x. They have no effect on the "middle" of the graph.

EXAMPLE 6 Graphing a Rational Function

Graph $f(x) = \dfrac{x^2 - 3x - 4}{x - 2}$.

SOLUTION

$$f(x) = \frac{x^2 - 3x - 4}{x - 2} = \frac{(x + 1)(x - 4)}{x - 2}$$ Factor numerator and denominator.

Step 1. *Intercepts*

$$(x + 1)(x - 4) = 0$$ Find zeros of the numerator.

$$x = -1, 4$$ Two x intercepts

$$f(0) = \frac{0 - 0 - 4}{0 - 2} = 2$$ Evaluate $f(0)$ to find y intercept.

Step 2. *Vertical Asymptotes*

$$\begin{aligned} x - 2 &= 0 && \text{Find zeros of the denominator.} \\ x &= 2 && \text{Vertical asymptote} \end{aligned}$$

The domain is $x \neq 2$, and f is discontinuous at $x = 2$.

Step 3. *Horizontal Asymptote*. The degree of the numerator is greater than that of the denominator, so by Theorem 3, part 3, there is no horizontal asymptote. However, we can still gain some useful information about the behavior of the graph as $x \to -\infty$ and as $x \to \infty$ if we first perform a long division (you'll see why in a minute):

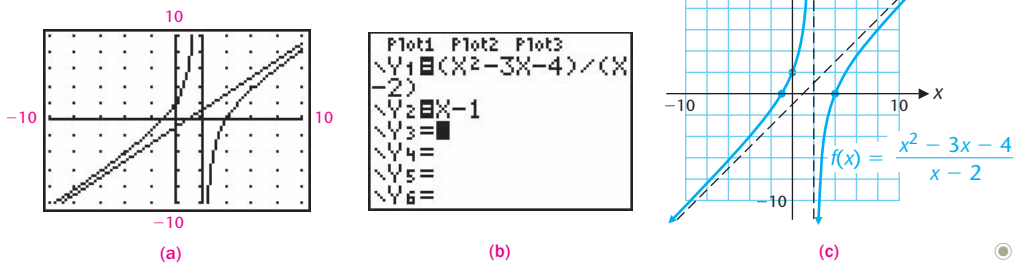
$$\begin{array}{r} x - 1 \\ x - 2 \overline{)\, x^2 - 3x - 4} \\ \underline{x^2 - 2x} \\ -x - 4 \\ \underline{-x + 2} \\ -6 \end{array} \qquad \begin{array}{l} \text{Quotient} \\ \\ \\ \\ \\ \text{Remainder} \end{array}$$

So we can rewrite $f(x)$ as

$$f(x) = \frac{x^2 - 3x - 4}{x - 2} = x - 1 - \frac{6}{x - 2}$$

Now notice that as $x \to \infty$ or $x \to -\infty$, $6/(x - 2)$ approaches 0, and the output of f approaches the output of $y = x - 1$. We can conclude that the graph of f approaches the graph of the line $y = x - 1$. This line is called an **oblique,** or **slant, asymptote** for the graph of f. A graphing calculator graph is shown in Figure 11(a). The line $y = x - 1$ was graphed as well [Fig. 11(b)]. Notice how the graph of $f(x)$ approaches that line as x gets large in both directions. The graph of f is sketched in Figure 11(c).

› **Figure 11**



MATCHED PROBLEM 6

Graph, including any slant asymptotes, $f(x) = \dfrac{x^2 + 5}{x + 1}$.

It turns out that rational functions have slant asymptotes exactly when the numerator has degree one more than the denominator.

› THEOREM 4 Oblique Asymptotes and Rational Functions

If $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials and the degree of $p(x)$ is 1 more than the degree of $q(x)$, then performing long division enables us to write $f(x)$ in the form

$$f(x) = mx + b + \frac{r(x)}{q(x)}$$

where the degree of $r(x)$ is less than the degree of $q(x)$. The line

$$y = mx + b$$

is an oblique (slant) asymptote for the graph of f.

At the beginning of this section we made the assumption that for a rational function $f(x) = p(x)/q(x)$, the polynomials $p(x)$ and $q(x)$ have no common real zero. Now we will look at a rational function where the numerator and denominator have common zeros. Suppose that $p(x)$ and $q(x)$ have one or more real zeros in common. Then, by the factor theorem, $p(x)$ and $q(x)$ have one or more linear factors in common. We proceed to cancel any common linear factors in

$$f(x) = \frac{p(x)}{q(x)}$$

until we obtain a rational function

$$f_r(x) = \frac{p_r(x)}{q_r(x)}$$

in which $p_r(x)$ and $q_r(x)$ have no common real zero. We analyze and graph $f_r(x)$, then insert "holes" as required in the graph of f_r to obtain the graph of f. Example 7 illustrates the details.

EXAMPLE 7 **Graphing Arbitrary Rational Functions**

Graph $f(x) = \dfrac{2x^5 - 4x^4 - 6x^3}{x^5 - 3x^4 - 3x^3 + 7x^2 + 6x}$.

SOLUTION

The real zeros of

$$p(x) = 2x^5 - 4x^4 - 6x^3$$

(obtained by graphing or factoring) are -1, 0, and 3.

The real zeros of

$$q(x) = x^5 - 3x^4 - 3x^3 + 7x^2 + 6x$$

are $-1, 0, 2$, and 3. The common zeros are $-1, 0$, and 3. Factoring and cancelling common linear factors gives

$$f(x) = \frac{2x^3(x+1)(x-3)}{x(x+1)^2(x-2)(x-3)} \quad \text{and} \quad f_r(x) = \frac{2x^2}{(x+1)(x-2)}$$

We analyze $f_r(x)$ as usual:

x intercept: $x = 0$
y intercept: $y = f_r(0) = 0$
Vertical asymptotes: $x = -1, x = 2$
Domain: $x \neq -1, 2$
Points of discontinuity: $x = -1, x = 2$
Horizontal asymptote: $y = 2$

The graph of f is identical to the graph of f_r except possibly at the common real zeros $-1, 0$, and 3. We consider each common zero separately.

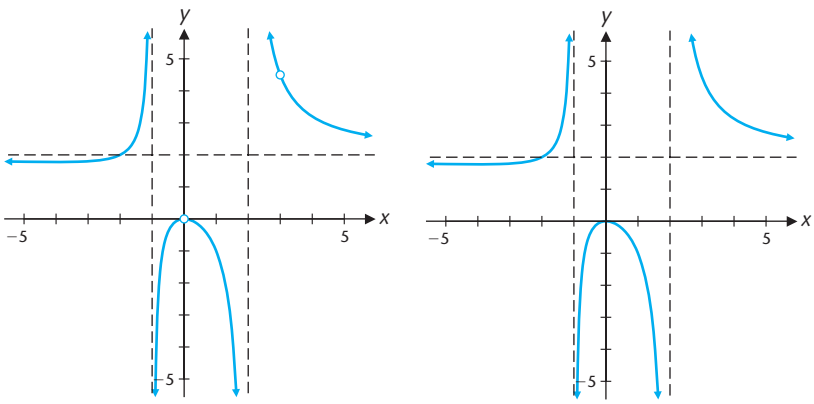
$x = -1$: Both f and f_r are undefined (no difference in their graphs).
$x = 0$: f is undefined but $f_r(0) = 0$, so the graph of f has a hole at $(0, 0)$.
$x = 3$: f is undefined but $f_r(3) = 4.5$, so the graph of f has a hole at $(3, 4.5)$.

Therefore, $f(x)$ has the following analysis:

x intercepts: none
y intercepts: none
Domain: $(-\infty, -1) \cup (-1, 0) \cup (0, 2) \cup (2, 3) \cup (3, \infty)$
Points of discontinuity: $x = -1, x = 0, x = 2, x = 3$
Vertical asymptotes: $x = -1, x = 2$
Horizontal asymptotes: $y = 2$
Holes: $(0, 0), (3, 4.5)$

Figure 12 shows the graphs of f and f_r.

› Figure 12



(a) $f(x) = \dfrac{2x^5 - 4x^4 - 6x^3}{x^5 - 3x^4 - 3x^3 + 7x^2 + 6x}$ (b) $f_r(x) = \dfrac{2x^2}{(x+1)(x-2)}$

MATCHED PROBLEM 7

Graph $f(x) = \dfrac{x^3 - x}{x^4 - x^2}$.

Solving Rational Inequalities

One of the things we can learn from studying the graphs of rational functions can be applied to solving inequalities. A rational function $f(x) = p(x)/q(x)$ can change sign at a real zero of $p(x)$ (where f has an x intercept) or at a real zero of $q(x)$ (where f is discontinuous), but nowhere else (because f is continuous except where it is not defined). Rational inequalities can therefore be solved in pretty much the same way as polynomial inequalities. The only real difference is that the intervals to be tested are bounded by both the zeros of the numerator and the zeros of denominator.

EXAMPLE 8 Solving Rational Inequalities

Solve $\dfrac{x^3 + 4x^2}{x^2 - 4} < 0$.

SOLUTION

Algebraic Solution

Let

$$f(x) = \frac{x^3 + 4x^2}{x^2 - 4}$$

Find the zeros of the numerator

$$x^3 + 4x^2 = x^2(x + 4);$$

the zeros are 0 and -4. Next, find the zeros of the denominator

$$x^2 - 4 = (x + 2)(x - 2);$$

the zeros are -2 and 2. These four zeros partition the x axis into the five intervals shown in the table. A test number is chosen from each interval as indicated to determine whether $f(x)$ is positive or negative.

Interval	Test number	Result
$(-\infty, -4)$	-5	$-25/21$; negative
$(-4, -2)$	-3	$9/5$; positive
$(-2, 0)$	-1	-1; negative
$(0, 2)$	1	$-5/3$; negative
$(2, \infty)$	3	$63/5$; positive

We conclude that the solution set of the inequality is

$$(-\infty, -4) \cup (-2, 0) \cup (0, 2)$$

Graphical Solution

Let

$$f(x) = \frac{x^3 - 4x^2}{x^2 - 4}.$$

The graph of $f(x)$ (Fig. 13) shows zeros at $x = -4$ and 0, and vertical asymptotes at $x = 2$ and $x = -2$. These are all of the x values where f can change sign, and they partition the x axis into five intervals.

By inspecting the graph of f, we see that f is below the x axis on the intervals $(-\infty, -4)$, $(-2, 0)$, and $(0, 2)$.

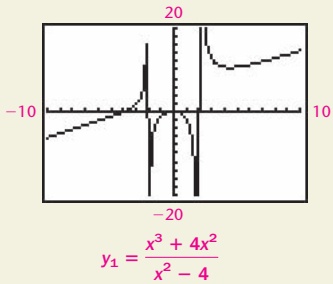


Figure 13

Note that $f(0) = 0$, so $x = 0$ is not a solution to the inequality. We conclude that the solution set is

$$(-\infty, -4) \cup (-2, 0) \cup (0, 2)$$

MATCHED PROBLEM 8

Solve $\dfrac{x^2 - 1}{x^2 - 9} \geq 0$.

EXAMPLE 9 Solving Rational Inequalities

Solve $1 \geq \dfrac{9x - 9}{x^2 + x - 3}$ to three decimal places.

SOLUTION

Our method of solving depends on the inequality being a statement about a function being positive or negative, so we first convert the inequality to an equivalent inequality in which one side is 0:

$$1 \geq \frac{9x - 9}{x^2 + x - 3}$$ Subtract $\frac{9x - 9}{x^2 + x - 3}$ from both sides.

$$1 - \frac{9x - 9}{x^2 + x - 3} \geq 0$$ Find a common denominator.

$$\frac{x^2 + x - 3}{x^2 + x - 3} - \frac{9x - 9}{x^2 + x - 3} \geq 0$$ Simplify.

$$\frac{x^2 - 8x + 6}{x^2 + x - 3} \geq 0$$

We can use either a graph or the quadratic formula to find the zeros of the numerator and denominator.

The zeros of $x^2 - 8x + 6$, to three decimal places, are 0.838 and 7.162 (details omitted). The zeros of $x^2 + x - 3$ are -2.303 and 1.303. These four zeros partition the x axis into five intervals:

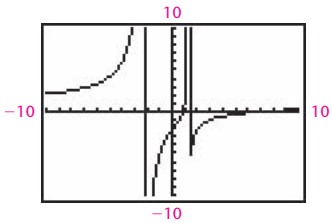
$$(-\infty, -2.303), (-2.303, 0.838), (0.838, 1.303), (1.303, 7.162), \text{ and } (7.162, \infty)$$

We graph

$$f(x) = \frac{x^2 - 8x + 6}{x^2 + x - 3}$$

(Fig. 14) and observe that the graph of f is above the x axis on the intervals $(-\infty, -2.303)$, $(0.838, 1.303)$, and $(7.162, \infty)$. The solution set of the inequality is thus

$$(-\infty, -2.303) \cup [0.838, 1.303) \cup [7.162, \infty)$$



› Figure 14

Note that the endpoints that are zeros of f, namely 0.838 and 7.162, make the inequality true, and are included in the solution set of the inequality. But the endpoints at which f is undefined (-2.303 and 1.303) make the inequality false, and are excluded.

MATCHED PROBLEM 9

Solve $\dfrac{x^3 + 3x^2 + 5x - 6}{x^2 - 5x - 1} \geq -1$ to three decimal places.

››› CAUTION ›››

When a rational inequality includes equality ($\geq$ or $\leq$), you should include the zeros of the numerator in the solution set, but never the zeros of the denominator.

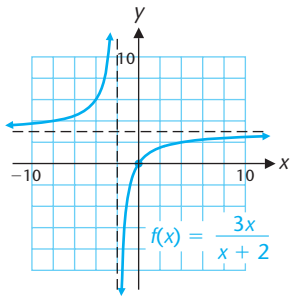
ANSWERS TO MATCHED PROBLEMS

1. Domain: $(-\infty, -3) \cup (-3, 1) \cup (1, \infty)$; x intercepts: $x = -2$, $x = 2$

2. (A) Properties 3 and 4 are not satisfied. (B) Property 1 is not satisfied.
(C) Properties 1 and 3 are not satisfied.

3. (A) Vertical asymptotes: $x = -3, x = 1$; horizontal asymptote: $y = 3$
(B) Vertical asymptotes: $x = -1, x = 0, x = 1$; horizontal asymptote: $y = 0$
(C) No asymptotes

4.



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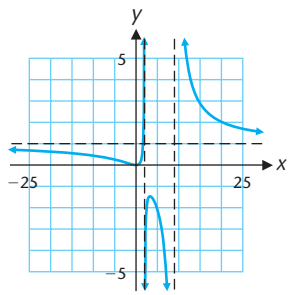
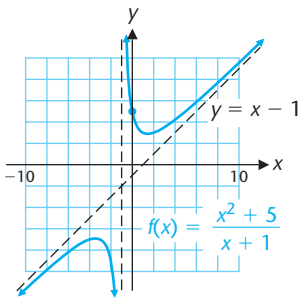


$f(x) = \dfrac{x^2}{x^2 - 11x + 18}$

6.



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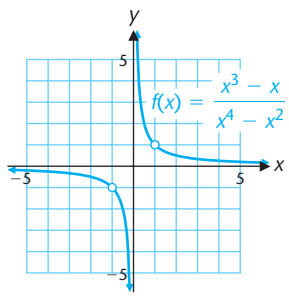


8. $(-\infty, -3) \cup [-1, 1] \cup (3, \infty)$

9. $[-3.391, -1.773] \cup (-0.193, 1.164] \cup (5.193, \infty)$

3-5 Exercises

1. What is a rational function?

2. Describe in your own words how to find the vertical asymptotes of a rational function.

3. Describe in your own words how to find the horizontal asymptotes of a rational function.

4. Is there a limit on the number of vertical asymptotes that the graph of a rational function can have?

5. Is it accurate to say that the graph of a function can never intersect an asymptote? Explain.

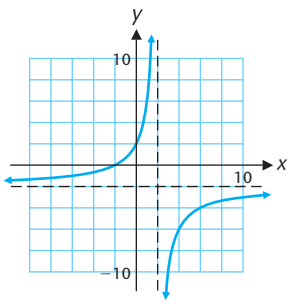
6. Is it accurate to say that the solution of a rational inequality with inequality sign $\leq$ or $\geq$ should always contain the endpoints of the intervals in the solution? Why or why not?

In Problems 7–10, match each graph with one of the following functions:

$$f(x) = \frac{2x - 4}{x + 2} \qquad g(x) = \frac{2x + 4}{2 - x}$$

$$h(x) = \frac{2x + 4}{x - 2} \qquad k(x) = \frac{4 - 2x}{x + 2}$$

7.



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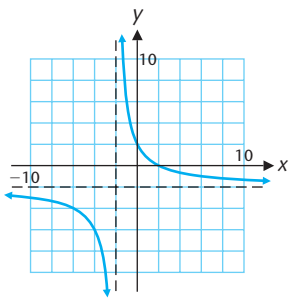

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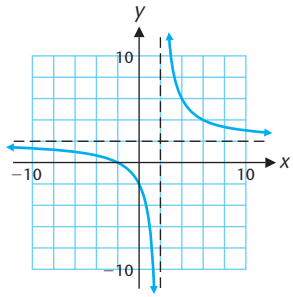

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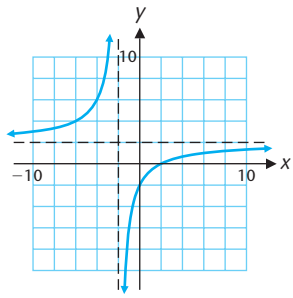


11. Let $f(x) = \dfrac{2x - 4}{x + 2}$. Complete each statement:

(A) As $x \to -2^-$, $f(x) \to ?$
(B) As $x \to -2^+$, $f(x) \to ?$
(C) As $x \to -\infty$, $f(x) \to ?$
(D) As $x \to \infty$, $f(x) \to ?$

12. Let $g(x) = \dfrac{2x + 4}{2 - x}$. Complete each statement:

(A) As $x \to 2^-$, $g(x) \to ?$
(B) As $x \to 2^+$, $g(x) \to ?$
(C) As $x \to -\infty$, $g(x) \to ?$
(D) As $x \to \infty$, $g(x) \to ?$

13. Let $h(x) = \dfrac{2x + 4}{x - 2}$. Complete each statement:

(A) As $x \to 2^-$, $h(x) \to ?$
(B) As $x \to 2^+$, $h(x) \to ?$
(C) As $x \to -\infty$, $h(x) \to ?$
(D) As $x \to \infty$, $h(x) \to ?$

14. Let $k(x) = \dfrac{4 - 2x}{x + 2}$. Complete each statement:

(A) As $x \to 2^-$, $k(x) \to$?
(B) As $x \to 2^+$, $k(x) \to$?
(C) As $x \to -\infty$, $k(x) \to$?
(D) As $x \to \infty$, $k(x) \to$?

In Problems 15–24, find the domain and x intercepts.

15. $f(x) = \dfrac{2x - 4}{x + 1}$

16. $g(x) = \dfrac{3x + 6}{x - 1}$

17. $h(x) = \dfrac{x^2 - 1}{x^2 - 16}$

18. $k(x) = \dfrac{x^2 - 36}{x^2 - 25}$

19. $r(x) = \dfrac{x^2 - x - 6}{x^2 - x - 12}$

20. $s(x) = \dfrac{x^2 + x - 12}{x^2 + x - 6}$

21. $F(x) = \dfrac{x}{x^2 + 4}$

22. $G(x) = \dfrac{x^2}{x^2 + 16}$

23. $u(x) = \dfrac{4 - x^2}{x}$

24. $v(x) = \dfrac{2 - 3x + x^2}{x^3}$

In Problems 25–34, find all vertical and horizontal asymptotes.

25. $f(x) = \dfrac{2x}{x - 4}$

26. $h(x) = \dfrac{3x}{x + 5}$

27. $s(x) = \dfrac{2x^2 + 3x}{3x^2 - 48}$

28. $r(x) = \dfrac{5x^2 - 7x}{2x^2 - 50}$

29. $p(x) = \dfrac{2x}{x^4 + 1}$

30. $q(x) = \dfrac{5x^4}{2x^2 + 3x - 2}$

31. $t(x) = \dfrac{6x^4}{3x^2 - 2x - 5}$

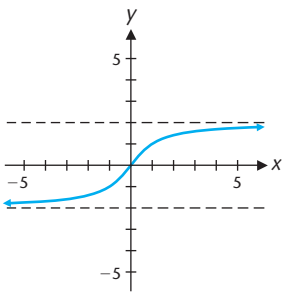
32. $g(x) = \dfrac{3x}{x^4 + 2x^2 + 1}$

33. $C(x) = \dfrac{4x^2 + 4x - 24}{x^2 - 2x}$

34. $D(x) = \dfrac{x^2 + 8x + 7}{8x^2 + 8x}$

In Problems 35–38, explain why each graph is not the graph of a rational function.

35.



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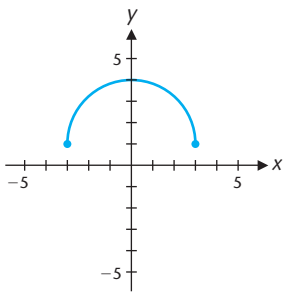

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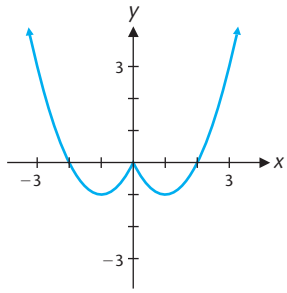

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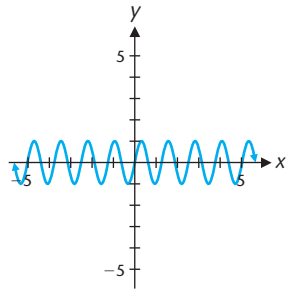


In Problems 39–42, explain how the graph of f differs from the graph of g.

39. $f(x) = \dfrac{x^2 + 2x}{x}$; $g(x) = x + 2$

40. $f(x) = \dfrac{x + 5}{x^2 - 25}$; $g(x) = \dfrac{1}{x - 5}$

41. $f(x) = \dfrac{x + 2}{x^2 + 10x + 16}$; $g(x) = \dfrac{1}{x + 8}$

42. $f(x) = \dfrac{x^2 - x - 12}{x - 4}$; $g(x) = x + 3$

In Problems 43–64, use the graphing strategy outlined in the text to sketch the graph of each function.

43. $f(x) = \dfrac{1}{x - 4}$

44. $g(x) = \dfrac{1}{x + 3}$

45. $f(x) = \dfrac{x}{x + 1}$

46. $f(x) = \dfrac{3x}{x - 3}$

47. $h(x) = \dfrac{x}{2x - 2}$

48. $p(x) = \dfrac{3x}{4x + 4}$

49. $f(x) = \dfrac{2x - 4}{x + 3}$

50. $f(x) = \dfrac{3x + 3}{2 - x}$

51. $g(x) = \dfrac{1 - x^2}{x^2}$

52. $f(x) = \dfrac{x^2 + 1}{x^2}$

53. $f(x) = \dfrac{9}{x^2 - 9}$

54. $g(x) = \dfrac{6}{x^2 - x - 6}$

55. $f(x) = \dfrac{x}{x^2 - 1}$

56. $p(x) = \dfrac{x}{1 - x^2}$

57. $g(x) = \dfrac{2}{x^2 + 1}$

58. $f(x) = \dfrac{x}{x^2 + 1}$

59. $f(x) = \dfrac{12x^2}{(3x + 5)^2}$

60. $f(x) = \dfrac{7x^2}{(2x - 3)^2}$

61. $f(x) = \dfrac{x^2 - 1}{x^2 + 7x + 10}$

62. $f(x) = \dfrac{x^2 + 6x + 8}{x^2 - x - 2}$

63. $f(x) = \dfrac{x^2 - 3x - 10}{x - 5}$ **64.** $f(x) = \dfrac{x + 2}{x^2 + 9x + 14}$

In Problems 65–68, give an example of a rational function that satisfies the given conditions.

65. Real zeros: $-2, -1, 1, 2$; vertical asymptotes: none; horizontal asymptote: $y = 3$

66. Real zeros: none; vertical asymptote: $x = 4$; horizontal asymptote: $y = -2$

67. Real zeros: none; vertical asymptote: $x = 10$; slant asymptote: $y = 2x + 5$

68. Real zeros: 1, 2, 3; vertical asymptotes: none; slant asymptote: $y = 2 - x$

In Problems 69–76, solve each rational inequality.

69. $\dfrac{x}{x - 2} \le 0$ **70.** $\dfrac{2x - 1}{x + 3} > 0$

71. $\dfrac{x^2 - 16}{5x - 2} > 0$ **72.** $\dfrac{x - 4}{x^2 - 9} \le 0$

73. $\dfrac{x^2 + 4x - 20}{3x} \ge 4$ **74.** $\dfrac{3x - 7}{x^2 + 6x} < 2$

75. $\dfrac{5x}{x^2 - 1} < \dfrac{9}{x}$ **76.** $\dfrac{1}{x^2 + 8x + 12} \ge \dfrac{1}{x}$

In Problems 77–86, solve each rational inequality to three decimal places.

77. $\dfrac{x^2 + 7x + 3}{x + 2} > 0$ **78.** $\dfrac{x^3 + 4}{x^2 + x - 3} \le 0$

79. $\dfrac{5}{x^2} - \dfrac{1}{x + 3} < 0$ **80.** $\dfrac{3x}{x + 4} - \dfrac{x + 1}{x^2} \ge 0$

81. $\dfrac{9}{x} - \dfrac{5}{x^2} \le 1$ **82.** $\dfrac{x + 4}{x^2 + 1} > 2$

83. $\dfrac{3x + 2}{x - 5} > 10$ **84.** $\dfrac{x}{x^2 + 5x - 6} \le 0.5$

85. $\dfrac{4}{x + 1} \ge \dfrac{7}{x}$ **86.** $\dfrac{1}{x^2 - 1} < \dfrac{x^2}{x^4 + 1}$

In Problems 87–92, find all vertical, horizontal, and slant asymptotes.

87. $f(x) = \dfrac{2x^2}{x - 1}$ **88.** $g(x) = \dfrac{3x^2}{x + 2}$

89. $p(x) = \dfrac{x^3}{x^2 + 1}$ **90.** $q(x) = \dfrac{x^5}{x^3 - 8}$

91. $r(x) = \dfrac{2x^2 - 3x + 5}{x}$ **92.** $s(x) = \dfrac{-3x^2 + 5x + 9}{x}$

In Problems 93–96, investigate the behavior of each function as $x \to \infty$ and as $x \to -\infty$, and find any horizontal asymptotes (note that these functions are not rational).

93. $f(x) = \dfrac{5x}{\sqrt{x^2 + 1}}$ **94.** $f(x) = \dfrac{2x}{\sqrt{x^2 - 1}}$

95. $f(x) = \dfrac{4\sqrt{x^2 - 4}}{x}$ **96.** $f(x) = \dfrac{3\sqrt{x^2 + 1}}{x - 1}$

In Problems 97–102, use the graphing strategy outlined in the text to sketch the graph of each function. Write the equations of all vertical, horizontal, and slant asymptotes.

97. $f(x) = \dfrac{x^2 + 1}{x}$ **98.** $g(x) = \dfrac{x^2 - 1}{x}$

99. $k(x) = \dfrac{x^2 - 4x + 3}{2x - 4}$ **100.** $h(x) = \dfrac{x^2 + x - 2}{2x - 4}$

101. $F(x) = \dfrac{8 - x^3}{4x^2}$ **102.** $G(x) = \dfrac{x^4 + 1}{x^3}$

In calculus, it is often necessary to consider rational functions in which the numerator and denominator have a common factor such as the functions given in Problems 103–106. For each function, state the domain. Write the equations of all vertical and horizontal asymptotes, and sketch the graph.

103. $f(x) = \dfrac{x^2 - 4}{x - 2}$ **104.** $g(x) = \dfrac{x^2 - 1}{x + 1}$

105. $r(x) = \dfrac{x + 2}{x^2 - 4}$ **106.** $s(x) = \dfrac{x - 1}{x^2 - 1}$

APPLICATIONS

107. EMPLOYEE TRAINING A company producing electronic components used in television sets has established that on average, a new employee can assemble $N(t)$ components per day after t days of on-the-job training, as given by

$$N(t) = \frac{50t}{t + 4} \qquad t \ge 0$$

Sketch the graph of N, including any vertical or horizontal asymptotes. What does N approach as $t \to \infty$? Explain the significance of this number.

108. PHYSIOLOGY In a study on the speed of muscle contraction in frogs under various loads, researchers W. O. Fems and J. Marsh found that the speed of contraction decreases with increasing loads. More precisely, they found that the relationship between speed of contraction S (in centimeters per second) and load w (in grams) is given approximately by

$$S(w) = \frac{26 + 0.06w}{w} \qquad w \geq 5$$

Sketch the graph of S, including any vertical or horizontal asymptotes. What does S approach as $w \to \infty$? Explain the significance of this number.

109. RETENTION An experiment on retention is conducted in a psychology class. Each student in the class is given 1 day to memorize the same list of 40 special characters. The lists are turned in at the end of the day, and for each succeeding day for 20 days each student is asked to turn in a list of as many of the symbols as can be recalled. Averages are taken, and it is found that a good approximation of the average number of symbols, $N(t)$, retained after t days is given by

$$N(t) = \frac{5t + 30}{t} \qquad t \geq 1$$

Sketch the graph of N, including any vertical or horizontal asymptotes. What does N approach as $t \to \infty$? Explain the significance of this number.

110. LEARNING THEORY In 1917, L. L. Thurstone, a pioneer in quantitative learning theory, proposed the function

$$f(x) = \frac{a(x + c)}{(x + c) + b}$$

to describe the number of successful acts per unit time that a person could accomplish after x practice sessions. Suppose that for a particular person enrolling in a typing class,

$$f(x) = \frac{50(x + 1)}{x + 5} \qquad x \geq 0$$

where $f(x)$ is the number of words per minute the person is able to type after x weeks of lessons. Sketch the graph of f, including any vertical or horizontal asymptotes. What does f approach as $x \to \infty$? Explain the significance of this number.

Problems 111–114 are calculus related.

111. REPLACEMENT TIME A desktop office copier has an initial price of \$2,500. A maintenance/service contract costs \$200 for the first year and increases \$50 per year thereafter. It can be shown that the total cost of the copier after n years is given by

$$C(n) = 2{,}500 + 175n + 25n^2$$

The average cost per year for n years is $\overline{C}(n) = C(n)/n$.
(A) Find the rational function $\overline{C}$.
(B) When is the average cost per year a minimum? (This is frequently referred to as the *replacement time* for this piece of equipment.)
(C) Sketch the graph of C, including any asymptotes.

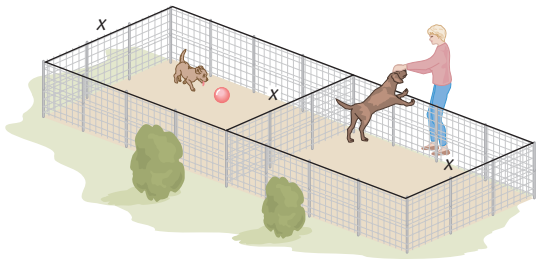
112. AVERAGE COST The total cost of producing x units of a certain product is given by

$$C(x) = \tfrac{1}{5}x^2 + 2x + 2{,}000$$

The average cost per unit for producing x units is $\overline{C}(x) = C(x)/x$.
(A) Find the rational function $\overline{C}$.
(B) At what production level will the average cost per unit be minimal?
(C) Sketch the graph of $\overline{C}$, including any asymptotes.

113. CONSTRUCTION A rectangular dog pen is to be made to enclose an area of 225 square feet.
(A) If x represents the width of the pen, express the total length L of the fencing material required for the pen in terms of x.
(B) Considering the physical limitations, what is the domain of the function L?
(C) Find the dimensions of the pen that will require the least amount of fencing material.
(D) Graph the function L, including any asymptotes.

114. CONSTRUCTION Rework Problem 113 with the added assumption that the pen is to be divided into two sections, as shown in the figure. (Approximate dimensions to three decimal places.)



3-6 Variation and Modeling

- Modeling with Direct Variation
- Modeling with Inverse Variation
- Modeling with Joint and Combined Variation

Very few things in our world happen without consequence. In most cases, things that change affect something else. If the number of hours you work at a part-time job goes up, the amount of money you earn goes up along with it. If the number of days you miss a certain class goes up, your grade will go down. Each of these is an example of *variation,* which is an important mathematical way to express a connection between two or more quantities. In this section, we will study different types of variation, and see how to use them to model real-life situations.

Modeling with Direct Variation

As just mentioned, if the number of hours you work goes up, your pay will go up as well. If the speed you drive goes down, the distance you cover in a certain amount of time will go down as well. These are examples of *direct variation*. We can often describe relationships like this with a simple equation.

DEFINITION 1 Direct Variation

Let x and y be variables. The statement ***y* is directly proportional to *x*** (or ***y* varies directly as *x***) means

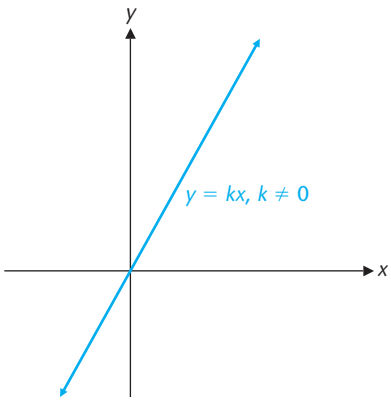
$$y = kx$$

for some nonzero constant k, called the **constant of proportionality** (or **constant of variation**).

Informally, when two quantities vary directly and the constant of proportionality is positive, if one increases, the other does as well.

In the first example above, if h represents hours worked and d is the number of dollars earned, we can write $d = kh$; the amount earned is directly proportional to the number of hours worked, and the constant of proportionality k is the hourly wage. In the second example, if r is speed and d is distance, we can write $d = kr$; in this case, the constant of proportionality is the amount of time passed.

Notice that any equation of direct variation $y = kx$, $k \neq 0$, gives a linear model with nonzero slope that passes through the origin (Fig. 1).



› **Figure 1** Direct variation.

EXAMPLE 1 **Direct Variation**

The force F exerted by a spring is directly proportional to the distance x that it is stretched. (This is known as Hooke's law.) Find the constant of proportionality and the equation of variation if $F = 12$ pounds when $x = \frac{1}{3}$ foot.

SOLUTION

Since F and x are directly proportional, the equation of variation has the form $F = kx$. To find the constant of proportionality, substitute $F = 12$ and $x = \frac{1}{3}$ and solve for k.

$$\begin{aligned} F &= kx && \text{Let } F = 12 \text{ and } x = \tfrac{1}{3}. \\ 12 &= k(\tfrac{1}{3}) && \text{Multiply both sides by 3.} \\ k &= 36 \end{aligned}$$

Therefore, the constant of proportionality is $k = 36$ and the equation of variation is

$$F = 36x$$

MATCHED PROBLEM 1

Find the constant of proportionality and the equation of variation if p is directly proportional to v, and $p = 200$ when $v = 8$.

››› EXPLORE-DISCUSS 1

All real numbers x are in the domain of the linear function $y = kx$, $k \neq 0$. Often in practice, however, $y = kx$ will not provide a good model for all values of x.

(A) Discuss factors that would make the spring model of Example 1 unreasonable for certain values of x.

(B) Explain why both positive and negative values of x could be allowed in the spring model of Example 1. What would be the physical interpretation of a negative force? [*Hint:* Consider a distance the spring is *compressed* to be negative.]

Sometimes a quantity varies directly not with another quantity, but with a certain power of that quantity, as in Example 2.

EXAMPLE 2 Direct Variation

The distance d covered by a falling object is directly proportional to the square of the length of time t since it began falling. If a coin dropped from a suspension bridge falls 144 feet in 3 seconds, find the equation of variation, and use it to find how far the coin would fall in 5 seconds.

SOLUTION

We were given that d varies directly with t^2, so the equation has the form

$$d = kt^2$$

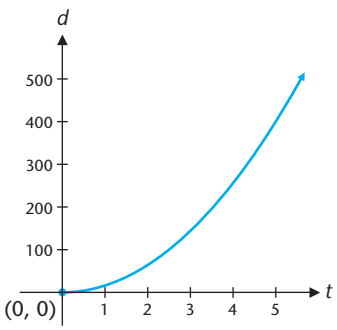
Substitute $d = 144$ and $t = 3$ to find the constant of proportionality:

$$144 = k(3)^2$$
$$144 = 9k$$
$$k = \frac{144}{9} = 16$$

The equation of variation is

$$d = 16t^2$$

and a graph of this relation is shown in Figure 2. (Note that the distance d increases as the coin falls.)



› Figure 2

After 5 seconds, the coin will have fallen

$$d = 16(5)^2 = 400 \text{ feet}$$

MATCHED PROBLEM 2

For an object falling toward the surface of the moon, the distance fallen d is directly proportional to the square of the length of time t since it began falling. An object falling toward the moon falls 41.6 feet in 4 seconds. Find the equation of variation, and the distance it will fall in 7 seconds.

Modeling with Inverse Variation

At the beginning of the section, we pointed out that as the number of class periods you miss goes up, your grade is likely to go down. When two quantities are related so that an increase in one goes along with a decrease in the other, we say that they are *inversely proportional*. In mathematical terms, this will correspond to the quantities being related by a constant multiple of the rational function $y = \frac{1}{x}$.

To understand that this makes sense, think about that equation. If x gets larger, it would make the fraction $\frac{1}{x}$ smaller, so that an increase in x corresponds to a decrease in y. Likewise, if x gets smaller, $\frac{1}{x}$ gets bigger, so a decrease in x corresponds to an increase in y.

DEFINITION 2 Inverse Variation

Let x and y be variables. The statement ***y* is inversely proportional to *x*** (or ***y* varies inversely as *x***) means

$$y = \frac{k}{x}$$

for some nonzero constant k, called the **constant of proportionality** (or **constant of variation**).

Informally, when two quantities are inversely proportional and the constant of proportionality is positive, if one increases, the other decreases.

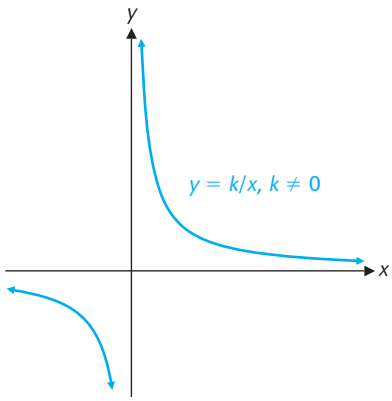
If you plan a trip of 100 miles, the faster you go, the less time it will take. Therefore, the rate r and time t it takes to travel a distance of 100 miles are inversely proportional (recall that distance equals rate times time, $d = rt$). The equation of variation is

$$t = \frac{100}{r}$$

and the constant of proportionality is 100.

The equation of inverse variation, $y = k/x$, determines a rational function having the y axis as a vertical asymptote and the x axis as a horizontal asymptote (Fig. 3). In most applications, the constant k of proportionality will be positive, and only the portion of the graph in Quadrant I will be relevant. Note that if x is very large, then y is close to 0; if x is close to 0, then y is very large.

› Figure 3 Inverse variation.



EXAMPLE 3 Inverse Variation

The note played by each pipe in a pipe organ is determined by the frequency of vibration of air in the pipe. The fundamental frequency f of vibration of air in an organ pipe is inversely proportional to the length L of the pipe. (This is why the low-frequency notes come from the very long pipes.)

(A) Find the constant of proportionality and the equation of variation if the fundamental frequency of an 8-foot pipe is 64 vibrations per second.

(B) Find the fundamental frequency of a 1.6-foot pipe.

SOLUTIONS

(A) Since the two quantities are inversely proportional, we know that the equation has the form $f = k/L$. To find the constant of proportionality, substitute $L = 8$ and $f = 64$ and solve for k.

$$f = \frac{k}{L} \qquad \text{Let } f = 64 \text{ and } L = 8.$$

$$64 = \frac{k}{8} \qquad \text{Multiply both sides by 8.}$$

$$k = 512$$

The constant of proportionality is $k = 512$ and the equation of variation is

$$f = \frac{512}{L}$$

(B) If $L = 1.6$, then $f = \frac{512}{1.6} = 320$ vibrations per second.

MATCHED PROBLEM 3

Find the constant of proportionality and the equation of variation if P is inversely proportional to V, and $P = 56$ when $V = 3.5$.

Modeling with Joint and Combined Variation

In some cases, the size of one quantity is related to two other quantities. For example, the area of a rectangle is determined by both its length and width. This is an example of *joint variation.*

DEFINITION 3 Joint Variation

Let x, y, and w be variables. The statement ***w* is jointly proportional to *x* and *y*** (or ***w* varies jointly as *x* and *y***) means

$$w = kxy$$

for some nonzero constant k, called the **constant of proportionality** (or **constant of variation**).

The area of a rectangle is jointly proportional to its length and width with constant of proportionality 1; the equation of variation is $A = LW$.

The concept of joint variation can be extended to apply to more than three variables. For example, the volume of a box is jointly proportional to its length, width, and height: $V = LWH$ (again with constant of proportionality 1). Similarly, the concepts of direct and inverse variation can be extended. For example, the area of a circle is directly proportional to the square of its radius; the constant of proportionality is π and the equation of variation is $A = \pi r^2$.

The three basic types of variation also can be combined. For example, Newton's law of gravitation, "The force of attraction F between two objects is jointly proportional to their masses m_1 and m_2 and inversely proportional to the square of the distance d between them," has the equation

$$F = k\frac{m_1 m_2}{d^2}$$

EXAMPLE 4 Joint Variation

The volume V of a right circular cone is jointly proportional to the square of its radius r and its height h. Find the constant of proportionality and the equation of variation if a cone of height 8 inches and radius 3 inches has a volume of 24π (about 75.4) cubic inches.

SOLUTION

Since V is jointly proportional to the square of r and h, the equation of variation has the form $V = kr^2h$. To find the constant of proportionality k, substitute $V = 24\pi$, $r = 3$, and $h = 8$.

$$\begin{aligned} V &= kr^2h && \text{Let } V = 24\pi,\ r = 3, \text{ and } h = 8. \\ 24\pi &= k(3)^2 8 && \text{Simplify.} \\ 24\pi &= 72k && \text{Divide both sides by } 72. \\ k &= \frac{\pi}{3} \end{aligned}$$

The constant of proportionality is $k = \frac{\pi}{3}$ and the equation of variation is

$$V = \frac{\pi}{3} r^2 h$$

MATCHED PROBLEM 4

The volume V of a box with a square base is jointly proportional to the square of a side x of the base and the height h. Find the constant of proportionality and the equation of variation if a box with volume 45 cubic inches has base 3 inches by 3 inches and a height of 5 inches.

EXAMPLE 5 Combined Variation

The note played by a string on a guitar is determined by the frequency at which the string vibrates. The frequency f of a vibrating string is directly proportional to the square root of the tension T and inversely proportional to the length L.

(A) If the tension of a guitar string is increased, how does the frequency change? What if the length is increased?

(B) What is the effect on the frequency if the length is doubled and the tension is quadrupled?

SOLUTIONS

(A) The frequency is directly proportional to the square root of the tension. If the tension increases, so does its square root, and consequently the frequency increases as well. On the other hand, the frequency is inversely proportional to length, so if the length is increased, the frequency will decrease.

(B) Since f is directly proportional to $\sqrt{T}$ and inversely proportional to L, the equation of variation has the form

$$f = k\frac{\sqrt{T}}{L}$$

Let f_1, T_1, and L_1 denote the initial frequency, tension, and length, respectively. Then doubling the length makes the new length L_2 satisfy $L_2 = 2L_1$. Likewise, quadrupling the tension makes the new tension T_2 satisfy $T_2 = 4T_1$. Substituting f_2, T_2, and L_2 in the equation of variation, we get

$$f_2 = k\frac{\sqrt{T_2}}{L_2}$$

To compare this to the original frequency, substitute $4T_1$ and $2L_1$ in for T_2 and L_2, respectively.

$$
\begin{aligned}
f_2 &= k\frac{\sqrt{T_2}}{L_2} && \text{Let } L_2 = 2L_1, \text{ and } T_2 = 4T_1.\\
&= k\frac{\sqrt{4T_1}}{2L_1} && \text{Simplify the radical.}\\
&= k\frac{2\sqrt{T_1}}{2L_1} && \text{Cancel and use the equation of variation.}\\
&= f_1
\end{aligned}
$$

We conclude that there is no effect on the frequency—the pitch remains the same.

MATCHED PROBLEM 5

Refer to Example 5. What is the effect on the frequency if the tension is quadrupled and the length is cut in half?

››› EXPLORE-DISCUSS 2

Refer to the equation of variation of Example 5. Explain why the frequency f, for fixed T, is a rational function of L, but f is *not*, for fixed L, a rational function of T.

ANSWERS TO MATCHED PROBLEMS

1. $k = 25$; $p = 25v$ **2.** $d = 2.6t^2$; 127.4 feet **3.** $k = 196$; $P = \dfrac{196}{V}$

4. $k = 1$; $V = x^2h$ **5.** The frequency is increased by a factor of 4.

3-6 Exercises

1. Give an example of two quantities that are directly proportional.

2. Give an example of two quantities that are inversely proportional.

3. If x varies directly as y, and y increases, what happens to x? How do you know?

4. If x varies inversely as y, and y increases, what happens to x? How do you know?

5. Explain why the distance traveled in a certain amount of time and the average speed are directly proportional.

6. Explain why the temperature setting of an oven and the amount of time it takes to heat an item to 170 degrees are inversely proportional.

In Problems 7–22, translate each statement into an equation using k as the constant of proportionality.

7. F is inversely proportional to x.

8. y is directly proportional to the square of x.

9. R is jointly proportional to S and T.

10. u is inversely proportional to v.

11. L is directly proportional to the cube of m.

12. W is jointly proportional to X, Y, and Z.

13. A varies jointly as the square of c and d.

14. q varies inversely as t.

15. P varies directly as x.

16. f varies directly as the square of b.

17. h varies inversely as the square root of s.

18. C varies jointly as the square of x and the cube of y.

19. R varies directly as m and inversely as the square of d.

20. T varies jointly as p and q and inversely as w.

21. D is jointly proportional to x and the square of y and inversely proportional to z.

22. S is directly proportional to the square root of u and inversely proportional to v.

23. Refer to Problem 7. If F increases, what happens to x?

24. Refer to Problem 10. If u increases, what happens to v?

25. Refer to Problem 11. If m increases, what happens to L?

26. Refer to Problem 8. If x increases, what happens to y?

27. u varies directly as the square root of v. If $u = 3$ when $v = 4$, find u when $v = 10$.

28. y varies directly as the cube of x. If $y = 48$ when $x = 4$, find y when $x = 8$.

29. L is inversely proportional to the square of M. If $L = 9$ when $M = 9$, find L when $M = 6$.

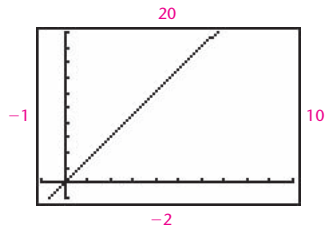
30. I is directly proportional to the cube root of y. If $I = 5$ when $y = 64$, find I when $y = 8$.

31. Q varies jointly as m and the square of n, and inversely as P. If $Q = 2$ when $m = 3$, $n = 6$, and $P = 12$, find Q when $m = 4$, $n = 18$, and $P = 2$.

32. w varies jointly as x, y, and z. If $w = 36$ when $x = 2$, $y = 8$, and $z = 12$, find w when $x = 1$, $y = 2$, and $z = 4$.

In problems 33–36, the graph of an equation of variation is shown. In each case, describe the type of variation as direct, inverse, or joint.

33.



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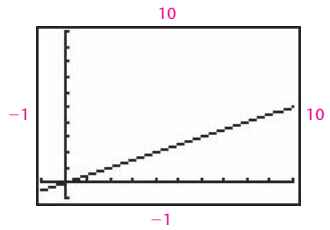

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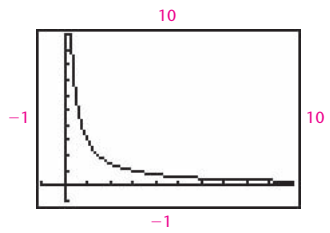

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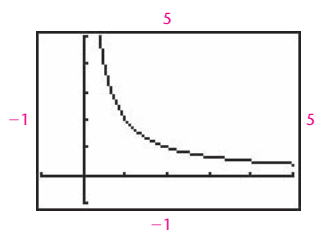


In Problems 37–42, translate each statement into an equation using k as the constant of variation.

37. The biologist René Réaumur suggested in 1735 that the length of time t that it takes fruit to ripen is inversely proportional to the sum T of the average daily temperatures during the growing season.

38. The erosive force P of a swiftly flowing stream is directly proportional to the sixth power of the velocity v of the water.

39. The maximum safe load L for a horizontal beam varies jointly as its width w and the square of its height h, and inversely as its length x.

40. The number N of long-distance phone calls between two cities varies jointly as the populations P_1 and P_2 of the two

cities, and inversely as the distance d between the two cities.

41. The f-stop numbers N on a camera, known as focal ratios, are directly proportional to the focal length F of the lens and inversely proportional to the diameter d of the effective lens opening.

42. The time t required for an elevator to lift a weight is jointly proportional to the weight w and the distance d through which it is lifted, and inversely proportional to the power P of the motor.

43. Suppose that f varies directly as x. Show that the ratio x_1/x_2 of two values of x is equal to f_1/f_2, the ratio of the corresponding values of f.

44. Suppose that f varies inversely as x. Show that the ratio x_1/x_2 of two values of x is equal to f_2/f_1, the reciprocal of the ratio of the corresponding values of f.

APPLICATIONS

45. PHYSICS The weight w of an object on or above the surface of the Earth varies inversely as the square of the distance d between the object and the center of Earth. If a girl weighs 100 pounds on the surface of Earth, how much would she weigh (to the nearest pound) 400 miles above Earth's surface? (Assume the radius of Earth is 4,000 miles.)

46. PHYSICS A child was struck by a car in a crosswalk. The driver of the car had slammed on his brakes and left skid marks 160 feet long. He told the police he had been driving at 30 miles per hour. The police know that the length of skid marks L (when brakes are applied) varies directly as the square of the speed of the car v, and that at 30 miles per hour (under ideal conditions) skid marks would be 40 feet long. How fast was the driver actually going before he applied his brakes?

47. ELECTRICITY Ohm's law states that the current I in a wire varies directly as the electromotive forces E and inversely as the resistance R. If $I = 22$ amperes when $E = 110$ volts and $R = 5$ ohms, find I if $E = 220$ volts and $R = 11$ ohms.

48. ANTHROPOLOGY Anthropologists, in their study of race and human genetic groupings, often use an index called the *cephalic index*. The cephalic index C varies directly as the width w of the head and inversely as the length l of the head (both when viewed from the top). If an Indian in Baja California (Mexico) has measurements of $C = 75$, $w = 6$ inches, and $l = 8$ inches, what is C for an Indian in northern California with $w = 8.1$ inches and $l = 9$ inches?

49. PHYSICS If the horsepower P required to drive a speedboat through water is directly proportional to the cube of the speed v of the boat, what change in horsepower is required to double the speed of the boat?

50. ILLUMINATION The intensity of illumination E on a surface is inversely proportional to the square of its distance d from a light source. What is the effect on the total illumination on a book if the distance between the light source and the book is doubled?

51. MUSIC The frequency of vibration f of a musical string is directly proportional to the square root of the tension T and inversely proportional to the length L of the string. If the tension of the string is increased by a factor of 4 and the length of the string is doubled, what is the effect on the frequency?

52. PHYSICS In an automobile accident, the destructive force F of a car is (approximately) jointly proportional to the weight w of the car and the square of the speed v of the car. (This is why accidents at high speeds are generally so serious.) What would be the effect on the destructive forces of a car if its weight were doubled and its speed were doubled?

53. SPACE SCIENCE The length of time t a satellite takes to complete a circular orbit of Earth varies directly as the radius r of the orbit and inversely as the orbital velocity v of the satellite. If $t = 1.42$ hours when $r = 4{,}050$ miles and $v = 18{,}000$ miles per hour (Sputnik I), find t to two decimal places for $r = 4{,}300$ miles and $v = 18{,}500$ miles per hour.

54. GENETICS The number N of gene mutations resulting from x-ray exposure varies directly as the size of the x-ray dose r. What is the effect on N if r is quadrupled?

55. BIOLOGY In biology there is an approximate rule, called the *bioclimatic rule* for temperate climates, which states that the difference d in time for fruit to ripen (or insects to appear) varies directly as the change in altitude h. If $d = 4$ days when $h = 500$ feet, find d when $h = 2{,}500$ feet.

56. PHYSICS Over a fixed distance d, speed r varies inversely as time t. Police use this relationship to set up speed traps. If in a given speed trap $r = 30$ mph when $t = 6$ seconds, what would be the speed of a car if $t = 4$ seconds?

57. PHYSICS The length L of skid marks of a car's tires (when the brakes are applied) is directly proportional to the square of the speed v of the car. How is the length of skid marks affected by doubling the speed?

58. PHOTOGRAPHY In taking pictures using flashbulbs, the lens opening (f-stop number) N is inversely proportional to the distance d from the object being photographed. What adjustment should you make on the f-stop number if the distance between the camera and the object is doubled?

59. ENGINEERING The total pressure P of the wind on a wall is jointly proportional to the area of the wall A and the square of the velocity of the wind v. If $P = 120$ pounds when $A = 100$ square feet and $v = 20$ miles per hour, find P if $A = 200$ square feet and $v = 30$ miles per hour.

60. **ENGINEERING** The thrust T of a given type of propeller is jointly proportional to the fourth power of its diameter d and the square of the number of revolutions per minute n it is turning. What happens to the thrust if the diameter is doubled and the number of revolutions per minute is cut in half?

61. **PSYCHOLOGY** In early psychological studies on sensory perception (hearing, seeing, feeling, and so on), the question was asked: "Given a certain level of stimulation S, what is the minimum amount of added stimulation ΔS that can be detected?" A German physiologist, E. H. Weber (1795–1878) formulated, after many experiments, the famous law that now bears his name: "The amount of change ΔS that will be just noticed varies directly as the magnitude S of the stimulus."
(A) Write the law as an equation of variation.
(B) If a person lifting weights can just notice a difference of 1 ounce at the 50-ounce level, what will be the least difference she will be able to notice at the 500-ounce level?
(C) Determine the just noticeable difference in illumination a person is able to perceive at 480 candlepower if he is just able to perceive a difference of 1 candlepower at the 60-candlepower level.

62. **PSYCHOLOGY** Psychologists in their study of intelligence often use an index called IQ. IQ varies directly as mental age MA and inversely as chronological age CA (up to the age of 15). If a 12-year-old boy with a mental age of 14.4 has an IQ of 120, what will be the IQ of an 11-year-old girl with a mental age of 15.4?

63. **GEOMETRY** The volume of a sphere varies directly as the cube of its radius r. What happens to the volume if the radius is doubled?

64. **GEOMETRY** The surface area S of a sphere varies directly as the square of its radius r. What happens to the area if the radius is cut in half?

65. **MUSIC** The frequency of vibration of air in an open organ pipe is inversely proportional to the length of the pipe. If the air column in an open 32-foot pipe vibrates 16 times per second (low C), then how fast would the air vibrate in a 16-foot pipe?

66. **MUSIC** The frequency of pitch f of a musical string is directly proportional to the square root of the tension T and inversely proportional to the length l and the diameter d. Write the equation of variation using k as the constant of variation. (It is interesting to note that if pitch depended on only length, then pianos would have to have strings varying from 3 inches to 38 feet.)

CHAPTER 3 Review

3-1 Polynomial Functions and Models

A function that can be written in the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $a_n \neq 0$, is a **polynomial function of degree *n*.** In this chapter, when not specified otherwise, the coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ are complex numbers and the domain of P is the set of complex numbers. A number r is said to be a **zero** (or **root**) of a function $P(x)$ if $P(r) = 0$. The zeros of $P(x)$ are thus the solutions of the equation $P(x) = 0$. The real zeros of $P(x)$ are just the x intercepts of the graph of $P(x)$. A point on a continuous graph that separates an increasing portion from a decreasing portion, or vice versa, is called a **turning point.** If $P(x)$ is a polynomial of degree $n > 0$ with real coefficients, then the graph of $P(x)$:

1. Is continuous for all real numbers

2. Has no sharp corners

3. Has at most n real zeros

4. Has at most $n - 1$ turning points

5. Increases or decreases without bound as $x \to \infty$ and as $x \to -\infty$

The left and right behavior of such a polynomial $P(x)$ is determined by its highest degree or **leading term:** As $x \to \pm\infty$, both $a_n x^n$ and $P(x)$ approach $\pm\infty$, with the sign depending on whether n is even or odd and the sign of a_n.

Many graphing calculators can find polynomial regression models to fit data sets; cubic (third-degree) and quartic (fourth-degree) models are often useful to model data with behavior too complicated to be modeled well with a linear or quadratic model.

3-2 Polynomial Division

For any polynomial $P(x)$ of degree n, we have the following important results:

Division Algorithm

For any other polynomial $D(x) \neq 0$, there are unique polynomials $Q(x)$ (the **quotient**) and $R(x)$ (the **remainder**) so that

$$P(x) = D(x)\,Q(x) + R(x)$$

and the degree of $R(x)$ is less than the degree of $D(x)$. $P(x)$ is called the **dividend,** $D(x)$ the **divisor.**

Remainder Theorem

$P(r) = R$, where R is the numeric remainder when P is divided by $x - r$.

Factor Theorem

$x - r$ is a factor of $P(x)$ if and only if the remainder is zero when P is divided by $x - r$.

Zeros Theorem

$P(x)$ has at most n zeros (recall that n is the degree of P).

Polynomials can be divided using a long-division process that is very similar to long division of numbers. **Synthetic division** is an efficient method for dividing polynomials by linear terms of the form $x - r$. Remember that synthetic division only works when the divisor is of the form $x - r$.

3-3 Real Zeros and Polynomial Inequalities

The following theorems are useful in locating and approximating all real zeros of a polynomial $P(x)$ of degree $n > 0$ with real coefficients, $a_n > 0$:

Upper and Lower Bound Theorem

1. Upper bound: A number $r > 0$ is an upper bound for the real zeros of $P(x)$ if, when $P(x)$ is divided by $x - r$ using synthetic division, all numbers in the quotient row, including the remainder, are nonnegative.
2. Lower bound: A number $r < 0$ is a lower bound for the real zeros of $P(x)$ if, when $P(x)$ is divided by $x - r$ using synthetic division, all numbers in the quotient row, including the remainder, alternate in sign.

Location Theorem

Suppose that a function f is continuous on an interval I that contains numbers a and b. If $f(a)$ and $f(b)$ have opposite signs, then the graph of f has at least one x intercept between a and b.

The **bisection method** uses the location theorem to find smaller and smaller intervals containing an x intercept. It can be used to approximate the real zeros of a polynomial.

When real zeros occur at turning points, the bisection method and the ZERO command on a graphing calculator may not be able to locate the zero. In this case, the MAXIMUM or MINIMUM commands should be used.

Polynomial inequalities can be solved using a method similar to the one used to solve quadratic inequalities in Section 2-7. They can also be solved by finding zeros and then examining the graph of an appropriate polynomial with real coefficients.

3-4 Complex Zeros and Rational Zeros of Polynomials

If $P(x)$ is a polynomial of degree $n > 0$, we have the following important theorems:

Fundamental Theorem of Algebra

$P(x)$ has at least one complex zero that may or may not be real.

***n* Linear Factors Theorem**

$P(x)$ can be factored as a product of n linear factors.

If $P(x)$ is factored as a product of linear factors, the number of linear factors that have zero r is said to be the **multiplicity** of r.

Imaginary Zeros Theorem

Imaginary zeros of polynomials with real coefficients, if they exist, occur in conjugate pairs.

Linear and Quadratic Factors Theorem

If $P(x)$ has real coefficients, then $P(x)$ can be factored as a product of linear factors (with real coefficients) and quadratic factors (with real coefficients and imaginary zeros).

Real Zeros and Polynomials of Odd Degree

If $P(x)$ has odd degree and real coefficients, then the graph of P has at least one x intercept.

Zeros of Even or Odd Multiplicity

Let $P(x)$ have real coefficients:

1. If r is a real zero of $P(x)$ of even multiplicity, then $P(x)$ has a turning point at r and does not change sign at r.
2. If r is a real zero of $P(x)$ of odd multiplicity, then $P(x)$ does not have a turning point at r and changes sign at r.

Rational Zero Theorem

If the rational number b/c, in lowest terms, is a zero of the polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad a_n \neq 0$$

with integer coefficients, then b must be an integer factor of a_0 and c must be an integer factor of a_n.

If $P(x) = (x - r)Q(x)$, then $Q(x)$ is called a **reduced polynomial** for $P(x)$.

3-5 Rational Functions and Inequalities

A function f is a **rational function** if it can be written in the form

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials of degrees m and n, respectively (we assume $p(x)$ and $q(x)$ have no common factor).

The graph of a rational function $f(x)$:

1. Is continuous with the exception of at most n real numbers

2. Has no sharp corners

3. Has at most m real zeros

4. Has at most $m + n - 1$ turning points

5. Has the same left and right behavior as the quotient of the leading terms of $p(x)$ and $q(x)$

The vertical line $x = a$ is a **vertical asymptote** for the graph of $f(x)$ if $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a^+$ or as $x \to a^-$. If a is a zero of the denominator, then the line $x = a$ is a vertical asymptote of the graph of f provided that the numerator and denominator have no common factors. The horizontal line $y = b$ is a **horizontal asymptote** for the graph of $f(x)$ if $f(x) \to b$ as $x \to -\infty$ or as $x \to \infty$.

$$\text{Let } f(x) = \frac{a_m x^m + \cdots + a_1 x + a_0}{b_n x^n + \cdots + b_1 x + b_0},\ a_m \neq 0,\ b_n \neq 0.$$

1. If $m < n$, the line $y = 0$ (the x axis) is a horizontal asymptote.

2. If $m = n$, the line $y = a_m/b_n$ is a horizontal asymptote.

3. If $m > n$, there is no horizontal asymptote.

The line $y = mx + b$ is an **oblique,** or **slant, asymptote** if the graph of f approaches the graph of that line as $x \to \pm\infty$. Slant asymptotes occur when the degree of the numerator is one more than the degree of the denominator, and can be found by long division.

Analyzing and Sketching the Graph of a Rational Function: $f(x) = p(x)/q(x)$

Step 1. *Intercepts.* Find the real solutions of the equation $p(x) = 0$ and use these solutions to plot any x intercepts of the graph of f. Evaluate $f(0)$, if it exists, and plot the y intercept.

Step 2. *Vertical Asymptotes.* Find the real solutions of the equation $q(x) = 0$ and use these solutions to determine the domain of f, the points of discontinuity, and the vertical asymptotes. Sketch any vertical asymptotes as dashed lines.

Step 3. *Horizontal Asymptotes.* Determine whether there is a horizontal asymptote and if so, sketch it as a dashed line.

Step 4. *Complete the Sketch.* Using a graphing calculator graph as an aid and the information determined in steps 1–3, sketch the graph.

Rational inequalities can be solved using a method similar to that for polynomial inequalities. The main difference is that the zeros of the denominator represent potential changes in sign, as well as the zeros of the function.

3-6 Variation and Modeling

Let x and y be variables. The statement:

1. y is directly proportional to x (or **y varies directly as x**) means

$$y = kx$$

for some nonzero constant k.

2. y is inversely proportional to x (or **y varies inversely as x**) means

$$y = \frac{k}{x}$$

for some nonzero constant k.

3. w is jointly proportional to x and y (or **w varies jointly as x and y**) means

$$w = kxy$$

for some nonzero constant k.

In each case the nonzero constant k is called the **constant of proportionality** (or **constant of variation**).

The three basic types of variation also can be combined. For example, Newton's law of gravitation, "The force of attraction F between two objects is jointly proportional to their masses m_1 and m_2 and inversely proportional to the square of the distance d between them" has the equation

$$F = k\frac{m_1 m_2}{d^2}$$

CHAPTER 3

Review Exercises

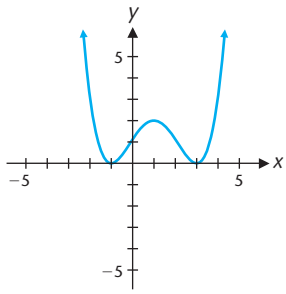
Work through all the problems in this chapter review, and check answers in the back of the book. Answers to all review problems are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

1. Determine whether the function is a polynomial. If it is, state the degree.

(A) $f(x) = 3x^{1/2} - 7x^3 + \dfrac{5}{x}$

(B) $g(x) = -4x^3 + 5x + 11$

2. List the real zeros and turning points, and state the left and right behavior, of the polynomial function that has the indicated graph.



3. Use synthetic division to divide $P(x) = 2x^3 + 3x^2 - 1$ by $D(x) = x + 2$, and write the answer in the form $P(x) = D(x)Q(x) + R$.

4. If $P(x) = x^5 - 4x^4 + 9x^2 - 8$, find $P(3)$ using the remainder theorem and synthetic division.

5. What are the zeros of $P(x) = 3(x - 2)(x + 4)(x + 1)$?

6. Sketch the graph of a polynomial with zeros $x = 5, -3$, and 2, odd degree, and positive leading coefficient.

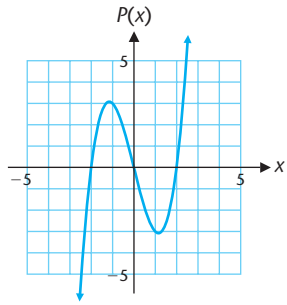
7. Find a polynomial of least degree that has zeros $x = 5, -3$, and 2. Write your answer in both factored form and standard form.

8. If $P(x) = x^2 - 2x + 2$ and $P(1 + i) = 0$, find another zero of $P(x)$.

9. Let $P(x)$ be the polynomial whose graph is shown in the figure at the top of the next column.

(A) Assuming that $P(x)$ has integer zeros and leading coefficient 1, find the lowest-degree equation that could produce this graph.

(B) Describe the left and right behavior of $P(x)$.



10. According to the upper and lower bound theorem, which of the following are upper or lower bounds of the zeros of $P(x) = x^3 - 4x^2 + 2$?

$$-2, -1, 3, 4$$

11. How do you know that $P(x) = 2x^3 - 3x^2 + x - 5$ has at least one real zero between 1 and 2?

12. Write the possible rational zeros for

$$P(x) = x^3 - 4x^2 + x + 6.$$

13. Find all rational zeros for $P(x) = x^3 - 4x^2 + x + 6$.

14. Find the domain and x intercept(s) for:

(A) $f(x) = \dfrac{2x - 3}{x + 4}$ (B) $g(x) = \dfrac{3x}{x^2 - x - 6}$

15. Find the horizontal and vertical asymptotes for the functions in Problem 14.

In Problems 16–21, translate each statement into an equation using k as the constant of proportionality.

16. F is directly proportional to the square root of x.

17. G is jointly proportional to x and the square of y.

18. H is inversely proportional to the cube of z.

19. R varies jointly as the square of x and the square of y.

20. S varies inversely as the square of u.

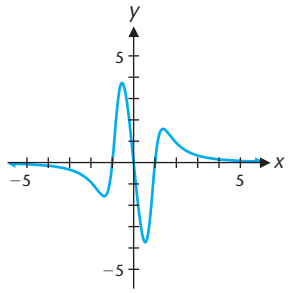
21. T varies directly as v and inversely as w.

22. The amount of light captured by a lens is directly proportional to the surface area of the lens. If a lens is exchanged for a similar one with a larger surface area, how will the amount of light captured change?

23. It has been suggested that the cleanliness of a home is inversely proportional to the number of children that live

there. If this is true, what would be the effect on cleanliness if another child is added to a home?

24. Explain why the graph is not the graph of a polynomial function.



25. Let $P(x) = x^3 - 3x^2 - 3x + 4$.

(A) Graph $P(x)$ and describe the graph verbally, including the number of x intercepts, the number of turning points, and the left and right behavior.

(B) Approximate the largest x intercept to two decimal places.

26. If $P(x) = 8x^4 - 14x^3 - 13x^2 - 4x + 7$, find $Q(x)$ and R such that $P(x) = (x - \frac{1}{4})Q(x) + R$. What is $P(\frac{1}{4})$?

27. Divide, using long division:
$(x^4 - 7x^3 - 5x + 1) \div (x^2 + 2x)$

28. If $P(x) = 4x^3 - 8x^2 - 3x - 3$, find $P(-\frac{1}{2})$ using the remainder theorem and synthetic division.

29. Use the quadratic formula and the factor theorem to factor $P(x) = x^2 - 2x - 1$.

30. Is $x + 1$ a factor of $P(x) = 9x^{26} - 11x^{17} + 8x^{11} - 5x^4 - 7$? Explain, without dividing or using synthetic division.

31. Given that $x = 2$ is a zero of $P(x) = 2x^3 - 9x^2 + 3x + 14$, find all other zeros.

32. Determine all rational zeros of $P(x) = 2x^3 - 3x^2 - 18x - 8$.

33. Factor the polynomial in Problem 32 into linear factors.

34. Find all rational zeros of $P(x) = x^3 - 3x^2 + 5$.

35. Find all zeros (rational, irrational, and imaginary) exactly for $P(x) = 2x^4 - x^3 + 2x - 1$.

36. Factor the polynomial in Problem 35 into linear factors.

37. If $P(x) = (x - 1)^2(x + 1)^3(x^2 - 1)(x^2 + 1)$, what is its degree? Write the zeros of $P(x)$, indicating the multiplicity of each if greater than 1.

38. Factor $P(x) = x^4 + 5x^2 - 36$ in two ways:

(A) As a product of linear factors (with real coefficients) and quadratic factors (with real coefficients and imaginary zeros)

(B) As a product of linear factors with complex coefficients

39. Let $P(x) = x^5 - 10x^4 + 30x^3 - 20x^2 - 15x - 2$.

(A) Approximate the zeros of $P(x)$ to two decimal places and state the multiplicity of each zero.

(B) Can any of these zeros be approximated with the bisection method? The MAXIMUM command? The MINIMUM command? Explain.

40. Let $P(x) = x^4 - 2x^3 - 30x^2 - 25$.

(A) Use the upper and lower bound theorem to find the smallest positive and largest negative integers that are upper and lower bounds, respectively, for the real zeros of $P(x)$.

(B) If $(k, k + 1)$, k an integer, is the interval containing the largest real zero of $P(x)$, determine how many additional intervals are required in the bisection method to approximate this zero to one decimal place.

(C) Approximate the real zeros of $P(x)$ to two decimal places.

41. Let $f(x) = \dfrac{x - 1}{2x + 2}$.

(A) Find the domain and the intercepts for f.

(B) Find the vertical and horizontal asymptotes for f.

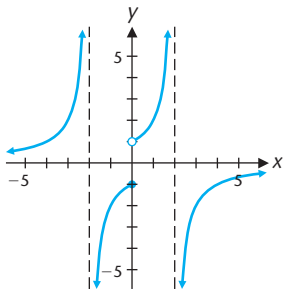
(C) Sketch a graph of f. Draw vertical and horizontal asymptotes with dashed lines.

42. Solve the polynomial inequality exactly: $3x^3 \geq 4x^2 + 15x$

43. Solve each polynomial inequality to three decimal places:

(A) $x^3 - 5x + 4 < 0$ (B) $x^3 - 5x + 4 < 2$

44. Explain why the graph is not the graph of a rational function.



45. B varies inversely as the square root of c. If $B = 5$ when $c = 4$, find B when $c = 25$.

46. D is jointly proportional to x and y. If $D = 10$ when $x = 3$ and $y = 2$, find D when $x = 9$ and $y = 8$.

47. Use synthetic division to divide $P(x) = x^3 + 3x + 2$ by $[x - (1 + i)]$, and write the answer in the form $P(x) = D(x)Q(x) + R$.

48. Find a polynomial of lowest degree with leading coefficient 1 that has zeros $-\frac{1}{2}$ (multiplicity 2), -3, and 1 (multiplicity 3). (Leave the answer in factored form.) What is the degree of the polynomial?

49. Repeat Problem 48 for a polynomial $P(x)$ with zeros -5, $2 - 3i$, and $2 + 3i$.

50. Find all zeros (rational, irrational, and imaginary) exactly for $P(x) = 2x^5 - 5x^4 - 8x^3 + 21x^2 - 4$.

51. Factor the polynomial in Problem 50 into linear factors.

52. Let $P(x) = x^4 + 16x^3 + 47x^2 - 137x + 73$. Approximate (to two decimal places) the x intercepts and the local extrema.

53. What is the minimal degree of a polynomial $P(x)$, given that $P(-1) = -4$, $P(0) = 2$, $P(1) = -5$, and $P(2) = 3$? Justify your conclusion.

54. If $P(x)$ is a cubic polynomial with integer coefficients and if $1 + 2i$ is a zero of $P(x)$, can $P(x)$ have an irrational zero? Explain.

55. The solutions to the equation $x^3 - 27 = 0$ are the cube roots of 27.

(A) How many cube roots of 27 are there?

(B) 3 is obviously a cube root of 27; find all others.

56. Let $P(x) = x^4 + 2x^3 - 500x^2 - 4{,}000$.

(A) Use the upper and lower bound theorem to find the smallest positive integer multiple of 10 and the largest negative integer multiple of 10 that are upper and lower bounds, respectively, for the real zeros of $P(x)$.

(B) Approximate the real zeros of $P(x)$ to two decimal places.

57. Graph

$$f(x) = \frac{x^2 + 2x + 3}{x + 1}$$

Indicate any vertical, horizontal, or oblique asymptotes with dashed lines.

58. Use a graphing calculator to find any horizontal asymptotes for

$$f(x) = \frac{2x}{\sqrt{x^2 + 3x + 4}}$$

59. Solve each rational inequality to three decimal places:

(A) $\dfrac{x^2 - 3}{x^3 - 3x + 1} \le 0$ (B) $\dfrac{x^2 - 3}{x^3 - 3x + 1} > \dfrac{5}{x^2}$

60. If $P(x) = x^3 - x^2 - 5x + 4$, determine the number of real zeros of $P(x)$ and explain why $P(x)$ has no rational zeros.

61. Give an example of a rational function $f(x)$ that satisfies the following conditions: the real zeros of f are -3, 0, and 2; the vertical asymptotes of f are the lines $x = -1$ and $x = 4$; and the line $y = 5$ is a horizontal asymptote.

APPLICATIONS

62. **PROFIT** An enterprising college student earns extra money by setting up a small coffee stand in the student union. She wants to figure out how many hours per week she should work the stand, so she tries a variety of hours and keeps track of profit, then uses regression analysis to model her results. She finds that the function $P(x) = -0.167x^3 + 2.50x^2 + 31.7x - 100$ $(0 \le x \le 20)$ describes her weekly profit in dollars, where x is hours per week the stand is open.

(A) How many hours should she work to break even?

(B) How many hours should she work to ensure that her weekly profit is at least \$325?

63. **PHYSICS** The centripetal force F of a body moving in a circular path at constant speed is inversely proportional to the radius r of the path. What happens to F if r is doubled?

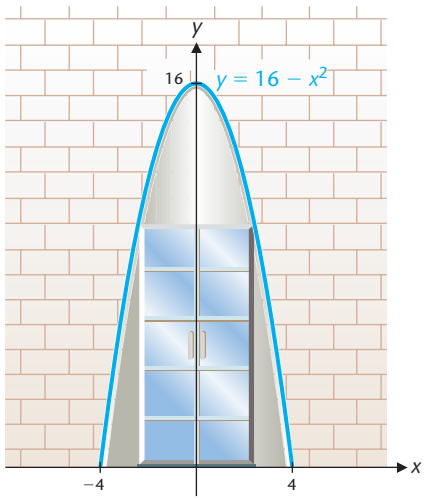
64. **PHYSICS** The Maxwell–Boltzmann equation says that the average velocity v of a molecule varies directly as the square root of the absolute temperature T and inversely as the square root of its molecular weight w. Write the equation of variation using k as the constant of variation.

65. **WORK** The amount A of work completed varies jointly as the number of workers W used and the time t they spend. If 10 workers can finish a job in 8 days, how long will it take 4 workers to do the same job?

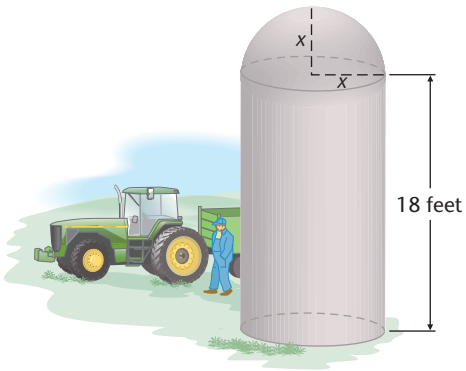
66. **SIMPLE INTEREST** The simple interest I earned in a given time is jointly proportional to the principal p and the interest rate r. If \$100 at 4% interest earns \$8, how much will \$150 at 3% interest earn in the same period?

In Problems 67–70, express the solutions as the roots of a polynomial equation of the form $P(x) = 0$. Find rational solutions exactly and irrational solutions to three decimal places.

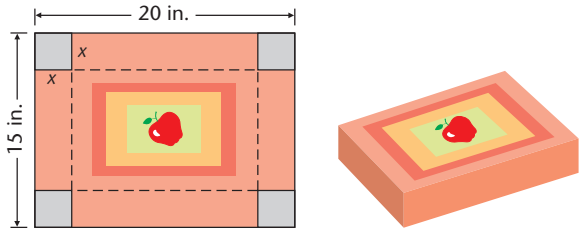
67. **ARCHITECTURE** An entryway is formed by placing a rectangular door inside an arch in the shape of the parabola with graph $y = 16 - x^2$, x and y in feet (see the figure). If the area of the door is 48 square feet, find the dimensions of the door.



68. **CONSTRUCTION** A grain silo is formed by attaching a hemisphere to the top of a right circular cylinder (see the figure). If the cylinder is 18 feet high and the volume of the silo is 486π cubic feet, find the common radius of the cylinder and the hemisphere.



69. **MANUFACTURING** A box is to be made out of a piece of cardboard that measures 15 by 20 inches. Squares, x inches on a side, will be cut from each corner, and then the ends and sides will be folded up (see the figure). Find the value of x that would result in a box with a volume of 300 cubic inches.



70. **GEOMETRY** Find all points on the graph of $y = x^2$ that are three units from the point (1, 4).

MODELING AND DATA ANALYSIS

71. **ADVERTISING** A chain of appliance stores uses television ads to promote the sale of refrigerators. Analyzing past records produced the data in Table 1, where x is the number of ads placed monthly and y is the number of refrigerators sold that month.

(A) Find a cubic regression equation for these data using the number of ads as the independent variable.

(B) Estimate (to the nearest integer) the number of refrigerators that would be sold if 15 ads are placed monthly.

(C) Estimate (to the nearest integer) the number of ads that should be placed to sell 750 refrigerators monthly.

Table 1

Number of Ads x	Number of Refrigerators y
10	270
20	430
25	525
30	630
45	890
48	915

72. **WOMEN IN THE WORKFORCE** It is reasonable to conjecture from the data given in Table 2 that many Japanese women tend to leave the workforce to marry and have children, but then reenter the workforce when the children are grown.

(A) Explain why you might expect cubic regression to provide a better fit to the data than linear or quadratic regression.

(B) Find a cubic regression model for these data using age as the independent variable.

(C) Use the regression equation to estimate (to the nearest year) the ages at which 65% of the women are in the workforce.

Table 2 Women in the Workforce in Japan (1997)

Age	Percentage of Women Employed
22	73
27	65
32	56
37	63
42	71
47	72
52	68
57	59
62	42

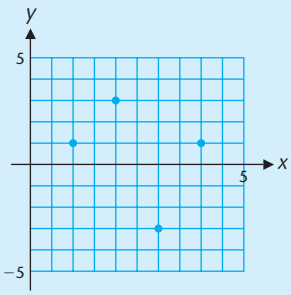
CHAPTER 3

››› GROUP ACTIVITY Interpolating Polynomials

Given two points in the plane, we can use the point–slope form of the equation of a line to find a polynomial whose graph passes through these two points. How can we proceed if we are given more than two points? For example, how can we find the equation of a polynomial $P(x)$ whose graph passes through the points listed in Table 1 and graphed in Figure 1?

Table 1

x	1	2	3	4
$P(x)$	1	3	−3	1



› **Figure 1**

The key to solving this problem is to write the unknown polynomial $P(x)$ in the following special form:

$$P(x) = a_0 + a_1(x - 1) + a_2(x - 1)(x - 2) + a_3(x - 1)(x - 2)(x - 3) \tag{1}$$

Because the graph of $P(x)$ is to pass through each point in Table 1, we can substitute each value of x in equation (1) to determine the coefficients a_0, a_1, a_2, and a_3. First we evaluate equation (1) at $x = 1$ to determine a_0:

$$\begin{aligned} 1 &= P(1) \\ &= a_0 \end{aligned}$$

All other terms in equation (1) are 0 when $x = 1$.

Using this value for a_0 in equation (1) and evaluating at $x = 2$, we have

$$3 = P(2) = 1 + a_1(1)$$

All other terms are 0.

$$2 = a_1$$

Continuing in this manner, we have

$$\begin{aligned} -3 &= P(3) = 1 + 2(2) + a_2(2)(1) \\ -8 &= 2a_2 \\ -4 &= a_2 \\ 1 &= P(4) = 1 + 2(3) - 4(3)(2) + a_3(3)(2)(1) \\ 18 &= 6a_3 \\ 3 &= a_3 \end{aligned}$$

We have now evaluated all the coefficients in equation (1) and can write

$$P(x) = 1 + 2(x - 1) - 4(x - 1)(x - 2) + 3(x - 1)(x - 2)(x - 3) \tag{2}$$

If we expand the products in equation (2) and collect like terms, we can express $P(x)$ in the more conventional form (verify this):

$$P(x) = 3x^3 - 22x^2 + 47x - 27$$

(A) To check these calculations, evaluate $P(x)$ at $x = 1, 2, 3$, and 4 and compare the results with Table 1. Then add the graph of $P(x)$ to Figure 1.
(B) Write a verbal description of the special form of $P(x)$ in equation (1).

In general, given a set of $n + 1$ points:

x	x_0	x_1	$\cdots$	x_n
y	y_0	y_1	$\cdots$	y_n

the **interpolating polynomial** for these points is the polynomial $P(x)$ of degree less than or equal to n that satisfies $P(x_k) = y_k$ for $k = 0, 1, \ldots, n$. The **general form** of the interpolating polynomial is

$$P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1) \cdot \cdots \cdot (x - x_{n-1})$$

(C) Summarize the procedure for using the points in the table to find the coefficients in the general form.
(D) Give an example to show that the interpolating polynomial can have degree strictly less than n.
(E) Could there be two different polynomials of degree less than or equal to n whose graph passes through the given $n + 1$ points? Justify your answer.
(F) Find the interpolating polynomial for each of Tables 2 and 3. Check your answers by evaluating the polynomial, and illustrate by graphing the points in the table and the polynomial in the same viewing window.

Table 2

x	-1	0	1	2
y	5	3	3	11

Table 3

x	-2	-1	0	1	2
y	-3	0	5	0	-3

(G) The student in Problem 62 of this chapter's Review Exercises recorded the data relating hours worked and weekly profit shown in Table 4.

Table 4 Weekly Profit from a Coffee Stand Open x Hours

x (hours)	0	5	10	15
P(\$)	-100	100	300	375

Find the interpolating polynomial for this data, then use a graphing calculator to find the cubic regression polynomial. Are they the same? Do they agree with the polynomial in Problem 62?

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CHAPTER

4

Exponential and Logarithmic Functions

MOST of the functions we have worked with so far have been polynomial or rational functions, with a few others involving roots. Functions that can be expressed in terms of addition, subtraction, multiplication, division, and roots of variables and constants are called *algebraic functions.* In Chapter 4 we will learn about *exponential and logarithmic functions.* These functions are not algebraic; they belong to the class of *transcendental functions.* Exponential and logarithmic functions are used to model a surprisingly wide variety of real-world phenomena: growth of populations of people, animals, and bacteria; radioactive decay; epidemics; and magnitudes of sounds and earthquakes. These and many other applications will be studied in this chapter.

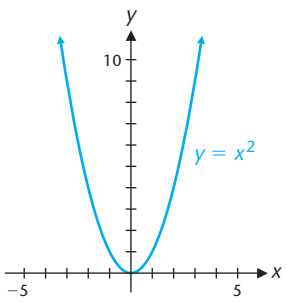
OUTLINE

4-1 Exponential Functions
4-2 Exponential Models
4-3 Logarithmic Functions
4-4 Logarithmic Models
4-5 Exponential and Logarithmic Equations
Chapter 4 Review
Chapter 4 Group Activity: Comparing Regression Models
Cumulative Review Chapters 3 and 4

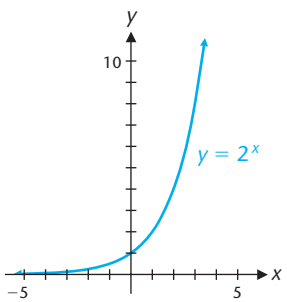
4-1 Exponential Functions

- Defining Exponential Functions
- Analyzing Graphs of Exponential Functions
- Additional Properties of Exponential Functions
- The Exponential Function with Base e
- Calculating Compound Interest
- Calculating Interest Compounded Continuously

Many of the functions we've studied so far have included exponents. But in every case, the exponent was a constant, and the base was often a variable. In this section, we will reverse those roles. In an *exponential function,* the variable appears in an exponent. As we'll see, this has a significant effect on the properties and graphs of these functions. A review of the basic properties of exponents in Appendix A, Section A-2, would be very helpful before moving on.



(a)



(b)

Figure 1

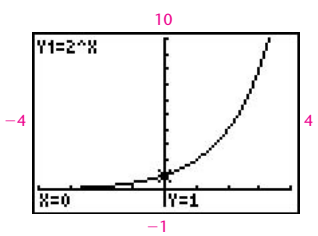


Figure 2

Defining Exponential Functions

Let's start by noting that the functions f and g given by

$$f(x) = 2^x \qquad \text{and} \qquad g(x) = x^2$$

are not the same function. Whether a variable appears as an exponent with a constant base or as a base with a constant exponent makes a big difference. The function g is a quadratic function, which we have already discussed. The function f is an *exponential function*.

The graphs of f and g are shown in Figure 1. As expected, they are very different.

We know how to define the values of 2^x for many types of inputs. For positive integers, it's simply repeated multiplication:

$$2^2 = 2 \cdot 2 = 4; \quad 2^3 = 2 \cdot 2 \cdot 2 = 8; \quad 2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$$

For negative integers, we use properties of negative exponents:

$$2^{-1} = \frac{1}{2}; \quad 2^{-2} = \frac{1}{2^2} = \frac{1}{4}; \quad 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

For rational numbers, a calculator comes in handy:

$$2^{\frac{1}{2}} = \sqrt{2} \approx 1.4; \quad 2^{\frac{3}{2}} = \sqrt{2^3} \approx 2.8; \quad 2^{\frac{9}{4}} = \sqrt[4]{2^9} \approx 4.8$$

A graphing calculator can be used to obtain the graph in Figure 1(b) [see Fig. 2].

The only catch is that we don't know how to define 2^x for *all* real numbers. For example, what does

$$2^{\sqrt{2}}$$

mean? Your calculator can give you a decimal approximation, but where does it come from? That question is not easy to answer at this point. In fact, a precise definition of $2^{\sqrt{2}}$ must wait for more advanced courses. For now, we will simply state that for any positive real number b, the expression b^x is defined for all real values of x, and the output is a real number as well. This enables us to draw the continuous graph for $f(x) = 2^x$ in Figure 1. In Problems 85 and 86 in Exercises 4-1, we will explore a method for defining b^x for irrational x values like $\sqrt{2}$.

› DEFINITION 1 Exponential Function

The equation

$$f(x) = b^x \qquad b > 0, b \neq 1$$

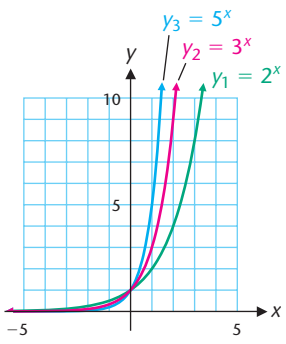
defines an **exponential function** for each different constant b, called the **base.** The independent variable x may assume any real value.

The domain of f is the set of all real numbers, and it can be shown that the range of f is the set of all positive real numbers. We require the base b to be positive to avoid imaginary numbers such as $(-2)^{1/2}$. Problems 57 and 58 in Exercises 4-1 explore why $b = 0$ and $b = 1$ are excluded.

› Analyzing Graphs of Exponential Functions

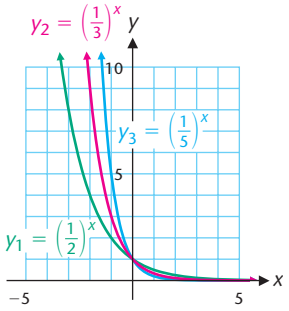
››› EXPLORE-DISCUSS 1

Compare the graphs of $f(x) = 3^x$ and $g(x) = 2^x$ by graphing both functions in the same viewing window. Find all points of intersection of the graphs. For which values of x is the graph of f above the graph of g? Below the graph of g? Are the graphs of f and g close together as $x \to \infty$? As $x \to -\infty$? Discuss.



› **Figure 3** $y = b^x$ for $b = 2, 3, 5$.

The graphs of $y = b^x$ for $b = 2, 3$, and 5 are shown in Figure 3. Note that all three have the same basic shape, and pass through the point (0, 1). Also, the x axis is a horizontal asymptote for each graph, but only as $x \to -\infty$. The main difference between the graphs is their steepness.



› **Figure 4** $y = b^x$ for $b = \frac{1}{2}, \frac{1}{3}, \frac{1}{5}$.

Next, let's look at the graphs of $y = b^x$ for $b = \frac{1}{2}, \frac{1}{3}$, and $\frac{1}{5}$ (Fig. 4). Again, all three have the same basic shape, pass through (0, 1), and have a horizontal asymptote $y = 0$, but we can see that for $b < 1$, the asymptote is only as $x \to \infty$. In general, for bases less than 1, the graph is a reflection through the y axis of the graphs for bases greater than 1.

The graphs in Figures 3 and 4 suggest that the graphs of exponential functions have the properties listed in Theorem 1, which we state without proof.

› THEOREM 1 Properties of Graphs of Exponential Functions

Let $f(x) = b^x$ be an exponential function, $b > 0, b \neq 1$. Then the graph of $f(x)$:

1. Is continuous for all real numbers
2. Has no sharp corners
3. Passes through the point (0, 1)
4. Lies above the x axis, which is a horizontal asymptote either as $x \to \infty$ or $x \to -\infty$, but not both
5. Increases as x increases if $b > 1$; decreases as x increases if $0 < b < 1$
6. Intersects any horizontal line at most once (that is, f is one-to-one)

These properties indicate that the graphs of exponential functions are distinct from the graphs we have already studied. (Actually, Property 4 is enough to ensure that graphs of exponential functions are different from graphs of polynomials and rational functions.) Property 6 is important because it guarantees that exponential functions have inverses. Those inverses, called *logarithmic functions,* are the subject of Section 4-3.

To begin a study of graphing exponentials, it's helpful to sketch a graph or two by hand after plotting points.

EXAMPLE 1 Drawing the Graph of an Exponential Function

Sketch the graph of each function after plotting at least seven points. Then confirm your result with a graphing calculator.

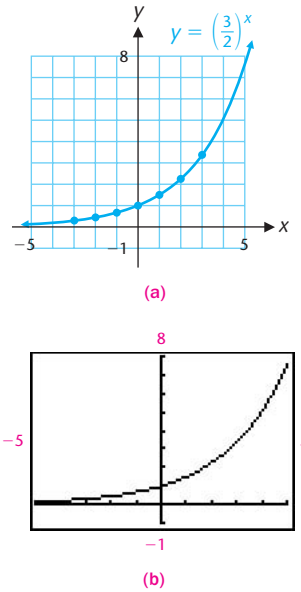
(A) $f(x) = \left(\frac{3}{2}\right)^x$ (B) $g(x) = \left(\frac{2}{3}\right)^x$

SOLUTIONS

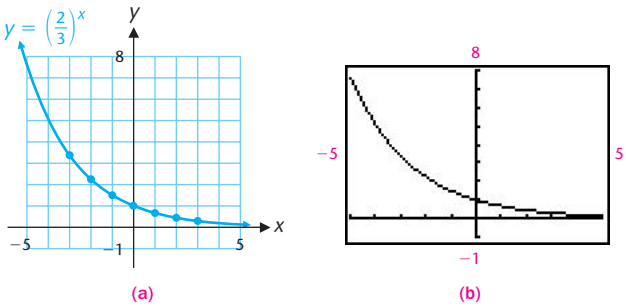
Make a table of values for f and g.

x	$\left(\frac{3}{2}\right)^x$	$\left(\frac{2}{3}\right)^x$
-3	$\left(\frac{3}{2}\right)^{-3} = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$	$\left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$
-2	$\left(\frac{3}{2}\right)^{-2} = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$	$\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$
-1	$\left(\frac{3}{2}\right)^{-1} = \left(\frac{2}{3}\right)^1 = \frac{2}{3}$	$\left(\frac{2}{3}\right)^{-1} = \left(\frac{3}{2}\right)^1 = \frac{3}{2}$
0	$\left(\frac{3}{2}\right)^0 = 1$	$\left(\frac{2}{3}\right)^0 = 1$
1	$\left(\frac{3}{2}\right)^1 = \frac{3}{2}$	$\left(\frac{2}{3}\right)^1 = \frac{2}{3}$
2	$\left(\frac{3}{2}\right)^2 = \frac{9}{4}$	$\left(\frac{2}{3}\right)^2 = \frac{4}{9}$
3	$\left(\frac{3}{2}\right)^3 = \frac{27}{8}$	$\left(\frac{2}{3}\right)^3 = \frac{8}{27}$

The hand sketches are shown in Figures 5 and 6, along with the corresponding graphing calculator graph.

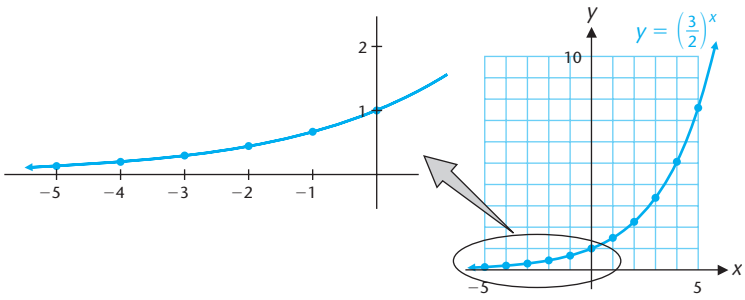


› Figure 5



› Figure 6

Notice that the outputs for $x < 0$ in Figure 5 and for $x > 0$ in Figure 6 get so small that it's hard to distinguish the graph from the x axis. Property 4 in Theorem 1 indicates that the graph of an exponential function is always above the x axis, and approaches height zero as $x \to \infty$ or $x \to -\infty$. (Zooming in on the graph, as in Figure 7, illustrates the behavior a bit better.)



› Figure 7

MATCHED PROBLEM 1

Sketch the graph of each function after plotting at least seven points. Then confirm your result with a graphing calculator.

(A) $f(x) = \left(\frac{3}{4}\right)^x$ (B) $g(x) = \left(\frac{4}{3}\right)^x$

››› EXPLORE-DISCUSS 2

Examine the graphs of $f(x) = (\frac{3}{2})^x$ and $g(x) = (\frac{2}{3})^x$ from Example 1.

(A) What is the relationship between the graphs?

(B) Rewrite $(\frac{2}{3})^x$ as an exponential with base $\frac{3}{2}$. How does this confirm your answer to part A?

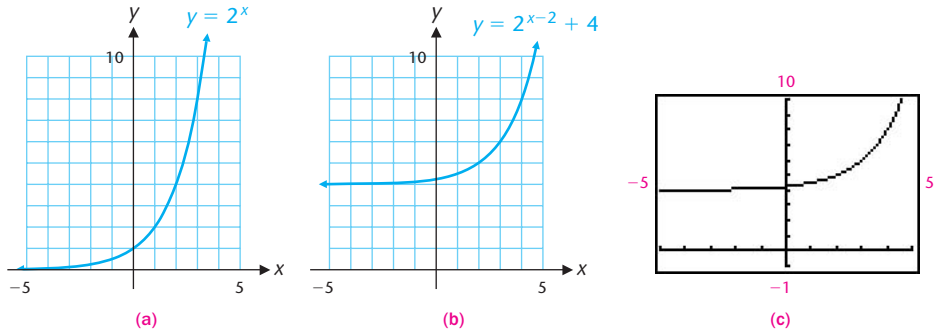
We can also use our knowledge of transformations to draw graphs of more complicated functions involving exponentials.

EXAMPLE 2 Drawing the Graph of an Exponential Function

Use transformations of $y = 2^x$ to graph $f(x) = 2^{x-2} + 4$.

SOLUTION

We start with a graph of $y = 2^x$ [Fig. 8(a)], then shift 2 units right and 4 units up [Fig. 8(b)]. A graphing calculator confirms our result [Fig. 8(c)]. Note that in this case, $y = 4$ is a horizontal asymptote.



› Figure 8

MATCHED PROBLEM 2

Use transformations of $y = (\frac{1}{2})^x$ to graph $f(x) = (\frac{1}{2})^{x+1} - 3$.

› Additional Properties of Exponential Functions

The properties of exponents you should be familiar with (see Appendix A, Section A-2) are often stated in terms of exponents that are rational numbers. But we're considering irrational exponents as well in defining exponential functions. Fortunately, these properties still apply. We will summarize the key properties we need in this chapter, and add two other useful properties.

› EXPONENTIAL FUNCTION PROPERTIES

For a and b positive, $a \neq 1$, $b \neq 1$, and for any real numbers x and y:

1. Exponent laws:

$$a^x a^y = a^{x+y} \qquad (a^x)^y = a^{xy} \qquad (ab)^x = a^x b^x$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \frac{a^x}{a^y} = a^{x-y} \qquad \frac{2^{5x}}{2^{7x}} \boxed{= 2^{5x-7x}}^{*} = 2^{-2x}$$

2. $a^x = a^y$ if and only if $x = y$. If $6^x = 6^3$, then $x = 3$.

3. For $x \neq 0$, $a^x = b^x$ if and only if $a = b$. If $a^4 = 3^4$, then $a = 3$.

*The dashed "think boxes" are used to enclose steps that may be performed mentally.

Property 2 is another way to express the fact that the exponential function $f(x) = a^x$ is one-to-one (see Property 6 of Theorem 1). Because all exponential functions pass through the point (0, 1) (see Property 3 of Theorem 1), property 3 indicates that the graphs of exponential functions with different bases do not intersect at any other points.

EXAMPLE 3 Using Exponential Function Properties

Find all solutions to $2^{10x-1} = 2^{5-2x}$.

SOLUTIONS

Algebraic Solution

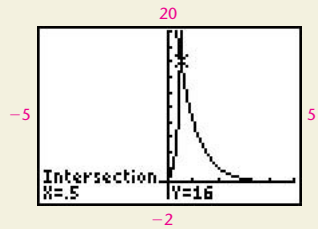
According to Property 2, $2^{10x-1} = 2^{5-2x}$ implies that

$$\begin{aligned} 10x - 1 &= 5 - 2x \\ 12x &= 6 \\ x &= \tfrac{6}{12} = \tfrac{1}{2} \end{aligned}$$

Check: $2^{10(1/2)-1} = 2^4$; $2^{5-2(1/2)} = 2^4$

Graphical Solution

Graph $y_1 = 2^{10x-1}$ and $y_2 = 2^{5-2x}$, then use the INTERSECT command to obtain $x = 0.5$ (Fig. 9).



› Figure 9

MATCHED PROBLEM 3

Find all solutions to $3^{3+y} = 3^{4y-9}$.

EXAMPLE 4 Using Exponential Function Properties

Find all solutions to $4^{x-3} = 8$.

SOLUTIONS

Algebraic Solution

Notice that the two bases, 4 and 8, can both be written as a power of 2. This will enable us to use Property 2 to equate exponents.

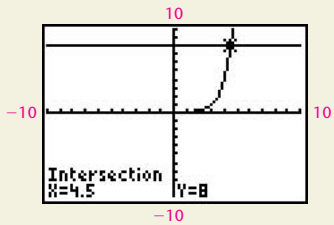
$$\begin{aligned} 4^{x-3} &= 8 && \text{Express 4 and 8 as powers of 2.} \\ (2^2)^{x-3} &= 2^3 && (a^x)^y = a^{xy} \\ 2^{2x-6} &= 2^3 && \text{Property 2} \\ 2x - 6 &= 3 && \text{Add 6 to both sides.} \\ 2x &= 9 && \text{Divide both sides by 2.} \\ x &= \tfrac{9}{2} \end{aligned}$$

CHECK

$$4^{(9/2)-3} = 4^{3/2} = (\sqrt{4})^3 = 2^3 \overset{\checkmark}{=} 8$$

Graphical Solution

Graph $y_1 = 4^{x-3}$ and $y_2 = 8$. Use the INTERSECT command to obtain $x = 4.5$ (Fig. 10).



› Figure 10

MATCHED PROBLEM 4

Solve $27^{x+1} = 9$ for x.

The Exponential Function with Base e

Surprisingly, among the exponential functions it is not the function $g(x) = 2^x$ with base 2 or the function $h(x) = 10^x$ with base 10 that is used most frequently in mathematics. Instead, the most commonly used base is a number that you may not be familiar with.

EXPLORE-DISCUSS 3

(A) Calculate the values of $[1 + (1/x)]^x$ for $x = 1, 2, 3, 4$, and 5. Are the values increasing or decreasing as x gets larger?

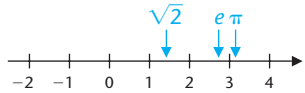
(B) Graph $y = [1 + (1/x)]^x$ and discuss the behavior of the graph as x increases without bound.

Table 1

x	$\left(1 + \frac{1}{x}\right)^x$
1	2
10	2.593 74 …
100	2.704 81 …
1,000	2.716 92 …
10,000	2.718 14 …
100,000	2.718 27 …
1,000,000	2.718 28 …

By calculating the value of $[1 + (1/x)]^x$ for larger and larger values of x (Table 1), it looks like $[1 + (1/x)]^x$ approaches a number close to 2.7183. In a calculus course, we can show that as x increases without bound, the value of $[1 + (1/x)]^x$ approaches an irrational number that we call e. Just as irrational numbers such as π and $\sqrt{2}$ have unending, nonrepeating decimal representations, e also has an unending, nonrepeating decimal representation. To 12 decimal places,

$$e = 2.718\,281\,828\,459$$



Don't let the symbol "e" intimidate you! It's just a number.

Exactly who discovered e is still being debated. It is named after the great Swiss mathematician Leonhard Euler (1707–1783), who computed e to 23 decimal places using $[1 + (1/x)]^x$.

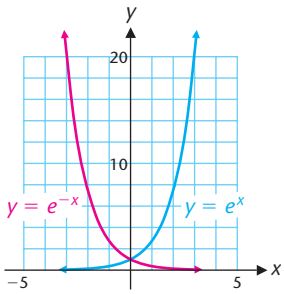
The constant e turns out to be an ideal base for an exponential function because in calculus and higher mathematics many operations take on their simplest form using this base. This is why you will see e used extensively in expressions and formulas that model real-world phenomena.

DEFINITION 2 Exponential Function with Base e

For x a real number, the equation

$$f(x) = e^x$$

defines the **exponential function with base *e*.**



› **Figure 11** Exponential functions.

The exponential function with base e is used so frequently that it is often referred to as *the* exponential function. The graphs of $y = e^x$ and $y = e^{-x}$ are shown in Figure 11.

››› EXPLORE-DISCUSS 4

(A) Graph $y_1 = e^x, y_2 = e^{0.5x}$, and $y_3 = e^{2x}$ in the same viewing window. How do these graphs compare with the graph of $y = b^x$ for $b > 1$?

(B) Graph $y_1 = e^{-x}, y_2 = e^{-0.5x}$, and $y_3 = e^{-2x}$ in the same viewing window. How do these graphs compare with the graph of $y = b^x$ for $0 < b < 1$?

(C) Use the properties of exponential functions to show that all of these functions can be written in the form $y = b^x$.

EXAMPLE 5 Analyzing an Exponential Graph

Describe the graph of $f(x) = 4 - e^{x/2}$, including x and y intercepts, increasing and decreasing properties, and horizontal asymptotes. Round any approximate values to two decimal places.

SOLUTION

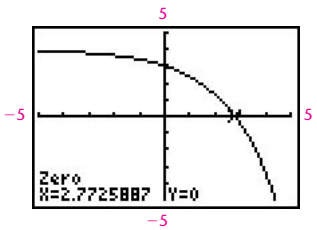
A graphing calculator graph of f is shown in Figure 12(a).

y intercept: $f(0) = 4 - e^0 = 4 - 1 = 3$

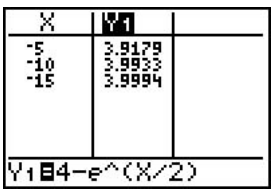
x intercept: $x \approx 2.77$

Graph is decreasing for all x.

Horizontal asymptote: We can write the exponential function $e^{x/2}$ as $(e^{1/2})^x$, and $e^{1/2} \approx 1.65 > 1$, so our earlier study of exponential graphs indicates that $e^{x/2} \to 0$ as $x \to -\infty$. But then, $4 - e^{x/2} \to 4$ as $x \to -\infty$, and $y = 4$ is a horizontal asymptote for the graph of f. The table in Figure 12(b) supports this conclusion.



(a)



(b)

› **Figure 12** $f(x) = 4 - e^{x/2}$.

MATCHED PROBLEM 5

Describe the graph of $f(x) = 2e^{x/2} - 5$, including x and y intercepts, increasing and decreasing properties, and horizontal asymptotes. Round any approximate values to two decimal places.

We will study a wide variety of applications of exponential functions in the next section. For now, it's a good start to examine how exponential functions apply very naturally to the world of finance, something relevant to almost everyone.

Calculating Compound Interest

The fee paid to use someone else's money is called **interest.** It is usually computed as a percentage, called the **interest rate,** of the original amount (or **principal**) over a given period of time. At the end of the payment period, the interest paid is usually added to the principal amount, so the interest in the next period is earned on both the original amount, as well as the interest previously earned. Interest paid on interest previously earned and reinvested in this manner is called **compound interest.**

Suppose you deposit \$1,000 in a bank that pays 8% interest compounded semiannually. How much will be in your account at the end of 2 years? "Compounded semiannually" means that the interest is paid to your account at the end of each 6-month period, and the interest will in turn earn more interest. To calculate the **interest rate per period,** we take the annual rate r, 8% (or 0.08), and divide by the number m of compounding periods per year, in this case 2. If A_1 represents the amount of money in the account after one compounding period (six months), then

Principal + 4% of principal

$$A_1 = \$1{,}000 + \$1{,}000\left(\frac{0.08}{2}\right) \qquad \text{Factor out \$1,000.}$$
$$= \$1{,}000(1 + 0.04)$$

We will next use A_2, A_3, and A_4 to represent the amounts at the end of the second, third, and fourth periods. (Note that the amount we're looking for is A_4.) A_2 is calculated by multiplying the amount at the beginning of the second compounding period (A_1) by 1.04.

$$A_2 = A_1(1 + 0.04) \qquad \text{Substitute our expression for } A_1.$$
$$= [\$1{,}000(1 + 0.04)](1 + 0.04) \qquad \text{Multiply.}$$
$$= \$1{,}000(1 + 0.04)^2 \qquad P\left(1 + \frac{r}{m}\right)^2$$
$$A_3 = A_2(1 + 0.04) \qquad \text{Substitute our expression for } A_2.$$
$$= [\$1{,}000(1 + 0.04)^2](1 + 0.04) \qquad \text{Multiply.}$$
$$= \$1{,}000(1 + 0.04)^3 \qquad P\left(1 + \frac{r}{m}\right)^3$$
$$A_4 = A_3(1 + 0.04) \qquad \text{Substitute our expression for } A_3.$$
$$= [\$1{,}000(1 + 0.04)^3](1 + 0.04) \qquad \text{Multiply.}$$
$$= \$1{,}000(1 + 0.04)^4 \qquad P\left(1 + \frac{r}{m}\right)^4$$
$$= \$1{,}169.86$$

What do you think the savings and loan will owe you at the end of 6 years (12 compounding periods)? If you guessed

$$A = \$1{,}000(1 + 0.04)^{12}$$

you have observed a pattern that is generalized in the following compound interest formula:

› COMPOUND INTEREST

If a **principal** P is invested at an annual **rate** r compounded m times a year, then the **amount** A in the account at the end of n compounding periods is given by

$$A = P\left(1 + \frac{r}{m}\right)^n$$

Note that the annual rate r must be expressed in decimal form, and that $n = mt$, where t is years.

EXAMPLE 6 Compound Interest

If you deposit \$5,000 in an account paying 9% compounded daily, how much will you have in the account in 5 years? Compute the answer to the nearest cent.

SOLUTIONS

Interest compounded daily will be compounded 365 times per year.*

Algebraic Solution

We use the compound interest formula with $P = 5{,}000$, $r = 0.09$, $m = 365$, and $n = 5(365) = 1{,}825$:

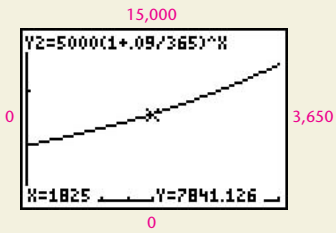
$$\begin{aligned} A &= P\left(1 + \frac{r}{m}\right)^n \\ &= 5{,}000\left(1 + \frac{0.09}{365}\right)^{1825} \\ &= \$7{,}841.13 \end{aligned}$$

Graphical Solution

Graphing

$$A = 5{,}000\left(1 + \frac{0.09}{365}\right)^x$$

and using the VALUE command (Fig. 13) shows that $A = \$7{,}841.13$.



› Figure 13

MATCHED PROBLEM 6

If \$1,000 is invested in an account paying 10% compounded monthly, how much will be in the account at the end of 10 years? Compute the answer to the nearest cent.

*In all problems involving interest compounded daily, we assume a 365-day year.

››› CAUTION ›››

When using the compound interest formula, don't forget to write the interest rate in decimal form.

EXAMPLE 7 Comparing Investments

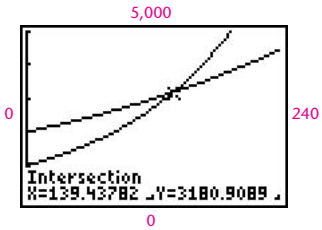
If $1,000 is deposited into an account earning 10% compounded monthly and, at the same time, $2,000 is deposited into an account earning 4% compounded monthly, will the first account ever be worth more than the second? If so, when?

SOLUTION

Let y_1 and y_2 represent the amounts in the first and second accounts, respectively, then

$$y_1 = 1{,}000(1 + 0.10/12)^x$$
$$y_2 = 2{,}000(1 + 0.04/12)^x$$

where x is the number of compounding periods (months). Using the INTERSECT command to analyze the graphs of y_1 and y_2 [Fig. 14(a)], we see that the graphs intersect at $x \approx 139.438$ months. Because compound interest is paid at the end of each compounding period, we should compare the amount in the accounts after 139 months and after 140 months [Fig. 14(b)]. The table shows that the first account is worth more than the second for $x \geq 140$ months, or 11 years and 8 months.



(a)

X	Y1	Y2
139	3169.4	3176.3
140	3195.8	3186.9

X=

(b)

› Figure 14

MATCHED PROBLEM 7

If $4,000 is deposited into an account earning 10% compounded quarterly and, at the same time, $5,000 is deposited into an account earning 6% compounded quarterly, when will the first account be worth more than the second?

› Calculating Interest Compounded Continuously

If $1,000 is deposited in an account that earns compound interest at an annual rate of 8% for 2 years, how will the amount A change if the number of compounding periods is increased? If m is the number of compounding periods per year, then

$$A = 1{,}000\left(1 + \frac{0.08}{m}\right)^{2m}$$

The amount A is computed for several values of m in Table 2. Notice that the largest gain appears in going from annually to semiannually. Then, the gains slow down as m increases. In fact, it appears that A might be approaching something close to \$1,173.50 as m gets larger and larger.

Table 2 Effect of Compounding Frequency

Compounding frequency	m	$A = 100\left(1 + \frac{0.08}{m}\right)^{2m}$
Annually	1	\$1,166.400
Semiannually	2	1,169.859
Quarterly	4	1,171.659
Weekly	52	1,173.367
Daily	365	1,173.490
Hourly	8,760	1,173.501

We now return to the general problem to see if we can determine what happens to $A = P[1 + (r/m)]^{mt}$ as m increases without bound. A little algebraic manipulation of the compound interest formula will lead to an answer and a significant result in the mathematics of finance:

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

Replace $\frac{r}{m}$ with $\frac{1}{m/r}$, and mt with $\frac{m}{r} \cdot rt$.

$$= P\left(1 + \frac{1}{m/r}\right)^{(m/r)rt}$$

Replace $\frac{m}{r}$ with variable x.

$$= P\left[\left(1 + \frac{1}{x}\right)^{x}\right]^{rt}$$

Does the expression within the square brackets look familiar? Recall from the first part of this section that

$$\left(1 + \frac{1}{x}\right)^{x} \to e \qquad \text{as} \qquad x \to \infty$$

Because the interest rate r is fixed, $x = m/r \to \infty$ as $m \to \infty$. So $(1 + \frac{1}{x})^{x} \to e$, and

$$P\left(1 + \frac{r}{m}\right)^{mt} = P\left[\left(1 + \frac{1}{x}\right)^{x}\right]^{rt} \to Pe^{rt} \qquad \text{as} \qquad m \to \infty$$

This is known as the **continuous compound interest formula,** a very important and widely used formula in business, banking, and economics.

› CONTINUOUS COMPOUND INTEREST FORMULA

If a principal P is invested at an annual rate r compounded continuously, then the amount A in the account at the end of t years is given by

$$A = Pe^{rt}$$

The annual rate r must be expressed as a decimal.

EXAMPLE 8 Interest Compounded Continuously

If \$1,000 is invested at an annual rate of 8% compounded continuously, what amount, to the nearest cent, will be in the account after 2 years?

SOLUTIONS

Algebraic Solution

Use the continuous compound interest formula to find A when $P = \$1{,}000$, $r = 0.08$, and $t = 2$:

$$\begin{aligned} A &= Pe^{rt} \\ &= \$1{,}000e^{(0.08)(2)} \qquad \text{8\% is equivalent to } r = 0.08. \\ &= \$1{,}173.51 \end{aligned}$$

Compare this result with the values calculated in Table 2.

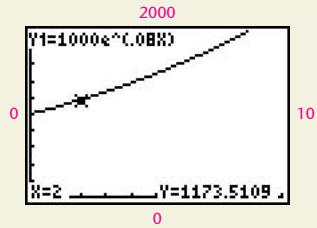
Graphical Solution

Graphing

$$A = 1{,}000e^{0.08x}$$

and using the VALUE command (Fig. 15) shows

$$A = \$1{,}173.51.$$



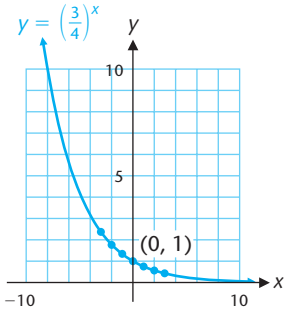
› Figure 15

MATCHED PROBLEM 8

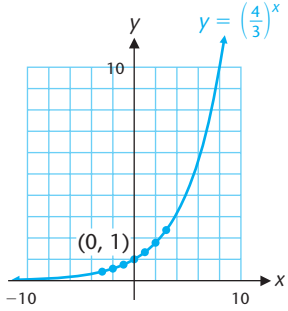
What amount will an account have after 5 years if \$1,000 is invested at an annual rate of 12% compounded annually? Quarterly? Continuously? Compute answers to the nearest cent.

ANSWERS TO MATCHED PROBLEMS

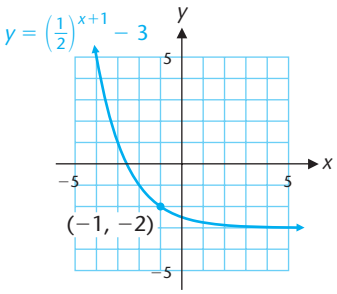
1. (A)



(B)



2.



3. $y = 4$ **4.** $x = -\frac{1}{3}$ **5.** y intercept: -3; x intercept: 1.83; increasing for all x; horizontal asymptote: $y = -5$ **6.** \$2,707.04 **7.** After 23 quarters **8.** Annually: \$1,762.34; quarterly: \$1,806.11; continuously: \$1,822.12

4-1 Exercises

1. What is an exponential function?

2. What is the significance of the symbol e in the study of exponential functions?

3. For a function $f(x) = b^x$, explain how you can tell if the graph increases or decreases without looking at the graph.

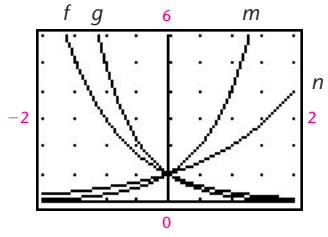
4. Explain why $f(x) = (1/4)^x$ and $g(x) = 4^{-x}$ are really the same function. Can you use this fact to add to your answer for Question 3?

5. How do we know that the equation $e^x = 0$ has no solution?

6. Define the following terms related to compound interest: principal, interest rate, compounding period.

7. Match each equation with the graph of f, g, m, or n in the figure.

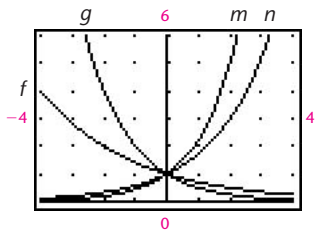
(A) $y = (0.2)^x$ (B) $y = 2^x$
(C) $y = (\frac{1}{3})^x$ (D) $y = 4^x$



8. Match each equation with the graph of f, g, m, or n in the figure.

(A) $y = e^{-1.2x}$ (B) $y = e^{0.7x}$

(C) $y = e^{-0.4x}$ (D) $y = e^{1.3x}$



In Problems 9–16, compute answers to four significant digits.

9. $5^{\sqrt{3}}$ **10.** $3^{-\sqrt{2}}$

11. $e^2 + e^{-2}$ **12.** $e - e^{-1}$

13. $\sqrt{e}$ **14.** $e^{\sqrt{2}}$

15. $\dfrac{2^{\pi} + 2^{-\pi}}{2}$ **16.** $\dfrac{3^{\pi} - 3^{-\pi}}{2}$

In Problems 17–20, sketch the graph of each function after plotting at least six points. Then confirm your result with a graphing calculator.

17. $y = 3^x$ **18.** $y = 5^x$

19. $y = (\frac{1}{3})^x = 3^{-x}$ **20.** $y = (\frac{1}{5})^x = 5^{-x}$

In Problems 21–28, use properties of exponents to simplify.

21. $10^{3x-1}10^{4-x}$ **22.** $(4^{3x})^{2y}$ **23.** $\dfrac{3^x}{3^{1-x}}$

24. $\dfrac{5^{x-3}}{5^{x-4}}$ **25.** $\left(\dfrac{4^x}{5^y}\right)^{3z}$ **26.** $(2^x3^y)^z$

27. $\dfrac{e^{5x}}{e^{2x+1}}$ **28.** $\dfrac{e^{4-3x}}{e^{2-5x}}$

29. (A) Explain what is wrong with the following reasoning about the expression $[1 + (1/x)]^x$: As x gets large, $1 + (1/x)$ approaches 1 because $1/x$ approaches 0, and 1 raised to any power is 1, so $[1 + 1/x]^x$ approaches 1.

(B) Which number does $[1 + (1/x)]^x$ approach as x approaches ∞?

30. (A) Explain what is wrong with the following reasoning about the expression $[1 + (1/x)]^x$: If $b > 1$, then the exponential function b^x approaches ∞ as x approaches ∞, and $1 + (1/x)$ is greater than 1, so $[1 + (1/x)]^x$ approaches infinity as $x \to \infty$.

(B) Which number does $[1 + (1/x)]^x$ approach as x approaches ∞?

In Problems 31–38, graph each function using transformations of an appropriate function of the form $y = b^x$.

31. $f(x) = 2^{x-3} - 1$ **32.** $g(x) = 3^{x+1} + 2$

33. $g(x) = \left(\dfrac{1}{3}\right)^{x+5} - 10$ **34.** $f(x) = \left(\dfrac{1}{2}\right)^{x-10} + 5$

35. $g(x) = e^x + 2$ **36.** $g(x) = e^x - 1$

37. $g(x) = 2e^{-(x+2)}$ **38.** $g(x) = 0.5e^{-(x-1)}$

In Problems 39–52, find all solutions to the equation.

39. $5^{3x} = 5^{4x-2}$ **40.** $10^{2-3x} = 10^{5x-6}$

41. $7^{x^2} = 7^{2x+3}$ **42.** $4^{5x-x^2} = 4^{-6}$

43. $(1 - x)^5 = (2x - 1)^5$ **44.** $5^3 = (x + 2)^3$

45. $9^{x^2} = 3^{3x-1}$ **46.** $4^{x^2} = 2^{x+3}$

47. $4^{x^2} = 8^x$ **48.** $9^{x^2} = 27^{x+3}$

49. $2xe^{-x} = 0$ **50.** $(x - 3)e^x = 0$

51. $x^2e^x - 5xe^x = 0$ **52.** $3xe^{-x} + x^2e^{-x} = 0$

53. Find all real numbers a such that $a^2 = a^{-2}$. Explain why this does not violate the second exponential function property in the box on page 387.

54. Find real numbers a and b such that $a \neq b$ but $a^4 = b^4$. Explain why this does not violate the third exponential function property in the box on page 387.

55. Examine the graph of $y = 1^x$ on a graphing calculator and explain why 1 cannot be the base for an exponential function.

56. Examine the graph of $y = 0^x$ on a graphing calculator and explain why 0 cannot be the base for an exponential function. [*Hint:* Turn the axes off before graphing.]

57. Evaluate $y = 1^x$ for $x = -3, -2, -1, 0, 1, 2$, and 3. Why is $b = 1$ excluded when defining the exponential function $y = b^x$?

58. Evaluate $y = 0^x$ for $x = -3, -2, -1, 0, 1, 2$, and 3. Why is $b = 0$ excluded when defining the exponential function $y = b^x$?

59. Explain why the graph of an exponential function cannot be the graph of a polynomial function.

60. Explain why the graph of an exponential function cannot be the graph of a rational function.

In Problems 61–64, simplify.

61. $\dfrac{-2x^3e^{-2x} - 3x^2e^{-2x}}{x^6}$ **62.** $\dfrac{5x^4e^{5x} - 4x^3e^{5x}}{x^8}$

63. $(e^x + e^{-x})^2 + (e^x - e^{-x})^2$

64. $e^x(e^{-x} + 1) - e^{-x}(e^x + 1)$

In Problems 65–76, use a graphing calculator to find local extrema, y intercepts, and x intercepts. Investigate the behavior as $x \to \infty$ and as $x \to -\infty$ and identify any horizontal asymptotes. Round any approximate values to two decimal places.

65. $f(x) = 2 + e^{x-2}$

66. $g(x) = -3 + e^{1+x}$

67. $m(x) = e^{|x|}$

68. $n(x) = e^{-|x|}$

69. $s(x) = e^{-x^2}$

70. $r(x) = e^{x^2}$

71. $F(x) = \dfrac{200}{1 + 3e^{-x}}$

72. $G(x) = \dfrac{100}{1 + e^{-x}}$

73. $m(x) = 2x(3^{-x}) + 2$

74. $h(x) = 3x(2^{-x}) - 1$

75. $f(x) = \dfrac{2^x + 2^{-x}}{2}$

76. $g(x) = \dfrac{3^x + 3^{-x}}{2}$

77. Use a graphing calculator to investigate the behavior of $f(x) = (1 + x)^{1/x}$ as x approaches 0.

78. Use a graphing calculator to investigate the behavior of $f(x) = (1 + x)^{1/x}$ as x approaches ∞.

It is common practice in many applications of mathematics to approximate nonpolynomial functions with appropriately selected polynomials. For example, the polynomials in Problems 79–82, called ***Taylor polynomials,*** *can be used to approximate the exponential function $f(x) = e^x$. To illustrate this approximation graphically, in each problem graph $f(x) = e^x$ and the indicated polynomial in the same viewing window, $-4 \le x \le 4$ and $-5 \le y \le 50$.*

79. $P_1(x) = 1 + x + \frac{1}{2}x^2$

80. $P_2(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

81. $P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$

82. $P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$

83. Investigate the behavior of the functions $f_1(x) = x/e^x$, $f_2(x) = x^2/e^x$, and $f_3(x) = x^3/e^x$ as $x \to \infty$ and as $x \to -\infty$, and find any horizontal asymptotes. Generalize to functions of the form $f_n(x) = x^n/e^x$, where n is any positive integer.

84. Investigate the behavior of the functions $g_1(x) = xe^x$, $g_2(x) = x^2e^x$, and $g_3(x) = x^3e^x$ as $x \to \infty$ and as $x \to -\infty$, and find any horizontal asymptotes. Generalize to functions of the form $g_n(x) = x^ne^x$, where n is any positive integer.

85. The irrational number $\sqrt{2}$ is approximated by 1.414214 to six decimal places. Each of $x = 1.4$, 1.41, 1.414, 1.4142, 1.41421, and 1.414214 is a rational number, so we know how to define 2^x for each. Compute the value of 2^x for each of these x values, and use your results to estimate the value of $2^{\sqrt{2}}$. Then compute $2^{\sqrt{2}}$ using your calculator to check your estimate.

86. The irrational number $\sqrt{3}$ is approximated by 1.732051 to six decimal places. Each of $x = 1.7$, 1.73, 1.732, 1.7321, 1.73205, and 1.732051 is a rational number, so we know how to define 3^x for each. Compute the value of 3^x for each of these x values, and use your results to estimate the value of $3^{\sqrt{3}}$. Then compute $3^{\sqrt{3}}$ using your calculator to check your estimate.

APPLICATIONS*

87. **FINANCE** Suppose \$4,000 is invested at 11% compounded weekly. How much money will be in the account in
(A) $\frac{1}{2}$ year? (B) 10 years?
Compute answers to the nearest cent.

88. **FINANCE** Suppose \$2,500 is invested at 7% compounded quarterly. How much money will be in the account in
(A) $\frac{3}{4}$ year? (B) 15 years?
Compute answers to the nearest cent.

89. **MONEY GROWTH** If you invest \$5,250 in an account paying 11.38% compounded continuously, how much money will be in the account at the end of
(A) 6.25 years? (B) 17 years?

90. **MONEY GROWTH** If you invest \$7,500 in an account paying 8.35% compounded continuously, how much money will be in the account at the end of
(A) 5.5 years? (B) 12 years?

91. **FINANCE** If \$3,000 is deposited into an account earning 8% compounded daily and, at the same time, \$5,000 is deposited into an account earning 5% compounded daily, will the first account ever be worth more than the second? If so, when?

92. **FINANCE** If \$4,000 is deposited into an account earning 9% compounded weekly and, at the same time, \$6,000 is deposited into an account earning 7% compounded weekly, will the first account ever be worth more than the second? If so, when?

93. **FINANCE** Will an investment of \$10,000 at 8.9% compounded daily ever be worth more at the end of a quarter than an investment of \$10,000 at 9% compounded quarterly? Explain.

94. **FINANCE** A sum of \$5,000 is invested at 13% compounded semiannually. Suppose that a second investment of \$5,000 is made at interest rate r compounded daily. For which values of r, to the nearest tenth of a percent, is the second investment better than the first? Discuss.

95. **PRESENT VALUE** A promissory note will pay \$30,000 at maturity 10 years from now. How much should you be willing to pay for the note now if the note gains value at a rate of 9% compounded continuously?

*Round monetary amounts to the nearest cent unless specified otherwise. In all problems involving interest that is compounded daily, assume a 365-day year.

96. **PRESENT VALUE** A promissory note will pay \$50,000 at maturity $5\frac{1}{2}$ years from now. How much should you be willing to pay for the note now if the note gains value at a rate of 10% compounded continuously?

97. **MONEY GROWTH** *Barron's,* a national business and financial weekly, published the following "Top Savings Deposit Yields" for $2\frac{1}{2}$-year certificate of deposit accounts:

Gill Savings	8.30% (CC)
Richardson Savings and Loan	8.40% (CQ)
USA Savings	8.25% (CD)

where CC represents compounded continuously, CQ compounded quarterly, and CD compounded daily. Compute the value of \$1,000 invested in each account at the end of $2\frac{1}{2}$ years.

98. **MONEY GROWTH** Refer to Problem 97. In another issue of *Barron's,* 1-year certificate of deposit accounts included:

Alamo Savings	8.25% (CQ)
Lamar Savings	8.05% (CC)

Compute the value of \$10,000 invested in each account at the end of 1 year.

99. **FINANCE** A couple just had a new child. How much should they invest now at 8.25% compounded daily to have \$100,000 for the child's education 17 years from now? Compute the answer to the nearest dollar.

100. **FINANCE** A person wishes to have \$15,000 cash for a new car 5 years from now. How much should be placed in an account now if the account pays 9.75% compounded weekly? Compute the answer to the nearest dollar.

4-2 Exponential Models

- Modeling Exponential Growth
- Modeling Negative Exponential Growth
- Carbon-14 Dating
- Modeling Limited Growth
- Data Analysis and Regression
- A Comparison of Exponential Growth Phenomena

One of the best reasons for studying exponential functions is the fact that many things that occur naturally in our world can be modeled accurately by these functions. In this section, we will study a wide variety of applications, including growth of populations of people, animals, and bacteria; radioactive decay; spread of epidemics; propagation of rumors; light intensity; atmospheric pressure; and electric circuits. The regression techniques we used in Chapter 2 to construct linear and quadratic models will be extended to construct exponential models.

Modeling Exponential Growth

What sort of function is likely to describe the growth of a population? We will consider this question in Explore-Discuss 1.

››› EXPLORE-DISCUSS 1

A certain species of fruit fly reproduces quickly, with a new generation appearing in about a week.

(A) Suppose that a population starts with 200 flies, and we assume that the population increases by 50 flies each week. Calculate the number of flies after 1, 2, 3, 4, and 5 weeks.

(B) Now suppose that the population increases by 25% of the current population each week. Calculate the number of flies after 1, 2, 3, 4, and 5 weeks.

(C) Which scenario do you think is more realistic? Why?

The population model described in part B of Explore-Discuss 1 is the more realistic one. As a population grows, there are more individuals to reproduce, so the rate of increase grows as well. This sounds an awful lot like compound interest: The percentage added to the population is in effect calculated on both the original amount and the number of new individuals. It should come as no surprise, then, that populations of organisms, from bacteria all the way to human beings, tend to grow exponentially.

One convenient and easily understood measure of growth rate is the **doubling time**—that is, the time it takes for a population to double. Over short periods the **doubling time growth model** is often used to model population growth:

$$A = A_0 2^{t/d}$$

where A = Population at time t
A_0 = Population at time $t = 0$
d = Doubling time

Note that when the amount of time passed is equal to the doubling time ($t = d$),

$$A = A_0 2^{t/d} = A_0 2$$

and the population is double the original, as it should be. We will use this model to solve a population growth problem in Example 1.

EXAMPLE 1 Population Growth

Mexico has a population of around 100 million people, and it is estimated that the population will double in 21 years. If population growth continues at the same rate, what will be the population:

(A) 15 years from now? (B) 30 years from now?

Calculate answers to three significant digits.

SOLUTIONS

Algebraic Solutions

We use the doubling time growth model with $A_0 = 100$ and $d = 21$:

$$A = A_0 2^{t/d} \qquad \text{Let } A_0 = 100,\ d = 21$$
$$A = 100(2^{t/21}) \qquad \text{Figure 1}$$

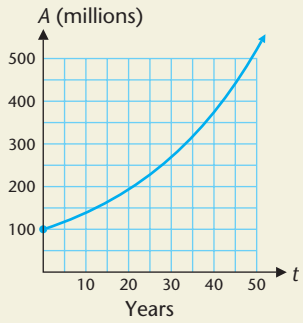
(A) Find A when $t = 15$ years:

$$A = 100(2^{15/21})$$
$$\approx 164 \text{ million people}$$

(B) Find A when $t = 30$ years:

$$A = 100(2^{30/21})$$
$$\approx 269 \text{ million people}$$

› **Figure 1** $A = 100(2^{t/21})$.



Graphical Solutions

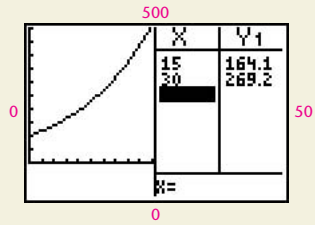
We graph

$$A = 100(2^{x/21})$$

and construct a table of values (Fig. 2).

(A) When $x = 15$ years, $A \approx 164$ million people.

(B) When $x = 30$ years, $A \approx 269$ million people.



› **Figure 2**

MATCHED PROBLEM 1

The bacterium *Escherichia coli* (*E. coli*) is found naturally in the intestines of many mammals. In a particular laboratory experiment, the doubling time for *E. coli* is found to be 25 minutes. If the experiment starts with a population of 1,000 *E. coli* and there is no change in the doubling time, how many bacteria will be present:

(A) In 10 minutes? (B) In 5 hours?

Write answers to three significant digits.

››› EXPLORE-DISCUSS 2

The doubling time growth model would *not* be expected to give accurate results over long periods. According to the doubling time growth model of Example 1, what was the population of Mexico 500 years ago at the height of Aztec civilization? What will the population of Mexico be 200 years from now? Explain why these results are unrealistic. Discuss factors that affect human populations that are not taken into account by the doubling time growth model.

The doubling time model is not the only one used to model populations. An alternative model based on the continuous compound interest formula will be used in Example 2. In this case, the formula is written as

$$A = A_0 e^{kt}$$

where A = Population at time t
A_0 = Population at time $t = 0$
k = Relative growth rate

The **relative growth rate** is written as a percentage in decimal form. For example, if a population is growing so that at any time the population is increasing at 3% of the current population per year, the relative growth rate k would be 0.03.

EXAMPLE 2 Medicine—Bacteria Growth

Cholera, an intestinal disease, is caused by a cholera bacterium that multiplies exponentially by cell division as modeled by

$$A = A_0 e^{1.386t}$$

where A is the number of bacteria present after t hours and A_0 is the number of bacteria present at $t = 0$. If we start with 1 bacterium, how many bacteria will be present in

(A) 5 hours? (B) 12 hours?

Compute the answers to three significant digits.

SOLUTIONS

Algebraic Solutions

(A) Use $A_0 = 1$ and $t = 5$:

$$\begin{aligned} A &= A_0 e^{1.386t} \\ &= e^{1.386(5)} \\ &= 1{,}020 \end{aligned}$$

(B) Use $A_0 = 1$ and $t = 12$:

$$\begin{aligned} A &= A_0 e^{1.386t} \\ &= e^{1.386(12)} \\ &= 16{,}700{,}000 \end{aligned}$$

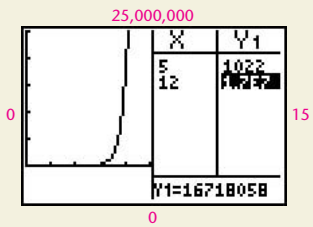
Graphical Solutions

We graph

$$A = e^{1.386x}$$

and construct a table of values (Fig. 3).

(A) When $x = 5$ hours, $A \approx 1{,}020$ bacteria.
(B) When $x = 12$ hours, $A \approx 16{,}700{,}000$ bacteria.



› Figure 3

MATCHED PROBLEM 2

Repeat Example 2 if $A = A_0e^{0.783t}$ and all other information remains the same.

Modeling Negative Exponential Growth

Exponential functions can also be used to model radioactive decay, which is sometimes referred to as **negative growth.** Radioactive materials are used extensively in medical diagnosis and therapy, as power sources in satellites, and as power sources in many countries. If we start with an amount A_0 of a particular radioactive substance, the amount declines exponentially over time. The rate of decay varies depending on the particular radioactive substance. A convenient and easily understood measure of the rate of decay is the **half-life** of the material—that is, the time it takes for half of a particular material to decay. We can use the following **half-life decay model:**

$$\begin{aligned} A &= A_0\left(\tfrac{1}{2}\right)^{t/h} \\ &= A_0 2^{-t/h} \end{aligned}$$

where A = Amount at time t
A_0 = Amount at time $t = 0$
h = Half-life

Note that when the amount of time passed is equal to the half-life ($t = h$),

$$A = A_0 2^{-h/h} = A_0 2^{-1} = A_0 \cdot \tfrac{1}{2}$$

and the amount of radioactive material is half the original amount, as it should be.

EXAMPLE 3 Radioactive Decay

The radioactive isotope gallium 67 (^{67}Ga), used in the diagnosis of malignant tumors, has a biological half-life of 46.5 hours. If we start with 100 milligrams of the isotope, how many milligrams will be left after

(A) 24 hours? (B) 1 week?

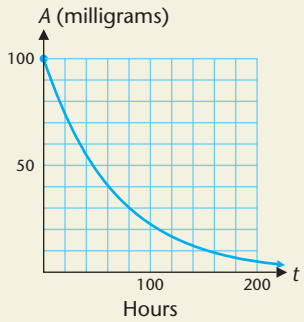
Compute answers to three significant digits.

SOLUTIONS

Algebraic Solutions

We use the half-life decay model with $A_0 = 100$ and $h = 46.5$:

$$A = A_0\left(\tfrac{1}{2}\right)^{t/h} = A_0 2^{-t/h} \qquad A_0 = 100,\ h = 46.5$$
$$A = 100(2^{-t/46.5}) \qquad \text{See Figure 4.}$$



› **Figure 4** $A = 100(2^{-t/46.5})$.

(A) Find A when $t = 24$ hours:

$$A = 100(2^{-24/46.5}) = 69.9 \text{ milligrams}$$

(B) Find A when $t = 168$ hours (1 week = 168 hours):

$$A = 100(2^{-168/46.5}) = 8.17 \text{ milligrams}$$

Graphical Solutions

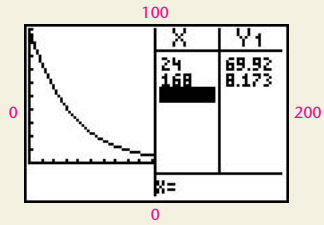
We graph

$$A = 100(2^{-x/46.5})$$

and construct a table of values (Fig. 5).

(A) When $x = 24$ hours, $A \approx 69.9$ milligrams.

(B) When $x = 168$ hours (1 week), $A \approx 8.17$ milligrams.



› **Figure 5**

MATCHED PROBLEM 3

Radioactive gold 198 (^{198}Au), used in imaging the structure of the liver, has a half-life of 2.67 days. If we start with 50 milligrams of the isotope, how many milligrams will be left after:

(A) $\frac{1}{2}$ day? (B) 1 week?

Compute answers to three significant digits.

›››› CAUTION ››››

When using exponential models, be aware of the units of time. In Example 3 the half-life was given in hours, so when time was provided in weeks, we had to first convert that into hours before using the half-life formula.

In Example 2, we saw that a base e exponential function can be used as an alternative to the doubling time model. Not surprisingly, the same can be said for the half-life model. In this case, the formula will be

$$A = A_0 e^{-kt}$$

where A = the amount of radioactive material at time t
A_0 = the amount at time $t = 0$
k = a positive constant specific to the type of material

Carbon-14 Dating

Our atmosphere is constantly being bombarded with cosmic rays. These rays produce neutrons, which in turn react with nitrogen to produce radioactive carbon-14. Radioactive carbon-14 enters all living tissues through carbon dioxide, which is first absorbed by plants. As long as a plant or animal is alive, carbon-14 is maintained in the living organism at a constant level. Once the organism dies, however, carbon-14 decays according to the equation

$$A = A_0 e^{-0.000124t} \qquad \textbf{Carbon-14 decay equation}$$

where A is the amount of carbon-14 present after t years and A_0 is the amount present at time $t = 0$. This can be used to calculate the approximate age of fossils.

EXAMPLE 4 Carbon-14 Dating

If 1,000 milligrams of carbon-14 are present in the tissue of a recently deceased animal, how many milligrams will be present in

(A) 10,000 years? (B) 50,000 years?

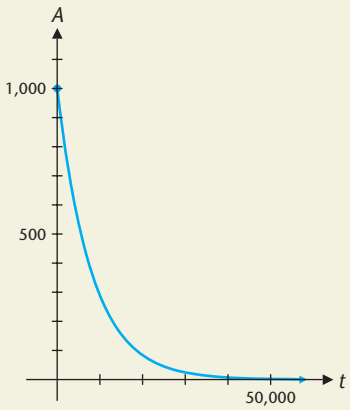
Compute answers to three significant digits.

SOLUTIONS

Algebraic Solutions

Substituting $A_0 = 1{,}000$ in the decay equation, we have

$$A = 1{,}000e^{-0.000124t}$$ Figure 6



› Figure 6

(A) Find A when $t = 10{,}000$:

$$\begin{aligned} A &= 1{,}000e^{-0.000124(10{,}000)} \\ &= 289 \text{ milligrams} \end{aligned}$$

(B) Find A when $t = 50{,}000$:

$$\begin{aligned} A &= 1{,}000e^{-0.000124(50{,}000)} \\ &= 2.03 \text{ milligrams} \end{aligned}$$

Graphical Solutions

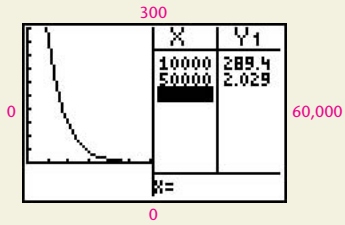
We graph

$$A = 1{,}000e^{-0.000124x}$$

and construct a table of values (Fig. 7).

(A) When $x = 10{,}000$ years, $A \approx 289$ milligrams.

(B) When $x = 50{,}000$ years, $A \approx 2.03$ milligrams.



› Figure 7

We will use the carbon-14 decay equation in Exercise 4-5, where we will be interested in solving for t after being given information about A and A_0.

MATCHED PROBLEM 4

Referring to Example 4, how many milligrams of carbon-14 would have to be present at the beginning to have 10 milligrams present after 20,000 years? Approximate the answer to four significant digits.

Modeling Limited Growth

One of the problems with using exponential functions to model things like population is that the growth is completely unlimited in the long term. But in real life, there is often some reasonable maximum value, like the largest population that space and resources allow. We can use modified versions of exponential functions to model such phenomena more realistically.

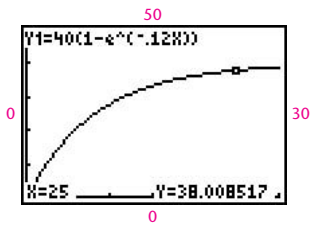
One such type of function is called a *learning curve* since it can be used to model the performance improvement of a person learning a new task. **Learning curves** are functions of the form $A = c(1 - e^{-kt})$, where c and k are positive constants.

EXAMPLE 5 Learning Curve

People assigned to assemble circuit boards for a computer manufacturing company undergo on-the-job training. From past experience, it was found that the learning curve for the average employee is given by

$$A = 40(1 - e^{-0.12t})$$

where A is the number of boards assembled per day after t days of training (Fig. 8).



› **Figure 8** Limited growth.

(A) How many boards can an average employee produce after 3 days of training? After 5 days of training? Round answers to the nearest integer.

(B) How many days of training will it take until an average employee can assemble 25 boards a day? Round answers to the nearest integer.

(C) Does A approach a limiting value as t increases without bound? Explain.

SOLUTIONS

(A) When $t = 3$,

$$A = 40(1 - e^{-0.12(3)}) = 12 \qquad \text{Rounded to nearest integer}$$

so the average employee can produce 12 boards after 3 days of training. Similarly, when $t = 5$,

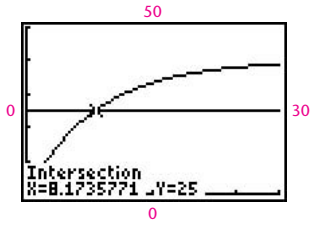
$$A = 40(1 - e^{-0.12(5)}) = 18 \qquad \text{Rounded to nearest integer}$$

so the average employee can produce 18 boards after 5 days of training.

(B) Solve the equation $40(1 - e^{-0.12t}) = 25$ for t by graphing

$$y_1 = 40(1 - e^{-0.12t}) \qquad \text{and} \qquad y_2 = 25$$

and using the INTERSECT command (Fig. 9). It will take about 8 days of training.



› **Figure 9**
$y_1 = 40(1 - e^{-0.12t})$, $y_2 = 25$.

(C) Because $e^{-0.12t}$ approaches 0 as t increases without bound,

$$A = 40(1 - e^{-0.12t}) \to 40(1 - 0) = 40$$

So the limiting value of A is 40 boards per day. (This can be supported by the graph.)

MATCHED PROBLEM 5

A company is trying to expose as many people as possible to a new product through television advertising in a large metropolitan area with 2 million potential viewers. A model for the number of people A, in millions, who are aware of the product after t days of advertising was found to be

$$A = 2(1 - e^{-0.037t})$$

(A) How many viewers are aware of the product after 2 days? After 10 days? Express answers as integers, rounded to three significant digits.

(B) How many days will it take until half of the potential viewers will become aware of the product? Round answer to the nearest integer.

(C) Does A approach a limiting value as t increases without bound? Explain.

Another limited-growth model is useful for phenomena such as the spread of an epidemic or the propagation of a rumor. It is called the *logistic equation,* and is given by

$$A = \frac{M}{(1 + ce^{-kt})}$$

where M, c, and k are positive constants. Logistic growth, illustrated in Example 6, also approaches a limiting value as t increases without bound.

EXAMPLE 6 Logistic Growth in an Epidemic

A certain community consists of 1,000 people. One individual who has just returned from another community has a particularly contagious strain of influenza. Assume the community has not had influenza shots and all are susceptible. The spread of the disease in the community is predicted to be given by the logistic curve

$$A(t) = \frac{1{,}000}{1 + 999e^{-0.3t}}$$

where A is the number of people who have contracted influenza after t days.

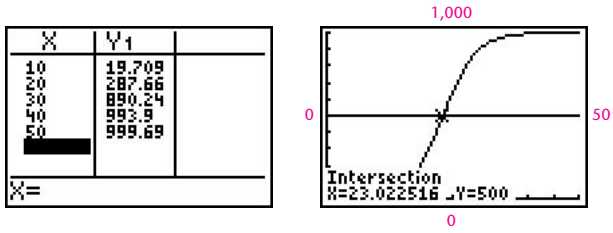
(A) How many people have contracted influenza after 10 days? After 20 days?

(B) How many days will it take until half the community has contracted influenza? Round answer to the nearest integer.

(C) Does A approach a limiting value as t increases without bound? Explain.

SOLUTIONS

(A) Enter $y_1 = 1{,}000/(1 + 999e^{-0.3t})$ into a graphing calculator. The table in Figure 10(a) shows that $A(10) \approx 20$ individuals and $A(20) \approx 288$ individuals.



› **Figure 10** Logistic growth.

(B) Figure 10(b) shows that the graph of $A(t)$ intersects the line $y = 500$ after approximately 23 days.

(C) The values in Figure 10(a) and the graph in Figure 10(b) both indicate that A approaches 1,000 as t increases without bound. We can confirm this algebraically by noting that because $999e^{-0.3t} \to 0$ as t increases without bound,

$$A(t) = \frac{1{,}000}{1 + 999e^{-0.3t}} \to \frac{1{,}000}{1 + 0} = 1{,}000$$

Thus, the upper limit on the growth of A is 1,000, the total number of people in the community.

MATCHED PROBLEM 6

A group of 400 parents, relatives, and friends are waiting anxiously at Kennedy Airport for a charter flight returning students after a year in Europe. It is stormy and the plane is late. A particular parent thought he had heard that the plane's radio had gone out and related this news to some friends, who in turn passed it on to others. The propagation of this rumor is predicted to be given by

$$A(t) = \frac{400}{1 + 399e^{-0.4t}}$$

where A is the number of people who have heard the rumor after t minutes.

(A) How many people have heard the rumor after 10 minutes? After 20 minutes?

(B) How many minutes will it take until half the group has heard the rumor? Round answer to the nearest integer.

(C) Does A approach a limiting value as t increases without bound? Explain.

› Data Analysis and Regression

Many graphing calculators with regression commands have options for exponential regression. We can use exponential regression to fit a function of the form $y = ab^x$ to a set of data points, and logistic regression to fit a function of the form

$$y = \frac{c}{1 + ae^{-bx}}$$

to a set of data points. The techniques are similar to those introduced in Chapter 2 for linear and quadratic functions.

EXAMPLE 7 Infectious Diseases

Table 1 Reported Cases of Infectious Diseases

Year	Mumps	Rubella
1970	104,953	56,552
1980	8,576	3,904
1990	5,292	1,125
1995	906	128
2000	323	152

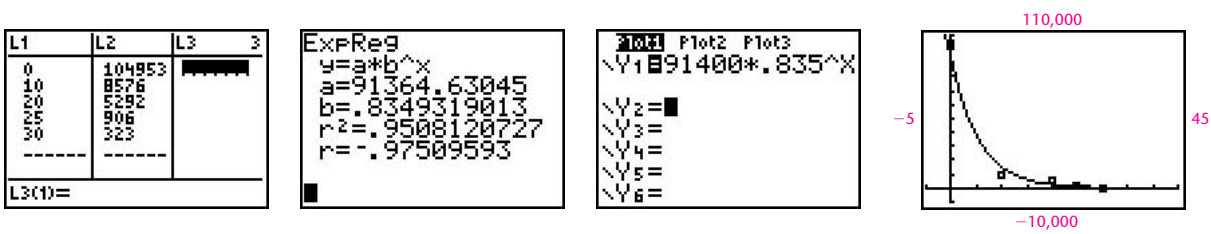
The U.S. Department of Health and Human Services published the data in Table 1.

(A) Let x represent time in years with $x = 0$ representing 1970, and let y represent the corresponding number of reported cases of mumps. Use regression analysis on a graphing calculator to find an exponential function of the form $y = ab^x$ that models the data. (Round the constants a and b to three significant digits.)

(B) Use the exponential regression function to predict the number of reported cases of mumps in 2010.

SOLUTIONS

(A) Figure 11 shows the details of constructing the model on a graphing calculator.



(a) Data (b) Regression equation (c) Regression equation entered in equation editor (d) Graph of data and regression equation

› Figure 11

(B) Evaluating $y_1 = 91{,}400(0.835)^x$ at $x = 40$ gives a prediction of 67 cases of mumps in 2010.

MATCHED PROBLEM 7

Repeat Example 7 for reported cases of rubella.

EXAMPLE 8 AIDS Cases and Deaths

Table 2 Acquired Immunodeficiency Syndrome (AIDS) Cases and Deaths in the United States

Year	Cases Diagnosed to Date	Known Deaths to Date
1988	107,755	62,468
1991	261,259	159,294
1994	493,713	296,507
1997	672,970	406,179
2000	774,467	447,648
2005	956,665	550,394

The U.S. Department of Health and Human Services published the data in Table 2.

(A) Let x represent time in years with $x = 0$ representing 1988, and let y represent the corresponding number of AIDS cases diagnosed to date. Use regression analysis on a graphing utility to find a logistic function of the form

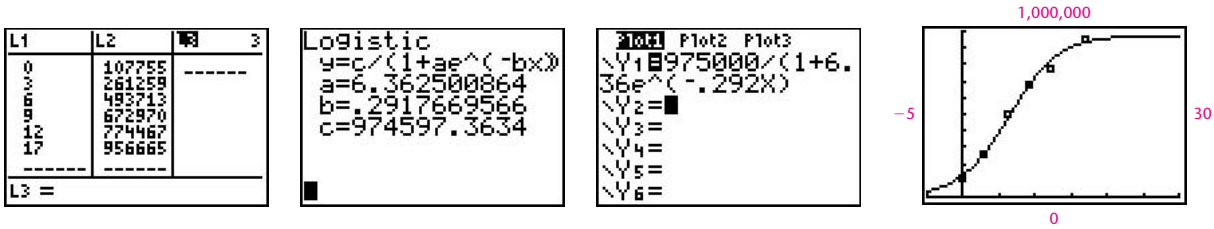
$$y = \frac{c}{1 + ae^{-bx}}$$

that models the data. (Round the constants a, b, and c to three significant digits.)

(B) Use the logistic regression function to predict the number of cases of AIDS diagnosed by 2015.

SOLUTIONS

(A) Figure 12 shows the details of constructing the model on a graphing calculator.



(a) Data (b) Regression equation (c) Regression equation entered in equation editor (d) Graph of data and regression equation

› Figure 12

(B) Evaluating

$$y_1 = \frac{975{,}000}{1 + 6.36e^{-0.292x}}$$

at $x = 27$ gives a prediction of approximately 973,000 cases of AIDS diagnosed by 2015.

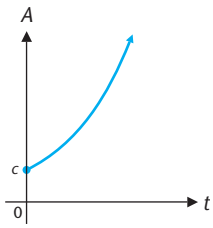
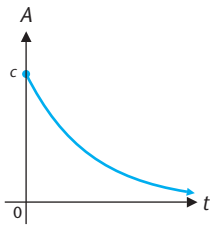
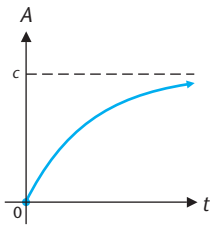
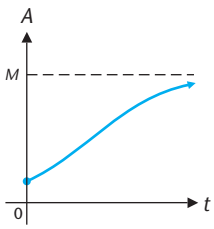
MATCHED PROBLEM 8

Repeat Example 8 for known deaths from AIDS to date.

A Comparison of Exponential Growth Phenomena

The equations and graphs given in Table 3 compare several widely used growth models. These are divided basically into two groups: unlimited growth and limited growth. Following each equation and graph is a short, incomplete list of areas in which the models are used. We have only touched on a subject that has been extensively developed and that you are likely to study in greater depth in the future.

Table 3 Exponential Growth and Decay

Description	Equation	Graph	Short List of Uses
Unlimited growth	$A = A_0e^{kt}$ $k > 0$		Short-term population growth (people, bacteria, etc.); growth of money at continuous compound interest
Exponential decay	$A = A_0e^{-kt}$ $k > 0$		Radioactive decay; light absorption in water, glass, and the like; atmospheric pressure; electric circuits
Limited growth	$A = c(1 - e^{-kt})$ $c, k > 0$		Learning skills; sales fads; company growth; electric circuits
Logistic growth	$A = \frac{M}{1 + ce^{-kt}}$ $c, k, M > 0$		Long-term population growth; epidemics; sales of new products; spread of rumors; company growth

ANSWERS TO MATCHED PROBLEMS

1. (A) 1,320 bacteria (B) 4,100,000 bacteria
2. (A) 50 bacteria (B) 12,000 bacteria **3.** (A) 43.9 milligrams (B) 8.12 milligrams
4. 119.4 milligrams **5.** (A) 143,000 viewers; 619,000 viewers (B) 19 days
(C) A approaches an upper limit of 2 million, the number of potential viewers
6. (A) 48 individuals; 353 individuals (B) 15 minutes (C) A approaches an upper limit of 400, the number of people in the entire group.
7. (A) $y = 44{,}500(0.815)^x$ (B) 12 cases
8. (A) $y = \dfrac{549{,}000}{1 + 6.14e^{-0.311x}}$ (B) 548,000 known deaths

4-2 Exercises

1. Define the terms "doubling time" and "half-life" in your own words.

2. One of the models below represents positive growth, and the other represents negative growth. Classify each, and explain how you decided on your answer. (Assume that $k > 0$.)

$$A = A_0e^{-kt} \qquad A = A_0e^{kt}$$

3. Explain the difference between exponential growth and limited growth.

4. Explain why a limited growth model would be more accurate than regular exponential growth in modeling the long-term population of birds on an island in Lake Erie.

In Problems 5–8, write an exponential equation describing the given population at any time t.

5. Initial population 200; doubling time 5 months

6. Initial population 5,000; doubling time 3 years

7. Initial population 2,000; continuous growth at 2% per year

8. Initial population 500; continuous growth at 3% per week

In Problems 9–12, write an exponential equation describing the amount of radioactive material present at any time t.

9. Initial amount 100 grams; half-life 6 hours

10. Initial amount 5 pounds; half-life 1,300 years

11. Initial amount 4 kilograms; continuous decay at 12.4% per year

12. Initial amount 50 milligrams; continuous decay at 0.03% per year

APPLICATIONS

13. **GAMING** A person bets on red and black on a roulette wheel using a *Martingale strategy*. That is, a $2 bet is placed on red, and the bet is doubled each time until a win occurs. The process is then repeated. If black occurs n times in a row, then $L = 2^n$ dollars is lost on the nth bet. Graph this function for $1 \le n \le 10$. Although the function is defined only for positive integers, points on this type of graph are usually joined with a smooth curve as a visual aid.

14. **BACTERIAL GROWTH** If bacteria in a certain culture double every $\frac{1}{2}$ hour, write an equation that gives the number of bacteria N in the culture after t hours, assuming the culture has 100 bacteria at the start. Graph the equation for $0 \le t \le 5$.

15. **POPULATION GROWTH** Because of its short life span and frequent breeding, the fruit fly *Drosophila* is used in some genetic studies. Raymond Pearl of Johns Hopkins University, for example, studied 300 successive generations of descendants of a single pair of *Drosophila* flies. In a laboratory situation with ample food supply and space, the doubling time for a particular population is 2.4 days. If we start with 5 male and 5 female flies, how many flies should we expect to have in
(A) 1 week? (B) 2 weeks?

16. **POPULATION GROWTH** If Kenya has a population of about 34,000,000 people and a doubling time of 27 years and if the growth continues at the same rate, find the population in
(A) 10 years (B) 30 years
Compute answers to two significant digits.

17. **COMPUTER DESIGN** In 1965, Gordon Moore, founder of Intel, predicted that the number of transistors that could be

placed on a computer chip would double every 2 years. This has come to be known as *Moore's law*. In 1970, 2,200 transistors could be placed on a chip. Use Moore's law to predict the number of transistors in
(A) 1990 (B) 2005

18. **HISTORY OF TECHNOLOGY** The earliest mechanical clocks appeared around 1350 in Europe, and would gain or lose an average of 30 minutes per day. After that, accuracy roughly doubled every 30 years. Find the predicted accuracy of clocks in
(A) 1700 (B) 2000

19. **INSECTICIDES** The use of the insecticide DDT is no longer allowed in many countries because of its long-term adverse effects. If a farmer uses 25 pounds of active DDT, assuming its half-life is 12 years, how much will still be active after
(A) 5 years? (B) 20 years?
Compute answers to two significant digits.

20. **RADIOACTIVE TRACERS** The radioactive isotope technetium-99m (^{99m}Tc) is used in imaging the brain. The isotope has a half-life of 6 hours. If 12 milligrams are used, how much will be present after
(A) 3 hours? (B) 24 hours?
Compute answers to three significant digits.

21. **POPULATION GROWTH** If the world population is about 6.5 billion people now and if the population grows continuously at a relative growth rate of 1.14%, what will the population be in 10 years? Compute the answer to two significant digits.

22. **POPULATION GROWTH** If the population of Mexico is around 106 million people now and if the population grows continuously at a relative growth rate of 1.17%, what will the population be in 8 years? Compute the answer to three significant digits.

23. **POPULATION GROWTH** In 2005 the population of Russia was 143 million and the population of Nigeria was 129 million. If the populations of Russia and Nigeria grow continuously at relative growth rates of -0.37% and 2.56%, respectively, in what year will Nigeria have a greater population than Russia?

24. **POPULATION GROWTH** In 2005 the population of Germany was 82 million and the population of Egypt was 78 million. If the populations of Germany and Egypt grow continuously at relative growth rates of 0% and 1.78%, respectively, in what year will Egypt have a greater population than Germany?

25. **SPACE SCIENCE** Radioactive isotopes, as well as solar cells, are used to supply power to space vehicles. The isotopes gradually lose power because of radioactive decay. On a particular space vehicle the nuclear energy source has a power output of P watts after t days of use as given by

$$P = 75e^{-0.0035t}$$

Graph this function for $0 \le t \le 100$.

26. **EARTH SCIENCE** The atmospheric pressure P, in pounds per square inch, decreases exponentially with altitude h, in miles above sea level, as given by

$$P = 14.7e^{-0.21h}$$

Graph this function for $0 \le h \le 10$.

27. **MARINE BIOLOGY** Marine life is dependent upon the microscopic plant life that exists in the *photic zone,* a zone that goes to a depth where about 1% of the surface light still remains. Light intensity I relative to depth d, in feet, for one of the clearest bodies of water in the world, the Sargasso Sea in the West Indies, can be approximated by

$$I = I_0e^{-0.00942d}$$

where I_0 is the intensity of light at the surface. To the nearest percent, what percentage of the surface light will reach a depth of
(A) 50 feet? (B) 100 feet?

28. **MARINE BIOLOGY** Refer to Problem 27. In some waters with a great deal of sediment, the photic zone may go down only 15 to 20 feet. In some murky harbors, the intensity of light d feet below the surface is given approximately by

$$I = I_0e^{-0.23d}$$

What percentage of the surface light will reach a depth of
(A) 10 feet? (B) 20 feet?

29. **AIDS EPIDEMIC** The World Health Organization estimated that 39.4 million people worldwide were living with HIV in 2004. Assuming that number continues to increase at a relative growth rate of 3.2% compounded continuously, estimate the number of people living with HIV in
(A) 2010 (B) 2015

30. **AIDS EPIDEMIC** The World Health Organization estimated that there were 3.1 million deaths worldwide from HIV/AIDS during the year 2004. Assuming that number continues to increase at a relative growth rate of 4.3% compounded continuously, estimate the number of deaths from HIV/AIDS during the year
(A) 2008 (B) 2012

31. **NEWTON'S LAW OF COOLING** This law states that the rate at which an object cools is proportional to the difference in temperature between the object and its surrounding medium. The temperature T of the object t hours later is given by

$$T = T_m + (T_0 - T_m)e^{-kt}$$

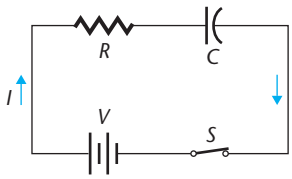
where T_m is the temperature of the surrounding medium and T_0 is the temperature of the object at $t = 0$. Suppose a bottle of wine at a room temperature of 72°F is placed in the refrigerator to cool before a dinner party. If the temperature in the refrigerator is kept at 40°F and $k = 0.4$, find the temperature of the wine, to the nearest degree, after 3 hours. (In Exercises 4-5 we will find out how to determine k.)

32. **NEWTON'S LAW OF COOLING** Refer to Problem 31. What is the temperature, to the nearest degree, of the wine after 5 hours in the refrigerator?

33. **PHOTOGRAPHY** An electronic flash unit for a camera is activated when a capacitor is discharged through a filament of wire. After the flash is triggered, and the capacitor is discharged, the circuit (see the figure) is connected and the battery pack generates a current to recharge the capacitor. The time it takes for the capacitor to recharge is called the *recycle time*. For a particular flash unit using a 12-volt battery pack, the charge q, in coulombs, on the capacitor t seconds after recharging has started is given by

$$q = 0.0009(1 - e^{-0.2t})$$

Find the value that q approaches as t increases without bound and interpret.



34. **MEDICINE** An electronic heart pacemaker uses the same type of circuit as the flash unit in Problem 33, but it is designed so that the capacitor discharges 72 times a minute. For a particular pacemaker, the charge on the capacitor t seconds after it starts recharging is given by

$$q = 0.000\,008(1 - e^{-2t})$$

Find the value that q approaches as t increases without bound and interpret.

35. **WILDLIFE MANAGEMENT** A herd of 20 white-tailed deer is introduced to a coastal island where there had been no deer before. Their population is predicted to increase according to the logistic curve

$$A = \frac{100}{1 + 4e^{-0.14t}}$$

where A is the number of deer expected in the herd after t years.
(A) How many deer will be present after 2 years? After 6 years? Round answers to the nearest integer.
(B) How many years will it take for the herd to grow to 50 deer? Round answer to the nearest integer.
(C) Does A approach a limiting value as t increases without bound? Explain.

36. **TRAINING** A trainee is hired by a computer manufacturing company to learn to test a particular model of a personal computer after it comes off the assembly line. The learning curve for an average trainee is given by

$$A = \frac{200}{4 + 21e^{-0.1t}}$$

where A is the number of computers an average trainee can test per day after t days of training.
(A) How many computers can an average trainee be expected to test after 3 days of training? After 6 days? Round answers to the nearest integer.
(B) How many days will it take until an average trainee can test 30 computers per day? Round answer to the nearest integer.
(C) Does A approach a limiting value as t increases without bound? Explain.

Problems 37–40 require a graphing calculator or a computer that can calculate exponential and logistic regression models for a given data set.

37. **DEPRECIATION** Table 4 gives the market value of a minivan (in dollars) x years after its purchase. Find an exponential regression model of the form $y = ab^x$ for this data set. Round to four significant digits. Estimate the purchase price of the van. Estimate the value of the van 10 years after its purchase. Round answers to the nearest dollar.

Table 4

x	Value ($)
1	12,575
2	9,455
3	8,115
4	6,845
5	5,225
6	4,485

Source: Kelley Blue Book

38. **DEPRECIATION** Table 5 gives the market value of a luxury sedan (in dollars) x years after its purchase. Find an exponential regression model of the form $y = ab^x$ for this data set. Estimate the purchase price of the sedan. Estimate the value of the sedan 10 years after its purchase. Round answers to the nearest dollar.

Table 5

x	Value ($)
1	23,125
2	19,050
3	15,625
4	11,875
5	9,450
6	7,125

Source: Kelley Blue Book

39. NUCLEAR POWER Table 6 gives data on nuclear power generation by region for the years 1980–1999.

Table 6 Nuclear Power Generation

	(Billion kilowatt-hours)	
Year	North America	Central and South America
1980	287.0	2.2
1985	440.8	8.4
1990	649.0	9.0
1995	774.4	9.5
1998	750.2	10.3
1999	807.5	10.5

Source: U.S. Energy Information Administration

(A) Let x represent time in years with $x = 0$ representing 1980. Find a logistic regression model

$$y = \frac{c}{1 + ae^{-bx}}$$

for the generation of nuclear power in North America. (Round the constants a, b, and c to three significant digits.)
(B) Use the logistic regression model to predict the generation of nuclear power in North America in 2010.

40. NUCLEAR POWER Refer to Table 6.
(A) Let x represent time in years with $x = 0$ representing 1980. Find a logistic regression model

$$y = \frac{c}{1 + ae^{-bx}}$$

for the generation of nuclear power in Central and South America. (Round the constants a, b, and c to three significant digits.)
(B) Use the logistic regression model to predict the generation of nuclear power in Central and South America in 2010.

4-3 Logarithmic Functions

- Defining Logarithmic Functions
- Converting Between Logarithmic Form and Exponential Form
- Properties of Logarithmic Functions
- Common and Natural Logarithms
- The Change-of-Base Formula

Solving an equation like $3^x = 9$ is easy: We know that $3^2 = 9$, so $x = 2$ is the solution. But what about an equation like $3^x = 20$? There probably is an exponent x between 2 and 3 for which 3^x is 20, but its exact value is not at all clear.

Compare this situation to an equation like $x^2 = 9$. This is easy to solve because we know that 3^2 and $(-3)^2$ are both 9. But what about $x^2 = 20$? To solve this equation, we needed to introduce a new function to be the opposite of the squaring function. This, of course, is the function $f(x) = \sqrt{x}$.

In this section, we will do something very similar with exponential functions. In the first section of this chapter, we learned that exponential functions are one-to-one, so we can define their inverses. These are known as the *logarithmic functions.*

Defining Logarithmic Functions

The exponential function $f(x) = b^x$ for $b > 0$, $b \neq 1$, is a one-to-one function, and therefore has an inverse. Its inverse, denoted $f^{-1}(x) = \log_b x$ (read "log to the base b of x") is called the *logarithmic function with base b*. Just like exponentials, there are different logarithmic functions for each positive base other than 1. A point (x, y)

is on the graph of $f^{-1} = \log_b x$ if and only if the point (y, x) is on the graph of $f = b^x$. In other words,

$$y = \log_b x \text{ if and only if } x = b^y$$

In a specific example,

$$y = \log_2 x \text{ if and only if } x = 2^y, \text{ and}$$

$\log_2 x$ is the power to which 2 must be raised to obtain x: $2^{\log_2 x} = 2^y = x$.

We can use this fact to learn some things about the logarithmic functions from our knowledge of exponential functions. For example, the graph of $f^{-1}(x) = \log_b x$ is the graph of $f(x) = b^x$ reflected through the line $y = x$. Also, the domain of $f^{-1}(x) = \log_b x$ is the range of $f(x) = b^x$, and vice versa.

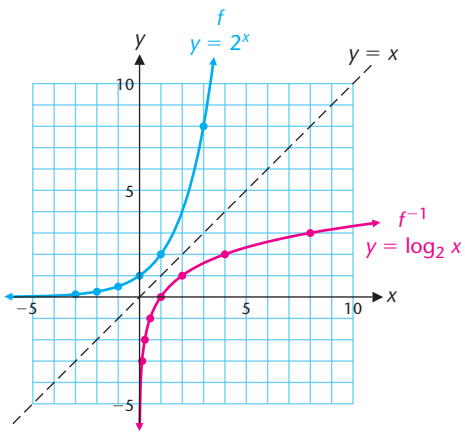
In Example 1, we will use information about $f(x) = 2^x$ to graph its inverse, $f^{-1}(x) = \log_2 x$.

EXAMPLE 1 Graphing a Logarithmic Function

Make a table of values for $f(x) = 2^x$ and reverse the ordered pairs to obtain a table of values for $f^{-1}(x) = \log_2 x$. Then use both tables to graph $f(x)$ and $f^{-1}(x)$ on the same set of axes.

SOLUTION

We chose to evaluate f for integer values from -3 to 3. The tables are shown here, along with the graph (Fig. 1). Note the important comments about domain and range below the graph.



DOMAIN of $f = (-\infty, \infty)$ = RANGE of f^{-1}
RANGE of $f = (0, \infty)$ = DOMAIN of f^{-1}

f		f^{-1}	
x	$y = 2^x$	x	$y = \log_2 x$
-3	$\frac{1}{8}$	$\frac{1}{8}$	-3
-2	$\frac{1}{4}$	$\frac{1}{4}$	-2
-1	$\frac{1}{2}$	$\frac{1}{2}$	-1
0	1	1	0
1	2	2	1
2	4	4	2
3	8	8	3

Ordered pairs reversed

› **Figure 1** Logarithmic function with base 2.

MATCHED PROBLEM 1

Repeat Example 1 for $f(x) = (\frac{1}{2})^x$ and $f^{-1}(x) = \log_{1/2} x$.

› DEFINITION 1 Logarithmic Function

For $b > 0, b \neq 1$, the inverse of $f(x) = b^x$, denoted $f^{-1}(x) = \log_b x$, is the **logarithmic function** with base b.

Logarithmic form		Exponential form
$y = \log_b x$	is equivalent to	$x = b^y$

The log to the base b of x is the exponent to which b must be raised to obtain x. For example,

$y = \log_{10} x$	is equivalent to	$x = 10^y$
$y = \log_e x$	is equivalent to	$x = e^y$

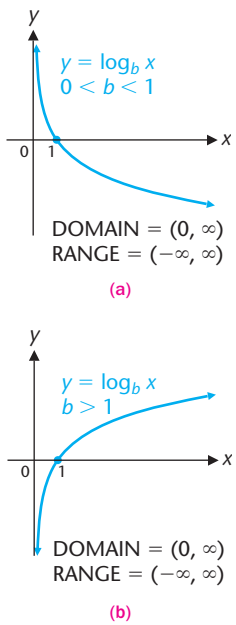
Remember: A logarithm is an exponent.

It is very important to remember that the equations $y = \log_b x$ and $x = b^y$ define the same function, and as such can be used interchangeably.

Because the domain of an exponential function includes all real numbers and its range is the set of positive real numbers, the domain of a logarithmic function is the set of all positive real numbers and its range is the set of all real numbers. Thus, $\log_{10} 3$ is defined, but $\log_{10} 0$ and $\log_{10}(-5)$ are not defined.

In short, the function $y = \log_b x$ for any b is only defined for positive x values. Typical logarithmic curves are shown in Figure 2. Notice that in each case, the y axis is a vertical asymptote for the graph.

The graphs in Example 1 and Figure 2 suggest that logarithmic graphs share some common properties. Several of these properties are listed in Theorem 1. It might be helpful in understanding them to review Theorem 1 in Section 4-1. Each of these properties is a consequence of a corresponding property of exponential graphs.



› **Figure 2** Typical logarithmic graphs.

› THEOREM 1 Properties of Graphs of Logarithmic Functions

Let $f(x) = \log_b x$ be a logarithmic function, $b > 0, b \neq 1$. Then the graph of $f(x)$:

1. Is continuous on its domain $(0, \infty)$
2. Has no sharp corners
3. Passes through the point $(1, 0)$
4. Lies to the right of the y axis, which is a vertical asymptote
5. Is increasing as x increases if $b > 1$; is decreasing as x increases if $0 < b < 1$
6. Intersects any horizontal line exactly once, so is one-to-one

››› EXPLORE-DISCUSS 1

For the exponential function $f(x) = (\frac{2}{3})^x$, graph f and $y = x$ on the same coordinate system. Then sketch the graph of f^{-1}. Use the DRAW INVERSE command on a graphing calculator to check your work. Discuss the domains and ranges of f and its inverse. By what other name is f^{-1} known?

› Converting Between Logarithmic Form and Exponential Form

We now look into the matter of converting logarithmic forms to equivalent exponential forms, and vice versa. Throughout the remainder of the chapter, it will be useful to sometimes convert a logarithmic expression into the equivalent exponential form. At other times, it will be useful to do the reverse.

EXAMPLE 2 **Logarithmic–Exponential Conversions**

Change each logarithmic form to an equivalent exponential form.

(A) $\log_2 8 = 3$ (B) $\log_{25} 5 = \frac{1}{2}$ (C) $\log_2 (\frac{1}{4}) = -2$

SOLUTIONS

(A) $\log_2 8 = 3$ is equivalent to $8 = 2^3$.
(B) $\log_{25} 5 = \frac{1}{2}$ is equivalent to $5 = 25^{1/2}$.
(C) $\log_2 (\frac{1}{4}) = -2$ is equivalent to $\frac{1}{4} = 2^{-2}$.

Note that in each case, the base of the logarithm matches the base of the corresponding exponent.

MATCHED PROBLEM 2

Change each logarithmic form to an equivalent exponential form.

(A) $\log_3 27 = 3$ (B) $\log_{36} 6 = \frac{1}{2}$ (C) $\log_3 (\frac{1}{9}) = -2$

EXAMPLE 3 Logarithmic–Exponential Conversions

Change each exponential form to an equivalent logarithmic form.

(A) $49 = 7^2$ (B) $3 = \sqrt{9}$ (C) $\frac{1}{5} = 5^{-1}$

SOLUTIONS

(A) $49 = 7^2$ is equivalent to $\log_7 49 = 2$.
(B) $3 = \sqrt{9}$ is equivalent to $\log_9 3 = \frac{1}{2}$.
(C) $\frac{1}{5} = 5^{-1}$ is equivalent to $\log_5 (\frac{1}{5}) = -1$.

Again, the bases match.

MATCHED PROBLEM 3

Change each exponential form to an equivalent logarithmic form.

(A) $64 = 4^3$ (B) $2 = \sqrt[3]{8}$ (C) $\frac{1}{16} = 4^{-2}$

To gain a little deeper understanding of logarithmic functions and their relationship to the exponential functions, we will consider a few problems where we want to find x, b, or y in $y = \log_b x$, given the other two values. All values were chosen so that the problems can be solved without a calculator. In each case, converting to the equivalent exponential form is useful.

EXAMPLE 4 Solutions of the Equation $y = \log_b x$

Find x, b, or y as indicated.

(A) Find y: $y = \log_4 8$. (B) Find x: $\log_3 x = -2$. (C) Find b: $\log_b 81 = 4$.

SOLUTIONS

(A) Write $y = \log_4 8$ in equivalent exponential form.

$$8 = 4^y \qquad \text{Write each number to the same base 2.}$$
$$2^3 = 2^{2y} \qquad \text{Recall that } b^m = b^n \text{ if and only if } m = n.$$
$$2y = 3$$
$$y = \tfrac{3}{2}$$

We conclude that $\frac{3}{2} = \log_4 8$.

(B) Write $\log_3 x = -2$ in equivalent exponential form.

$$x = 3^{-2}$$
$$= \frac{1}{3^2} = \frac{1}{9}$$

We conclude that $\log_3 (\frac{1}{9}) = -2$.

(C) Write $\log_b 81 = 4$ in equivalent exponential form:

$$81 = b^4 \qquad \text{Write 81 as a fourth power.}$$
$$3^4 = b^4 \qquad b \text{ could be 3 or } -3\text{, but the base of a logarithm must be positive.}$$
$$b = 3$$

We conclude that $\log_3 81 = 4$.

MATCHED PROBLEM 4

Find x, b, or y as indicated.

(A) Find y: $y = \log_9 27$. (B) Find x: $\log_2 x = -3$. (C) Find b: $\log_b 100 = 2$.

Properties of Logarithmic Functions

Some of the properties of exponential functions that we studied in Section 4-1 can be used to develop corresponding properties of logarithmic functions. Several of these important properties of logarithmic functions are listed in Theorem 1. We will justify them individually.

THEOREM 2 Properties of Logarithmic Functions

If b, M, and N are positive real numbers, $b \neq 1$, and p and x are real numbers, then

1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\log_b b^x = x$
4. $b^{\log_b x} = x, x > 0$
5. $\log_b M = \log_b N$ if and only if $M = N$
6. $\log_b MN = \log_b M + \log_b N$
7. $\log_b \dfrac{M}{N} = \log_b M - \log_b N$
8. $\log_b M^p = p \log_b M$

››› CAUTION ›››

1. In Properties 3 and 4, it's essential that the base of the exponential and the base of the logarithm are the same.
2. Properties 6 and 7 are often misinterpreted, so you should examine them carefully.

$$\frac{\log_b M}{\log_b N} \neq \log_b M - \log_b N$$

$\log_b M - \log_b N = \log_b \frac{M}{N}$; $\frac{\log_b M}{\log_b N}$ cannot be simplified.

$$\log_b (M + N) \neq \log_b M + \log_b N$$

$\log_b M + \log_b N = \log_b MN$; $\log_b (M + N)$ cannot be simplified.

Now we will justify properties in Theorem 2.

1. $\log_b 1 = 0$ because $b^0 = 1$
2. $\log_b b = 1$ because $b^1 = b$
3. and 4. These are simply another way to state that $f(x) = b^x$ and $f^{-1}(x) = \log_b x$ are inverse functions. Property 3 can be written as $f^{-1}(f(x)) = x$ for all x in the domain of f. Property 4 can be written as $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}. This matches our characterization of inverse functions in Theorem 5, Section 1-6. Collectively, these properties say that if you apply an exponential function and a logarithmic function with the same base consecutively (in either order) you end up with the same value you started with.
5. This follows from the fact that logarithmic functions are one-to-one.

Properties 6, 7, and 8 are used often in manipulating logarithmic expressions. We will justify them in Problems 111 and 112 in Exercises 4-3, and Problem 68 in the Chapter 4 Review Exercises.

EXAMPLE 5 Using Logarithmic Properties

Simplify, using the properties in Theorem 2.

(A) $\log_e 1$ (B) $\log_{10} 10$ (C) $\log_e e^{2x+1}$
(D) $\log_{10} 0.01$ (E) $10^{\log_{10} 7}$ (F) $e^{\log_e x^2}$

SOLUTIONS

(A) $\log_e 1 = 0$ Property 1
(B) $\log_{10} 10 = 1$ Property 2
(C) $\log_e e^{2x+1} = 2x + 1$ Property 3
(D) $\log_{10} 0.01 = \log_{10} 10^{-2} = -2$ Property 3
(E) $10^{\log_{10} 7} = 7$ Property 4
(F) $e^{\log_e x^2} = x^2$ Property 4 ◉

MATCHED PROBLEM 5

Simplify, using the properties in Theorem 2.

(A) $\log_{10} 10^{-5}$ (B) $\log_5 25$ (C) $\log_{10} 1$
(D) $\log_e e^{m+n}$ (E) $10^{\log_{10} 4}$ (F) $e^{\log_e (x^4+1)}$

Common and Natural Logarithms

To work with logarithms effectively, we will need to be able to calculate (or at least approximate) the logarithms of any positive number to a variety of bases. Historically, tables were used for this purpose, but now calculators are used because they are faster and can find far more values than any table can possibly include.

Of all possible bases, there are two that are used most often. **Common logarithms** are logarithms with base 10. **Natural logarithms** are logarithms with base e. Most calculators have a function key labeled "log" and a function key labeled "ln." The former represents the common logarithmic function and the latter the natural logarithmic function. In fact, "log" and "ln" are both used in most math books, and whenever you see either used in this book without a base indicated, they should be interpreted as follows:

LOGARITHMIC FUNCTIONS

$$y = \log x = \log_{10} x \quad \text{Common logarithmic function}$$
$$y = \ln x = \log_e x \quad \text{Natural logarithmic function}$$

EXPLORE-DISCUSS 2

(A) Sketch the graph of $y = 10^x$, $y = \log x$, and $y = x$ in the same coordinate system and state the domain and range of the common logarithmic function.

(B) Sketch the graph of $y = e^x$, $y = \ln x$, and $y = x$ in the same coordinate system and state the domain and range of the natural logarithmic function.

EXAMPLE 6 Calculator Evaluation of Logarithms

Use a calculator to evaluate each to six decimal places.

(A) $\log 3{,}184$ (B) $\ln 0.000\ 349$ (C) $\log(-3.24)$

SOLUTIONS

(A) $\log 3{,}184 = 3.502\ 973$ (B) $\ln 0.000\ 349 = -7.960\ 439$

(C) $\log(-3.24) =$ Error

Why is an error indicated in part C? Because -3.24 is not in the domain of the log function. [*Note:* Calculators display error messages in various ways. Some calculators use a more advanced definition of logarithmic functions that involves complex numbers. They will display an ordered pair, representing a complex number, as the value of $\log(-3.24)$, rather than an error message. You should interpret such a display as indicating that the number entered is not in the domain of the logarithmic function as we have defined it.]

MATCHED PROBLEM 6

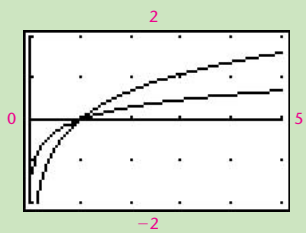
Use a calculator to evaluate each to six decimal places.

(A) $\log 0.013\ 529$ (B) $\ln 28.693\ 28$ (C) $\ln(-0.438)$

When working with common and natural logarithms, we will follow the common practice of using the equal sign "=" where it might be technically correct to use the approximately equal sign "≈." No harm is done as long as we keep in mind that in a statement such as $\log 3.184 = 0.503$, the number on the right is only assumed accurate to three decimal places and is not exact.

››› EXPLORE-DISCUSS 3

Graphs of the functions $f(x) = \log x$ and $g(x) = \ln x$ are shown in the graphing calculator display of Figure 3. Which graph belongs to which function? It appears from the display that one of the functions may be a constant multiple of the other. Is that true? Find and discuss the evidence for your answer.



› Figure 3

EXAMPLE 7 Calculator Evaluation of Logarithms

Use a calculator to evaluate each expression to three decimal places.

(A) $\dfrac{\log 2}{\log 1.1}$ (B) $\log \dfrac{2}{1.1}$ (C) $\log 2 - \log 1.1$

SOLUTIONS

(A) $\dfrac{\log 2}{\log 1.1} = 7.273$ Enter as (log 2) ÷ (log 1.1).

(B) $\log \dfrac{2}{1.1} = 0.260$ Enter as log (2 ÷ 1.1).

(C) $\log 2 - \log 1.1 = 0.260$. Note that $\dfrac{\log 2}{\log 1.1} \neq \log 2 - \log 1.1$, but $\log \dfrac{2}{1.1} = \log 2 - \log 1.1$ (see Theorem 2).

MATCHED PROBLEM 7

Use a calculator to evaluate each to three decimal places.

(A) $\dfrac{\ln 3}{\ln 1.08}$ (B) $\ln \dfrac{3}{1.08}$ (C) $\ln 3 - \ln 1.08$

We now turn to the second problem: Given the logarithm of a number, find the number. To solve this problem, we make direct use of the logarithmic–exponential relationships, and change logarithmic expressions into exponential form.

LOGARITHMIC–EXPONENTIAL RELATIONSHIPS

$\log x = y$	is equivalent to	$x = 10^y$
$\ln x = y$	is equivalent to	$x = e^y$

EXAMPLE 8 Solving $\log_b x = y$ for x

Find x to three significant digits, given the indicated logarithms.

(A) $\log x = -9.315$ (B) $\ln x = 2.386$

SOLUTIONS

(A) $\log x = -9.315$ Change to exponential form (Definition 1).

$x = 10^{-9.315}$

$= 4.84 \times 10^{-10}$

Notice that the answer is displayed in scientific notation in the calculator.

(B) $\ln x = 2.386$ Change to exponential form (Definition 1).

$x = e^{2.386}$

$= 10.9$

MATCHED PROBLEM 8

Find x to four significant digits, given the indicated logarithms.

(A) $\ln x = -5.062$ (B) $\log x = 12.0821$

››› EXPLORE-DISCUSS 4

Example 8 was solved algebraically using logarithmic–exponential relationships. Use the INTERSECT command on a graphing calculator to solve this problem graphically. Discuss the relative merits of the two approaches.

The Change-of-Base Formula

How would you find the logarithm of a positive number to a base other than 10 or e? For example, how would you find $\log_3 5.2$? In Example 9 we evaluate this logarithm using several properties of logarithms. Then we develop a change-of-base formula to find such logarithms more easily.

EXAMPLE 9 Evaluating a Base 3 Logarithm

Evaluate $\log_3 5.2$ to four decimal places.

SOLUTION

Let $y = \log_3 5.2$ and proceed as follows:

$$\log_3 5.2 = y \quad \text{Change to exponential form.}$$

$$5.2 = 3^y \quad \text{Apply the natural log (or common log) to each side.}$$

$$\ln 5.2 = \ln 3^y \quad \text{Use } \log_b M^p = p \log_b M.$$

$$\ln 5.2 = y \ln 3 \quad \text{Solve for } y.$$

$$y = \frac{\ln 5.2}{\ln 3}$$

Replace y with $\log_3 5.2$ from the first step, and use a calculator to evaluate the right side:

$$\log_3 5.2 = \frac{\ln 5.2}{\ln 3} = 1.5007$$

MATCHED PROBLEM 9

Evaluate $\log_{0.5} 0.0372$ to four decimal places.

If we repeat the process we used in Example 9 on a generic logarithm, something interesting happens. The goal is to evaluate $\log_b N$, where b is any acceptable base, and N is any positive real number. As in Example 9, let $y = \log_b N$.

$$\begin{aligned}
\log_b N &= y && \text{Write in exponential form.}\\
N &= b^y && \text{Apply natural log to each side.}\\
\ln N &= \ln b^y && \text{Use } \ln b^y = y \ln b \text{ (Property 8, Theorem 2).}\\
\ln N &= y \ln b && \text{Solve for } y.\\
y &= \frac{\ln N}{\ln b}
\end{aligned}$$

This provides a formula for evaluating a logarithm to any base by using natural log:

$$\log_b N = \frac{\ln N}{\ln b}$$

We could also have used log base 10 rather than natural log, and developed an alternative formula:

$$\log_b N = \frac{\log N}{\log b}$$

In fact, the same approach would enable us to rewrite $\log_b N$ in terms of a logarithm with any base we choose!

› THE CHANGE-OF-BASE FORMULA

For any $b > 0$, $b \neq 1$, and any positive real number N,

$$\log_b N = \frac{\log_a N}{\log_a b}$$

where a is any positive number other than 1.

››› EXPLORE-DISCUSS 5

If b is any positive real number different from 1, the change-of-base formula implies that the function $y = \log_b x$ is a constant multiple of the natural logarithmic function; that is, $\log_b x = k \ln x$ for some k.

(A) Graph the functions $y = \ln x$, $y = 2 \ln x$, $y = 0.5 \ln x$, and $y = -3 \ln x$.

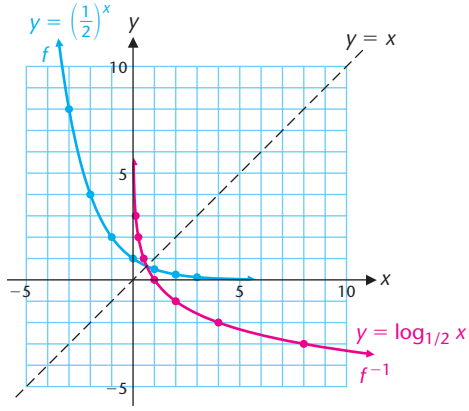
(B) Write each function of part A in the form $y = \log_b x$ by finding the base b to two decimal places.

(C) Is every exponential function $y = b^x$ a constant multiple of $y = e^x$? Explain.

ANSWERS TO MATCHED PROBLEMS

1.

f		f^{-1}	
x	$y = \left(\frac{1}{2}\right)^x$	x	$y = \log_{1/2} x$
-3	8	8	-3
-2	4	4	-2
-1	2	2	-1
0	1	1	0
1	$\frac{1}{2}$	$\frac{1}{2}$	1
2	$\frac{1}{4}$	$\frac{1}{4}$	2
3	$\frac{1}{8}$	$\frac{1}{8}$	3



2. (A) $27 = 3^3$ (B) $6 = 36^{1/2}$ (C) $\frac{1}{9} = 3^{-2}$
3. (A) $\log_4 64 = 3$ (B) $\log_8 2 = \frac{1}{3}$ (C) $\log_4 (\frac{1}{16}) = -2$
4. (A) $y = \frac{3}{2}$ (B) $x = \frac{1}{8}$ (C) $b = 10$
5. (A) -5 (B) 2 (C) 0 (D) $m + n$ (E) 4 (F) $x^4 + 1$
6. (A) $-1.868\,734$ (B) $3.356\,663$ (C) Not possible
7. (A) 14.275 (B) 1.022 (C) 1.022
8. (A) $x = 0.006\,333$ (B) $x = 1.208 \times 10^{12}$ **9.** 4.7486

4-3 Exercises

1. Describe the relationship between logarithmic functions and exponential functions in your own words.

2. Explain why there are infinitely many different logarithmic functions.

3. Why are logarithmic functions undefined for zero and negative inputs?

4. Why is $\log_b 1 = 0$ for any base?

5. Explain how to calculate $\log_5 3$ on a calculator that only has log buttons for base 10 and base e.

6. Using the word "inverse," explain why $\log_b b^x = x$ for any x and any acceptable base b.

Rewrite Problems 7–12 in equivalent exponential form.

7. $\log_3 81 = 4$

8. $\log_5 125 = 3$

9. $\log_{10} 0.001 = -3$

10. $\log_{10} 1{,}000 = 3$

11. $\log_6 \frac{1}{36} = -2$

12. $\log_2 \frac{1}{64} = -6$

Rewrite Problems 13–18 in equivalent logarithmic form.

13. $8 = 4^{3/2}$

14. $9 = 27^{2/3}$

15. $\frac{1}{2} = 32^{-1/5}$

16. $\frac{1}{8} = 2^{-3}$

17. $(\frac{2}{3})^3 = \frac{8}{27}$

18. $(\frac{5}{2})^{-2} = 0.16$

In Problems 19–22, make a table of values similar to the one in Example 1, then use it to graph both functions by hand.

19. $f(x) = 3^x$ $\quad f^{-1}(x) = \log_3 x$

20. $f(x) = (\frac{1}{3})^x$ $\quad f^{-1}(x) = \log_{1/3} x$

21. $f(x) = (\frac{2}{3})^x$ $\quad f^{-1}(x) = \log_{2/3} x$

22. $f(x) = 10^x$ $\quad f^{-1}(x) = \log x$

In Problems 23–38, simplify each expression using Theorem 2.

23. $\log_{16} 1$

24. $\log_{25} 1$

25. $\log_{0.5} 0.5$

26. $\log_7 7$

27. $\log_e e^4$

28. $\log_{10} 10^5$

29. $\log_{10} 0.01$

30. $\log_{10} 100$

31. $\log_3 27$

32. $\log_4 256$

33. $\log_{1/2} 2$

34. $\log_{1/5} (\frac{1}{25})$

35. $e^{\log_e 5}$

36. $e^{\log_e 10}$

37. $\log_5 \sqrt[3]{5}$

38. $\log_2 \sqrt{8}$

In Problems 39–46, evaluate to four decimal places.

39. $\log 49{,}236$

40. $\log 691{,}450$

41. $\ln 54.081$

42. $\ln 19.722$

43. $\log_7 13$

44. $\log_9 78$

45. $\log_5 120.24$

46. $\log_{17} 304.66$

In Problems 47–54, evaluate x to four significant digits.

47. $\log x = 5.3027$

48. $\log x = 1.9168$

49. $\log x = -3.1773$

50. $\log x = -2.0411$

51. $\ln x = 3.8655$

52. $\ln x = 5.0884$

53. $\ln x = -0.3916$

54. $\ln x = -4.1083$

Find x, y, or b, as indicated in Problems 55–72.

55. $\log_2 x = 2$

56. $\log_3 x = 3$

57. $\log_4 16 = y$

58. $\log_8 64 = y$

59. $\log_b 16 = 2$

60. $\log_b 10^{-3} = -3$

61. $\log_b 1 = 0$

62. $\log_b b = 1$

63. $\log_4 x = \frac{1}{2}$

64. $\log_8 x = \frac{1}{3}$

65. $\log_{1/3} 9 = y$

66. $\log_{49} (\frac{1}{7}) = y$

67. $\log_b 1{,}000 = \frac{3}{2}$

68. $\log_b 4 = \frac{2}{3}$

69. $\log_8 x = -\frac{4}{3}$

70. $\log_{25} x = -\frac{3}{2}$

71. $\log_{16} 8 = y$

72. $\log_9 27 = y$

In Problems 73–78, evaluate to three decimal places.

73. $\dfrac{\log 2}{\log 1.15}$

74. $\dfrac{\log 2}{\log 1.12}$

75. $\dfrac{\ln 3}{\ln 1.15}$

76. $\dfrac{\ln 4}{\ln 1.2}$

77. $\dfrac{\ln 150}{2 \ln 3}$

78. $\dfrac{\log 200}{3 \log 2}$

In Problems 79–82, rewrite the expression in terms of $\log x$ and $\log y$.

79. $\log \left(\dfrac{x}{y}\right)$

80. $\log (xy)$

81. $\log (x^4 y^3)$

82. $\log \left(\dfrac{x^2}{\sqrt{y}}\right)$

In Problems 83–86, rewrite the expression as a single log.

83. $\ln x - \ln y$ **84.** $\log_3 x + \log_3 y$

85. $2 \ln x + 5 \ln y - \ln z$ **86.** $\log a - 2 \log b + 3 \log c$

In Problems 87–90, given that $\log x = -2$ and $\log y = 3$, find:

87. $\log (xy)$ **88.** $\log \left(\frac{x}{y}\right)$ **89.** $\log \left(\frac{\sqrt{x}}{y^3}\right)$ **90.** $\log (x^5 y^3)$

In Problems 91–98, use transformations to explain how the graph of g is related to the graph of the given logarithmic function f. Determine whether g is increasing or decreasing, find its domain and asymptote, and sketch the graph of g.

91. $g(x) = 3 + \log_2 x; f(x) = \log_2 x$

92. $g(x) = -4 + \log_3 x; f(x) = \log_3 x$

93. $g(x) = \log_{1/3} (x - 2); f(x) = \log_{1/3} x$

94. $g(x) = \log_{1/2} (x + 3); f(x) = \log_{1/2} x$

95. $g(x) = -1 - \log x; f(x) = \log x$

96. $g(x) = 2 - \log x; f(x) = \log x$

97. $g(x) = 5 - 3 \ln x; f(x) = \ln x$

98. $g(x) = -3 - 2 \ln x; f(x) = \ln x$

In Problems 99–102, find f^{-1}.

99. $f(x) = \log_5 x$ **100.** $f(x) = \log_{1/3} x$

101. $f(x) = 4 \log_3 (x + 3)$ **102.** $f(x) = 2 \log_2 (x - 5)$

103. Let $f(x) = \log_3 (2 - x)$.
(A) Find f^{-1}. (B) Graph f^{-1}.
(C) Reflect the graph of f^{-1} in the line $y = x$ to obtain the graph of f.

104. Let $f(x) = \log_2 (-3 - x)$.
(A) Find f^{-1}. (B) Graph f^{-1}.
(C) Reflect the graph of f^{-1} in the line $y = x$ to obtain the graph of f.

105. What is wrong with the following "proof" that 3 is less than 2?

$$
\begin{aligned}
1 &< 3 && \text{Divide both sides by 27.}\\
\tfrac{1}{27} &< \tfrac{3}{27}\\
\tfrac{1}{27} &< \tfrac{1}{9}\\
(\tfrac{1}{3})^3 &< (\tfrac{1}{3})^2\\
\log (\tfrac{1}{3})^3 &< \log (\tfrac{1}{3})^2\\
3 \log \tfrac{1}{3} &< 2 \log \tfrac{1}{3} && \text{Divide both sides by } \log \tfrac{1}{3}.\\
3 &< 2
\end{aligned}
$$

106. What is wrong with the following "proof" that 1 is greater than 2?

$$
\begin{aligned}
3 &> 2 && \text{Multiply both sides by } \log \tfrac{1}{2}.\\
3 \log \tfrac{1}{2} &> 2 \log \tfrac{1}{2}\\
\log (\tfrac{1}{2})^3 &> \log (\tfrac{1}{2})^2\\
(\tfrac{1}{2})^3 &> (\tfrac{1}{2})^2\\
\tfrac{1}{8} &> \tfrac{1}{4} && \text{Multiply both sides by 8.}\\
1 &> 2
\end{aligned}
$$

*The polynomials in Problems 107–110, called **Taylor polynomials**, can be used to approximate the function $g(x) = \ln (1 + x)$. To illustrate this approximation graphically, in each problem, graph $g(x) = \ln (1 + x)$ and the indicated polynomial in the same viewing window, $-1 \le x \le 3$ and $-2 \le y \le 2$.*

107. $P_1(x) = x - \frac{1}{2}x^2$ **108.** $P_2(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$

109. $P_3(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$

110. $P_4(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5$

111. Prove that for any positive M, N, and b ($b \ne 1$), $\log_b (\frac{M}{N}) = \log_b M - \log_b N$. (*Hint:* Start by writing $u = \log_b M$ and $v = \log_b N$ and changing each to exponential form.)

112. Prove that for any positive integer p and any positive b and M ($b \ne 1$), $\log_b M^p = p \log_b M$. [*Hint:* Write M^p as $M \cdot M \cdot \cdots M$ (p factors).]

4-4 Logarithmic Models

- Logarithmic Scales
- Data Analysis and Regression

Logarithmic functions occur naturally as the inverses of exponential functions. But that's not to say that they are not useful in their own right. Some of these uses are probably familiar to you, but you might not have realized that they involved logarithmic functions.

In this section, we will study logarithmic scales that are used to compare the intensity of sounds, the severity of earthquakes, and the brightness of distant stars. We will also look at using regression to model data with a logarithmic function, and discuss what sort of data is likely to fit such a model.

Logarithmic Scales

SOUND INTENSITY: The human ear is able to hear sound over an incredible range of intensities. The loudest sound a healthy person can hear without damage to the eardrum has an intensity 1 trillion (1,000,000,000,000) times that of the softest sound a person can hear. If we were to use these intensities as a scale for measuring volume, we would be stuck using numbers from zero all the way to the trillions, which seems cumbersome, if not downright silly. In the last section, we saw that logarithmic functions increase very slowly. We can take advantage of this to create a scale for sound intensity that is much more condensed, and therefore more manageable.

The decibel scale for sound intensity is an example of such a scale. The **decibel,** named after the inventor of the telephone, Alexander Graham Bell (1847–1922), is defined as follows:

$$D = 10 \log \frac{I}{I_0} \qquad \text{Decibel scale} \tag{1}$$

where D is the **decibel level** of the sound, I is the **intensity** of the sound measured in watts per square meter (W/m^2), and I_0 is the intensity of the least audible sound that an average healthy young person can hear. The latter is standardized to be $I_0 = 10^{-12}$ watts per square meter. Table 1 lists some typical sound intensities from familiar sources. In Example 1 and Problems 1 and 2 in Exercises 4-4, we will calculate the decibel levels for these sounds.

Table 1 Typical Sound Intensities

Sound Intensity (W/m^2)	Sound
1.0×10^{-12}	Threshold of hearing
5.2×10^{-10}	Whisper
3.2×10^{-6}	Normal conversation
8.5×10^{-4}	Heavy traffic
3.2×10^{-3}	Jackhammer
1.0×10^{0}	Threshold of pain
8.3×10^{2}	Jet plane with afterburner

EXAMPLE 1 Sound Intensity

(A) Find the number of decibels from a whisper with sound intensity 5.20×10^{-10} watts per square meter, then from heavy traffic at 8.5×10^{-4} watts per square meter. Round your answers to two decimal places.

(B) How many times larger is the sound intensity of heavy traffic compared to a whisper?

SOLUTIONS

(A) We can use the decibel formula (1) with $I_0 = 10^{-12}$. First, we use $I = 5.2 \times 10^{-10}$:

$$\begin{aligned} D &= 10 \log \frac{I}{I_0} && \text{Substitute } I = 5.2 \times 10^{-10},\ I_0 = 10^{-12}. \\ &= 10 \log \frac{5.2 \times 10^{-10}}{10^{-12}} && \text{Simplify the fraction.} \\ &= 10 \log 520 \\ &= 27.16 \text{ decibels} \end{aligned}$$

Next, for heavy traffic:

$$\begin{aligned} D &= 10 \log \frac{I}{I_0} && \text{Substitute } I = 8.5 \times 10^{-4},\ I_0 = 10^{-12}. \\ &= 10 \log \frac{8.5 \times 10^{-4}}{10^{-12}} && \text{Simplify the fraction.} \\ &= 10 \log 850{,}000{,}000 \\ &= 89.29 \text{ decibels} \end{aligned}$$

(B) Dividing the larger intensity by the smaller,

$$\frac{8.5 \times 10^{-4}}{5.2 \times 10^{-10}} = 1{,}634{,}615.4$$

we see that the sound intensity of heavy traffic is more than 1.6 million times as great as the intensity of a whisper!

MATCHED PROBLEM 1

Find the number of decibels from a jackhammer with sound intensity 3.2×10^{-3} watts per square meter. Compute the answer to two decimal places.

››› EXPLORE-DISCUSS 1

Suppose that you are asked to draw a graph of the data in Table 1, with sound intensities on the x axis, and the corresponding decibel levels on the y axis.

(A) What would be the coordinates of the point corresponding to a jackhammer (see Matched Problem 1)?

(B) Suppose the axes of this graph are labeled as follows: Each tick mark on the x axis corresponds to the intensity of the least audible sound (10^{-12} watts per square meter), and each tick mark on the y axis corresponds to 1 decibel. If there is $\frac{1}{8}$ inch between all tick marks, how far away from the x axis is the point you found in part A? From the y axis? (Give the first answer in inches and the second in miles!) Discuss your result.

EARTHQUAKE INTENSITY: The energy released by the largest earthquake recorded, measured in joules, is about 100 billion (100,000,000,000) times the energy released by a small earthquake that is barely felt. Over the past 150 years several people from various countries have devised different types of measures of earthquake magnitudes so that their severity could be compared without using tremendously large numbers. In 1935 the California seismologist Charles Richter devised a logarithmic scale that bears his name and is still widely used in the United States. The **magnitude** of an earthquake M on the **Richter scale*** is given as follows:

$$M = \frac{2}{3}\log\frac{E}{E_0} \quad \text{Richter scale} \tag{2}$$

where E is the energy released by the earthquake, measured in joules, and E_0 is the energy released by a very small reference earthquake, which has been standardized to be

$$E_0 = 10^{4.40} \text{ joules}$$

The destructive power of earthquakes relative to magnitudes on the Richter scale is indicated in Table 2.

Table 2 The Richter Scale

Magnitude on Richter Scale	Destructive Power
$M < 4.5$	Small
$4.5 < M < 5.5$	Moderate
$5.5 < M < 6.5$	Large
$6.5 < M < 7.5$	Major
$7.5 < M$	Greatest

*Originally, Richter defined the magnitude of an earthquake in terms of logarithms of the maximum seismic wave amplitude, in thousandths of a millimeter, measured on a standard seismograph. Equation (2) gives essentially the same magnitude that Richter obtained for a given earthquake but in terms of logarithms of the energy released by the earthquake.

EXAMPLE 2 Earthquake Intensity

The 1906 San Francisco earthquake released approximately 5.96×10^{16} joules of energy. Another quake struck the Bay Area just before game 3 of the 1989 World Series, releasing 1.12×10^{15} joules of energy.

(A) Find the magnitude of each earthquake on the Richter scale. Round your answers to two decimal places.

(B) How many times more energy did the 1906 earthquake release than the one in 1989?

SOLUTIONS

(A) We can use the magnitude formula (2) with $E_0 = 10^{4.40}$. First, for the 1906 earthquake, we use $E = 5.96 \times 10^{16}$:

$$\begin{aligned} M &= \frac{2}{3}\log\frac{E}{E_0} && \text{Substitute } E = 5.96 \times 10^{16},\ E_0 = 10^{4.40}. \\ &= \frac{2}{3}\log\frac{5.96 \times 10^{16}}{10^{4.40}} \\ &= 8.25 \end{aligned}$$

Next, for the 1989 earthquake:

$$\begin{aligned} M &= \frac{2}{3}\log\frac{E}{E_0} && \text{Substitute } E = 1.12 \times 10^{15},\ E_0 = 10^{4.40}. \\ &= \frac{2}{3}\log\frac{1.12 \times 10^{15}}{10^{4.40}} \\ &= 7.1 \end{aligned}$$

(B) Dividing the larger energy release by the smaller,

$$\frac{5.96 \times 10^{16}}{1.12 \times 10^{15}} = 53.2$$

we see that the 1906 earthquake released 53.2 times as much energy as the 1989 quake.

MATCHED PROBLEM 2

The 1985 earthquake in central Chile released approximately 1.26×10^{16} joules of energy. What was its magnitude on the Richter scale? Compute the answer to two decimal places.

EXAMPLE 3 Earthquake Intensity

If the energy release of one earthquake is 1,000 times that of another, how much larger is the Richter scale reading of the larger than the smaller?

SOLUTION

Let

$$M_1 = \frac{2}{3}\log\frac{E_1}{E_0} \quad \text{and} \quad M_2 = \frac{2}{3}\log\frac{E_2}{E_0}$$

be the Richter equations for the smaller and larger earthquakes, respectively. Since the larger earthquake released 1,000 times as much energy, we can write $E_2 = 1{,}000E_1$.

$$\begin{aligned}
M_2 &= \frac{2}{3}\log\frac{E_2}{E_0} && \text{Substitute } 1{,}000E_1 \text{ for } E_2.\\
&= \frac{2}{3}\log\frac{1{,}000E_1}{E_0} && \text{Use } \log(MN) = \log M + \log N.\\
&= \frac{2}{3}\left(\log 1{,}000 + \log\frac{E_1}{E_0}\right) && \log 1{,}000 = \log 10^3 = 3\\
&= \frac{2}{3}\left(3 + \log\frac{E_1}{E_0}\right) && \text{Distribute.}\\
&= \frac{2}{3}(3) + \frac{2}{3}\log\frac{E_1}{E_0} && \tfrac{2}{3}\log\tfrac{E_1}{E_0} \text{ is } M_1!\\
&= 2 + M_1
\end{aligned}$$

Thus, an earthquake with 1,000 times the energy of another has a Richter scale reading of 2 more than the other.

MATCHED PROBLEM 3

If the energy release of one earthquake is 10,000 times that of another, how much larger is the Richter scale reading of the larger than the smaller?

ROCKET FLIGHT: The theory of rocket flight uses advanced mathematics and physics to show that the **velocity** v of a rocket at burnout (depletion of fuel supply) is given by

$$v = c \ln\frac{W_t}{W_b} \quad \textbf{Rocket equation} \tag{3}$$

where c is the exhaust velocity of the rocket engine, W_t is the takeoff weight (fuel, structure, and payload), and W_b is the burnout weight (structure and payload).

Because of the Earth's atmospheric resistance, a launch vehicle velocity of at least 9.0 kilometers per second is required to achieve the minimum altitude needed for a stable orbit. Formula (3) indicates that to increase velocity v, either the weight ratio W_t/W_b must be increased or the exhaust velocity c must be increased. The weight ratio can be increased by the use of solid fuels, and the exhaust velocity can be increased by improving the fuels, solid or liquid.

EXAMPLE 4 **Rocket Flight Theory**

A typical single-stage, solid-fuel rocket may have a weight ratio $W_t/W_b = 18.7$ and an exhaust velocity $c = 2.38$ kilometers per second. Would this rocket reach a launch velocity of 9.0 kilometers per second?

SOLUTION

We can use the rocket equation (3) with $c = 2.38$ and $W_t/W_b = 18.7$:

$$\begin{aligned} v &= c \ln \frac{W_t}{W_b} \\ &= 2.38 \ln 18.7 \\ &= 6.97 \text{ kilometers per second} \end{aligned}$$

The velocity of the launch vehicle is far short of the 9.0 kilometers per second required to achieve orbit. This is why multiple-stage launchers are used—the dead-weight from a preceding stage can be jettisoned into the ocean when the next stage takes over.

MATCHED PROBLEM 4

A launch vehicle using liquid fuel, such as a mixture of liquid hydrogen and liquid oxygen, can produce an exhaust velocity of $c = 4.7$ kilometers per second. However, the weight ratio W_t/W_b must be low—around 5.5 for some vehicles—because of the increased structural weight to accommodate the liquid fuel. How much more or less than the 9.0 kilometers per second required to reach orbit will be achieved by this vehicle?

› Data Analysis and Regression

Based on the logarithmic graphs we studied in the last section, when a quantity increases relatively rapidly at first, but then levels off and increases very slowly, it might be a good candidate to be modeled by a logarithmic function. Most graphing calculators with regression commands can fit functions of the form $y = a + b \ln x$ to a set of data points using the same techniques we used earlier for other types of regression.

EXAMPLE 5 Home Ownership Rates

Table 3 Home Ownership Rates

Year	Home Ownership Rate (%)
1940	43.6
1950	55.0
1960	61.9
1970	62.9
1980	64.4
1990	64.2
2000	67.4

The U.S. Census Bureau published the data in Table 3 on home ownership rates.

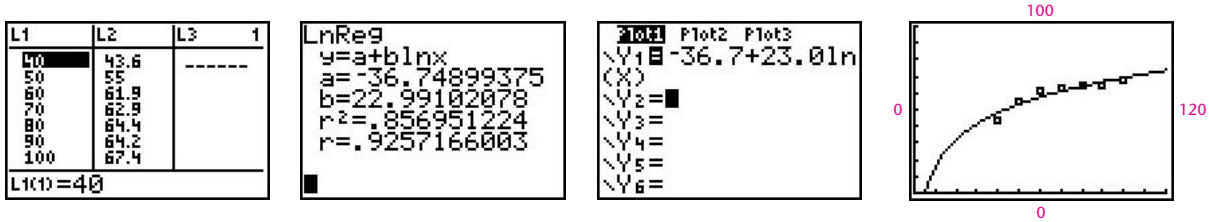
(A) Let x represent time in years with $x = 0$ representing 1900, and let y represent the corresponding home ownership rate. Use regression analysis on a graphing calculator to find a logarithmic function of the form $y = a + b \ln x$ that models the data. (Round the constants a and b to three significant digits.)

(B) Use your logarithmic function to predict the home ownership rate in 2010.

SOLUTIONS

(A) Figure 1 shows the details of constructing the model on a graphing calculator.

(B) The year 2010 corresponds to $x = 110$. Evaluating $y_1 = -36.7 + 23.0 \ln x$ at $x = 110$ predicts a home ownership rate of 71.4% in 2010.



(a) Data (b) Regression equation (c) Regression equation entered in equation editor (d) Graph of data and regression equation

› Figure 1

MATCHED PROBLEM 5

Refer to Example 5. The home ownership rate in 1995 was 64.7%.

(A) Find a logarithmic regression equation for the expanded data set.

(B) Predict the home ownership rate in 2010.

ANSWERS TO MATCHED PROBLEMS

1. 95.05 decibels **2.** 7.80 **3.** 2.67 **4.** 1 kilometer per second less

5. (A) $-31.5 + 21.7 \ln x$ (B) 70.5%

4-4 Exercises

1. Describe the decibel scale in your own words.

2. Describe the Richter scale in your own words.

3. Explain why logarithms are a good choice for describing sound intensity and earthquake magnitude.

4. Think of a real-life quantity that is likely to be modeled well by a logarithmic function, and explain your reasoning.

APPLICATIONS

5. SOUND What is the decibel level of
(A) The threshold of hearing, 1.0×10^{-12} watts per square meter?
(B) The threshold of pain, 1.0 watt per square meter?
Compute answers to two significant digits.

6. SOUND What is the decibel level of
(A) A normal conversation, 3.2×10^{-6} watts per square meter?
(B) A jet plane with an afterburner, 8.3×10^{2} watts per square meter?
Compute answers to two significant digits.

7. SOUND If the intensity of a sound from one source is 1,000 times that of another, how much more is the decibel level of the louder sound than the quieter one?

8. SOUND If the intensity of a sound from one source is 10,000 times that of another, how much more is the decibel level of the louder sound than the quieter one?

9. EARTHQUAKES One of the strongest recorded earthquakes to date was in Colombia in 1906, with an energy release of 1.99×10^{17} joules. What was its magnitude on the Richter scale? Compute the answer to one decimal place.

10. EARTHQUAKES Anchorage, Alaska, had a major earthquake in 1964 that released 7.08×10^{16} joules of energy. What was its magnitude on the Richter scale? Compute the answer to one decimal place.

11. EARTHQUAKES The 1933 Long Beach, California, earthquake had a Richter scale reading of 6.3, and the 1964 Anchorage, Alaska, earthquake had a Richter scale reading of 8.3. How many times more powerful was the Anchorage earthquake than the Long Beach earthquake?

12. EARTHQUAKES Generally, an earthquake requires a magnitude of over 5.6 on the Richter scale to inflict serious damage. How many times more powerful than this was the great 1906 Colombia earthquake, which registered a magnitude of 8.6 on the Richter scale?

13. EXPLOSIVE ENERGY The atomic bomb dropped on Nagasaki, Japan, on August 9, 1945, released about 1.34×10^{14} joules of energy. What would be the magnitude of an earthquake that released that much energy?

14. EXPLOSIVE ENERGY The largest and most powerful nuclear weapon ever detonated was tested by the Soviet Union on October 30, 1961, on an island in the Arctic Sea. The blast was so powerful there were reports of windows breaking in Finland, over 700 miles away. The detonation released about 2.1×10^{17} joules of energy. What would be the magnitude of an earthquake that released that much energy?

15. ASTRONOMY A moderate-size solar flare observed on the sun on July 9, 1996, released enough energy to power the United States for almost 23,000 years at 2001 consumption levels, 2.38×10^{21} joules. What would be the magnitude of an earthquake that released that much energy?

16. CONSTRUCTION The energy released by a typical construction site explosion is about 7.94×10^{5} joules. What would be the magnitude of an earthquake that released that much energy?

17. SPACE VEHICLES A new solid-fuel rocket has a weight ratio $W_t/W_b = 19.8$ and an exhaust velocity $c = 2.57$ kilometers per second. What is its velocity at burnout? Compute the answer to two decimal places.

18. SPACE VEHICLES A liquid-fuel rocket has a weight ratio $W_t/W_b = 6.2$ and an exhaust velocity $c = 5.2$ kilometers per second. What is its velocity at burnout? Compute the answer to two decimal places.

19. CHEMISTRY The hydrogen ion concentration of a substance is related to its acidity and basicity. Because hydrogen ion concentrations vary over a very wide range, logarithms are used to create a compressed **pH scale,** which is defined as follows:

$$\text{pH} = -\log [\text{H}^+]$$

where $[\text{H}^+]$ is the hydrogen ion concentration, in moles per liter. Pure water has a pH of 7, which means it is neutral. Substances with a pH less than 7 are acidic, and those with a pH

greater than 7 are basic. Compute the pH of each substance listed, given the indicated hydrogen ion concentration. Also, indicate whether each substance is acidic or basic. Compute answers to one decimal place.
(A) Seawater, 4.63×10^{-9}
(B) Vinegar, 9.32×10^{-4}

20. CHEMISTRY Refer to Problem 19. Compute the pH of each substance below, given the indicated hydrogen ion concentration. Also, indicate whether it is acidic or basic. Compute answers to one decimal place.
(A) Milk, 2.83×10^{-7}
(B) Garden mulch, 3.78×10^{-6}

21. ECOLOGY Refer to Problem 19. Many lakes in Canada and the United States will no longer sustain some forms of wildlife because of the increase in acidity of the water from acid rain and snow caused by sulfur dioxide emissions from industry. If the pH of a sample of rainwater is 5.2, what is its hydrogen ion concentration in moles per liter? Compute the answer to two significant digits.

22. ECOLOGY Refer to Problem 19. If normal rainwater has a pH of 5.7, what is its hydrogen ion concentration in moles per liter? Compute the answer to two significant digits.

23. ASTRONOMY The brightness of stars is expressed in terms of magnitudes on a numerical scale that increases as the brightness decreases. The magnitude m is given by the formula

$$m = 6 - 2.5 \log \frac{L}{L_0}$$

where L is the light flux of the star and L_0 is the light flux of the dimmest stars visible to the naked eye.
(A) What is the magnitude of the dimmest stars visible to the naked eye?
(B) How many times brighter is a star of magnitude 1 than a star of magnitude 6?

24. ASTRONOMY An optical instrument is required to observe stars beyond the sixth magnitude, the limit of ordinary vision. However, even optical instruments have their limitations. The limiting magnitude L of any optical telescope with lens diameter D, in inches, is given by

$$L = 8.8 + 5.1 \log D$$

(A) Find the limiting magnitude for a homemade 6-inch reflecting telescope.
(B) Find the diameter of a lens that would have a limiting magnitude of 20.6.
Compute answers to three significant digits.

25. AGRICULTURE Table 4 shows the yield (bushels per acre) and the total production (millions of bushels) for corn in the United States for selected years since 1950. Let x represent years since 1900.

Table 4 United States Corn Production

Year	Yield (bushels per acre)	Total Production (million bushels)
1950	37.6	2,782
1960	55.6	3,479
1970	81.4	4,802
1980	97.7	6,867
1990	115.6	7,802
2000	137.0	9,915

Source: U.S. Department of Agriculture

(A) Find a logarithmic regression model ($y = a + b \ln x$) for the yield. Estimate (to one decimal place) the yield in 2003 and in 2010.
(B) The actual yield in 2003 was 142 bushels per acre. How does this compare with the estimated yield in part A? What effect with this additional 2003 information have on the estimate for 2010? Explain.

26. AGRICULTURE Refer to Table 4.
(A) Find a logarithmic regression model ($y = a + b \ln x$) for the total production. Estimate (to the nearest million) the production in 2003 and in 2010.
(B) The actual production in 2003 was 10,114 million bushels. How does this compare with the estimated production in part A? What effect will this 2003 production information have on the estimate for 2010? Explain.

4-5 Exponential and Logarithmic Equations

› Solving Exponential Equations
› Solving Logarithmic Equations

When quantities are modeled by exponential or logarithmic functions, it's not a surprise that solving equations involving expressions of these types is useful in studying those quantities.

Equations involving exponential and logarithmic functions, such as

$$2^{3x-2} = 5 \qquad \text{and} \qquad \log(x + 3) + \log x = 1$$

are called **exponential** and **logarithmic equations,** respectively. The properties of logarithms that we studied in Section 4-3 play a central role in their solution. Of course, a graphing calculator can be used to find approximate solutions for many exponential and logarithmic equations. However, there are situations in which the algebraic solution is necessary. In this section, we will emphasize algebraic solutions, but will still consider graphical solutions in many cases.

› Solving Exponential Equations

The distinguishing feature of exponential equations is that the variable appears in an exponent. Before defining logarithms, we didn't have a reliable method for removing variables from an exponent: Now we do. To illustrate the idea, we return to the equation we considered at the beginning of Section 4-3, $3^x = 20$.

EXAMPLE 1 Solving an Exponential Equation

Solve $3^x = 20$. Round your answer to four decimal places.

SOLUTIONS

Algebraic Solution

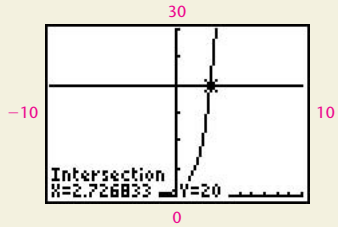
The key is to apply a logarithmic function to each side, then use one of the properties of logs from Section 4-3.

$$3^x = 20 \qquad \text{Apply common or natural log to both sides.}$$
$$\ln 3^x = \ln 20 \qquad \text{Use } \log_b N^p = p \log_b N.$$
$$x \ln 3 = \ln 20 \qquad \text{Solve for } x.$$
$$x = \frac{\ln 20}{\ln 3} \approx 2.7268$$

Notice that the solution is between 2 and 3, as we surmised at the beginning of Section 4-3 (since $3^2 = 9$ and $3^3 = 27$).

Graphical Solution

Graph $y_1 = 3^x$ and $y_2 = 20$ and use the INTERSECT command (Fig. 1).



› **Figure 1** $y_1 = 3^x, y_2 = 20.$

The solution is $x = 2.7268$ to four decimal places.

MATCHED PROBLEM 1

Solve $5^x = 30$. Round your answer to four decimal places.

In Example 1, the choice of natural log to apply to both sides of the equation was unimportant. We could have chosen common log, or really log with any base. We'll usually choose either natural or common log because those are easiest to compute using a calculator.

In Example 2, we will use the technique of Example 1 on a slightly more complicated equation.

EXAMPLE 2 Solving an Exponential Equation

Solve $2^{3x-2} = 5$ for x to four decimal places.

SOLUTIONS

Algebraic Solution

Again, we will use logs to get x out of the exponent.

$$2^{3x-2} = 5$$ Take the common or natural log of both sides.

$$\log 2^{3x-2} = \log 5$$ Use $\log_b N^p = p \log_b N$ to get $3x - 2$ out of the exponent position.

$$(3x - 2)\log 2 = \log 5$$ Solve.

$$3x - 2 = \frac{\log 5}{\log 2}$$ *Remember:* $\frac{\log 5}{\log 2} \neq \log 5 - \log 2$.

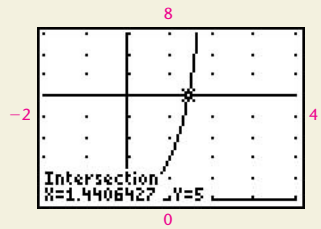
$$3x = 2 + \frac{\log 5}{\log 2}$$ Multiply both sides by $\frac{1}{3}$.

$$x = \frac{1}{3}\left(2 + \frac{\log 5}{\log 2}\right)$$

$$= 1.4406$$ To four decimal places

Graphical Solution

Graph $y_1 = 2^{3x-2}$ and $y_2 = 5$ and use the INTERSECT command (Fig. 2).



› **Figure 2** $y_1 = 2^{3x-2}$, $y_2 = 5$.

MATCHED PROBLEM 2

Solve $35^{1-2x} = 7$ for x to four decimal places.

Being able to solve exponential equations comes in handy when working with quantities that can be modeled with exponential functions.

EXAMPLE 3 Compound Interest

A certain amount of money P (principal) is invested at an annual rate r compounded annually. The amount of money A in the account after t years, assuming no withdrawals, is given by

$$A = P\left(1 + \frac{r}{m}\right)^n = P(1 + r)^n \quad m = 1 \text{ for annual compounding}$$

How many years to the nearest year will it take the money to double if it is invested at 6% compounded annually?

SOLUTIONS

Algebraic Solution

We don't know the original amount, so we'll have to just use P to represent it. We can substitute $r = 0.6$ to get

$$A = P(1.06)^n$$

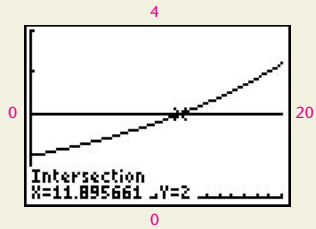
We are asked to find the number of years (n) when the amount (A) equals twice the original amount ($2P$). So we substitute $2P$ for A and solve for n.

$$2P = P(1.06)^n \quad \text{Divide both sides by } P.$$
$$2 = 1.06^n \quad \text{Take the common or natural log of both sides.}$$
$$\log 2 = \log 1.06^n \quad \text{Note how log properties are used to get } n \text{ out of the exponent position.}$$
$$\log 2 = n \log 1.06 \quad \text{Solve for } n.$$
$$n = \frac{\log 2}{\log 1.06}$$
$$= 12 \text{ years} \quad \text{To the nearest year}$$

Graphical Solution

From the first part of the algebraic solution, we need to solve the equation $2 = 1.06^n$.

Graph $y_1 = 1.06^x$ and $y_2 = 2$ and use the INTERSECT command (Fig. 3).



› Figure 3 $y_1 = 1.06^x, y_2 = 2$.

The solution (rounded to the nearest year) is 12.

MATCHED PROBLEM 3

Repeat Example 3, changing the interest rate to 9% compounded annually.

››› CAUTION ›››

When solving exponential equations, it is crucial to first isolate the exponential expression before applying a log function to each side. [In Example 3, this entailed dividing both sides by P to isolate the exponential expression $(1.06)^n$.]

EXAMPLE 4 Atmospheric Pressure

The atmospheric pressure P, in pounds per square inch, at x miles above sea level is given approximately by

$$P = 14.7e^{-0.21x}$$

At what height will the atmospheric pressure be half the sea-level pressure? Compute the answer to two significant digits.

SOLUTIONS

Algebraic Solution

Since x represents miles above sea level, sea-level pressure is the pressure at $x = 0$:

$$P = 14.7e^{0} = 14.7$$

One-half of sea-level pressure is $14.7/2 = 7.35$. Now our problem is to find x so that $P = 7.35$; that is, we solve $7.35 = 14.7e^{-0.21x}$ for x:

$$7.35 = 14.7e^{-0.21x}$$ Divide both sides by 14.7 to isolate the exponential expression.

$$0.5 = e^{-0.21x}$$ Because the base is e, take the natural log of both sides.

$$\ln 0.5 = \ln e^{-0.21x}$$ Use the property $\ln e^a = a$.

$$\ln 0.5 = -0.21x$$ Solve for x.

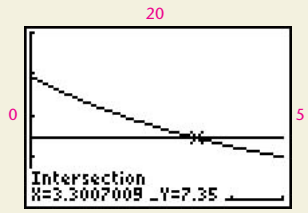
$$x = \frac{\ln 0.5}{-0.21}$$

$$= 3.3 \text{ miles}$$ To two significant digits.

Graphical Solution

From the first part of the algebraic solution, we need to solve $7.35 = 14.7e^{-0.21x}$.

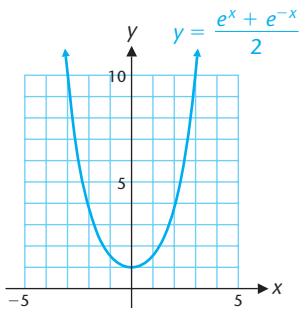
Graph $y_1 = 14.7e^{-0.21x}$ and $y_2 = 7.35$ and use the INTERSECT command (Fig. 4).



› **Figure 4** $y_1 = 14.7e^{-0.21x}$, $y_2 = 7.35$.

MATCHED PROBLEM 4

Using the formula in Example 4, find the altitude in miles so that the atmospheric pressure will be one-eighth that at sea level. Compute the answer to two significant digits.



› **Figure 5** Catenary.

Many people assume that a cable hanging between two fixed points (think of utility wires between two poles) are parabolas, but actually they are not. Instead, they follow the shape of the graph in Figure 5, known as a **catenary.** Catenaries are important in engineering and architecture, and are often studied in calculus. The graph of the equation

$$y = \frac{e^x + e^{-x}}{2} \tag{1}$$

is an example of a catenary.

EXAMPLE 5 Solving an Exponential Equation

Given equation (1), find x for $y = 2.5$. Compute the answer to four decimal places.

SOLUTIONS

Algebraic Solution

$$y = \frac{e^x + e^{-x}}{2}$$ Substitute $y = 2.5$.

$$2.5 = \frac{e^x + e^{-x}}{2}$$ Multiply both sides by 2 to clear fractions.

$$5 = e^x + e^{-x}$$ Multiply both sides by e^x to eliminate negative exponents.

$$5e^x = e^{2x} + 1$$ Rearrange so that zero is on one side.

$$e^{2x} - 5e^x + 1 = 0$$ Use the quadratic formula.

Let $u = e^x$, then

$$u^2 - 5u + 1 = 0$$

$$u = \frac{5 \pm \sqrt{25 - 4(1)(1)}}{2}$$

$$= \frac{5 \pm \sqrt{21}}{2}$$ Replace u with e^x and solve for x.

$$e^x = \frac{5 \pm \sqrt{21}}{2}$$ Take the natural log of both sides (both values on the right are positive).

$$\ln e^x = \ln \frac{5 \pm \sqrt{21}}{2}$$ Use $\ln e^x = x$.

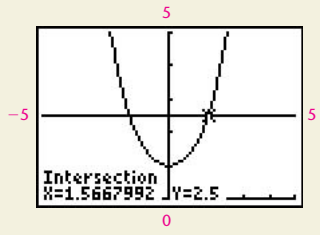
$$x = \ln \frac{5 \pm \sqrt{21}}{2}$$

$$= -1.5668, 1.5668$$

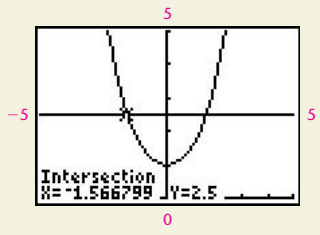
Note that the algebraic method produces exact solutions, an important consideration in certain calculus applications (see Problems 69–72 in Exercises 4-5).

Graphical Solution

Graph $y_1 = (e^x + e^{-x})/2$ and $y_2 = 2.5$ and use the INTERSECT command (Fig. 6).



(a)



(b)

› Figure 6 $y_1 = \frac{e^x + e^{-x}}{2}$, $y_2 = 2.5$.

The two solutions are $x = -1.5668$ and $x = 1.5668$ to four decimal places.

MATCHED PROBLEM 5

Given $y = (e^x - e^{-x})/2$, find x for $y = 1.5$. Compute the answer to three decimal places.

››› EXPLORE-DISCUSS 1

Let $y = e^{2x} + 3e^{x} + e^{-x}$

(A) Try to find x when $y = 7$ using the method of Example 5. Explain the difficulty that arises.

(B) Use a graphing calculator to find x when $y = 7$.

› Solving Logarithmic Equations

We will begin our study of solving logarithmic equations with a key observation. For equations of the form

$$\log_b x = a$$

changing to exponential form solves the equation, as in Example 6.

EXAMPLE 6 Solve $\log_5 x = 3$.

SOLUTION

Change to exponential form:

$$5^3 = x$$
$$x = 125$$

MATCHED PROBLEM 6

Solve $\log_2 x = -4$.

Obviously, this is a very simple example, but it provides some valuable insight in solving logarithmic equations. If we can reduce an equation to the form $\log_b(\text{expression}) = a$, where "expression" is something involving the variable, then changing to exponential form should result in an equation we already know how to solve.

EXAMPLE 7 Solving a Logarithmic Equation

Solve $\log(x + 3) + \log x = 1$, and check.

SOLUTIONS

Algebraic Solution

First use properties of logarithms to express the left side as a single logarithm, then convert to exponential form and solve for x, as in Example 6.

$$\log(x + 3) + \log x = 1$$ Combine left side using $\log M + \log N = \log MN$.

$$\log[x(x + 3)] = 1$$ Change to equivalent exponential form.

$$x(x + 3) = 10^1$$ Write in $ax^2 + bx + c = 0$ form and solve.

$$x^2 + 3x - 10 = 0$$ Factor.

$$(x + 5)(x - 2) = 0$$

$$x = -5, 2$$

CHECK

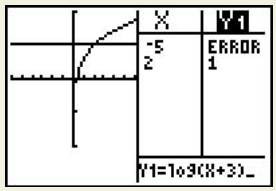
$x = -5$: $\log(-5 + 3) + \log(-5)$ is not defined because the domain of the log function is $(0, \infty)$.

$x = 2$: $\log(2 + 3) + \log 2 = \log 5 + \log 2$
$= \log(5 \cdot 2) = \log 10 \overset{\checkmark}{=} 1$

The only solution to the original equation is $x = 2$. Remember, solutions should be checked in the original equation to see whether any should be discarded.

Graphical Solution

Graph $y_1 = \log(x + 3) + \log x$ and $y_2 = 1$ and use the INTERSECT command. Figure 7 shows that $x = 2$ is a solution, and also shows that y_1 (the left side of the original equation) is not defined at $x = -5$, the extraneous solution produced by the algebraic method.



› **Figure 7**
$y_1 = \log(x + 3) + \log x$,
$y_2 = 1$.

MATCHED PROBLEM 7

Solve $\log(x - 15) = 2 - \log x$, and check.

››› CAUTION ›››

It's important to check your answer when solving logarithmic equations. Because log functions are undefined for negative inputs, extraneous solutions are common.

EXAMPLE 8 Solving a Logarithmic Equation

Solve $(\ln x)^2 = \ln x^2$.

SOLUTIONS

Algebraic Solution

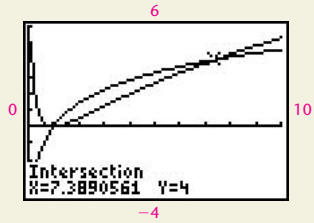
There are no logarithmic properties for simplifying $(\ln x)^2$. However, we can simplify $\ln x^2$, obtaining an equation involving $\ln x$ and $(\ln x)^2$.

$$(\ln x)^2 = \ln x^2 \qquad \text{Use } \log_b N^p = p \log_b N.$$
$$(\ln x)^2 = 2 \ln x \qquad \text{Rearrange so that zero is on one side.}$$
$$(\ln x)^2 - 2 \ln x = 0 \qquad \text{Factor out } \ln x.$$
$$(\ln x)(\ln x - 2) = 0 \qquad \text{Set each factor equal to zero.}$$
$$\ln x = 0 \quad \text{or} \quad \ln x - 2 = 0$$
$$\ln x = 0 \quad \text{or} \quad \ln x = 2 \qquad \text{Change to exponential form. Recall that } \ln x = \log_e x.$$
$$e^0 = x \quad \text{or} \quad e^2 = x$$
$$x = 1, e^2$$

Checking that both $x = 1$ and $x = e^2$ are solutions to the original equation is left to you. Don't let us down.

Graphical Solution

Graph $y_1 = (\ln x)^2$ and $y_2 = \ln x^2$ and use the INTERSECT command to obtain the solutions $x = 1$ and $x = 7.3890561$ (Fig. 8). The second solution is not exact; it is an approximation to e^2.



› Figure 8

MATCHED PROBLEM 8

Solve $\log x^2 = (\log x)^2$.

››› CAUTION ›››

Note that

$$(\log_b x)^2 \neq \log_b x^2 \qquad (\log_b x)^2 = (\log_b x)(\log_b x); \quad \log_b x^2 = 2 \log_b x$$

You might find it helpful to keep these straight by writing $\log_b x^2$ as $\log_b (x^2)$.

EXAMPLE 9 Earthquake Intensity

Recall from the last section that the magnitude of an earthquake on the Richter scale is given by

$$M = \frac{2}{3}\log\frac{E}{E_0}$$

Solve for E in terms of the other symbols.

SOLUTION

$$M = \frac{2}{3}\log\frac{E}{E_0}$$ Multiply both sides by $\frac{3}{2}$.

$$\log\frac{E}{E_0} = \frac{3M}{2}$$ Change to exponential form.

$$\frac{E}{E_0} = 10^{3M/2}$$ Multiply both sides by E_0.

$$E = E_0 10^{3M/2}$$

MATCHED PROBLEM 9

Solve the rocket equation from the last section for W_b in terms of the other symbols:

$$v = c\ln\frac{W_t}{W_b}$$

ANSWERS TO MATCHED PROBLEMS

1. 2.1133 **2.** $x = 0.2263$
3. More than double in 9 years, but not quite double in 8 years
4. 9.9 miles **5.** $x = 1.195$ **6.** $x = \frac{1}{16}$
7. $x = 20$ **8.** $x = 1{,}100$ **9.** $W_b = W_t e^{-v/c}$

4-5 Exercises

1. Which property of logarithms do you think is most useful in solving exponential equations? Explain.

2. Which properties of logarithms do you think are most useful in solving equations with more than one logarithm? Explain.

3. If u and v represent expressions with variable x, how can you solve equations of the form $\log_b u = \log_b v$ for x? Explain why this works.

4. Why is it especially important to check answers when solving logarithmic equations?

5. Explain the difference between $(\ln x)^2$ and $\ln x^2$.

6. When solving logarithmic and exponential equations, what is the advantage of solving algebraically, rather than graphically?

Solve Problems 7–22 algebraically and graphically. Round answers to three significant digits.

7. $10^{-x} = 0.0347$

8. $10^{x} = 14.3$

9. $10^{3x+1} = 92$

10. $10^{5x-2} = 348$

11. $e^{x} = 3.65$

12. $e^{-x} = 0.0142$

13. $e^{2x-1} = 405$

14. $e^{3x+5} = 23.8$

15. $5^{x} = 18$

16. $3^{x} = 4$

17. $2^{-x} = 0.238$

18. $3^{-x} = 0.074$

19. $\log_5 (2x - 7) = 2$

20. $\log_2 (4 - x) = 4$

21. $\log_3 (x^2 - 8x) = 2$

22. $\log_2 (x^2 + 5) = 3$

Solve Problems 23–32 exactly.

23. $\log 5 + \log x = 2$

24. $\log x - \log 8 = 1$

25. $\log x + \log (x - 3) = 1$

26. $\log (x - 9) + \log 100x = 3$

27. $\log (x + 1) - \log (x - 1) = 1$

28. $\log (2x + 1) = 1 + \log (x - 2)$

29. $\ln (4x - 3) = \ln (x + 1)$

30. $\log_5 (2 - x) = \log_5 (3x + 8)$

31. $\log_2 (x^2 - 2x) = \log_2 (3x - 6)$

32. $\log_7 (x + 1) = \log_7 (2x^2 - x - 3)$

Solve Problems 33–44 algebraically and graphically. Round answers to three significant digits.

33. $2 = 1.05^{x}$

34. $3 = 1.06^{x}$

35. $e^{-1.4x} = 13$

36. $e^{0.32x} = 632$

37. $5 + 3^{x} = 10$

38. $-3 = (\frac{1}{2})^{x} - 12$

39. $10^{2x+5} - 7 = 13$

40. $3 - 4^{7-x} = -16$

41. $123 = 500e^{-0.12x}$

42. $438 = 200e^{0.25x}$

43. $e^{-x^2} = 0.23$

44. $e^{x^2} = 125$

Solve Problems 45–56 exactly.

45. $\log x - \log 5 = \log 2 - \log (x - 3)$

46. $\log (6x + 5) - \log 3 = \log 2 - \log x$

47. $\ln x = \ln (2x - 1) - \ln (x - 2)$

48. $\ln (x + 1) = \ln (3x + 1) - \ln x$

49. $\log (2x + 1) = 1 - \log (x - 1)$

50. $1 - \log (x - 2) = \log (3x + 1)$

51. $(\ln x)^3 = \ln x^4$

52. $(\log x)^3 = \log x^4$

53. $\ln (\ln x) = 1$

54. $\log (\log x) = 1$

55. $x^{\log x} = 100x$

56. $3^{\log x} = 3x$

In Problems 57–60,
(A) Explain the difficulty in solving the equation exactly.
(B) Determine the number of solutions by graphing the functions on each side of the equation.

57. $e^{x/2} = 5 \ln x$

58. $\ln (\ln x) + \ln x = 2$

59. $3^{x} + 2 = 7 + x - e^{-x}$

60. $e^{x/4} = 5 \log x + 4 \ln x$

Solve Problems 61–68 for the indicated variable in terms of the remaining symbols. Use the natural log for solving exponential equations.

61. $A = Pe^{rt}$ for r (finance)

62. $A = P\left(1 + \frac{r}{n}\right)^{nt}$ for t (finance)

63. $D = 10 \log \frac{I}{I_0}$ for I (sound)

64. $t = \frac{-1}{k}(\ln A - \ln A_0)$ for A (decay)

65. $M = 6 - 2.5 \log \dfrac{I}{I_0}$ for I (astronomy)

66. $L = 8.8 + 5.1 \log D$ for D (astronomy)

67. $I = \dfrac{E}{R}(1 - e^{-Rt/L})$ for t (circuitry)

68. $S = R\dfrac{(1+i)^n - 1}{i}$ for n (annuity)

The following combinations of exponential functions define four of six ***hyperbolic functions,*** *an important class of functions in calculus and higher mathematics. Solve Problems 69–72 for x in terms of y. The results are used to define* ***inverse hyperbolic functions,*** *another important class of functions in calculus and higher mathematics.*

69. $y = \dfrac{e^x + e^{-x}}{2}$

70. $y = \dfrac{e^x - e^{-x}}{2}$

71. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

72. $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

In Problems 73–84, use a graphing calculator to approximate to two decimal places any solutions of the equation in the interval $0 \le x \le 1$. None of these equations can be solved exactly using any step-by-step algebraic process.

73. $2^{-x} - 2x = 0$

74. $3^{-x} - 3x = 0$

75. $x3^x - 1 = 0$

76. $x2^x - 1 = 0$

77. $e^{-x} - x = 0$

78. $xe^{2x} - 1 = 0$

79. $xe^x - 2 = 0$

80. $e^{-x} - 2x = 0$

81. $\ln x + 2x = 0$

82. $\ln x + x^2 = 0$

83. $\ln x + e^x = 0$

84. $\ln x + x = 0$

APPLICATIONS

85. **COMPOUND INTEREST** How many years, to the nearest year, will it take a sum of money to double if it is invested at 15% compounded annually?

86. **COMPOUND INTEREST** How many years, to the nearest year, will it take money to quadruple if it is invested at 20% compounded annually?

87. **COMPOUND INTEREST** At what annual rate compounded continuously will \$1,000 have to be invested to grow to \$2,500 in 10 years? Compute the answer to three significant digits.

88. **COMPOUND INTEREST** How many years will it take \$5,000 to grow to \$8,000 if it is invested at an annual rate of 9% compounded continuously? Compute the answer to three significant digits.

89. **IMMIGRATION** According to the U.S. Office of Immigration Statistics, there were 10.5 million illegal immigrants in the United States in May 2005, and that number had grown to 11.3 million by May 2007.
(A) Find the relative growth rate if we use the $P = P_0e^{rt}$ model for population growth. Round to three significant digits.
(B) Use your answer from part A to write a function describing the illegal immigrant population in millions in terms of years after May 2005, and use it to predict when the illegal immigrant population should reach 20 million.

90. **POPULATION GROWTH** According to U.S. Census Bureau estimates, the population of the United States was 227.2 million on July 1, 1980, and 249.5 million on July 1, 1990.
(A) Find the relative growth rate if we use the $P = P_0e^{rt}$ model for population growth. Round to three significant digits.
(B) Use your answer from part A to write a function describing the population of the United States in millions in terms of years after July 1980, and use it to predict when the population should reach 400 million.
(C) Use your function from part B to estimate the population of the United States today, then compare your estimate to the one found at www.census.gov/population/www/popclockus.html.

91. **WORLD POPULATION** A mathematical model for world population growth over short periods is given by

$$P = P_0e^{rt}$$

where P is the population after t years, P_0 is the population at $t = 0$, and the population is assumed to grow continuously at the annual rate r. How many years, to the nearest year, will it take the world population to double if it grows continuously at an annual rate of 1.14%?

92. **WORLD POPULATION** Refer to Problem 91. Starting with a world population of 6.5 billion people and assuming that the population grows continuously at an annual rate of 1.14%, how many years, to the nearest year, will it be before there is only 1 square yard of land per person? Earth contains approximately 1.7×10^{14} square yards of land.

93. **MEDICAL RESEARCH** A medical researcher is testing a radioactive isotope for use in a new imaging process. She finds that an original sample of 5 grams decays to 1 gram in 6 hours. Find the half-life of the sample to three significant digits. [Recall that the half-life model is $A = A_0(\frac{1}{2})^{t/h}$, where A_0 is the original amount and h is the half-life.]

94. **CARBON-14 DATING** If 90% of a sample of carbon-14 remains after 866 years, what is the half-life of carbon-14? (See Problem 93 for the half-life model.)

As long as a plant or animal remains alive, carbon-14 is maintained in a constant amount in its tissues. Once dead, however, the plant or animal ceases taking in carbon, and carbon-14 diminishes by radioactive decay. The amount remaining can be modeled by the equation $A = A_0e^{-0.000124t}$, where A is the amount after t years, and A_0 is the amount at time $t = 0$. Use this model to solve Problems 95–98.

95. **CARBON-14 DATING** In 2003, Japanese scientists announced the beginning of an effort to bring the long-extinct woolly mammoth back to life using modern cloning techniques. Their efforts were focused on an especially well-preserved specimen discovered frozen in the Siberian ice. Nearby samples

of plant material were found to have 28.9% of the amount of carbon-14 in a living sample. What was the approximate age of these samples?

96. **CARBON-14 DATING** In 2004, archaeologist Al Goodyear discovered a site in South Carolina that contains evidence of the earliest human settlement in North America. Carbon dating of burned plant material indicated 0.2% of the amount of carbon-14 in a live sample. How old was that sample?

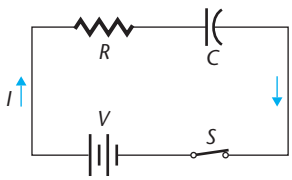
97. **CARBON-14 DATING** Many scholars believe that the earliest nonnative settlers of North America were Vikings who sailed from Iceland. If a fragment of a wooden tool found and dated in 2004 had 88.3% of the amount of carbon-14 in a living sample, when was this tool made?

98. **CARBON-14 DATING** In 1998, the Shroud of Turin was examined by researchers, who found that plant fibers in the fabric had 92.1% of the amount of carbon-14 in a living sample. If this is accurate, when was the fabric made?

99. **PHOTOGRAPHY** An electronic flash unit for a camera is activated when a capacitor is discharged through a filament of wire. After the flash is triggered and the capacitor is discharged, the circuit (see the figure) is connected and the battery pack generates a current to recharge the capacitor. The time it takes for the capacitor to recharge is called the *recycle time.* For a particular flash unit using a 12-volt battery pack, the charge q, in coulombs, on the capacitor t seconds after recharging has started is given by

$$q = 0.0009(1 - e^{-0.2t})$$

How many seconds will it take the capacitor to reach a charge of 0.0007 coulomb? Compute the answer to three significant digits.



100. **ADVERTISING** A company is trying to expose as many people as possible to a new product through television advertising in a large metropolitan area with 2 million possible viewers. A model for the number of people N, in millions, who are aware of the product after t days of advertising was found to be

$$N = 2(1 - e^{-0.037t})$$

How many days, to the nearest day, will the advertising campaign have to last so that 80% of the possible viewers will be aware of the product?

101. **NEWTON'S LAW OF COOLING** This law states that the rate at which an object cools is proportional to the difference in temperature between the object and its surrounding medium. The temperature T of the object t hours later is given by

$$T = T_m + (T_0 - T_m)e^{-kt}$$

where T_m is the temperature of the surrounding medium and T_0 is the temperature of the object at $t = 0$. Suppose a bottle of wine at a room temperature of 72°F is placed in a refrigerator at 40°F to cool before a dinner party. After an hour the temperature of the wine is found to be 61.5°F. Find the constant k, to two decimal places, and the time, to one decimal place, it will take the wine to cool from 72 to 50°F.

102. **MARINE BIOLOGY** Marine life is dependent upon the microscopic plant life that exists in the *photic zone,* a zone that goes to a depth where about 1% of the surface light still remains. Light intensity is reduced according to the exponential function

$$I = I_0 e^{-kd}$$

where I is the intensity d feet below the surface and I_0 is the intensity at the surface. The constant k is called the *coefficient of extinction*. At Crystal Lake in Wisconsin it was found that half the surface light remained at a depth of 14.3 feet. Find k, and find the depth of the photic zone. Compute answers to three significant digits.

Problems 103–106 are based on the Richter scale equation from Section 4-4, $M = \frac{2}{3}\log\frac{E}{10^{4.40}}$, where M is the magnitude and E is the amount of energy in joules released by the earthquake. Round all calculations to three significant digits.

103. **EARTHQUAKES** There were 11 earthquakes recorded worldwide in 2005 with magnitude at least 7.0.
(A) How much energy is released by a magnitude 7.0 earthquake?
(B) The total average daily consumption of energy for the entire United States in 2006 was 2.88×10^{14} joules. How many days could the energy released by a magnitude 7.0 earthquake power the United States?

104. **EARTHQUAKES** On December 26, 2004, a magnitude 9.0 earthquake struck in the Indian Ocean, causing a massive tsunami that resulted in over 230,000 deaths.
(A) How much energy was released by this earthquake?
(B) The total average daily consumption of energy for the entire United States in 2006 was 2.88×10^{14} joules. How many days could the energy released by a magnitude 9.0 earthquake power the United States?

105. **EARTHQUAKES** There were 10 earthquakes worldwide in 2005 with magnitudes between 7.0 and 7.9. Assume that these earthquakes had an average magnitude of 7.5. How long could the total energy released by these ten earthquakes power the United States, which had a total energy consumption of 1.05×10^{17} joules in 2006?

106. **EARTHQUAKES** There were 144 earthquakes worldwide in 2005 with magnitudes between 6.0 and 6.9. Assume that these earthquakes had an average magnitude of 6.5. How long could the total energy released by these 144 earthquakes power the United States, which had a total energy consumption of 1.05×10^{17} joules in 2006?

CHAPTER 4 Review

4-1 Exponential Functions

The equation $f(x) = b^x$, $b > 0$, $b \neq 1$, defines an **exponential function** with **base *b***. The domain of f is $(-\infty, \infty)$ and the range is $(0, \infty)$. The graph of f is a continuous curve that has no sharp corners; passes through $(0, 1)$; lies above the x axis, which is a horizontal asymptote; increases as x increases if $b > 1$; decreases as x increases if $b < 1$; and intersects any horizontal line at most once. The function f is one-to-one and has an inverse. The following **exponential function properties** are useful in working with these functions.

1. $a^x a^y = a^{x+y}$ $\quad (a^x)^y = a^{xy}$ $\quad (ab)^x = a^x b^x$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \frac{a^x}{a^y} = a^{x-y}$$

2. $a^x = a^y$ if and only if $x = y$.

3. For $x \neq 0$, $a^x = b^x$ if and only if $a = b$.

As x approaches ∞, the expression $[1 + (1/x)]^x$ approaches the irrational number $e \approx 2.718\,281\,828\,459$. The function $f(x) = e^x$ is called the **exponential function with base *e*.** The growth of money in an account paying **compound interest** is described by $A = P(1 + r/m)^n$, where P is the **principal,** r is the annual **rate,** m is the number of compounding periods in 1 year, and A is the **amount** in the account after n compounding periods.

If the account pays **interest compounded continuously,** the amount A in the account after t years is given by $A = Pe^{rt}$.

4-2 Exponential Models

Exponential functions are used to model various types of growth:

1. *Population growth* can be modeled by using the **doubling time growth model** $A = A_0 2^{t/d}$, where A is the population at time t, A_0 is the population at time $t = 0$, and d is the **doubling time**—the time it takes for the population to double. Another model of population growth, $A = A_0 e^{kt}$, where A_0 is the population at time zero and k is a positive constant called the **relative growth rate**, uses the exponential function with base e. This model is used for many other types of quantities that exhibit exponential growth as well.

2. *Radioactive decay* can be modeled by using the **half-life decay model** $A = A_0(\frac{1}{2})^{t/h} = A_0 2^{-t/h}$, where A is the amount at time t, A_0 is the amount at time $t = 0$, and h is the **half-life**—the time it takes for half the material to decay. Another model of radioactive decay, $A = A_0 e^{-kt}$, where A_0 is the amount at time zero and k is a positive constant, uses the exponential function with base e. This model can be used for other types of quantities that exhibit negative exponential growth as well.

3. *Limited growth*—the growth of a company or proficiency at learning a skill, for example—can often be modeled by the equation $y = A(1 - e^{-kt})$, where A and k are positive constants.

Logistic growth is another limited growth model that is useful for modeling phenomena like the spread of an epidemic, or sales of a new product. The logistic model is $y = M/(1 + ce^{-kt})$, where c, k, and M are positive constants. A good comparison of these different exponential models can be found in Table 3 at the end of Section 4-2.

Exponential regression can be used to fit a function of the form $y = ab^x$ to a set of data points. Logistic regression can be use to find a function of the form $y = c/(1 + ae^{-bx})$.

4-3 Logarithmic Functions

The **logarithmic function with base *b*** is defined to be the inverse of the exponential function with base b and is denoted by $y = \log_b x$. Thus, $y = \log_b x$ if and only if $x = b^y$, $b > 0$, $b \neq 1$. This relationship can be used to convert an expression from logarithmic to exponential form, and vice versa.

The domain of a logarithmic function is $(0, \infty)$ and the range is $(-\infty, \infty)$. The graph of a logarithmic function is a continuous curve that always passes through the point $(1, 0)$ and has the y axis as a vertical asymptote. The following **properties of logarithmic functions** are useful in working with these functions:

1. $\log_b 1 = 0$

2. $\log_b b = 1$

3. $\log_b b^x = x$

4. $b^{\log_b x} = x, x > 0$

5. $\log_b M = \log_b N$ if and only if $M = N$

6. $\log_b MN = \log_b M + \log_b N$

7. $\log_b \dfrac{M}{N} = \log_b M - \log_b N$

8. $\log_b M^p = p \log_b M$

Logarithms to the base 10 are called **common logarithms** and are denoted by $\log x$. Logarithms to the base e are called **natural**

logarithms and are denoted by $\ln x$. Thus, $\log x = y$ is equivalent to $x = 10^y$, and $\ln x = y$ is equivalent to $x = e^y$.

The **change-of-base formula,** $\log_b N = (\log_a N)/(\log_a b)$, relates logarithms to two different bases and can be used, along with a calculator, to evaluate logarithms to bases other than e or 10.

4-4 Logarithmic Models

Logarithmic functions increase very slowly as the input gets very large, so they can be used to scale down quantities that involve very large numbers, like the intensity of sound waves and the energy released by earthquakes.

The following applications involve logarithmic functions:

1. The **decibel** is defined by $D = 10 \log (I/I_0)$, where D is the **decibel level** of a sound, I is the **intensity** of the sound, and $I_0 = 10^{-12}$ watts per square meter is a standardized sound level.
2. The **magnitude** M of an earthquake on the **Richter scale** is given by $M = \frac{2}{3} \log (E/E_0)$, where E is the energy released by the earthquake and $E_0 = 10^{4.40}$ joules is a standardized energy level.
3. The **velocity** v of a rocket at burnout is given by the **rocket equation** $v = c \ln(W_t/W_b)$, where c is the exhaust velocity, W_t is the takeoff weight, and W_b is the burnout weight.

Logarithmic regression is used to fit a function of the form $y = a + b \ln x$ to a set of data points.

4-5 Exponential and Logarithmic Equations

Exponential equations are equations in which the variable appears in an exponent. If the exponential expression is isolated, applying a logarithmic function to both sides and using the property $\log_b N^p = p \log_b N$ will enable you to remove the variable from the exponent. If the exponential expression is not isolated, we can use previously developed techniques to first solve for the exponential, then solve as above.

Logarithmic equations are equations in which the variable appears inside a logarithmic function. In most cases, the key to solving them is to change the equation to the equivalent exponential expression. For equations with multiple log expressions, properties of logarithms can be used to combine the expressions before solving.

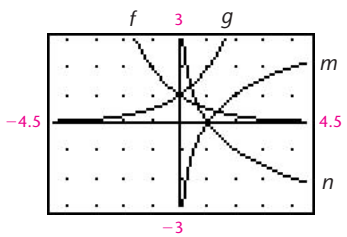
CHAPTER 4 Review Exercises

Work through all the problems in this chapter review and check answers in the back of the book. Answers to all review problems are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

1. Match each equation with the graph of f, g, m, or n in the figure.

(A) $y = \log_2 x$ (B) $y = 0.5^x$

(C) $y = \log_{0.5} x$ (D) $y = 2^x$



2. Write in logarithmic form using base 10: $m = 10^n$.

3. Write in logarithmic form using base e: $x = e^y$.

Write Problems 4 and 5 in exponential form.

4. $\log x = y$ **5.** $\ln y = x$

6. (A) Plot at least five points, then draw a hand sketch of the graph of $y = (\frac{4}{3})^x$.

(B) Use your result from part A to sketch the graph of $y = \log_{4/3} x$.

In Problems 7 and 8, simplify using properties of exponents.

7. $\dfrac{7^{x+2}}{7^{2-x}}$ **8.** $\left(\dfrac{e^x}{e^{-x}}\right)^x$

Solve Problems 9–11 for x exactly. Do not use a calculator or table.

9. $\log_2 x = 3$ **10.** $\log_x 25 = 2$ **11.** $\log_3 27 = x$

Solve Problems 12–15 for x to three significant digits.

12. $10^x = 17.5$ **13.** $e^x = 143{,}000$

14. $\ln x = -0.015\,73$ **15.** $\log x = 2.013$

Evaluate Problems 16–19 to four significant digits using a calculator.

16. $\ln \pi$ **17.** $\log(-e)$

18. $\pi^{\ln 2}$ **19.** $\dfrac{e^{\pi} + e^{-\pi}}{2}$

20. Write as a single log: $\log x + 3 \log y - \frac{1}{2} \log z$

21. Write in terms of $\ln x$ and $\ln y$: $\ln \frac{x^3}{y}$

Solve Problems 22–34 for x exactly. Do not use a calculator or table.

22. $\ln(2x - 1) = \ln(x + 3)$

23. $\log(x^2 - 3) = 2 \log(x - 1)$

24. $e^{x^2-3} = e^{2x}$ **25.** $4^{x-1} = 2^{1-x}$

26. $4 - 3^x = 2$ **27.** $5 + \frac{1}{2}e^x = \frac{17}{2}$

28. $2x^2e^{-x} = 18e^{-x}$ **29.** $\log_{1/4} 16 = x$

30. $\log_x 9 = -2$ **31.** $\log_{16} x = \frac{3}{2}$

32. $\log_x e^5 = 5$ **33.** $10^{\log_{10} x} = 33$

34. $\ln x = 0$

Solve Problems 35–44 for x to three significant digits.

35. $x = 2(10^{1.32})$ **36.** $x = \log_5 23$

37. $\ln x = -3.218$ **38.** $x = \log(2.156 \times 10^{-7})$

39. $x = \dfrac{\ln 4}{\ln 2.31}$ **40.** $25 = 5(2^x)$

41. $4{,}000 = 2{,}500(e^{0.12x})$ **42.** $0.01 = e^{-0.05x}$

43. $5^{2x-3} = 7.08$ **44.** $\dfrac{e^x - e^{-x}}{2} = 1$

Solve Problems 45–50 for x exactly. Do not use a calculator or table.

45. $\log 3x^2 - \log 9x = 2$

46. $\log x - \log 3 = \log 4 - \log(x + 4)$

47. $\ln(x + 3) - \ln x = 2 \ln 2$

48. $\ln(2x + 1) - \ln(x - 1) = \ln x$

49. $(\log x)^3 = \log x^9$ **50.** $\ln(\log x) = 1$

In Problems 51 and 52, simplify.

51. $(e^x + 1)(e^{-x} - 1) - e^x(e^{-x} - 1)$

52. $(e^x + e^{-x})(e^x - e^{-x}) - (e^x - e^{-x})^2$

In Problems 53–56, use a graphing calculator to help you draw the graph of each function. Then find the domain and range, intercepts, and asymptotes. Round all approximate values to two decimal places.

53. $y = 2^{x-1}$ **54.** $f(t) = 10e^{-0.08t}$

55. $y = \ln(x - 1)$ **56.** $N = \dfrac{100}{1 + 3e^{-t}}$

57. If the graph of $y = e^x$ is reflected through the line $y = x$, the graph of what function is obtained? Discuss the functions that are obtained by reflecting the graph of $y = e^x$ through the x axis and the y axis.

58. Approximate all real zeros of $f(x) = 4 - x^2 + \ln x$ to three decimal places.

59. Find the coordinates of the points of intersection of $f(x) = 10^{x-3}$ and $g(x) = 8 \log x$ to three decimal places.

Solve Problems 60–63 for the indicated variable in terms of the remaining symbols.

60. $D = 10 \log \dfrac{I}{I_0}$ for I (sound intensity)

61. $y = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$ for x (probability)

62. $x = -\dfrac{1}{k} \ln \dfrac{I}{I_0}$ for I (x-ray intensity)

63. $r = P \dfrac{i}{1 - (1 - i)^{-n}}$ for n (finance)

64. (A) Explain why the equation $e^{-x/3} = 4 \ln(x + 1)$ has exactly one solution.

(B) Find the solution of the equation to three decimal places.

65. Write $\ln y = -5t + \ln c$ in an exponential form free of logarithms; then solve for y in terms of the remaining symbols.

66. For $f(x) = \log_2 x$, graph f and f^{-1} on the same coordinate system. What are the domains and ranges for f and f^{-1}?

67. Explain why 1 cannot be used as a logarithmic base.

68. Prove that for any positive M, N, and b $(b \neq 1)$, $\log_b MN = \log_b M + \log_b N$. (*Hint:* Start by writing $u = \log_b M$ and $v = \log_b N$ and changing each to exponential form.)

APPLICATIONS

Solve these application problems algebraically or graphically, whichever seems more appropriate.

69. **POPULATION GROWTH** Many countries have a population growth rate of 3% (or more) per year. At this rate, how many years will it take a population to double? Use the annual compounding growth model $P = P_0(1 + r)^t$. Compute the answer to three significant digits.

70. **POPULATION GROWTH** Repeat Problem 69 using the continuous compounding growth model $P = P_0e^{rt}$.

71. **CARBON 14-DATING** How many years will it take for carbon-14 to diminish to 1% of the original amount after the death of a plant or animal? Use the formula $A = A_0e^{-0.000124t}$. Compute the answer to three significant digits.

72. **MEDICINE** One leukemic cell injected into a healthy mouse will divide into two cells in about $\frac{1}{2}$ day. At the end of the day these two cells will divide into four. This doubling continues until 1 billion cells are formed; then the animal dies with leukemic cells in every part of the body.

(A) Write an equation that will give the number N of leukemic cells at the end of t days.

(B) When, to the nearest day, will the mouse die?

73. **MONEY GROWTH** Assume \$1 had been invested at an annual rate of 3% compounded continuously in the year A.D. 1. What would be the value of the account in the year 2011? Compute the answer to two significant digits.

74. **PRESENT VALUE** Solving $A = Pe^{rt}$ for P, we obtain $P = Ae^{-rt}$, which is the **present value** of the amount A due in t years if money is invested at a rate r compounded continuously.

(A) Graph $P = 1{,}000(e^{-0.08t})$, $0 \le t \le 30$.

(B) What does it appear that P tends to as t tends to infinity? [*Conclusion:* The longer the time until the amount A is due, the smaller its present value, as we would expect.]

75. **EARTHQUAKES** The 1971 San Fernando, California, earthquake released 1.99×10^{14} joules of energy. Compute its magnitude on the Richter scale using the formula $M = \frac{2}{3}\log(E/E_0)$, where $E_0 = 10^{4.40}$ joules. Compute the answer to one decimal place.

76. **EARTHQUAKES** Refer to Problem 75. If the 1906 San Francisco earthquake had a magnitude of 8.3 on the Richter scale, how much energy was released? Compute the answer to three significant digits.

77. **SOUND** If the intensity of a sound from one source is 100,000 times that of another, how much more is the decibel level of the louder sound than the softer one? Use the formula $D = 10\log(I/I_0)$.

78. **MARINE BIOLOGY** The intensity of light entering water is reduced according to the exponential function

$$I = I_0e^{-kd}$$

where I is the intensity d feet below the surface, I_0 is the intensity at the surface, and k is the coefficient of extinction. Measurements in the Sargasso Sea in the West Indies have indicated that half the surface light reaches a depth of 73.6 feet. Find k, and find the depth at which 1% of the surface light remains. Compute answers to three significant digits.

79. **WILDLIFE MANAGEMENT** A lake formed by a newly constructed dam is stocked with 1,000 fish. Their population is expected to increase according to the logistic curve

$$N = \frac{30}{1 + 29e^{-1.35t}}$$

where N is the number of fish, in thousands, expected after t years. The lake will be open to fishing when the number of fish reaches 20,000. How many years, to the nearest year, will this take?

MODELING AND DATA ANALYSIS

80. **MEDICARE** The annual expenditures for Medicare (in billions of dollars) by the U.S. government for selected years since 1980 are shown in Table 1. Let x represent years since 1980.

(A) Find an exponential regression model of the form $y = ab^x$ for these data. Round to three significant digits. Estimate (to the nearest billion) the total expenditures in 2010 and in 2020.

(B) When (to the nearest year) will the total expenditures reach \$900 billion?

Table 1 Medicare Expenditures

Year	Billion \$
1980	37
1985	72
1990	111
1995	181
2000	225
2005	342

Source: U.S. Bureau of the Census

81. AGRICULTURE The total U.S. corn consumption (in millions of bushels) is shown in Table 2 for selected years since 1975. Let x represent years since 1900.

(A) Find a logarithmic regression model of the form $y = a + b \ln x$ for these data. Round to four significant digits. Estimate (to the nearest million) the total consumption in 1996 and in 2010.

(B) The actual consumption in 1996 was 1,583 million bushels. How does this compare with the estimated consumption in part A? What effect will this additional 1996 information have on the estimate for 2010? Explain.

Table 2 Corn Consumption

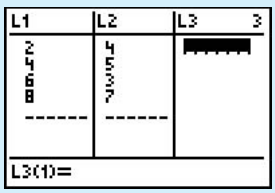
Year	Total Consumption (million bushels)
1975	522
1980	659
1985	1,152
1990	1,373
1995	1,690

Source: U.S. Department of Agriculture

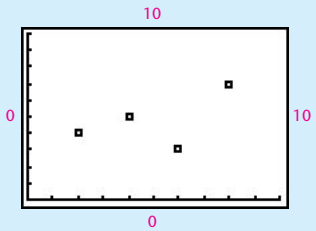
CHAPTER 4

››› GROUP ACTIVITY Comparing Regression Models

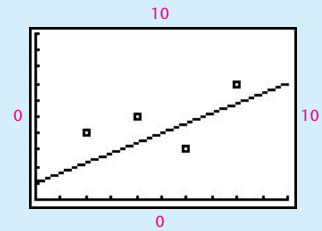
We have used polynomial, exponential, and logarithmic regression models to fit curves to data sets. And there are other equations that can be used for curve fitting (the TI-84 graphing calculator has 12 different equations on its STAT-CALC menu). How can we determine which equation provides the best fit for a given set of data? There are two principal ways to select models. The first is to use information about the type of data to help make a choice. For example, we expect the weight of a fish to be related to the cube of its length. And we expect most populations to grow exponentially, at least over the short term. The second method for choosing among equations involves developing a measure of how closely an equation fits a given data set. This is best introduced through an example. Consider the data set in Figure 1, where L1 represents the x coordinates and L2 represents the y coordinates. The graph of this data set is shown in Figure 2. Suppose we arbitrarily choose the equation $y_1 = 0.6x + 1$ to model these data (Fig. 3).



› Figure 1



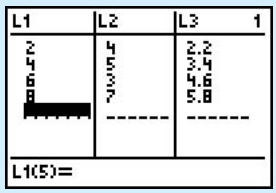
› Figure 2



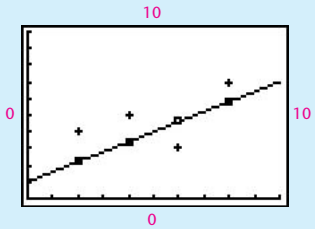
› Figure 3 $y_1 = 0.6x + 1$.

To measure how well the graph of y_1 fits these data, we examine the difference between the y coordinates in the data set and the corresponding y coordinates on the graph of y_1 (L3 in Figs. 4 and 5). Each of these differ-

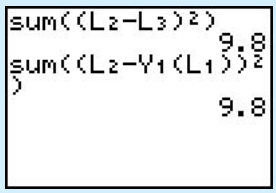
ences is called a **residual.** The most commonly accepted measure of the fit provided by a given model is the **sum of the squares of the residuals (SSR).** Computing this quantity is a simple matter on a graphing calculator (Fig. 6).



› **Figure 4**



› **Figure 5** Here + is L2 and □ is L3.



› **Figure 6** Two ways to calculate SSR.

(A) Find the linear regression model for the data in Figure 1, compute the SSR for this equation, and compare it with the one we computed for y_1.

It turns out that among all possible linear polynomials, **the linear regression model minimizes the sum of the squares of the residuals.** For this reason, the linear regression model is often called the **least-squares line.** A similar statement can be made for polynomials of any fixed degree. That is, the quadratic regression model minimizes the SSR over all quadratic polynomials, the cubic regression model minimizes the SSR over all cubic polynomials, and so on. The same statement cannot be made for exponential or logarithmic regression models. Nevertheless, the SSR can still be used to compare exponential, logarithmic, and polynomial models.

(B) Find the exponential and logarithmic regression models for the data in Figure 1, compute their SSRs, and compare with the linear model.

(C) National annual advertising expenditures for selected years since 1950 are shown in Table 1 where x is years since 1950 and y is total expenditures in billions of dollars. Which regression model would fit this data best: a quadratic model, a cubic model, or an exponential model? Use the SSRs to support your choice.

Table 1 Annual Advertising Expenditures, 1950–2005

x (years)	0	5	10	15	20	25	30	35	40	45	50	55
y (billion $)	5.7	9.2	12.0	15.3	19.6	27.9	53.6	94.8	128.6	160.9	243.3	271.1

Source: U.S. Bureau of the Census

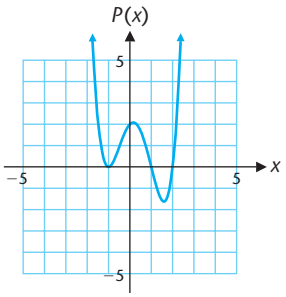
CHAPTERS 3–4 Cumulative Review

Work through all the problems in this cumulative review and check answers in the back of the book. Answers to all review problems are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

1. Let $P(x)$ be the polynomial whose graph is shown in the figure on the next page.

(A) Assuming that $P(x)$ has integer zeros and leading coefficient 1, find the lowest-degree equation that could produce this graph.

(B) Describe the left and right behavior of $P(x)$.

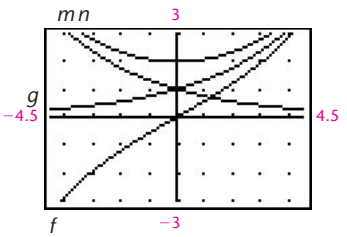


2. Draw the graph of a polynomial with lowest possible degree that has zeros -5, 1, and 6, and has a negative leading coefficient.

3. Match each equation with the graph of f, g, m, or n in the figure.

(A) $y = (\frac{3}{4})^x$ (B) $y = (\frac{4}{3})^x$

(C) $y = (\frac{3}{4})^x + (\frac{4}{3})^x$ (D) $y = (\frac{4}{3})^x - (\frac{3}{4})^x$



4. For $P(x) = 3x^3 + 5x^2 - 18x - 3$ and $D(x) = x + 3$, use synthetic division to divide $P(x)$ by $D(x)$, and write the answer in the form $P(x) = D(x)Q(x) + R$.

5. Let $P(x) = 2(x + 2)(x - 3)(x - 5)$. What are the zeros of $P(x)$?

6. Let $P(x) = 4x^3 - 5x^2 - 3x - 1$. How do you know that $P(x)$ has at least one real zero between 1 and 2?

7. Let $P(x) = x^3 + x^2 - 10x + 8$. Find all rational zeros for $P(x)$.

8. Solve for x.

(A) $y = 10^x$ (B) $y = \ln x$

9. Simplify using properties of exponents.

(A) $(2e^x)^3$ (B) $\dfrac{e^{3x}}{e^{-2x}}$

10. Solve for x exactly. Do not use a calculator or a table.

(A) $\log_3 x = 2$ (B) $\log_3 81 = x$ (C) $\log_x 4 = -2$

11. Solve for x to three significant digits.

(A) $10^x = 2.35$ (B) $e^x = 87{,}500$

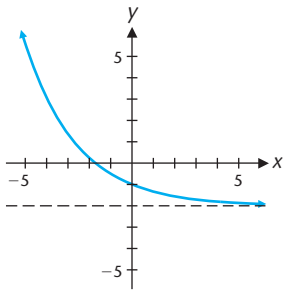
(C) $\log x = -1.25$ (D) $\ln x = 2.75$

In Problems 12 and 13, translate each statement into an equation using k as the constant of proportionality.

12. E varies directly as p and inversely as the cube of x.

13. F is jointly proportional to q_1 and q_2 and inversely proportional to the square of r.

14. Explain why the graph in the figure is not the graph of a polynomial function.



15. Explain why the graph in the figure is not the graph of a rational function.

16. The function f subtracts the square root of the domain element from three times the natural log of the domain element. Write an algebraic definition of f.

17. Write a verbal description of the function

$$f(x) = 100e^{0.5x} - 50.$$

18. Let $f(x) = \dfrac{2x + 8}{x + 2}$.

(A) Find the domain and the intercepts for f.

(B) Find the vertical and horizontal asymptotes for f.

(C) Sketch the graph of f. Draw vertical and horizontal asymptotes with dashed lines.

19. Find all zeros of $P(x) = (x^3 + 4x)(x + 4)$, and specify those zeros that are x intercepts.

20. Solve $(x^3 + 4x)(x + 4) \le 0$.

21. If $P(x) = 2x^3 - 5x^2 + 3x + 2$, find $P(\frac{1}{2})$ using the remainder theorem and synthetic division.

22. One of the zeros of $P(x) = 3x^3 - 7x^2 - 18x - 8$ is $x = -1$. Find all others.

23. Which of the following is a factor of

$$P(x) = x^{25} - x^{20} + x^{15} + x^{10} - x^5 + 1$$

(A) $x - 1$ (B) $x + 1$

24. Let $P(x) = x^4 - 8x^2 + 3$.

(A) Graph $P(x)$ and describe the graph verbally, including the number of x intercepts, the number of turning points, and the left and right behavior.

(B) Approximate the largest x intercept to two decimal places.

25. Let $P(x) = x^5 - 8x^4 + 17x^3 + 2x^2 - 20x - 8$.

(A) Approximate the zeros of $P(x)$ to two decimal places and state the multiplicity of each zero.

(B) Can any of these zeros be approximated with the bisection method? The MAXIMUM command? The MINIMUM command? Explain.

26. Let $P(x) = x^4 + 2x^3 - 20x^2 - 30$.

(A) Use the upper and lower bound theorem to find the smallest positive and largest negative integers that are upper and lower bounds, respectively, for the real zeros of $P(x)$.

(B) If $(k, k + 1)$, k an integer, is the interval containing the largest real zero of $P(x)$, determine how many additional intervals are required in the bisection method to approximate this zero to one decimal place.

(C) Approximate the real zeros of $P(x)$ to two decimal places.

27. Find all zeros (rational, irrational, and imaginary) exactly for $P(x) = 4x^3 - 20x^2 + 29x - 15$.

28. Find all zeros (rational, irrational, and imaginary) exactly for $P(x) = x^4 + 5x^3 + x^2 - 15x - 12$, and factor $P(x)$ into linear factors.

Solve Problems 29–39 for x exactly. Do not use a calculator or a table.

29. $2^{x^2} = 4^{x+4}$ **30.** $\frac{13}{2} - 3^x = \frac{1}{2}$

31. $2x^2e^{-x} + xe^{-x} = e^{-x}$ **32.** $e^{\ln x} = 2.5$

33. $\log_x 10^4 = 4$ **34.** $\log_9 x = -\frac{3}{2}$

35. $\ln(x + 4) - \ln(x - 4) = 2 \ln 3$

36. $\ln(2x^2 + 2) = 2 \ln(2x - 4)$

37. $\log x + \log(x + 15) = 2$

38. $\log(\ln x) = -1$ **39.** $4(\ln x)^2 = \ln x^2$

Solve Problems 40–44 for x to three significant digits.

40. $x = \log_3 41$ **41.** $\ln x = 1.45$

42. $4(2^x) = 20$ **43.** $10e^{-0.5x} = 1.6$

44. $\dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{1}{2}$

In Problems 45–49, use a graphing calculator to draw the graph of each function. The find the domain and range, intercepts, and asymptotes. Round all approximate values to two decimal places.

45. $f(x) = 3^{1-x}$ **46.** $g(x) = \ln(2 - x)$

47. $A(t) = 100e^{-0.3t}$ **48.** $h(x) = -2e^{-x} + 3$

49. $N(t) = \dfrac{6}{2 + e^{-0.1t}}$

50. If the graph of $y = \ln x$ is reflected through the line $y = x$, the graph of what function is obtained? Discuss the functions that are obtained by reflecting the graph of $y = \ln x$ in the x axis and in the y axis.

51. (A) Explain why the equation $e^{-x} = \ln x$ has exactly one solution.

(B) Approximate the solution of the equation to two decimal places.

In Problems 52 and 53, factor each polynomial in two ways:
(A) As a product of linear factors (with real coefficients) and quadratic factors (with real coefficients and imaginary zeros)
(B) As a product of linear factors with complex coefficients

52. $P(x) = x^4 + 9x^2 + 18$

53. $P(x) = x^4 - 23x^2 - 50$

54. G is directly proportional to the square of x. If $G = 10$ when $x = 5$, find G when $x = 7$.

55. H varies inversely as the cube of r. If $H = 162$ when $r = 2$, find H when $r = 3$.

56. Graph f and indicate any horizontal, vertical, or slant asymptotes with dashed lines:

$$f(x) = \frac{x^2 + 4x + 8}{x + 2}$$

57. Solve $\dfrac{x^3 - x}{x^3 - 8} \geq 0$.

58. Let $P(x) = x^4 - 28x^3 + 262x^2 - 922x + 1{,}083$. Approximate (to two decimal places) the x intercepts and the local extrema.

59. Find a polynomial of lowest degree with leading coefficient 1 that has zeros -1 (multiplicity 2), 0 (multiplicity 3), and $3 - 5i$. Leave the answer in factored form. What is the degree of the polynomial?

60. If $P(x)$ is a fourth-degree polynomial with integer coefficients and if i is a zero of $P(x)$, can $P(x)$ have any irrational zeros? Explain.

61. Let $P(x) = x^4 + 9x^3 - 500x^2 + 20{,}000$.

(A) Use the upper and lower bound theorem to find the smallest positive integer multiple of 10 and the largest negative integer

multiple of 10 that are upper and lower bounds, respectively, for the real zeros of $P(x)$.

(B) Approximate the real zeros of $P(x)$ to two decimal places.

62. Find all zeros (rational, irrational, and imaginary) exactly for

$$P(x) = x^5 - 4x^4 + 3x^3 + 10x^2 - 10x - 12$$

and factor $P(x)$ into linear factors.

63. Find rational roots exactly and irrational roots to two decimal places for

$$P(x) = x^5 + 4x^4 + x^3 - 11x^2 - 8x + 4$$

64. Give an example of a rational function $f(x)$ that satisfies the following conditions: the real zeros of f are 5 and 8; $x = 1$ is the only vertical asymptote; and the line $y = 3$ is a horizontal asymptote.

65. Use natural logarithms to solve for n.

$$A = P\frac{(1 + i)^n - 1}{i}$$

66. Solve $\ln y = 5x + \ln A$ for y. Express the answer in a form that is free of logarithms.

67. Solve for x.

$$y = \frac{e^x - 2e^{-x}}{2}$$

68. Solve (to three decimal places)

$$\frac{4x}{x^2 - 1} < 3$$

APPLICATIONS

69. PROFIT ANALYSIS The daily profit in dollars made by the snack bar at a small college can be modeled by the function

$$P(x) = -4.8x^3 + 47x^2 - 35x - 40 \quad (0 \le x \le 12)$$

where x is the number of hours the snack bar is open per day.

(A) How many hours should the snack bar be open to maximize its profit?

(B) How long will the snack bar need to stay open to make a profit of $300?

(C) For what range of hours will the snack bar at least break even?

70. EFFICIENCY After learning how to solve a Rubik's Cube puzzle, a student practices for 2 hours each week, trying to decrease her best time to solve the puzzle. Suppose that the function

$$T(w) = 540 - \frac{450w}{w + 2}$$

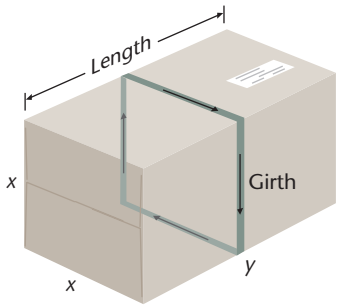
describes her best time in seconds after w weeks of practice.

(A) What was her best time after one week of practice?

(B) Find the horizontal asymptote of this rational function. What does it tell you about this student's performance?

(C) Explain why the vertical asymptote is not relevant to this problem.

71. SHIPPING A mailing service provides customers with rectangular shipping containers. The length plus the girth of one of these containers is 10 feet (see the figure). If the end of the container is square and the volume is 8 cubic feet, find the dimensions. Find rational solutions exactly and irrational solutions to two decimal places.



72. GEOMETRY The diagonal of a rectangle is 2 feet longer than one of the sides, and the area of the rectangle is 6 square feet. Find the dimensions of the rectangle. Find rational solutions exactly and irrational solutions to two decimal places.

73. ASTRONOMY The square of the time t required for a planet to make one orbit around the sun varies directly as the cube of its mean (average) distance d from the sun. Write the equation of variation, using k as the constant of variation.

74. PHYSICS Atoms and molecules that make up the air constantly fly about like microscopic missiles. The velocity v of a particle at a fixed temperature varies inversely as the square root of its molecular weight w. If an oxygen molecule in air at room temperature has an average velocity of 0.3 mile/second, what will be the average velocity of a hydrogen molecule, given that the hydrogen molecule is one-sixteenth as heavy as the oxygen molecule?

75. POPULATION GROWTH If the Democratic Republic of the Congo has a population of about 60 million people and a doubling time of 23 years, find the population in

(A) 5 years (B) 30 years

Compute answers to three significant digits.

76. COMPOUND INTEREST How long will it take money invested in an account earning 7% compounded annually to double? Use the annual compounding growth model $P = P_0(1 + r)^t$, and compute the answer to three significant digits.

77. **COMPOUND INTEREST** Repeat Problem 76 using the continuous compound interest model $P = P_0e^{rt}$.

78. **EARTHQUAKES** If the 1906 and 1989 San Francisco earthquakes registered 8.3 and 7.1, respectively, on the Richter scale, how many times more powerful was the 1906 earthquake than the 1989 earthquake? Use the formula $M = \frac{2}{3}\log(E/E_0)$, where $E_0 = 10^{4.40}$ joules, and compute the answer to one decimal place.

79. **SOUND** If the decibel level at a rock concert is 88, find the intensity of the sound at the concert. Use the formula $D = 10\log(I/I_0)$, where $I_0 = 10^{-12}$ watts per square meter, and compute the answer to two significant digits.

MODELING AND DATA ANALYSIS

80. Table 1 shows the life expectancy (in years) at birth for residents of the United States from 1970 to 1995. Let x represent years since 1970. Use the indicated regression model to estimate the life expectancy (to the nearest tenth of a year) for a U.S. resident born in 2010.

(A) Linear regression

(B) Quadratic regression

(C) Cubic regression

(D) Exponential regression

Table 1

Year	Life expectancy
1970	70.8
1975	72.6
1980	73.7
1985	74.7
1990	75.4
1995	75.9
2000	77.0
2005	77.7

Source: U.S. Census Bureau

81. Refer to Problem 80. The Census Bureau projected the life expectancy for a U.S. resident born in 2010 to be 77.9 years. Which of the models in Problem 80 is closest to the Census Bureau projection?

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Trigonometric Functions

TRIGONOMETRIC functions seem to have had their origins with the Greeks' investigation of the indirect measurement of distances and angles in the "celestial sphere." (The ancient Egyptians had used some elementary geometry to build the pyramids and remeasure lands flooded by the Nile, but neither they nor the ancient Babylonians had developed the concept of angle measure.) The word *trigonometry,* based on the Greek words for "triangle measurement," was first used as the title for a text by the German mathematician Pitiscus in A.D. 1600.

Modern applications of the trigonometric functions range over many types of problems that have little or nothing to do with angles or triangles—applications involving periodic phenomena such as sound, light, and electrical waves; business cycles; and planetary motion.

CHAPTER

5

OUTLINE

5-1 Angles and Their Measure

5-2 Trigonometric Functions: A Unit Circle Approach

5-3 Solving Right Triangles

5-4 Properties of Trigonometric Functions

5-5 More General Trigonometric Functions and Models

5-6 Inverse Trigonometric Functions

Chapter 5 Review

Chapter 5 Group Activity: A Predator–Prey Analysis Involving Mountain Lions and Deer

5-1 Angles and Their Measure

- Angles
- Degree and Radian Measure
- Converting Degrees to Radians and Vice Versa
- Linear and Angular Speed

In this section we introduce the concept of angle and two measures of angles, *degree* and *radian*.

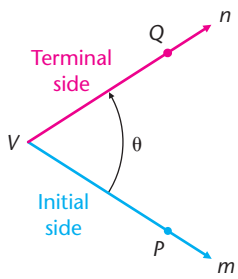


Figure 1 Angle θ or angle *PVQ* or ∠*V*.

Angles

The study of trigonometry depends on the concept of angle. An **angle** is formed by rotating (in a plane) a ray *m*, called the **initial side** of the angle, around its endpoint until it coincides with a ray *n*, called the **terminal side** of the angle. The common endpoint *V* of *m* and *n* is called the **vertex** (Fig. 1).

A counterclockwise rotation produces a **positive angle,** and a clockwise rotation produces a **negative angle,** as shown in Figures 2(a) and 2(b). The amount of rotation in either direction is not restricted. Two different angles may have the same initial and terminal sides, as shown in Figure 2(c). Such angles are said to be **coterminal.**

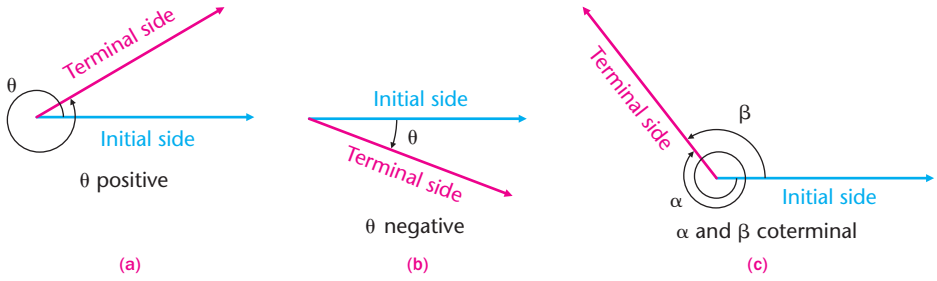
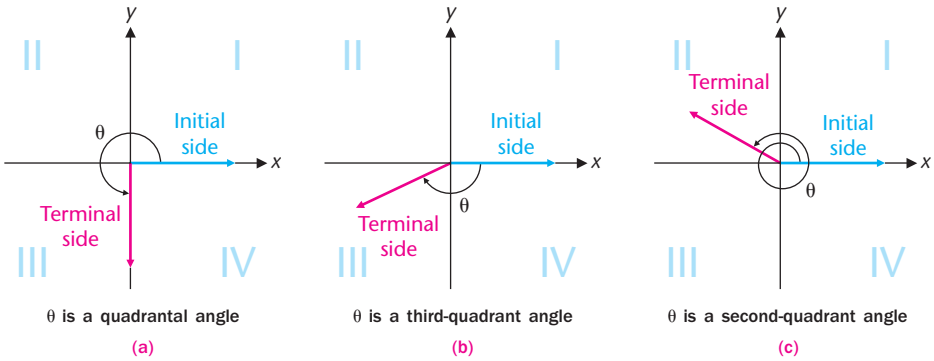


Figure 2 Angles and rotation.

An angle in a rectangular coordinate system is said to be in **standard position** if its vertex is at the origin and the initial side is along the positive *x* axis. If the terminal side of an angle in standard position lies along a coordinate axis, the angle is said to be a **quadrantal angle.** If the terminal side does not lie along a coordinate axis, then the angle is often referred to in terms of the quadrant in which the terminal side lies (Fig. 3).

› **Figure 3** Angles in standard positions.



› Degree and Radian Measure

Just as line segments are measured in centimeters, meters, inches, or miles, angles are measured in different units. The two most commonly used units for angle measure are *degree* and *radian*.

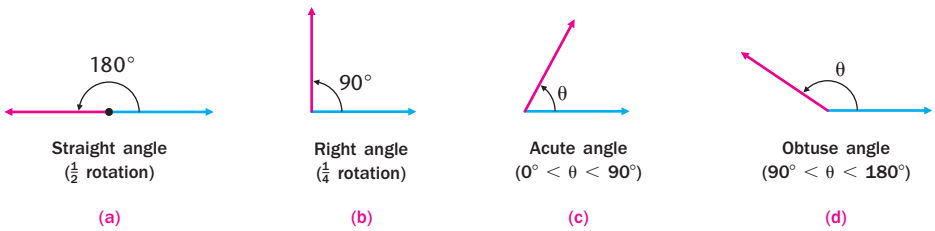
› DEFINITION 1 Degree Measure

A positive angle formed by one complete rotation is said to have a measure of *360 degrees* (360°). A positive angle formed by $\frac{1}{360}$ of a complete rotation is said to have a measure of *1 degree* (1°). The symbol ° denotes degrees.

Definition 1 is extended to all angles, not just the positive (counterclockwise) ones, in the obvious way. So, for example, a negative angle formed by $\frac{1}{4}$ of a complete clockwise rotation has a measure of −90°, and an angle for which the initial and terminal sides coincide, without rotation, has a measure of 0°.

Certain angles have special names that indicate their degree measure. Figure 4 shows a **straight angle,** a **right angle,** an **acute angle,** and an **obtuse angle.**

› **Figure 4** Types of angles.



Two positive angles are **complementary** if their sum is 90°; they are **supplementary** if their sum is 180°.

A degree can be divided further using decimal notation. For example, 42.75° represents an angle of degree measure 42 plus three-quarters of 1 degree. A degree can also be divided further using minutes and seconds just as an hour is divided into

minutes and seconds. Each degree is divided into 60 equal parts called **minutes,** and each minute is divided into 60 equal parts called **seconds.** Symbolically, minutes are represented by $'$ and seconds by $''$. Thus,

$$12°23'14''$$

is a concise way of writing 12 degrees, 23 minutes, and 14 seconds.

Decimal degrees (DD) are useful in some instances and degrees–minutes–seconds (DMS) are useful in others. You should be able to go from one form to the other as demonstrated in Example 1.

› CONVERSION ACCURACY

If an angle is measured to the nearest second, the converted decimal form should not go beyond three decimal places, and vice versa.

EXAMPLE 1 From DMS to DD and Back

(A) Convert $21°47'12''$ to decimal degrees.

(B) Convert $105.183°$ to degree–minute–second form.

SOLUTIONS

(A) $$21°47'12'' = \left(21 + \frac{47}{60} + \frac{12}{3{,}600}\right)^{\circ} = 21.787°$$

(B) $$\begin{aligned} 105.183° &= 105°\,(0.183 \cdot 60)' \\ &= 105°\,10.98' \\ &= 105°\,10'\,(0.98 \cdot 60)'' \\ &= 105°10'59'' \end{aligned}$$ *

MATCHED PROBLEM 1

(A) Convert $193°17'34''$ to DD form.

(B) Convert $237.615°$ to DMS form.

Some scientific and some graphing calculators can convert the DD and DMS forms automatically, but the process differs significantly among the various types of calculators. Check your owner's manual for your particular calculator. The conversion methods outlined in Example 1 show you the reasoning behind the process, and are sometimes easier to use than the "automatic" methods for some calculators.

*The dashed "think boxes" are used to enclose steps that may be performed mentally.

Degree measure of angles is used extensively in engineering, surveying, and navigation. Another unit of angle measure, called the *radian,* is often preferred in mathematics, scientific work, and engineering applications.

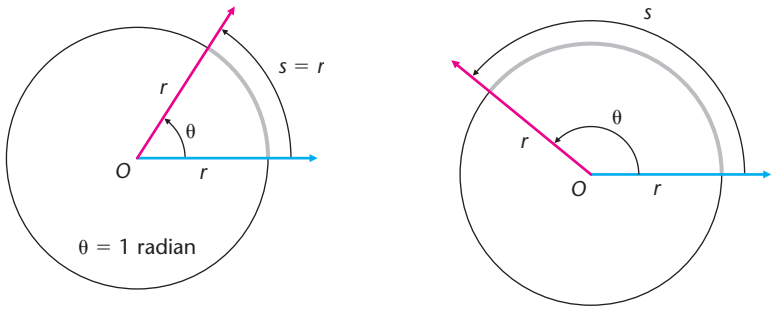
› DEFINITION 2 Radian Measure

A positive angle θ formed by a central angle of a circle has measure **1 radian** if the length s of the arc opposite θ is equal to the radius r of the circle.

More generally, if θ is any positive angle formed by the central angle of a circle, then the **radian measure** of θ is given by

$$\theta = \frac{s}{r} \text{ radians}$$

where s is the length of the arc opposite θ and r is the radius of the circle. [*Note:* s and r must be measured in the same units.]



REMARKS:

1. Because the circumference of a circle is proportional to its radius, circles of different radii will give the same radian measure for an angle θ.
2. If the radius $r = 1$, then the radian measure of angle θ is simply the arc length s.

The circumference of a circle of radius r is $2\pi r$, so the radian measure of a positive angle formed by one complete rotation is

$$\theta = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi \approx 6.283 \text{ radians}$$

Just as for degree measure, the definition is extended to apply to all angles; if θ is a negative angle, its radian measure is given by $\theta = -\frac{s}{r}$. Note that in the preceding sentence, as well as in Definition 2, the symbol θ is used in two ways: as the name of the angle and as the measure of the angle. The context indicates the meaning.

EXAMPLE 2 Computing Radian Measure

What is the radian measure of a central angle θ opposite an arc of 24 meters in a circle of radius 6 meters?

SOLUTION

$$\theta = \frac{s}{r} = \frac{24 \text{ meters}}{6 \text{ meters}} = 4 \text{ radians}$$

MATCHED PROBLEM 2

What is the radian measure of a central angle θ opposite an arc of 60 feet in a circle of radius 12 feet?

REMARK: It is customary to omit the word *radians* when giving the radian measure of an angle. But if an angle is measured in any other units, the units must be stated explicitly. For example, if $\theta = 17$, then $\theta \approx 974°$.

››› EXPLORE-DISCUSS 1

Discuss why the radian measure of an angle is independent of the size of the circle having the angle as a central angle.

› Converting Degrees to Radians and Vice Versa

What is the radian measure of an angle of 180°? Let θ be a central angle of 180° in a circle of radius r. Then the length s of the arc opposite θ is $\frac{1}{2}$ the circumference C of the circle. Therefore,

$$s = \frac{C}{2} = \frac{2\pi r}{2} = \pi r \qquad \text{and} \qquad \theta = \frac{s}{r} = \frac{\pi r}{r} = \pi \text{ radians}$$

In other words, 180° corresponds to π* radians. This is important to remember, because the radian measures of many special angles can be obtained from this correspondence. For example, 90° is 180°/2; therefore, 90° corresponds to $\pi/2$ radians.

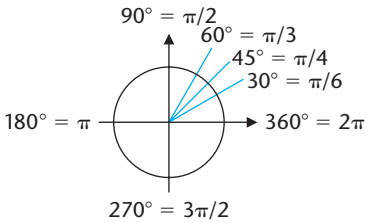
*The constant π has a long and interesting history; a few important dates are listed below:

1650 B.C.	Rhind Papyrus	$\pi \approx \frac{256}{81} = 3.16049\ldots$
240 B.C.	Archimedes	$3\frac{10}{71} < \pi < 3\frac{1}{7}$ $(3.1408\cdots < \pi < 3.1428\ldots)$
A.D. 264	Liu Hui	$\pi \approx 3.14159$
A.D. 470	Tsu Ch'ung-chih	$\pi \approx \frac{355}{113} = 3.1415929\ldots$
A.D. 1674	Leibniz	$\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots)$ $\approx 3.141592653589793238 4626$ (This and other series can be used to compute π to any decimal accuracy desired.)
A.D. 1761	Johann Lambert	Showed π to be irrational (π as a decimal is nonrepeating and nonterminating.)

››› EXPLORE-DISCUSS 2

Write the radian measure of each of the following angles in the form $\frac{a}{b}\pi$, where a and b are positive integers and fraction $\frac{a}{b}$ is reduced to lowest terms: 15°, 30°, 45°, 60°, 75°, 90°, 105°, 120°, 135°, 150°, 165°, 180°.

Some key results from Explore-Discuss 2 are summarized in Figure 5 for easy reference. These correspondences and multiples of them will be used extensively in work that follows.



› **Figure 5** Radian–degree correspondences.

The following proportion can be used to convert degree measure to radian measure and vice versa.

› RADIAN–DEGREE CONVERSION FORMULAS

$$\frac{\theta_{\text{deg}}}{180^\circ} = \frac{\theta_{\text{rad}}}{\pi \text{ radians}} \qquad \text{Basic proportion}$$

$$\theta_{\text{deg}} = \frac{180^\circ}{\pi \text{ radians}}\,\theta_{\text{rad}} \qquad \text{Radians to degrees}$$

$$\theta_{\text{rad}} = \frac{\pi \text{ radians}}{180^\circ}\,\theta_{\text{deg}} \qquad \text{Degrees to radians}$$

[*Note:* We will omit units in calculations until the final answer. If your calculator does not have a key labeled π, use $\pi \approx 3.14159$.]

Some scientific and graphing calculators can automatically convert radian measure to degree measure, and vice versa. Check the owner's manual for your particular calculator.

EXAMPLE 3 Radian–Degree Conversions

(A) Find the radian measure, exact and to three significant digits, of an angle of 75°.

(B) Find the degree measure, exact and to four significant digits, of an angle of 5 radians.

(C) Find the radian measure to two decimal places of an angle of 41°12′.

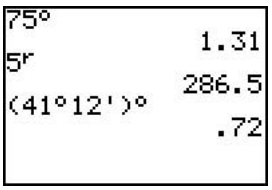
SOLUTIONS

(A) $\theta_{\text{rad}} = \dfrac{\pi \text{ radians}}{180^\circ}\theta_{\text{deg}} = \dfrac{\pi}{180}(75) = \dfrac{5\pi}{12} = 1.31$ Exact; Three significant digits

(B) $\theta_{\text{deg}} = \dfrac{180^\circ}{\pi \text{ radians}}\theta_{\text{rad}} = \dfrac{180}{\pi}(5) = \dfrac{900}{\pi} = 286.5^\circ$ Exact; Four significant digits

(C) $41^\circ 12' = \left(41 + \dfrac{12}{60}\right)^\circ = 41.2^\circ$ Change 41°12′ to DD first.

$\theta_{\text{rad}} = \dfrac{\pi \text{ radians}}{180^\circ}\theta_{\text{deg}} = \dfrac{\pi}{180}(41.2) = 0.72$ To two decimal places



› **Figure 6** Automatic conversion.

Figure 6 shows the three preceding conversions done automatically on a graphing calculator by selecting the appropriate angle mode.

MATCHED PROBLEM 3

(A) Find the radian measure, exact and to three significant digits, of an angle of 240°.

(B) Find the degree measure, exact and to three significant digits, of an angle of 1 radian.

(C) Find the radian measure to three decimal places of an angle of 125°23′.

REMARK: We will write θ in place of θ_{deg} and θ_{rad} when it is clear from the context whether we are dealing with degree or radian measure.

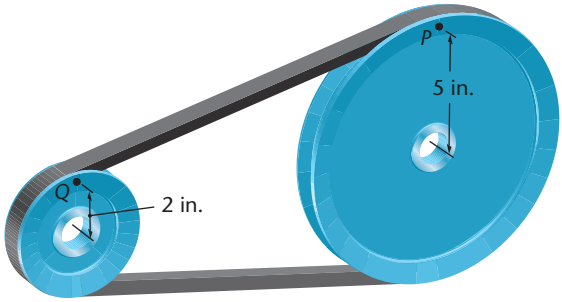
EXAMPLE 4 Engineering

A belt connects a pulley of 2-inch radius with a pulley of 5-inch radius. If the larger pulley turns through 10 radians, through how many radians will the smaller pulley turn?

SOLUTION

First we draw a sketch (Fig. 7).

› Figure 7



When the larger pulley turns through 10 radians, the point P on its circumference will travel the same distance (arc length) that point Q on the smaller circle travels. For the larger pulley,

$$\theta = \frac{s}{r}$$

$$s = r\theta = (5)(10) = 50 \text{ inches}$$

For the smaller pulley,

$$\theta = \frac{s}{r} = \frac{50}{2} = 25 \text{ radians}$$

MATCHED PROBLEM 4

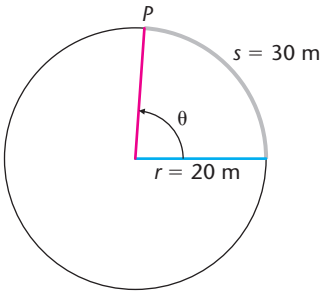
In Example 4, through how many radians will the larger pulley turn if the smaller pulley turns through 4 radians?

› Linear and Angular Speed

The *average speed* v of an object that travels a distance $d = 30$ meters in time $t = 3$ seconds is given by

$$v = \frac{d}{t} = \frac{30 \text{ meters}}{3 \text{ seconds}} = 10 \text{ meters per second}$$

Suppose that a point P moves an arc length of $s = 30$ meters in $t = 3$ seconds on the circumference of a circle of radius $r = 20$ meters (Fig. 8). Then, in those 3 seconds, the point P has moved through an angle of



› Figure 8

$$\theta = \frac{s}{r} = \frac{30}{20} = 1.5 \text{ radians}$$

We call the average speed of point P, given by

$$v = \frac{s}{t} = 10 \text{ meters per second}$$

the (average) *linear speed* to distinguish it from the (average) *angular speed* that is given by

$$\omega = \frac{\theta}{t} = \frac{1.5}{3} = 0.5 \text{ radians per second}$$

The simple formula, $v = r\omega$, which relates linear and angular speed, is obtained from the definition of radian measure as follows:

$$\frac{s}{r} = \theta \qquad \text{Multiply both sides by } r.$$

$$s = r\theta \qquad \text{Divide both sides by } t.$$

$$\frac{s}{t} = r\frac{\theta}{t} \qquad \text{Substitute } v = \frac{s}{t} \text{ and } \omega = \frac{\theta}{t}.$$

$$v = r\omega$$

These concepts are summarized in the box.

› LINEAR SPEED AND ANGULAR SPEED

Suppose a point P moves through an angle θ and arc length s, in time t, on the circumference of a circle of radius r. The (average) **linear speed** of P is

$$v = \frac{s}{t}$$

and the (average) **angular speed** is

$$\omega = \frac{\theta}{t}$$

Furthermore, $v = r\omega$.

EXAMPLE 5 Wind Power

A wind turbine of rotor diameter 15 meters makes 62 revolutions per minute. Find the angular speed (in radians per second) and the linear speed (in meters per second) of the rotor tip.

SOLUTION

The radius of the rotor is $15/2 = 7.5$ meters. In 1 minute the rotor moves through an angle of $62(2\pi) = 124\pi$ radians. Therefore, the angular speed is

$$\omega = \frac{\theta}{t} = \frac{124\pi \text{ radians}}{60 \text{ seconds}} \approx 6.49 \text{ radians per second}$$

and the linear speed of the rotor tip is

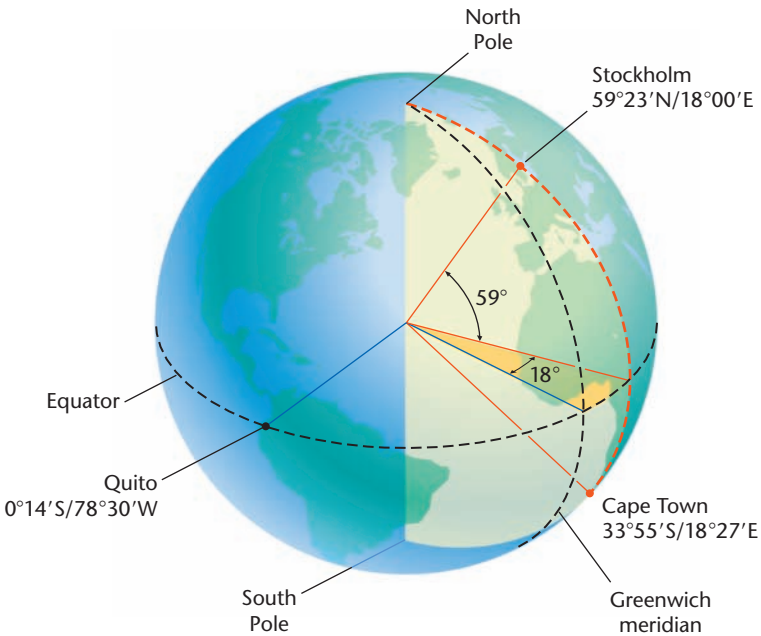
$$v = r\omega = 7.5\,\frac{124\pi}{60} \approx 48.69 \text{ meters per second}$$

MATCHED PROBLEM 5

A wind turbine of rotor diameter 12 meters has a rotor tip speed of 34.2 meters per second. Find the angular speed of the rotor (in radians per second) and the number of revolutions per minute.

EXAMPLE 6 Navigation

The traditional unit of distance for air and sea travel is the *nautical mile*. One **nautical mile** is the length of one minute of arc on the Earth's equator. Because there are $360 \times 60 = 21{,}600$ minutes in a complete circle, the Earth's circumference is 21,600 nautical miles. (One nautical mile is approximately 1.151 miles or 6,076 feet; ordinary miles are also called *statute miles* to distinguish them from nautical miles). The traditional unit of speed for air and sea travel is the *knot:* a speed of 1 **knot** is 1 nautical mile per hour. Recall that any point on the surface of the Earth can be specified by giving its latitude, measured in degrees north or south from the equator, and longitude, measured in degrees east or west from the Greenwich meridian (see Fig. 9). Stockholm, Sweden [59°23′N/18°00′E], and Cape Town, South Africa [33°55′S/18°27′E], have nearly the same longitude. If a plane flies from Stockholm to Cape Town in 11 hours, find its linear speed (to the nearest knot) and angular speed (to the nearest tenth of a degree per hour).



› Figure 9

SOLUTION

We consider Earth to be a sphere, so any great circle (that is, any circle on the surface of the sphere having the same center as the equator) has circumference 21,600 nautical miles. In particular, the circumference of the meridian passing through Stockholm and Cape Town is 21,600 nautical miles (we ignore the small difference in longitude between the two cities). The central angle between Stockholm and Cape Town (see Fig. 9) is

$$\theta = 59°23' + 33°55' = 93°18' \quad \text{or} \quad 93.3°$$

Let s denote the arc length between Stockholm and Cape Town. Because arc lengths on a circle are in the same proportion as their central angles (see Problems 101 and 102 in Exercises 5-1)

$$\frac{s}{21{,}600} = \frac{93.3}{360} \qquad \text{Multiply both sides by 21,600.}$$

$$s = 5{,}598 \text{ nautical miles}$$

Therefore, the linear speed of the plane is

$$v = \frac{s}{t} = \frac{5{,}598}{11} = 509 \text{ knots}$$

and the angular speed is

$$\omega = \frac{\theta}{t} = \frac{93.3°}{11} = 8.5° \text{ per hour.}$$

MATCHED PROBLEM 6

A plane flies from Quito, Ecuador [0°14′S/78°30′W], to Kampala, Uganda [0°19′N/32°35′E], in 14 hours. Find the plane's linear speed (to the nearest knot) and angular speed (to the nearest tenth of a degree per hour). Ignore the small difference in latitude between the two cities; Quito and Kampala are both close to the equator.

ANSWERS TO MATCHED PROBLEMS

1. (A) 193.293° (B) 237°36′54″ **2.** 5 radians

3. (A) $\frac{4\pi}{3} = 4.19$ (B) $\frac{180}{\pi} = 57.3°$ (C) 2.188

4. 1.6 radians **5.** 5.7 radians per second; 54.43 revolutions per minute

6. 476 knots; 7.9° per hour

5-1 Exercises

In all problems, if angle measure is expressed by a number that is not in degrees, it is assumed to be in radians.

1. Explain the difference between a positive angle and a negative angle.

2. Explain the difference between complementary angles and supplementary angles.

3. Would it be better to measure angles by dividing the circumference of a circle into 100 equal parts, rather than 360 equal parts as in degree measure? Explain.

4. Explain the connection between an angle of 1 *radian,* and the *radius* of a circle.

5. You are watching your nieces ride a Ferris wheel. Explain how you could do a mental calculation to estimate their angular speed.

6. Refer to Problem 5. Explain how you could do a mental calculation to estimate their linear speed.

Find the degree measure of each of the angles in Problems 7–12, keeping in mind that an angle of one complete rotation corresponds to 360°.

7. $\frac{1}{9}$ rotation **8.** $\frac{1}{5}$ rotation **9.** $\frac{3}{4}$ rotation

10. $\frac{3}{8}$ rotation **11.** $\frac{9}{8}$ rotations **12.** $\frac{7}{6}$ rotations

Find the radian measure of a central angle θ opposite an arc s in a circle of radius r, where r and s are as given in Problems 13–18.

13. $r = 4$ centimeters, $s = 24$ centimeters

14. $r = 8$ inches, $s = 16$ inches

15. $r = 12$ feet, $s = 30$ feet

16. $r = 18$ meters, $s = 27$ meters

17. $r = 1$ m, $s = 5$ cm

18. $r = 2$ ft, $s = 3$ inches

Find the radian measure of each angle in Problems 19–24, keeping in mind that an angle of one complete rotation corresponds to 2π radians.

19. $\frac{1}{8}$ rotation **20.** $\frac{1}{6}$ rotation **21.** $\frac{3}{4}$ rotation

22. $\frac{5}{12}$ rotation **23.** $\frac{13}{12}$ rotations **24.** $\frac{11}{8}$ rotations

Find the exact radian measure, in terms of π, of each angle in Problems 25–30.

25. 30°, 60°, 90°, 120°, 150°, 180°

26. 60°, 120°, 180°, 240°, 300°, 360°

27. −45°, −90°, −135°, −180°

28. −90°, −180°, −270°, −360°

29. 72°, 144°, 216°, 288°, 360°

30. 36°, 72°, 108°, 144°, 180°

Find the exact degree measure of each angle in Problems 31–36.

31. $\frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$

32. $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$

33. $-\frac{\pi}{2}, -\pi, -\frac{3\pi}{2}, -2\pi$

34. $-\frac{\pi}{4}, -\frac{\pi}{2}, -\frac{3\pi}{4}, -\pi$

35. $\frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi$

36. $\frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}, 2\pi$

In Problems 37–42, determine whether the statement is true or false. If true, explain why. If false, give a counterexample.

37. If two angles in standard position have the same measure, then they are coterminal.

38. If two angles in standard position are coterminal, then they have the same measure.

39. If two positive angles are complementary, then both are acute.

40. If two positive angles are supplementary, then one is obtuse and the other is acute.

41. If the terminal side of an angle in standard position lies in quadrant I, then the angle is positive.

42. If the initial and terminal sides of an angle coincide, then the measure of the angle is zero.

Convert each angle in Problems 43–48 to decimal degrees to three decimal places.

43. 38°41′ **44.** 95°7′ **45.** 5°51′33″

46. 14°18′37″ **47.** 354°8′29″ **48.** 184°31′7″

Convert each angle in Problems 49–54 to degree–minute–second form.

49. 27.6° **50.** 64.35° **51.** 3.042°

52. 49.715° **53.** 403.223° **54.** 156.808°

Find the radian measure to three decimal places for each angle in Problems 55–60.

55. 64° **56.** 25° **57.** 108.413°

58. 203.097° **59.** 13°25′14″ **60.** 56°11′52″

Find the degree measure to two decimal places for each angle in Problems 61–66.

61. 0.93 **62.** 0.08 **63.** 1.13

64. 3.07 **65.** −2.35 **66.** −1.72

Indicate whether each angle in Problems 67–86 is a first-, second-, third-, or fourth-quadrant angle or a quadrantal angle. All angles are in standard position in a rectangular coordinate system. (A sketch may be of help in some problems.)

67. 187° **68.** 135° **69.** −200°

70. −60° **71.** 4 **72.** 3

73. 270° **74.** 360° **75.** −1

76. −6 **77.** $\frac{5\pi}{3}$ **78.** $\frac{2\pi}{3}$

79. $-\frac{7\pi}{6}$ **80.** $-\frac{3\pi}{4}$ **81.** $-\pi$

82. $-\frac{3\pi}{2}$ **83.** 820° **84.** −565°

85. $\frac{13\pi}{4}$ **86.** $\frac{23\pi}{3}$

87. Verbally describe the meaning of a central angle in a circle with radian measure 1.

88. Verbally describe the meaning of an angle with degree measure 1.

In Problems 89–94, find all angles θ in degree measure that satisfy the given conditions.

89. $360° \le \theta \le 720°$ and θ is coterminal with 150°

90. $360° \le \theta \le 720°$ and θ is coterminal with 240°

91. $0° \le \theta \le 360°$ and θ is coterminal with −80°

92. $0° \le \theta \le 360°$ and θ is coterminal with −310°

93. $-900° \le \theta \le -180°$ and θ is coterminal with 210°

94. $-900° \le \theta \le -180°$ and θ is coterminal with 135°

In Problems 95–100, find all angles θ in radian measure that satisfy the given conditions.

95. $2\pi \le \theta \le 6\pi$ and θ is coterminal with $\pi/4$

96. $2\pi \le \theta \le 6\pi$ and θ is coterminal with $5\pi/6$

97. $0 \le \theta \le 5\pi$ and θ is coterminal with $-7\pi/6$

98. $0 \le \theta \le 5\pi$ and θ is coterminal with $-2\pi/3$

99. $-3\pi \le \theta \le \pi$ and θ is coterminal with $\pi/2$

100. $-3\pi \le \theta \le \pi$ and θ is coterminal with $3\pi/2$

101. An arc of a circle of radius r has length s_1 and central angle of radian measure θ_1. A second arc of the same circle has length s_2 and central angle of radian measure θ_2. Show that

$$\frac{s_1}{s_2} = \frac{\theta_1}{\theta_2}$$

102. Refer to Problem 101. If θ_1 and θ_2 are measured in degrees, is the equation

$$\frac{s_1}{s_2} = \frac{\theta_1}{\theta_2}$$

valid? Explain.

103. Justify the following rule of thumb: Seven nautical miles equal 8 statute miles. [*Hint:* See Example 6.]

104. Justify the following rule of thumb: A speed of 1500 yards in 3 minutes equals 15 knots [*Hint:* See Example 6.]

APPLICATIONS

105. **ANGULAR SPEED** A wheel with diameter 6 feet makes 200 revolutions per minute. Find the angular speed (in radians per second) and the linear speed (in feet per second) of a point on the rim.

106. **ANGULAR SPEED** A point on the rim of a wheel with diameter 6 feet has a linear speed of 100 feet per second. Find the angular speed (in radians per second) and the number of revolutions per minute.

107. **RADIAN MEASURE** What is the radian measure of the larger angle made by the hands of a clock at 4:30? Express the answer exactly in terms of π.

108. **RADIAN MEASURE** What is the radian measure of the smaller angle made by the hands of a clock at 1:30? Express the answer exactly in terms of π.

109. **ENGINEERING** Through how many radians does a pulley of 10-centimeter diameter turn when 10 meters of rope are pulled through it without slippage?

110. **ENGINEERING** Through how many radians does a pulley of 6-inch diameter turn when 4 feet of rope are pulled through it without slippage?

111. **ASTRONOMY** A line from the sun to the Earth sweeps out an angle of how many radians in 1 week? Assume the Earth's orbit is circular and there are 52 weeks in a year. Express the answer in terms of π and as a decimal to two decimal places.

112. **ASTRONOMY** A line from the center of the Earth to the equator sweeps out an angle of how many radians in 9 hours? Express the answer in terms of π and as a decimal to two decimal places.

113. **ENGINEERING** A trail bike has a front wheel with a diameter of 40 centimeters and a back wheel of diameter 60 centimeters. Through what angle in radians does the front wheel turn if the back wheel turns through 8 radians?

114. **ENGINEERING** In Problem 113, through what angle in radians will the back wheel turn if the front wheel turns through 15 radians?

115. **ANGULAR SPEED** If the trail bike of Problem 113 travels at a speed of 10 kilometers per hour, find the angular speed (in radians per second) of each wheel.

116. **ANGULAR SPEED** If a car travels at a speed of 60 miles per hour, find the angular speed (in radians per second) of a tire that has a diameter of 2 feet.

In Problems 117–120, each pair of cities lies nearly on the same meridian. Ignore the small difference in longitude.

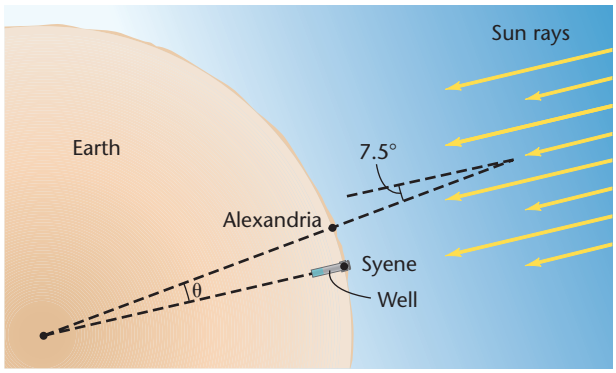
117. **NAVIGATION** Find the distance (to the nearest nautical mile) from Havana, Cuba [23°08′N/82°23′W], to Cleveland, Ohio [41°30′N/81°41′W].

118. **NAVIGATION** Find the distance (to the nearest nautical mile) from Indianapolis, Indiana [39°44′N/86°17′W], to Managua, Nicaragua [12°06′N/86°18′W].

119. **NAVIGATION** A plane flies from Lima, Peru [12°06′S/76°55′W], to Washington, D.C. [38°53′N/77°02′W], in 6 hours. Find the plane's linear speed (to the nearest knot) and angular speed (to the nearest tenth of a degree per hour).

120. **NAVIGATION** A plane flies from Jakarta, Indonesia [6°08′S/106°45′E], to Ulaanbaatar, Mongolia [47°55′N/106°53′E], in 7 hours. Find the plane's linear speed (to the nearest knot) and angular speed (to the nearest tenth of a degree per hour).

121. **CIRCUMFERENCE OF THE EARTH** The early Greeks used the proportion $s/C = \theta°/360°$, where s is an arc length on a circle, $\theta°$ is degree measure of the corresponding central angle, and C is the circumference of the circle ($C = 2\pi r$). Eratosthenes (240 B.C.), in his famous calculation of the circumference of the Earth, reasoned as follows: He knew at Syene (now Aswan) during the summer solstice the noon sun was directly overhead and shined on the water straight down a deep well. On the same day at the same time, 5,000 stadia (approx. 500 miles) due north in Alexandria, sun rays crossed a vertical pole at an angle of 7.5° as indicated in the figure. Carry out Eratosthenes' calculation for the circumference of the Earth to the nearest thousand miles. (The current calculation for the equatorial circumference is 24,902 miles.)

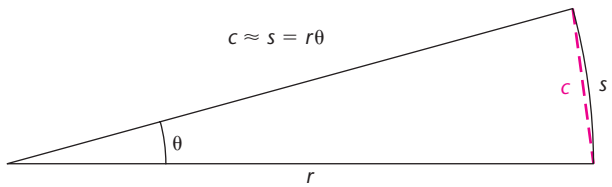


122. **CIRCUMFERENCE OF THE EARTH** Repeat Problem 121 with the sun crossing the vertical pole in Alexandria at 7°12′.

123. **CIRCUMFERENCE OF THE EARTH** In Problem 121, verbally explain how θ in the figure was determined.

124. **CIRCUMFERENCE OF THE EARTH** Verbally explain how the radius, surface area, and volume of the Earth can be determined from the result of Problem 121.

The arc length on a circle is easy to compute if the corresponding central angle is given in radians and the radius of the circle is known ($s = r\theta$). If the radius of a circle is large and a central angle is small, then an arc length is often used to approximate the length of the corresponding chord as shown in the figure. If an angle is given in degree measure, converting to radian measure first may be helpful in certain problems. This information will be useful in Problems 125–128.



125. **ASTRONOMY** The sun is about 9.3×10^7 mi from the Earth. If the angle subtended by the diameter of the sun on the surface of the Earth is 9.3×10^{-3} radians, approximately what is the diameter of the sun to the nearest thousand miles in standard decimal notation?

126. **ASTRONOMY** The moon is about 381,000 kilometers from the Earth. If the angle subtended by the diameter of the moon on

the surface of the Earth is 0.0092 radians, approximately what is the diameter of the moon to the nearest hundred kilometers?

127. PHOTOGRAPHY The angle of view of a 1,000-millimeter telephoto lens is 2.5°. At 750 feet, what is the width of the field of view to the nearest foot?

128. PHOTOGRAPHY The angle of view of a 300-millimeter lens is 8°. At 500 feet, what is the width of the field of view to the nearest foot?

5-2 Trigonometric Functions: A Unit Circle Approach

- The Wrapping Function
- Defining the Trigonometric Functions
- Graphing the Trigonometric Functions

In this section we introduce the six trigonometric functions in terms of the coordinates of points on the unit circle.

The Wrapping Function

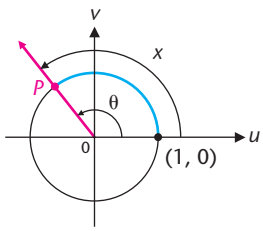


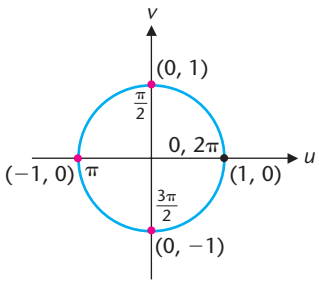
Figure 1

Consider a positive angle θ in standard position, and let P denote the point of intersection of the terminal side of θ with the unit circle $u^2 + v^2 = 1$ (Fig. 1).* Let x denote the length of the arc opposite θ on the unit circle. Because the unit circle has radius $r = 1$, the radian measure of θ is given by

$$\theta = \frac{x}{r} = \frac{x}{1} = x \text{ radians}$$

In other words, on the unit circle, the radian measure of a positive angle is equal to the length of the intercepted arc; similarly, on the unit circle, the radian measure of a negative angle is equal to the negative of the length of the intercepted arc. Because $\theta = x$, we may consider the real number x to be the name of the angle θ, when convenient. The function W that associates with each real number x the point $W(x) = P$ is called the **wrapping function**. The point P is called a **circular point**.

*We use the variables u and v instead of x and y so that x can be used without ambiguity as an independent variable in defining the wrapping function and the trigonometric functions.



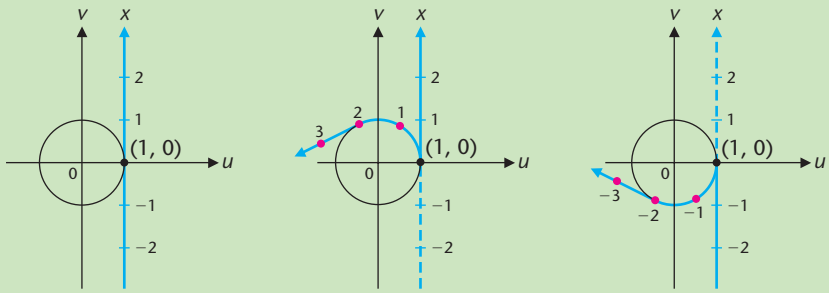
› **Figure 2** Circular points on the coordinate axes.

Consider, for example, the angle in standard position that has radian measure $\pi/2$. Its terminal side intersects the unit circle at the point $(0, 1)$. Therefore, $W(\pi/2) = (0, 1)$. Similarly, we can find the circular point associated with any angle that is an integer multiple of $\pi/2$ (Fig. 2).

$$W(0) = (1, 0)$$
$$W\left(\frac{\pi}{2}\right) = (0, 1)$$
$$W(\pi) = (-1, 0)$$
$$W\left(\frac{3\pi}{2}\right) = (0, -1)$$
$$W(2\pi) = (1, 0)$$

››› EXPLORE-DISCUSS 1

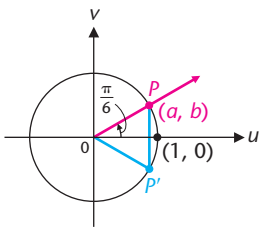
The name *wrapping function* stems from visualizing the correspondence as a wrapping of the real number line (shown in blue in Fig. 3) around the unit circle—the positive real axis is wrapped counterclockwise, and the negative real axis is wrapped clockwise—so that each real number is paired with a unique circular point.



› **Figure 3** The wrapping function.

(A) Explain why the wrapping function is not one-to-one.

(B) In which quadrant is the circular point $W(1)$? $W(-10)$? $W(100)$?



› **Figure 4**

Given a real number x, it is difficult, in general, to find the coordinates (a, b) of the circular point $W(x)$ that is associated with x. (It is trigonometry that overcomes this difficulty.) For certain real numbers x, however, we can find the coordinates (a, b) of $W(x)$ by using simple geometric facts. For example, consider $x = \pi/6$ and let P denote the circular point $W(x) = (a, b)$ that is associated with x. Let P' be the reflection of P through the u axis (Fig. 4).

Then triangle OPP' is equiangular (each angle has measure $\pi/3$ radians or 60°) and thus equilateral. Therefore $pp' = 1$, so $b = 1/2$. Because (a, b) lies on the unit circle, we solve for a:

$$a^2 + b^2 = 1 \qquad \text{Substitute } b = \frac{1}{2}.$$

$$a^2 + \left(\frac{1}{2}\right)^2 = 1 \qquad \text{Subtract } \frac{1}{4} \text{ from both sides.}$$

$$a^2 = \frac{3}{4} \qquad \text{Take square roots.}$$

$$a = \pm\frac{\sqrt{3}}{2} \qquad a = -\frac{\sqrt{3}}{2} \text{ must be discarded. (Why?)}$$

Therefore,

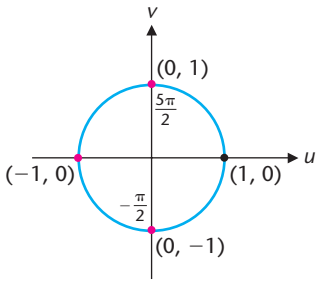
$$W\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

EXAMPLE 1 Finding Coordinates of Circular Points

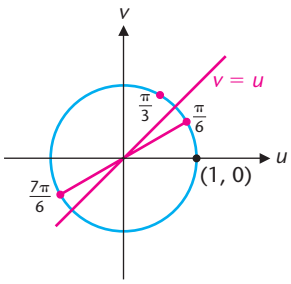
Find the coordinates of the following circular points:

(A) $W(-\pi/2)$ (B) $W(5\pi/2)$ (C) $W(\pi/3)$
(D) $W(7\pi/6)$ (E) $W(\pi/4)$

SOLUTIONS



› Figure 5



› Figure 6

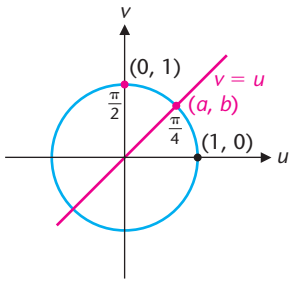
(A) Because the circumference of the unit circle is 2π, $-\pi/2$ is the radian measure of a negative angle that is $\frac{1}{4}$ of a complete clockwise rotation. Therefore, $W(-\pi/2) = (0, -1)$ (Fig. 5).

(B) Starting at (1, 0) and proceeding counterclockwise, we count quarter-circle steps, $\pi/2, 2\pi/2, 3\pi/2, 4\pi/2$, and end at $5\pi/2$. Therefore, the circular point is on the positive vertical axis, and $W(5\pi/2) = (0, 1)$ (see Fig. 5).

(C) The circular point $W(\pi/3)$ is the reflection of the point $W(\pi/6) = (\sqrt{3}/2, 1/2)$ through the line $u = v$. To reflect a point through the line $u = v$, you interchange its coordinates (see Problem 145, in Exercises 5-2). Therefore, $W(\pi/3) = (1/2, \sqrt{3}/2)$ (Fig. 6).

(D) The circular point $W(7\pi/6)$ is the reflection of the point $W(\pi/6) = (\sqrt{3}/2, 1/2)$ through the origin. To reflect a point through the origin you change the sign of each coordinate (see Section 1-4). Therefore, $W(7\pi/6) = (-\sqrt{3}/2, -1/2)$ (see Fig. 6).

(E) The circular point $W(\pi/4) = (a, b)$ lies on the line $u = v$, so $a = b$ (see Fig. 7 on the next page).



› Figure 7

Since (a, b) lies on the unit circle,

$$a^2 + b^2 = 1 \qquad \text{Substitute } b = a.$$
$$2a^2 = 1 \qquad \text{Divide both sides by 2.}$$
$$a^2 = \frac{1}{2} \qquad \text{Take the square root of both sides.}$$
$$a = \frac{1}{\sqrt{2}} \qquad a = -\frac{1}{\sqrt{2}} \text{ is impossible. (Why?)}$$

Therefore, $W(\pi/4) = (1/\sqrt{2}, 1/\sqrt{2})$ (Fig. 7).

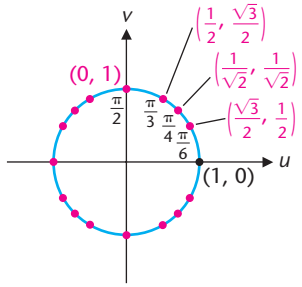
MATCHED PROBLEM 1

Find the coordinates of the following circular points:

(A) $W(3\pi)$ (B) $W(-7\pi/2)$ (C) $W(5\pi/6)$
(D) $W(-\pi/3)$ (E) $W(5\pi/4)$

Some key results from Example 1 are summarized in Figure 8. If x is any integer multiple of $\pi/6$ or $\pi/4$, then the coordinates of $W(x)$ can be determined easily from Figure 8 by using symmetry properties. For example, change the sign of the first coordinate of the three points in Quadrant I to obtain the coordinates of their reflections through the v axis in Quadrant II. Similarly, change the sign of both coordinates of the three points in Quadrant I to obtain the coordinates of their reflections through the origin in Quadrant III. And change the sign of the second coordinate of the three points in Quadrant I to obtain the coordinates of their reflections through the u axis in Quadrant IV.

› COORDINATES OF KEY CIRCULAR POINTS



› Figure 8

››› EXPLORE-DISCUSS 2

An effective memory aid for recalling the coordinates of the key circular points in Figure 8 can be created by writing the coordinates of the circular points $W(0)$, $W(\pi/6)$, $W(\pi/4)$, $W(\pi/3)$, and $W(\pi/2)$, keeping this order, in a form where each numerator is the square root of an appropriate number and each denominator is 2. For example, $W(0) = (1, 0) = (\sqrt{4}/2, \sqrt{0}/2)$. Describe the pattern that results.

› Defining the Trigonometric Functions

We define the trigonometric functions in terms of the coordinates of points on the unit circle. This suggests that the trigonometric functions are useful in analyzing circular motion, for example, of satellites, DVD players, generators, wheels, and propellers. While true, we will also discover that these functions have many applications that are apparently unrelated to rotary motion. There are six **trigonometric functions: sine, cosine, tangent, cotangent, secant,** and **cosecant.** The values of these functions at a real number x are denoted by **sin** x**, cos** x**, tan** x**, cot** x**, sec** x**,** and **csc** x**,** respectively.

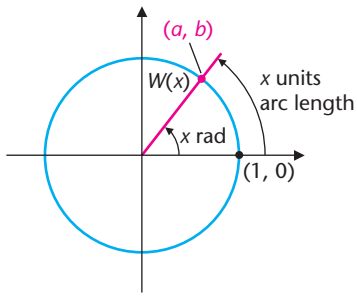
› DEFINITION 1 Trigonometric Functions

Let x be a real number and let (a, b) be the coordinates of the circular point $W(x)$ that lies on the terminal side of the angle with radian measure x. Then:

$$\sin x = b \qquad \csc x = \frac{1}{b} \quad b \neq 0$$

$$\cos x = a \qquad \sec x = \frac{1}{a} \quad a \neq 0$$

$$\tan x = \frac{b}{a} \quad a \neq 0 \qquad \cot x = \frac{a}{b} \quad b \neq 0$$



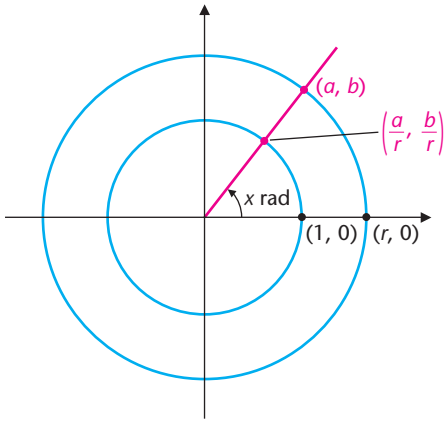
REMARKS:

1. Note that sin x and cos x are the second and first coordinates, respectively, of the point (a, b) on the unit circle.
2. We assume in Definition 1 that (a, b) is the point on the *unit circle* that lies on the terminal side of the angle with radian measure x. More generally, however, if (a, b) is the point on that terminal side that lies on the *circle of radius* $r > 0$, then:

$$a^2 + b^2 = r^2 \qquad \text{Divide both sides by } r^2.$$

$$\left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 = 1 \qquad \text{So } \left(\frac{a}{r}, \frac{b}{r}\right) \text{ lies on the unit circle.}$$

Therefore $(a/r, b/r)$ is the point on the terminal side of the angle with radian measure x (see Problems 147 and 148 in Exercises 5-2) that lies on the unit circle (Fig. 9).



› Figure 9

By Definition 1,

$$\sin x = \frac{b}{r} \qquad \csc x = \frac{r}{b} \quad b \neq 0$$

$$\cos x = \frac{a}{r} \qquad \sec x = \frac{r}{a} \quad a \neq 0$$

$$\tan x = \frac{b}{a} \quad a \neq 0 \qquad \cot x = \frac{a}{b} \quad b \neq 0$$

Note that these formulas coincide with those of Definition 1 when $r = 1$.

EXAMPLE 2 **Evaluating Trigonometric Functions**

Find the values of all six trigonometric functions of the angle x if

(A) $W(x) = \left(\frac{3}{5}, -\frac{4}{5}\right)$.

(B) The terminal side of x contains the point $(-60, -11)$.

SOLUTIONS

(A) Note that $W(x)$ is indeed on the unit circle because

$$\sqrt{\left(\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = 1$$

Using Definition 1, with $a = \frac{3}{5}$ and $b = -\frac{4}{5}$,

$$\sin x = b = -\frac{4}{5} \qquad \csc x = \frac{1}{b} = -\frac{5}{4}$$

$$\cos x = a = \frac{3}{5} \qquad \sec x = \frac{1}{a} = \frac{5}{3}$$

$$\tan x = \frac{b}{a} = \frac{-4/5}{3/5} = -\frac{4}{3} \qquad \cot x = \frac{a}{b} = \frac{3/5}{-4/5} = -\frac{3}{4}$$

(B) The distance r from $(-60, -11)$ to $(0, 0)$ is

$$\sqrt{(-60)^2 + (-11)^2} = \sqrt{3{,}600 + 121} = \sqrt{3{,}721} = 61$$

Using Remark 2 following Definition 1, with $a = -60$, $b = -11$, and $r = 61$:

$$\sin x = \frac{b}{r} = -\frac{11}{61} \qquad \csc x = \frac{r}{b} = -\frac{61}{11}$$

$$\cos x = \frac{a}{r} = -\frac{60}{61} \qquad \sec x = \frac{r}{a} = -\frac{61}{60}$$

$$\tan x = \frac{b}{a} = \frac{11}{60} \qquad \cot x = \frac{a}{b} = \frac{60}{11}$$

››› CAUTION ›››

Always check that values of sin x and cos x are numbers that are between (or equal to) -1 and 1, as implied by Definition 1. Note in particular that this is the case in Example 1.

MATCHED PROBLEM 2

Find the values of all six trigonometric functions of the angle x if

(A) $W(x) = \left(-\frac{12}{13}, \frac{5}{13}\right)$.

(B) The terminal side of x contains the point $(13, 84)$.

The domain of both the sine and cosine functions is the set of real numbers R. The range of both the sine and cosine functions is $[-1, 1]$. This is the set of numbers assumed by b, for sine, and a, for cosine, as the circular point (a, b) moves around the unit circle. The domain of cosecant is the set of real numbers x such that b in

$W(x) = (a, b)$ is not 0. Similar restrictions are made on the domains of the other three trigonometric functions. We will have more to say about the domains and ranges of all six trigonometric functions in subsequent sections.

Note from Definition 1 that csc x is the reciprocal of sin x, provided that sin $x \neq 0$. Therefore sin x is the reciprocal of csc x. Similarly, cos x and sec x are reciprocals of each other, as are tan x and cot x. We call these useful facts the *reciprocal identities*.

RECIPROCAL IDENTITIES

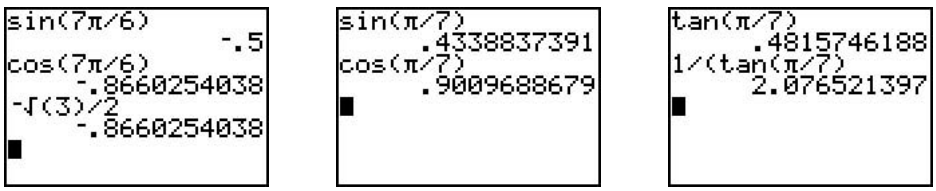
For x any real number:

$$\csc x = \frac{1}{\sin x} \qquad \sin x \neq 0$$

$$\sec x = \frac{1}{\cos x} \qquad \cos x \neq 0$$

$$\cot x = \frac{1}{\tan x} \qquad \tan x \neq 0$$

In Example 1 we were able to give a simple geometric argument to find, for example, that the coordinates of $W(7\pi/6)$ are $(-\sqrt{3}/2, -1/2)$. Therefore, $\sin(7\pi/6) = -1/2$ and $\cos(7\pi/6) = -\sqrt{3}/2$. These exact values correspond to the approximations given by a calculator [Fig. 10(a)]. For most values of x, however, simple geometric arguments fail to give the exact coordinates of $W(x)$. But a calculator, set in radian mode, can be used to give approximations. For example, if $x = \pi/7$, then $W(\pi/7) \approx (0.901, 0.434)$ [Fig. 10(b)].



(a) (b) (c)

Figure 10

Most calculators have function keys for the sine, cosine, and tangent functions, but not for the cotangent, secant, and cosecant. Because the cotangent, secant, and cosecant are the reciprocals of the tangent, cosine, and sine, respectively, they can be evaluated easily. For example, $\cot(\pi/7) = 1/\tan(\pi/7) \approx 2.077$ [Fig. 10(c)]. Do not use the calculator function keys marked $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$ for this purpose—these keys are used to evaluate the inverse trigonometric functions of Section 5-6, not reciprocals.

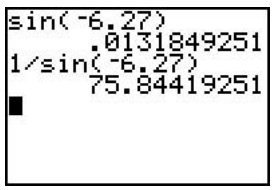
EXAMPLE 3 **Calculator Evaluation**

Evaluate to four significant digits.

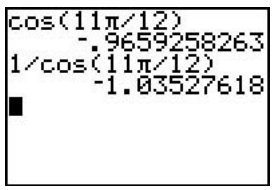
(A) tan 1.5 (B) csc (−6.27)
(C) sec (11π/12) (D) The coordinates (a, b) of $W(1)$

SOLUTIONS

(A) tan 1.5 = 14.10
(B) csc (−6.27) = 1/sin (−6.27) = 75.84



(C) sec (11π/12) = 1/cos (11π/12) = −1.035



(D) $W(1) = (\cos 1, \sin 1) = (0.5403, 0.8415)$

MATCHED PROBLEM 3

Evaluate to four significant digits.

(A) cot (−8.25) (B) sec (7π/8)
(C) csc (4.67) (D) The coordinates (a, b) of $W(100)$

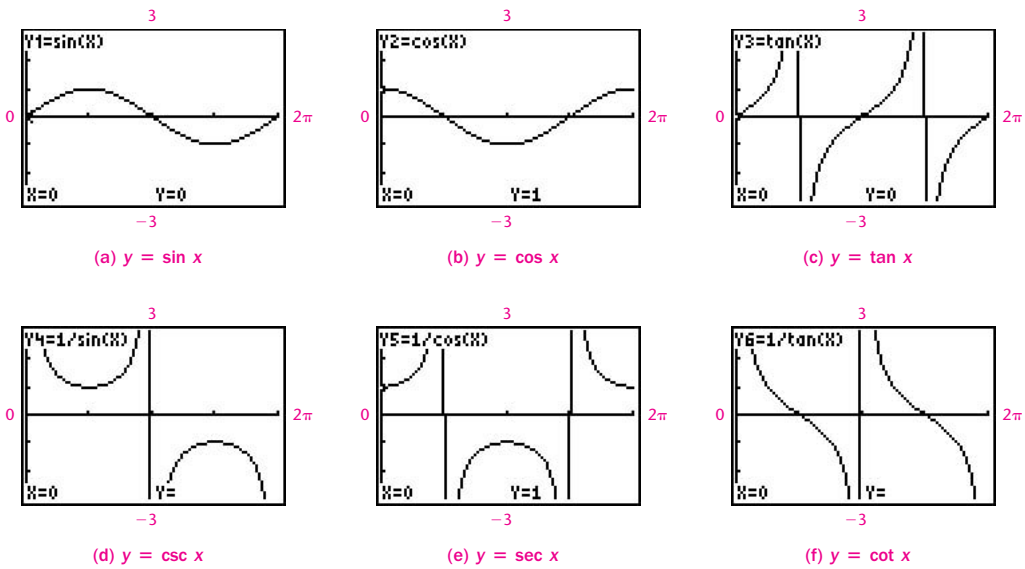
› Graphing the Trigonometric Functions

The graph of $y = \sin x$ is the set of all ordered pairs (x, y) of real numbers that satisfy the equation. Because $\sin x$, by Definition 1, is the second coordinate of the circular point $W(x)$, our knowledge of the coordinates of certain circular points (Table 1) gives the following solutions to $y = \sin x$: $x = 0, y = 0; x = \pi/2, y = 1; x = \pi, y = 0;$ and $x = 3\pi/2, y = -1$.

Table 1

x	0	$\pi/2$	π	$3\pi/2$
$W(x)$	$(0, \mathbf{0})$	$(0, \mathbf{1})$	$(-1, \mathbf{0})$	$(0, \mathbf{-1})$
$\sin x$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{-1}$

As x increases from 0 to $\pi/2$, the circular point $W(x)$ moves on the circumference of the unit circle from (0, 0) to (0, 1), and so $\sin x$ [the second coordinate of $W(x)$] increases from 0 to 1. [But this increase is *not* linear: At $\pi/4$ (the midpoint of 0 and $\pi/2$) the second coordinate of $W(x)$ is $1/\sqrt{2}$, *not* 1/2 (the midpoint of 0 and 1)]. Similarly, as x increases from $\pi/2$ to π, the circular point $W(x)$ moves on the circumference of the unit circle from (0, 1) to $(-1, 0)$, and so $\sin x$ decreases from 1 to 0. These observations are in agreement with the graph of $y = \sin x$, obtained from a graphing calculator in radian mode [Fig. 11(a)].



(a) $y = \sin x$ (b) $y = \cos x$ (c) $y = \tan x$

(d) $y = \csc x$ (e) $y = \sec x$ (f) $y = \cot x$

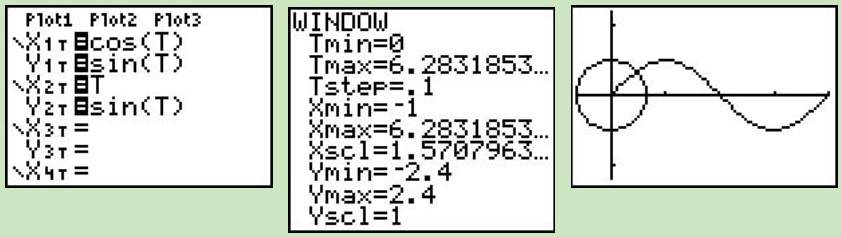
› **Figure 11**

Figure 11 shows the graphs of all six trigonometric functions from $x = 0$ to $x = 2\pi$. Because the circular point $W(2\pi)$ coincides with the circular point $W(0)$, the graphs of the six trigonometric functions from $x = 2\pi$ to $x = 4\pi$ would be identical to the graphs shown in Figure 11. The functions $y = \sin x$ and $y = \cos x$ are bounded; their maximum values are 1 and their minimum values are -1. The functions $y = \tan x$, $y = \cot x$, $y = \sec x$, and $y = \csc x$ are unbounded; they have vertical asymptotes at the values of x for which they are undefined. It is instructive to study and compare the graphs of reciprocal pairs, for example, $y = \cos x$ and $y = \sec x$. Note that $\sec x$ is undefined when $\cos x$ equals 0, and that because the maximum positive value of $\cos x$ is 1, the minimum positive value of $\sec x$ is 1. We will study the properties of trigonometric functions and their graphs in Section 5-4.

A graphing calculator can be used to interactively explore the relationship between the unit circle definition of the sine function and the graph of the sine function. Explore-Discuss 3 provides the details.

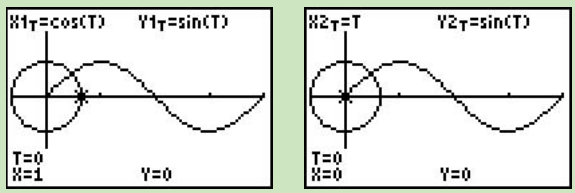
››› EXPLORE-DISCUSS 3

With your graphing calculator, you can illuminate the connection between the unit circle definition of the sine function and the graph of the sine function. Set your graphing calculator in radian and parametric modes (parametric equations are discussed in Section B-3, Appendix B). Make the entries as indicated in Figure 12 to obtain the indicated graph (2π is entered as Tmax and Xmax, $\pi/2$ is entered as Xscl).



› Figure 12

Use TRACE and move back and forth between the unit circle and the graph of the sine function for various values of T as T increases from 0 to 2π. Discuss what happens in each case. Figure 13 illustrates the case for $T = 0$.



› Figure 13

Repeat the exploration with $Y_{2T} = \cos(T)$

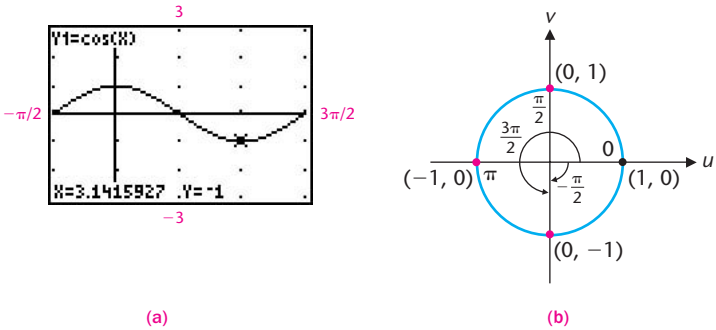
EXAMPLE 4 Zeros and Turning Points

Find the zeros and turning points of $y = \cos x$ on the interval $[-\pi/2, 3\pi/2]$.

SOLUTION

Recall that a *turning point* is a point on a graph that separates an increasing portion from a decreasing portion, or vice versa. A visual inspection of the graph of $y = \cos x$ [Fig. 14(a)] suggests that $(0, 1)$ and $(\pi, -1)$ are turning points, and that $-\pi/2$, $\pi/2$,

and $3\pi/2$ are zeros. These observations are confirmed by noting that as x increases from $-\pi/2$ to $3\pi/2$, the first coordinate of the circular point $W(x)$ (that is, $\cos x$) has a maximum value of 1 (when $x = 0$), a minimum value of -1 (when $x = \pi$), and has the value 0 when $x = -\pi/2, \pi/2$, and $3\pi/2$ [Fig. 14(b)].



› Figure 14

MATCHED PROBLEM 4

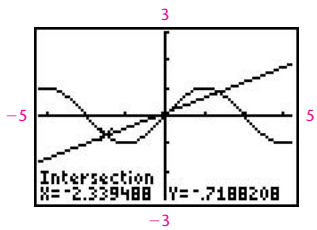
Find all zeros and turning points of $y = \csc x$ on the interval $(0, 4\pi)$.

EXAMPLE 5 Solving a Trigonometric Equation

Find all solutions of the equation $\sin x = 0.35x + 0.1$ to three decimal places.

SOLUTION

Graph $y_1 = \sin x$ and $y_2 = 0.35x + 0.1$ and use the intersect command (Fig. 15). The solutions are $x = -2.339, 0.155,$ and 2.132.



› Figure 15

MATCHED PROBLEM 5

Find all solutions of the equation $\cot x = x$ on the interval $(0, 2\pi)$ to three decimal places.

››› CAUTION ›››

A common cause of error is to forget to set a calculator in the correct mode, degree or radian, before graphing or evaluating a function. In radian mode, a calculator will give 1 as the value of $\sin(\pi/2)$; in degree mode, it will give 0.0274 as the value of $\sin(\pi/2)$ [because $(\pi/2)^\circ \approx 1.5708^\circ$].

ANSWERS TO MATCHED PROBLEMS

1. (A) $(-1, 0)$ (B) $(0, 1)$ (C) $(-\sqrt{3}/2, 1/2)$
(D) $(1/2, -\sqrt{3}/2)$ (E) $(-1/\sqrt{2}, -1/\sqrt{2})$

2. (A) $\sin x = \frac{5}{13}$ $\csc x = \frac{13}{5}$ $\cos x = -\frac{12}{13}$ $\sec x = -\frac{13}{12}$ $\tan x = -\frac{5}{12}$ $\cot x = -\frac{12}{5}$

(B) $\sin x = \frac{84}{85}$ $\csc x = \frac{85}{84}$ $\cos x = \frac{13}{85}$ $\sec x = \frac{85}{13}$ $\tan x = \frac{84}{13}$ $\cot x = \frac{13}{84}$

3. (A) 0.4181 (B) -1.082 (C) -1.001 (D) $(0.8623, -0.5064)$

4. Zeros: none; turning points: $(\pi/2, 1)$, $(3\pi/2, -1)$, $(5\pi/2, 1)$, $(7\pi/2, -1)$

5. 0.860, 3.426

5-2 Exercises

1. What is the unit circle?

2. Describe the wrapping function, including its domain and range.

3. Explain the connection between points on the unit circle and the six trigonometric functions.

4. Explain why the function $y = \sec x$ is undefined for certain values of x.

5. Explain why the graph of $y = \tan x$ has vertical asymptotes at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$.

6. Explain why every point on the graph of $y = \cos x$ lies on or between the lines $y = -1$ and $y = 1$.

In Problems 7–22, find the coordinates of each circular point.

7. $W(3\pi/2)$ **8.** $W(-5\pi)$ **9.** $W(-6\pi)$

10. $W(-15\pi/2)$ **11.** $W(\pi/4)$ **12.** $W(\pi/3)$

13. $W(\pi/6)$ **14.** $W(-\pi/6)$ **15.** $W(-\pi/3)$

16. $W(-\pi/4)$ **17.** $W(2\pi/3)$ **18.** $W(11\pi/6)$

19. $W(-3\pi/4)$ **20.** $W(-7\pi/6)$ **21.** $W(13\pi/4)$

22. $W(-10\pi/3)$

In Problems 23–38, use your answers to Problems 7–22 to give the exact value of the expression (if it exists).

23. $\sin(3\pi/2)$ **24.** $\tan(-5\pi)$ **25.** $\cos(-6\pi)$

26. $\cot(-15\pi/2)$ **27.** $\sec(\pi/4)$ **28.** $\csc(\pi/3)$

29. $\tan(\pi/6)$ **30.** $\cos(-\pi/6)$ **31.** $\sin(-\pi/3)$

32. $\sec(-\pi/4)$ **33.** $\csc(2\pi/3)$ **34.** $\cot(11\pi/6)$

35. $\cos(-3\pi/4)$ **36.** $\tan(-7\pi/6)$ **37.** $\cot(13\pi/4)$

38. $\sin(-10\pi/3)$

In Problems 39–42, find the exact value of the expression given that $W(x) = (\frac{5}{13}, \frac{12}{13})$.

39. $\sin x$ **40.** $\cot x$

41. $\sec x$ **42.** $\csc x$

In Problems 43–46, find the exact value of the expression given that $W(x) = (-\frac{15}{17}, \frac{8}{17})$.

43. $\cot x$ **44.** $\cos x$

45. $\csc x$ **46.** $\tan x$

In Problems 47–50, find the exact value of the expression given that $W(x) = (0, -1)$.

47. $\cos x$ **48.** $\sec x$

49. $\tan x$ **50.** $\csc x$

In Problems 51–62, find the exact value of the expression given that x is an angle in standard position and the terminal side of x contains the indicated point.

51. $\cos x$; $(0, 5)$ **52.** $\sin x$; $(-9, 0)$

53. $\tan x$; $(4, 3)$ **54.** $\cot x$; $(-6, 8)$

55. csc x; (0.4, 0.09)

56. sec x; (0.12, 0.35)

57. cot x; $(-8, 15)$

58. tan x; $(-3, 4)$

59. sec x; $(-9, -40)$

60. csc x; $(-35, 12)$

61. sin x; $(2, -1)$

62. cos x; $(-1, -3)$

In Problems 63–68, in which quadrants must $W(x)$ lie so that:

63. $\cos x < 0$

64. $\tan x > 0$

65. $\sin x > 0$

66. $\sec x > 0$

67. $\cot x < 0$

68. $\csc x < 0$

Evaluate Problems 69–78 to four significant digits using a calculator set in radian mode.

69. cos 2.288

70. sin 3.104

71. tan (-4.644)

72. sec (-1.555)

73. csc 1.571

74. cot 0.7854

75. sin (cos 0.3157)

76. cos (tan 5.183)

77. cos [csc (-1.408)]

78. sec [cot (-3.566)]

Evaluate Problems 79–88 to four significant digits using a calculator. Make sure your calculator is in the correct mode (degree or radian) for each problem.

79. sin 25°

80. tan 89°

81. cot 12

82. csc 13

83. sin 2.137

84. tan 4.327

85. cot $(-431.41°)$

86. sec $(-247.39°)$

87. sin 113°27′13″

88. cos 235°12′47″

In Problems 89–94, determine whether the statement about the wrapping function W is true or false. Explain.

89. The domain of the wrapping function is the set of all points on the unit circle.

90. The domain of the wrapping function is the set of all real numbers.

91. If $W(x) = W(y)$, then $x = y$.

92. If $x = y$, then $W(x) = W(y)$.

93. If a and b are real numbers and $a^2 + b^2 = 1$, then there exists a real number x such that $W(x) = (a, b)$.

94. If a and b are real numbers and $a^2 + b^2 = 1$, then there exists a unique real number x such that $W(x) = (a, b)$.

In Problems 95–104, determine whether the statement about the trigonometric functions is true or false. Explain.

95. If x is a real number, then cos x is the reciprocal of sin x.

96. If x is a real number, then $(\cot x)(\tan x) = 1$.

97. If $\sec x = \sec y$, then $x = y$.

98. If $x = y$, then $\cos x = \cos y$.

99. The functions sin x and csc x have the same domain.

100. The functions sin x and cos x have the same domain.

101. The graph of the function cos x has infinitely many turning points.

102. The graph of the function tan x has infinitely many turning points.

103. The graph of the function cot x has infinitely many zeros.

104. The graph of the function csc x has infinitely many zeros.

In Problems 105–108, find all zeros and turning points of each function on $[0, 4\pi]$.

105. $y = \sec x$

106. $y = \sin x$

107. $y = \tan x$

108. $y = \cot x$

Determine the signs of a and b for the coordinates (a, b) of each circular point indicated in Problems 109–118. First determine the quadrant in which each circular point lies. [Note: $\pi/2 \approx 1.57$, $\pi \approx 3.14$, $3\pi/2 \approx 4.71$, and $2\pi \approx 6.28$.]

109. $W(2)$

110. $W(1)$

111. $W(3)$

112. $W(4)$

113. $W(5)$

114. $W(7)$

115. $W(-2.5)$

116. $W(-4.5)$

117. $W(-6.1)$

118. $W(-1.8)$

In Problems 119–122, for each equation find all solutions for $0 \le x \le 2\pi$, then write an expression that represents all solutions for the equation without any restrictions on x.

119. $W(x) = (1, 0)$

120. $W(x) = (-1, 0)$

121. $W(x) = (-1/\sqrt{2}, 1/\sqrt{2})$

122. $W(x) = (1/\sqrt{2}, -1/\sqrt{2})$

123. Describe in words why $W(x) = W(x + 4\pi)$ for every real number x.

124. Describe in words why $W(x) = W(x - 6\pi)$ for every real number x.

If $W(x) = (a, b)$, indicate whether the statements in Problems 125–130 are true or false. Sketching figures should help you decide.

125. $W(x + \pi) = (-a, -b)$

126. $W(x + \pi) = (a, b)$

127. $W(-x) = (-a, b)$

128. $W(-x) = (a, -b)$

129. $W(x + 2\pi) = (a, b)$

130. $W(x + 2\pi) = (-a, -b)$

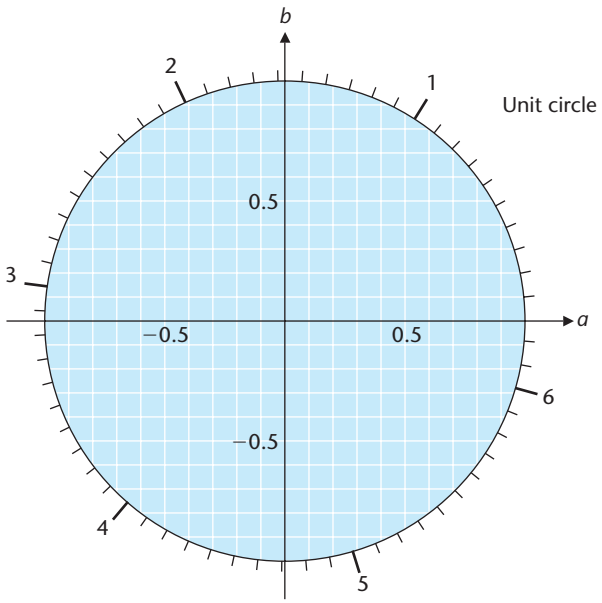
In Problems 131–134, find the value of each expression to one significant digit. Use only the accompanying figure below, Definition 1, and a calculator as necessary for multiplication and division. Check your results by evaluating each directly on a calculator.

131. (A) sin 0.4 (B) cos 0.4 (C) tan 0.4

132. (A) sin 0.8 (B) cos 0.8 (C) cot 0.8

133. (A) sec 2.2 (B) tan 5.9 (C) cot 3.8

134. (A) csc 2.5 (B) cot 5.6 (C) tan 4.3



In Problems 135–138, in which quadrants are the statements true and why?

135. $\sin x < 0$ and $\cot x < 0$

136. $\cos x > 0$ and $\tan x < 0$

137. $\cos x < 0$ and $\sec x > 0$

138. $\sin x > 0$ and $\csc x < 0$

For which values of x, $0 \le x \le 2\pi$, is each of Problems 139–144 not defined?

139. $\cos x$ **140.** $\sin x$ **141.** $\tan x$

142. $\cot x$ **143.** $\sec x$ **144.** $\csc x$

145. Prove that the reflection of the point (a, b) through the line $y = x$ is the point (b, a) by verifying statements (A) and (B):
(A) The line through (a, b) and (b, a) is perpendicular to the line $y = x$.
(B) The midpoint of (a, b) and (b, a) lies on the line $y = x$.

146. Prove that the reflection of the point (a, b) through the line $y = -x$ is the point $(-b, -a)$ by verifying statements (A) and (B):
(A) The line through (a, b) and $(-b, -a)$ is perpendicular to the line $y = -x$.
(B) The midpoint of (a, b) and $(-b, -a)$ lies on the line $y = -x$.

In Problems 147 and 148, consider the point $P = (a, b)$, where a and b are not both zero, and let $O = (0, 0)$. Ray $\overrightarrow{OP}$ is defined by

$$\overrightarrow{OP} = \{(ka, kb) \mid k \ge 0\}$$

147. Show that $bx - ay = 0$ is the equation of the line through O and P.

148. Refer to Problem 147. Show that every point on $\overrightarrow{OP}$ satisfies the equation of the line through O and P.

APPLICATIONS

If an n-sided regular polygon is inscribed in a circle of radius r, then it can be shown that the area of the polygon is given by

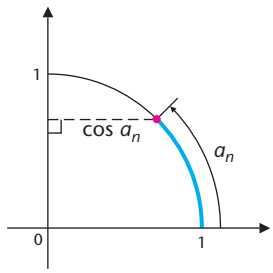
$$A = \frac{1}{2} nr^2 \sin \frac{2\pi}{n}$$

In Problems 149–152, compute each area exactly and then to four significant digits.

149. $n = 12, r = 5$ meters **150.** $n = 4, r = 3$ inches

151. $n = 3, r = 4$ inches **152.** $n = 8, r = 10$ centimeters

APPROXIMATING π *Problems 153 and 154 refer to a sequence of numbers generated as follows:*

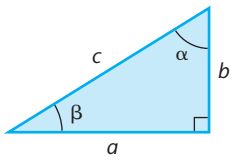


$$\begin{aligned} a_1 & \\ a_2 &= a_1 + \cos a_1 \\ a_3 &= a_2 + \cos a_2 \\ &\vdots \\ a_{n+1} &= a_n + \cos a_n \end{aligned}$$

153. Let $a_1 = 0.5$, and compute the first five terms of the sequence to six decimal places and compare the fifth term with $\pi/2$ computed to six decimal places.

154. Repeat Problem 153, starting with $a_1 = 1$.

5-3 Solving Right Triangles*

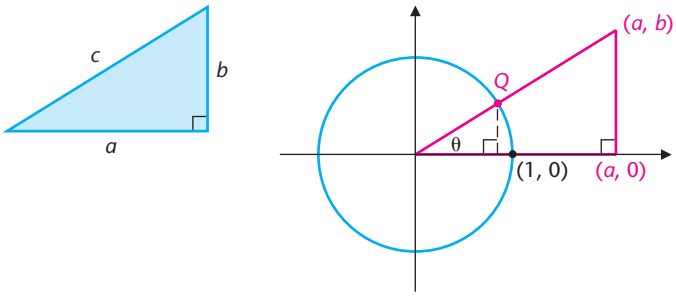


› Figure 1

A **right triangle** is a triangle with one 90° angle (Fig. 1).

If only the angles of a right triangle are known, it is impossible to solve for the sides. (Why?) But if we are given two sides, or one acute angle and a side, then it is possible to solve for the remaining three quantities. This process is called **solving the right triangle.** We use the trigonometric functions to solve right triangles.

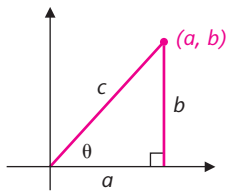
If a right triangle is located in the first quadrant as indicated by Figure 2, then, by similar triangles, the coordinates of the circular point Q are $(a/c, b/c)$.



› Figure 2

Therefore, using the definition of the trigonometric functions, $\sin\theta = b/c$ and $\cos\theta = a/c$. (Calculations using such trigonometric ratios are valid if θ is measured in either degrees or radians, provided your calculator is set in the correct mode—in this section, we use degree measure.) All six trigonometric ratios are displayed in the box.

› TRIGONOMETRIC RATIOS



$0° < \theta < 90°$

$$\sin\theta = \frac{b}{c} \qquad \csc\theta = \frac{c}{b}$$

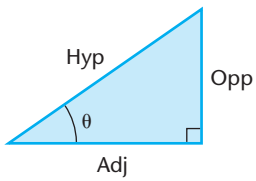
$$\cos\theta = \frac{a}{c} \qquad \sec\theta = \frac{c}{a}$$

$$\tan\theta = \frac{b}{a} \qquad \cot\theta = \frac{a}{b}$$

*This section provides a significant application of trigonometric functions to real-world problems. However, it may be postponed or omitted without loss of continuity, if desired. Some may want to cover the section just before Sections 7-1 and 7-2.

Side b is often referred to as the **side opposite** angle θ, a as the **side adjacent** to angle θ, and c as the **hypotenuse.** Using these designations for an arbitrary right triangle removed from a coordinate system, we have the following:

› RIGHT TRIANGLE RATIOS



$0° < \theta < 90°$

$$\sin\theta = \frac{\text{Opp}}{\text{Hyp}} \qquad \csc\theta = \frac{\text{Hyp}}{\text{Opp}}$$

$$\cos\theta = \frac{\text{Adj}}{\text{Hyp}} \qquad \sec\theta = \frac{\text{Hyp}}{\text{Adj}}$$

$$\tan\theta = \frac{\text{Opp}}{\text{Adj}} \qquad \cot\theta = \frac{\text{Adj}}{\text{Opp}}$$

››› EXPLORE-DISCUSS 1

For a given value of θ, $0 < \theta < 90°$, explain why the value of each of the six trigonometric functions is independent of the size of the right triangle that contains θ.

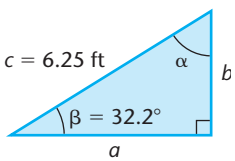
Table 1

Angle to nearest	Significant digits for side measure
1°	2
10′ or 0.1°	3
1′ or 0.01°	4
10″ or 0.001°	5

The use of the trigonometric ratios for right triangles is made clear in Examples 1 through 4. Regarding computational accuracy, we use Table 1 as a guide. (The table is also printed inside the cover of this book for easy reference.) We will use $=$ rather than $\approx$ in many places, realizing the accuracy indicated in Table 1 is all that is assumed. Another word of caution: When using your calculator be sure it is set in degree mode.

EXAMPLE 1 Solving a Right Triangle

Solve the right triangle with $c = 6.25$ feet and $\beta = 32.2°$.



› Figure 3

SOLUTION

First draw a figure and label the parts (Fig. 3):

SOLVE FOR α

$$\alpha = 90° - 32.2° = 57.8°$$ α and β are complementary.

SOLVE FOR b

$$\sin \beta = \frac{b}{c} \qquad \text{Substitute } \beta = 32.2°,\ c = 6.25$$

$$\sin 32.2° = \frac{b}{6.25} \qquad \text{Multiply both sides by 6.25.}$$

$$b = 6.25 \sin 32.2° \qquad \text{Calculate.}$$

$$= 3.33 \text{ feet}$$

SOLVE FOR a

$$\cos \beta = \frac{a}{c} \qquad \text{Substitute } \beta = 32.2°,\ c = 6.25$$

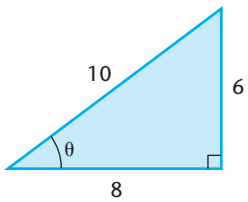
$$\cos 32.2° = \frac{a}{6.25} \qquad \text{Multiply both sides by 6.25.}$$

$$a = 6.25 \cos 32.2° \qquad \text{Calculate.}$$

$$= 5.29 \text{ feet}$$

MATCHED PROBLEM 1

Solve the right triangle with $c = 27.3$ meters and $\alpha = 47.8°$.



› Figure 4

In Example 1 we solved a right triangle given a side and one of the acute angles (32.2°). We used trigonometric functions of 32.2° to find the remaining sides.

What if only the sides of a right triangle are known (Fig. 4)? How can we find the angle θ?

Because $\sin \theta = 6/10 = 0.6$, we want to solve the equation

$$\sin \theta = 0.6$$

for θ. The solution to the equation is the angle θ whose sine is 0.6, that is, θ is the *inverse sine* of 0.6. Using the inverse sine function, denoted $\sin^{-1}$, on a calculator in degree mode, we obtain

$$\theta = \sin^{-1} 0.6$$

$$= 36.87° \qquad \text{To nearest hundredth degree.}$$

$$= 36°52' \qquad \text{To nearest minute.}$$

CHECK:

$$\sin 36.87° = 0.6$$

››› CAUTION ›››

It is important to note that $\sin^{-1} 0.6$ does *not* mean $1/(\sin 0.6)$. The superscript -1 is part of a function symbol, and *$\sin^{-1}$ represents the inverse sine function,* in accordance with the notation for inverse functions introduced in Section 1-5. Some books use arcsin in place of $\sin^{-1}$.

Inverse trigonometric functions are developed in detail in the last section of this chapter. For now, we will use the calculator function keys SIN^{-1}, COS^{-1}, and TAN^{-1} or their equivalents (check your manual) to find the angle with a given trigonometric ratio.

››› EXPLORE-DISCUSS 2

Solve each of the following for θ to the nearest hundredth of a degree using a calculator. Explain why an error message occurs in one of the problems.

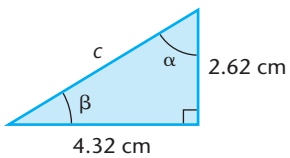
(A) $\cos\theta = 0.2044$ **(B)** $\tan\theta = 1.4138$ **(C)** $\sin\theta = 1.4138$

EXAMPLE 2 Solving a Right Triangle

Solve the right triangle with $a = 4.32$ centimeters and $b = 2.62$ centimeters. Compute the angle measures to the nearest $10'$.

SOLUTION

Draw a figure and label the known parts (Fig. 5):



› Figure 5

SOLVE FOR β

$$\tan\beta = \frac{2.62}{4.32} \qquad \text{Use } \tan^{-1} \text{ to solve for } \beta.$$

$$\beta = \tan^{-1}\frac{2.62}{4.32} \qquad \text{Calculate.}$$

$$= 31.2^\circ \text{ or } 31^\circ 10' \qquad 0.2^\circ = [(0.2)(60)]' = 12' \approx 10' \text{ to nearest } 10'$$

SOLVE FOR α

$$\alpha = 90^\circ - 31^\circ 10' \boxed{= 89^\circ 60' - 31^\circ 10'} = 58^\circ 50'$$

SOLVE FOR c

$$\sin \beta = \frac{2.62}{c} \qquad \text{Substitute } \beta = 31.2°, \text{ and solve for } c.$$

$$c = \frac{2.62}{\sin 31.2°} = 5.06 \text{ centimeters}$$

or, using the Pythagorean theorem,

$$c = \sqrt{4.32^2 + 2.62^2} = 5.05 \text{ centimeters}$$

Note the slight difference in the values obtained for c (5.05 versus 5.06). This was caused by rounding β to the nearest $10'$ in the first calculation for c.

MATCHED PROBLEM 2

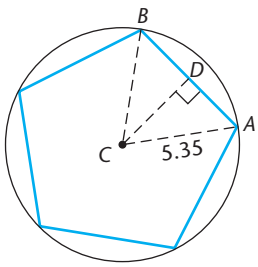
Solve the right triangle with $a = 1.38$ kilometers and $b = 6.73$ kilometers.

EXAMPLE 3 Geometry

If a regular pentagon (a five-sided regular polygon) is inscribed in a circle of radius 5.35 centimeters, find the length of one side of the pentagon.

SOLUTION

Sketch a figure and insert triangle ACB with C at the center (Fig. 6). Add the auxiliary line CD as indicated. We will find AD and double it to find the length of the side wanted.



› Figure 6

$$\text{Angle } ACB = \frac{360°}{5} = 72° \qquad \text{Exact}$$

$$\text{Angle } ACD = \frac{72°}{2} = 36° \qquad \text{Exact}$$

$$\sin(\text{angle } ACD) = \frac{AD}{AC}$$

$$AD = AC \sin(\text{angle } ACD)$$

$$= 5.35 \sin 36°$$

$$= 3.14 \text{ centimeters}$$

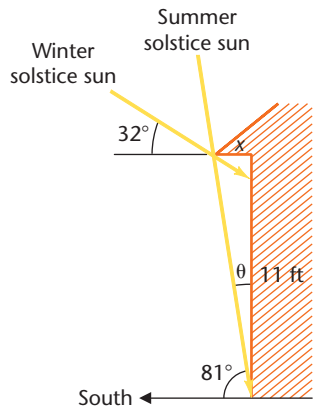
$$AB = 2AD = 6.28 \text{ centimeters}$$

MATCHED PROBLEM 3

If a square of side 43.6 meters is inscribed in a circle, what is the radius of the circle?

EXAMPLE 4 Architecture

In designing a house an architect wishes to determine the amount of overhang of a roof so that it shades the entire south wall at noon during the summer solstice when the angle of elevation of the sun is 81° [the *angle of elevation* is the acute angle between the horizontal and the line of sight to the sun (Fig. 7)]. Minimally, how much overhang should be provided for this purpose?



› Figure 7

SOLUTION

Using Figure 7, we consider the right triangle with angle θ and sides x (the overhang) and 11 feet, and solve for x:

$$\theta = 90° - 81° = 9°$$

$$\tan\theta = \frac{x}{11}$$ Substitute $\theta = 9°$ and multiply both sides by 11.

$$x = 11\tan 9° = 1.7 \text{ feet}$$

MATCHED PROBLEM 4

With the overhang found in Example 4, how far will the shadow of the overhang come down the wall at noon during the winter solstice when the angle of elevation of the sun is 32°?

ANSWERS TO MATCHED PROBLEMS

1. $\beta = 42.2°$, $a = 20.2$ meters, $b = 18.3$ meters
2. $\alpha = 11°40'$, $\beta = 78°20'$, $c = 6.87$ kilometers **3.** 30.8 meters **4.** 1.1 feet

5-3 Exercises

1. Can every rectangle be partitioned into two right triangles? Explain.

2. Can every triangle be partitioned into two right triangles? Explain.

3. Explain why it is not possible to solve for the sides of a triangle if only its angles are known.

4. Explain why the cosine of an acute angle of a right triangle is equal to the sine of the complementary angle.

5. Explain why the cotangent of an acute angle of a right triangle is equal to the tangent of the complementary angle.

6. Explain why the cosecant of an acute angle of a right triangle is equal to the secant of the complementary angle.

In Problems 7–12, use the figure to write the ratio of sides that corresponds to each trigonometric function.

7. sin θ **8.** cot θ **9.** csc θ

10. cos θ **11.** tan θ **12.** sec θ

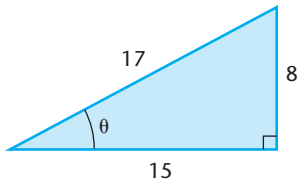


Figure for Problems 7–12

In Problems 13–18, indicate which trigonometric function of θ is equal to the given ratio (refer to the figure for Problems 7–12).

13. $\frac{15}{17}$ **14.** $\frac{8}{15}$ **15.** $\frac{17}{15}$

16. $\frac{8}{17}$ **17.** $\frac{15}{8}$ **18.** $\frac{17}{8}$

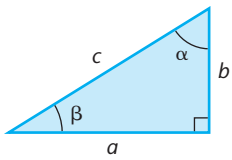
In Problems 19–24, find each acute angle θ in degree measure to two decimal places using a calculator.

19. $\cos\theta = 0.4917$ **20.** $\sin\theta = 0.0859$

21. $\theta = \tan^{-1} 8.031$ **22.** $\theta = \cos^{-1} 0.5097$

23. $\sin\theta = 0.6031$ **24.** $\tan\theta = 1.993$

In Problems 25–36, use the figure and the given information to solve each triangle.



25. $\beta = 17.8°, c = 3.45$ **26.** $\beta = 33.7°, b = 22.4$

27. $\beta = 43°20', a = 123$ **28.** $\beta = 62°30', c = 42.5$

29. $\alpha = 23°0', a = 54.0$ **30.** $\alpha = 54°, c = 4.3$

31. $\alpha = 53.21°, b = 23.82$ **32.** $\alpha = 35.73°, b = 6.482$

33. $a = 6.00, b = 8.46$ **34.** $a = 22.0, b = 46.2$

35. $b = 10.0, c = 12.6$ **36.** $b = 50.0, c = 165$

In Problems 37–42, determine whether the statement is true or false. If true, explain why. If false, give a counterexample.

37. If any two angles of a right triangle are known, then it is possible to solve for the remaining angle and the three sides.

38. If any two sides of a right triangle are known, then it is possible to solve for the remaining side and the three angles.

39. If α and β are the acute angles of a right triangle, then $\cos\alpha = \sin\beta$.

40. If α and β are the acute angles of a right triangle, then $\tan\alpha = \cot\beta$.

41. If α and β are the acute angles of a right triangle, then $\sec\alpha = \cos\beta$.

42. If α and β are the acute angles of a right triangle, then $\csc\alpha = \sec\beta$.

In Problems 43–48, find the degree measure to one decimal place of the acute angle between the given line and the x axis.

43. $y = \frac{1}{2}x + 3$ **44.** $y = \frac{1}{3}x - \frac{1}{4}$

45. $y = 5x + 21$ **46.** $y = 4x - 16$

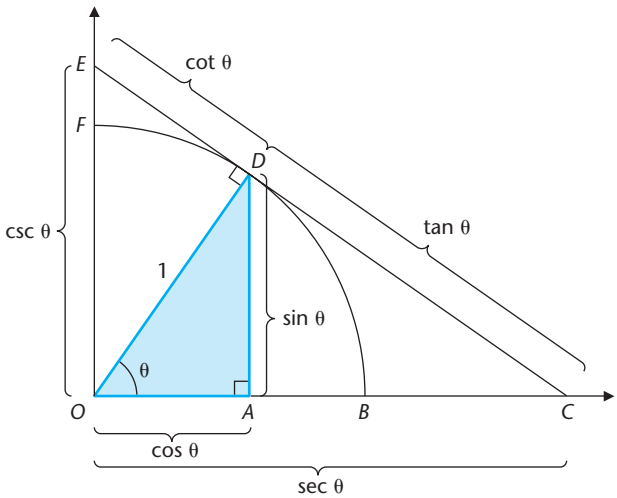
47. $y = -2x + 7$ **48.** $y = -3x - 1$

In Problems 49–54, find the slope to two decimal places of each line for which there is an angle of measure θ between the line and the x axis. [Hint: Note that there is an angle of measure 45° between the line $y = x$ and the x axis, and also between the line $y = -x$ and the x axis.]

49. $\theta = 20°$ **50.** $\theta = 40°$ **51.** $\theta = 80°$

52. $\theta = 70°$ **53.** $\theta = \pi/30$ **54.** $\theta = \pi/20$

*Problems 55–60 give a **geometric interpretation of the trigonometric ratios.** Refer to the figure, where O is the center of a circle of radius 1, θ is the acute angle AOD, D is the intersection point of the terminal side of angle θ with the circle, and EC is tangent to the circle at D.*



55. Explain why
(A) $\cos\theta = OA$ (B) $\cot\theta = DE$ (C) $\sec\theta = OC$

56. Explain why
(A) $\sin\theta = AD$ (B) $\tan\theta = DC$ (C) $\csc\theta = OE$

57. Explain what happens to each of the following as the acute angle θ approaches 90°.
(A) cos θ (B) cot θ (C) sec θ

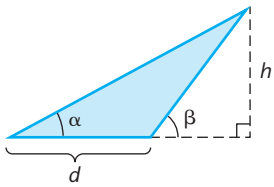
58. Explain what happens to each of the following as the acute angle θ approaches 90°.
(A) sin θ (B) tan θ (C) csc θ

59. Explain what happens to each of the following as the acute angle θ approaches 0°.
(A) sin θ (B) tan θ (C) csc θ

60. Explain what happens to each of the following as the acute angle θ approaches 0°.
(A) cos θ (B) cot θ (C) sec θ

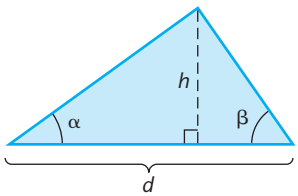
61. Show that

$$h = \frac{d}{\cot \alpha - \cot \beta}$$



62. Show that

$$h = \frac{d}{\cot \alpha + \cot \beta}$$



APPLICATIONS

63. **SURVEYING** Find the height of a tree (growing on level ground) if at a point 105 feet from the base of the tree, the angle to its top relative to the horizontal is found to be 65.3°.

64. **AIR SAFETY** To measure the height of a cloud ceiling over an airport, a searchlight is directed straight upward to produce a lighted spot on the clouds. Five hundred meters away an observer reports the angle of the spot relative to the horizontal to be 32.2°. How high (to the nearest meter) are the clouds above the airport?

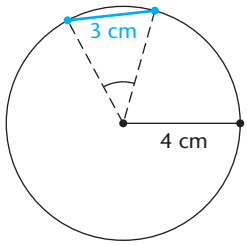
65. **ENGINEERING** If a train climbs at a constant angle of 1°23′, how many vertical feet has it climbed after going 1 mile? (1 mile = 5,280 feet)

66. **AIR SAFETY** If a jet airliner climbs at an angle of 15°30′ with a constant speed of 315 miles per hour, how long will it take (to the nearest minute) to reach an altitude of 8.00 miles? Assume there is no wind.

67. **ASTRONOMY** Find the diameter of the moon (to the nearest mile) if at 239,000 miles from Earth it produces an angle of 32′ relative to an observer on Earth.

68. **ASTRONOMY** If the sun is 93,000,000 miles from Earth and its diameter is opposite an angle of 32′ relative to an observer on Earth, what is the diameter of the sun (to two significant digits)?

69. **GEOMETRY** If a circle of radius 4 centimeters has a chord of length 3 centimeters, find the central angle that is opposite this chord (to the nearest degree).



70. **GEOMETRY** Find the length of one side of a nine-sided regular polygon inscribed in a circle of radius 4.06 inches.

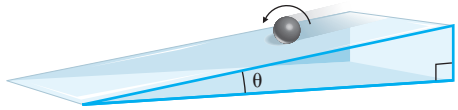
71. **PHYSICS** In a course in physics it is shown that the velocity v of a ball rolling down an inclined plane (neglecting air resistance and friction) is given by

$$v = gt \sin \theta$$

where g is a gravitational constant (acceleration due to gravity), t is time, and θ is the angle of inclination of the plane (see the following figure). Galileo (1564–1642) used this equation in the form

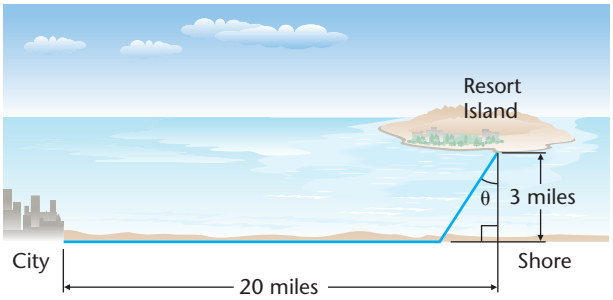
$$g = \frac{v}{t \sin \theta}$$

to estimate g after measuring v experimentally. (At that time, no timing devices existed to measure the velocity of a free-falling body, so Galileo used the inclined plane to slow the motion down.) A steel ball is rolled down a glass plane inclined at 8.0°. Approximate g to one decimal place if at the end of 3.0 seconds the ball has a measured velocity of 4.2 meters per second.



72. **PHYSICS** Refer to Problem 71. A steel ball is rolled down a glass plane inclined at 4.0°. Approximate g to one decimal place if at the end of 4.0 seconds the ball has a measured velocity of 9.0 feet per second.

73. **ENGINEERING—COST ANALYSIS** A cable television company wishes to run a cable from a city to a resort island 3 miles offshore. The cable is to go along the shore, then to the island underwater, as indicated in the accompanying figure. The cost of running the cable along the shore is $15,000 per mile and underwater, $25,000 per mile.



(A) Referring to the figure, show that the cost in terms of θ is given by

$$C(\theta) = 75{,}000 \sec\theta - 45{,}000 \tan\theta + 300{,}000$$

(B) Calculate a table of costs, each cost to the nearest dollar, for the following values of θ: 10°, 20°, 30°, 40°, and 50°. (Notice how the costs vary with θ. In a course in calculus, students are asked to find θ so that the cost is minimized.)

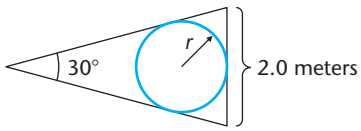
74. ENGINEERING—COST ANALYSIS Refer to Problem 73. Suppose the island is 4 miles offshore and the cost of running the cable along the shore is $20,000 per mile and underwater, $30,000 per mile.

(A) Referring to the figure for Problem 73 with appropriate changes, show that the cost in terms of u is given by

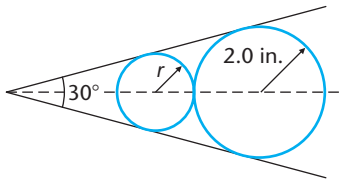
$$C(\theta) = 120{,}000 \sec\theta - 80{,}000 \tan\theta + 400{,}000$$

(B) Calculate a table of costs, each cost to the nearest dollar, for the following values of θ: 10°, 20°, 30°, 40°, and 50°.

75. GEOMETRY Find r in the accompanying figure (to two significant digits) so that the circle is tangent to all three sides of the isosceles triangle. [*Hint:* The radius of a circle is perpendicular to a tangent line at the point of tangency.]



76. GEOMETRY Find r in the accompanying figure (to two significant digits) so that the smaller circle is tangent to the larger circle and the two sides of the angle. [See the hint in Problem 75.]



5-4 Properties of Trigonometric Functions

- Basic Identities
- Sign Properties
- Reference Triangles
- Periodic Functions

In this section we study properties of the trigonometric functions that distinguish them from the polynomial, rational, exponential, and logarithmic functions. The trigonometric functions are *periodic,* and as a consequence, have infinitely many zeros, or infinitely many turning points, or both.

Basic Identities

The definition of trigonometric functions provides several useful relationships among these functions. For convenience, we restate that definition.

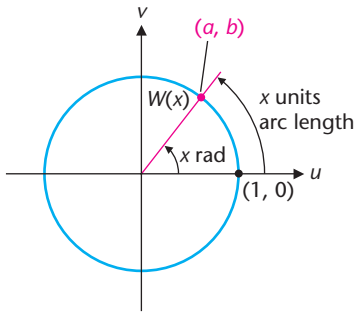
› DEFINITION 1 Trigonometric Functions

Let x be a real number and let (a, b) be the coordinates of the circular point $W(x)$ that lies on the terminal side of the angle with radian measure x. Then:

$$\sin x = b \qquad \csc x = \frac{1}{b} \quad b \neq 0$$

$$\cos x = a \qquad \sec x = \frac{1}{a} \quad a \neq 0$$

$$\tan x = \frac{b}{a} \quad a \neq 0 \qquad \cot x = \frac{a}{b} \quad b \neq 0$$



Because $\sin x = b$ and $\cos x = a$, we obtain the following equations:

$$\csc x = \frac{1}{b} = \frac{1}{\sin x} \tag{1}$$

$$\sec x = \frac{1}{a} = \frac{1}{\cos x} \tag{2}$$

$$\cot x = \frac{a}{b} = \frac{1}{b/a} = \frac{1}{\tan x} \tag{3}$$

$$\tan x = \frac{b}{a} = \frac{\sin x}{\cos x} \tag{4}$$

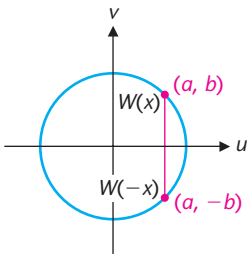
$$\cot x = \frac{a}{b} = \frac{\cos x}{\sin x} \tag{5}$$

Because the circular points $W(x)$ and $W(-x)$ are symmetrical with respect to the horizontal axis (Fig. 1), we have the following sign properties:

$$\sin(-x) = -b = -\sin x \tag{6}$$

$$\cos(-x) = a = \cos x \tag{7}$$

$$\tan(-x) = \frac{-b}{a} = -\frac{b}{a} = -\tan x \tag{8}$$



› **Figure 1** Symmetry property.

Finally, because $(a, b) = (\cos x, \sin x)$ is on the unit circle $u^2 + v^2 = 1$, it follows that

$$(\cos x)^2 + (\sin x)^2 = 1$$

which is usually written as

$$\sin^2 x + \cos^2 x = 1 \tag{9}$$

where $\sin^2 x$ and $\cos^2 x$ are concise ways of writing $(\sin x)^2$ and $(\cos x)^2$, respectively.

››› CAUTION ›››

$$(\sin x)^2 \neq \sin x^2$$
$$(\cos x)^2 \neq \cos x^2$$

Equations (1)–(9) are called **basic identities.** They hold true for all replacements of x by real numbers for which both sides of an equation are defined. These basic identities should be memorized along with the definitions of the six trigonometric functions. Note that most of Chapter 6 is devoted to trigonometric identities.

We summarize the basic identities for convenient reference in Theorem 1.

› THEOREM 1 Basic Trigonometric Identities

For x any real number (in all cases restricted so that both sides of an equation are defined),

Reciprocal identities

$$\overset{(1)}{\csc x = \frac{1}{\sin x}} \qquad \overset{(2)}{\sec x = \frac{1}{\cos x}} \qquad \overset{(3)}{\cot x = \frac{1}{\tan x}}$$

Quotient identities

$$\overset{(4)}{\tan x = \frac{\sin x}{\cos x}} \qquad \overset{(5)}{\cot x = \frac{\cos x}{\sin x}}$$

Identities for negatives

$$\overset{(6)}{\sin(-x) = -\sin x} \qquad \overset{(7)}{\cos(-x) = \cos x} \qquad \overset{(8)}{\tan(-x) = -\tan x}$$

Pythagorean identity

$$\overset{(9)}{\sin^2 x + \cos^2 x = 1}$$

EXAMPLE 1 Using Basic Identities

Use the basic identities to find the values of the other five trigonometric functions given $\sin x = -\frac{2}{3}$ and $\tan x > 0$.

SOLUTION

We first note that the circular point $W(x)$ is in quadrant III, because that is the only quadrant in which $\sin x < 0$ and $\tan x > 0$. We next find $\cos x$ using identity (9):

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 && \text{Substitute } \sin x = -\tfrac{2}{3}. \\ (-\tfrac{2}{3})^2 + \cos^2 x &= 1 && \text{Subtract } \tfrac{4}{9} \text{ from both sides.} \\ \cos^2 x &= \tfrac{5}{9} && \text{Take square roots of both sides.} \\ \cos x &= -\frac{\sqrt{5}}{3} && \text{Choose the negative square root since } W(x) \text{ is in quadrant III.} \end{aligned}$$

Now, because we have values for $\sin x$ and $\cos x$, we can find values for the other four trigonometric functions using identities (1), (2), (4), and (5):

$$\csc x = \frac{1}{\sin x} = \frac{1}{-\frac{2}{3}} = -\frac{3}{2} \qquad \text{Reciprocal identity (1)}$$

$$\sec x = \frac{1}{\cos x} = \frac{1}{-\sqrt{5}/3} = -\frac{3}{\sqrt{5}} \qquad \text{Reciprocal identity (2)}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{-\frac{2}{3}}{-\sqrt{5}/3} = \frac{2}{\sqrt{5}} \qquad \text{Quotient identity (4)}$$

$$\cot x = \frac{\cos x}{\sin x} = \frac{-\sqrt{5}/3}{-\frac{2}{3}} = \frac{\sqrt{5}}{2} \qquad \text{Quotient identity (5)}$$

[*Note:* We could also use identity (3).]

It is important to note that we were able to find the values of the other five trigonometric functions without finding x.

MATCHED PROBLEM 1

Use the basic identities to find the values of the other five trigonometric functions given $\cos x = 1/\sqrt{2}$ and $\cot x < 0$.

››› EXPLORE-DISCUSS 1

Suppose that $\sin x = -\frac{2}{3}$ and $\tan x > 0$, as in Example 1. Using basic identities and the results in Example 1, find each of the following:

(A) $\sin(-x)$ **(B)** $\sec(-x)$ **(C)** $\tan(-x)$

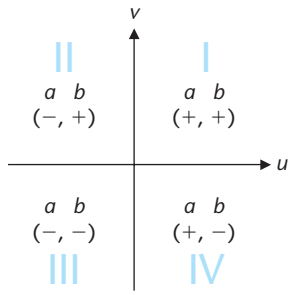
Verbally justify each step in your solution process.

Sign Properties

As a circular point $W(x)$ moves from quadrant to quadrant, its coordinates (a, b) undergo sign changes. Therefore, the trigonometric functions also undergo sign changes. It is important to know the sign of each trigonometric function in each quadrant. Table 1 shows the sign behavior for each function. It is not necessary to memorize Table 1, because the sign of each function for each quadrant is easily determined from its definition (which *should* be memorized).

Table 1 Sign Properties

Trigonometric function	Sign in quadrant I	II	III	IV
$\sin x = b$	+	+	−	−
$\csc x = 1/b$	+	+	−	−
$\cos x = a$	+	−	−	+
$\sec x = 1/a$	+	−	−	+
$\tan x = b/a$	+	−	+	−
$\cot x = a/b$	+	−	+	−



Reference Triangles

Consider an angle θ in standard position. Let $P = (a, b)$ be the point of intersection of the terminal side of θ with a circle of radius $r > 0$ (Fig. 2).

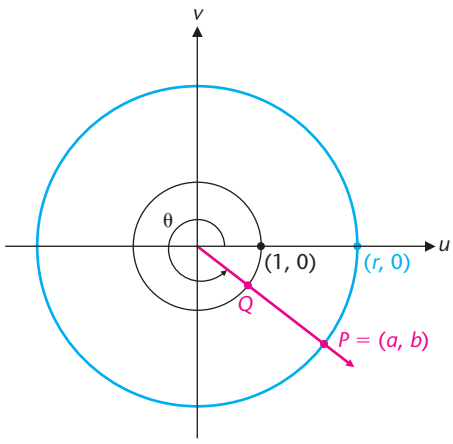


Figure 2

Recall (see Definition 1 and remarks on p. 482) that the six trigonometric functions of θ are given by

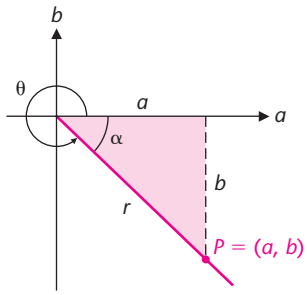
$$\begin{aligned} \sin\theta &= \frac{b}{r} & \csc\theta &= \frac{r}{b}, \quad b \neq 0 \\ \cos\theta &= \frac{a}{r} & \sec\theta &= \frac{r}{a}, \quad a \neq 0 \\ \tan\theta &= \frac{b}{a}, \quad a \neq 0 & \cot\theta &= \frac{a}{b}, \quad b \neq 0 \end{aligned} \tag{10}$$

To simplify the use of formulas (10), it is often convenient to associate a reference triangle and reference angle with θ, and to label the horizontal side, vertical side, and hypotenuse of the reference triangle with a, b, and r, respectively.

› REFERENCE TRIANGLE AND REFERENCE ANGLE

1. To form a **reference triangle** for θ, draw a perpendicular from a point $P = (a, b)$ on the terminal side of θ to the horizontal axis.

2. The **reference angle** α is the acute angle (always taken positive) between the terminal side of θ and the horizontal axis.



$(a, b) \neq (0, 0)$
α is always positive.

If Adj and Opp denote the labels a and b (possibly negative) on the horizontal and vertical sides of the reference triangle, and Hyp denotes the length r (always positive) of the hypotenuse, then formulas (10) become

$$\begin{aligned} \sin\theta &= \frac{\text{Opp}}{\text{Hyp}} & \csc\theta &= \frac{\text{Hyp}}{\text{Opp}} \\ \cos\theta &= \frac{\text{Adj}}{\text{Hyp}} & \sec\theta &= \frac{\text{Hyp}}{\text{Adj}} \\ \tan\theta &= \frac{\text{Opp}}{\text{Adj}} & \cot\theta &= \frac{\text{Adj}}{\text{Opp}} \end{aligned} \tag{11}$$

Formulas (11) are easy to remember, because, if the signs of Adj and Opp are ignored, the formulas coincide with the right triangle ratios (see the last section) for the angle α of the reference triangle.

In Example 2, a reference triangle is used to find values of the trigonometric functions. This method provides a simple alternative to the approach of Example 1, which emphasized basic identities.

EXAMPLE 2 Finding Values of the Trigonometric Functions

If $\sin\theta = 4/7$ and $\cos\theta < 0$, find the values of each of the other five trigonometric functions of θ.

SOLUTION

Because the sine of θ is positive and the cosine is negative, the angle θ is in quadrant II. We sketch a reference triangle (Fig. 3) and use the Pythagorean theorem to calculate the length of the horizontal side:

$$\sqrt{7^2 - 4^2} = \sqrt{33}$$

Therefore Adj $= -\sqrt{33}$, Opp $= 4$, Hyp $= 7$. The values of the other five trigonometric functions are:

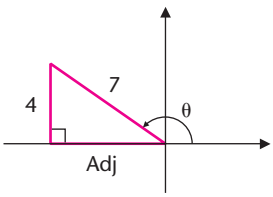
$$\cos\theta = \frac{-\sqrt{33}}{7}$$

$$\tan\theta = \frac{4}{-\sqrt{33}}$$

$$\csc\theta = \frac{7}{4}$$

$$\sec\theta = \frac{7}{-\sqrt{33}}$$

$$\cot\theta = \frac{-\sqrt{33}}{4}$$



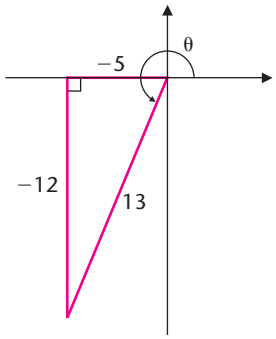
› Figure 3

MATCHED PROBLEM 2

If $\tan\theta = 10$ and $\sin\theta < 0$, find the values of each of the other five trigonometric functions of θ.

EXAMPLE 3 Finding Values of the Trigonometric Functions

Find the value of each of the six trigonometric functions for an angle θ whose terminal side contains the point $(-5, -12)$.



› Figure 4

SOLUTION

Because the sine and cosine of θ are both negative, the angle θ is in quadrant III. We sketch a reference triangle (Fig. 4) and use the Pythagorean theorem to calculate the length of the hypotenuse:

$$\sqrt{(-5)^2 + (-12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

Therefore, Adj $= -5$, Opp $= -12$, and Hyp $= 13$. The values of the six trigonometric functions are

$$\sin\theta = -\frac{12}{13} \qquad \csc\theta = -\frac{13}{12}$$
$$\cos\theta = -\frac{5}{13} \qquad \sec\theta = -\frac{13}{5}$$
$$\tan\theta = \frac{12}{5} \qquad \cot\theta = \frac{5}{12}$$

MATCHED PROBLEM 3

Find the value of each of the six trigonometric functions for an angle θ whose terminal side contains the point $(15, -8)$.

››› CAUTION ›››

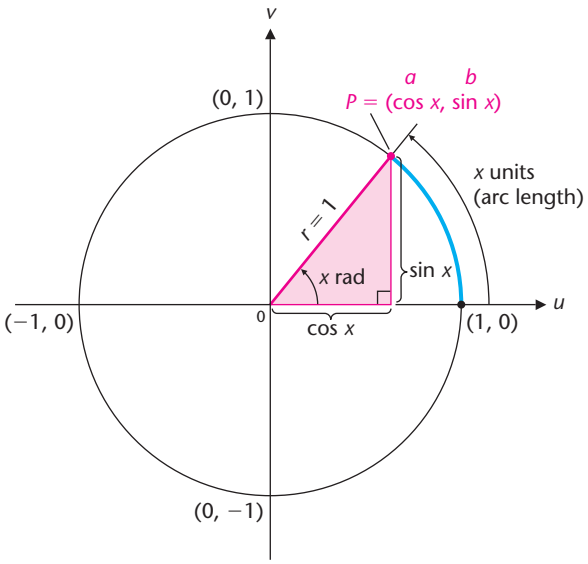
When using a reference triangle, the label Hyp on the hypotenuse is always positive. The label Adj on the horizontal leg of the reference triangle is positive or negative depending on whether that leg lies on the positive or negative horizontal axis, respectively. The label Opp on the vertical leg is positive or negative depending on whether that leg is above or below the horizontal axis, respectively.

› Periodic Functions

Because the unit circle has a circumference of 2π, we find that for a given value of x (Fig. 5) we will return to the circular point $W(x) = (a, b)$ if we add any integer multiple of 2π to x. Think of a point P moving around the unit circle in either direction. Every time P covers a distance of 2π, the circumference of the circle, it is back at the point where it started. Therefore, for x any real number,

$$\sin(x + 2k\pi) = \sin x \qquad k \text{ any integer}$$
$$\cos(x + 2k\pi) = \cos x \qquad k \text{ any integer}$$

Functions with this kind of repetitive behavior are called **periodic functions.**



› Figure 5

› **DEFINITION 2 Periodic Functions**

A function f is **periodic** if there exists a positive real number p such that

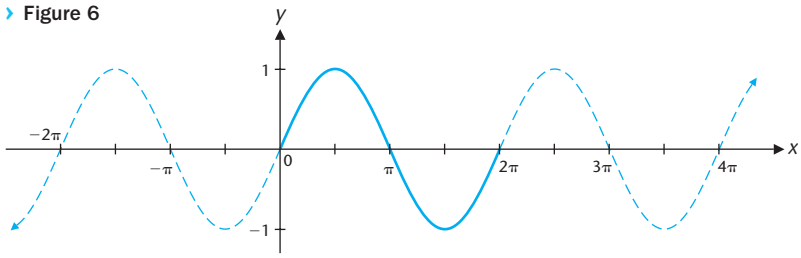
$$f(x + p) = f(x)$$

for all x in the domain of f. The smallest such positive p, if it exists, is called the **fundamental period of f** (or often just the **period of f**).

Both the sine and cosine functions are periodic with period 2π. Once the graph for one period is known, the entire graph is obtained by repetition. The domain of both functions is the set of all real numbers, and the range of both is $[-1, 1]$. Because $b = 0$ at the circular points $(1, 0)$ and $(-1, 0)$, the zeros of the sine function are $k\pi$, k any integer. Because $a = 0$ at the circular points $(0, 1)$ and $(0, -1)$, the zeros of the cosine function are $\pi/2 + k\pi$, k any integer. Both the sine and cosine functions possess symmetry properties (see Section 1-4). By the basic identity $\sin(-x) = -\sin x$, the sine function is symmetric with respect to the origin, so it is an odd function. Because $\cos(-x) = \cos x$, the cosine function is symmetric with respect to the y axis, and so is an even function. Figures 6 and 7 summarize these properties and show the graphs of the sine and cosine functions, respectively.

GRAPH OF $y = \sin x$

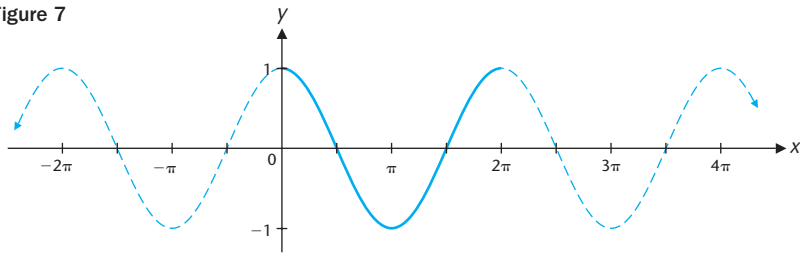
Figure 6



Period: 2π Domain: All real numbers Range: $[-1, 1]$
Symmetric with respect to the origin

GRAPH OF $y = \cos x$

Figure 7



Period: 2π Domain: All real numbers Range: $[-1, 1]$
Symmetric with respect to the y axis

EXAMPLE 4 Symmetry

Determine whether the function $f(x) = \dfrac{\sin x}{x}$ is even, odd, or neither.

SOLUTION

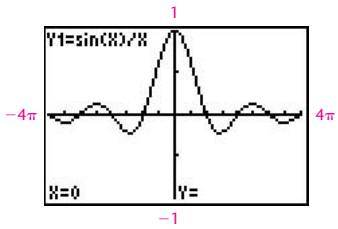
$$f(-x) = \frac{\sin(-x)}{-x}$$ Sine function is odd so $\sin(-x) = -\sin x$.

$$= \frac{-\sin x}{-x}$$ Multiply by $\frac{-1}{-1} = 1$.

$$= \frac{\sin x}{x}$$ Use expression for $f(x)$.

$$= f(x)$$

› Figure 8 $y_1 = \dfrac{\sin x}{x}$



Therefore $f(x)$ is symmetric with respect to the y axis and is an even function. This fact is confirmed by the graph of $f(x)$ (Fig. 8). Note that although $f(x)$ is undefined at $x = 0$, it appears that $f(x)$ approaches 1 as x approaches 0 from either side.

MATCHED PROBLEM 4

Determine whether the function $g(x) = \dfrac{\cos x}{x}$ is even, odd, or neither.

Because the tangent function is the quotient of the sine and cosine functions, you might expect that it would also be periodic with period 2π. Surprisingly, the tangent function is periodic with period π. To see this, note that if (a, b) is the circular point associated with x, then $(-a, -b)$ is the circular point associated with $x + \pi$. Therefore,

$$\tan(x + \pi) = \frac{-b}{-a} = \frac{b}{a} = \tan x$$

The tangent function is symmetric with respect to the origin because

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\tan x$$

Because $\tan x = \sin x/\cos x$, the zeros of the tangent function are the zeros of the sine function, namely, $k\pi$, k any integer, and the tangent function is undefined at the zeros of the cosine function, namely, $\pi/2 + k\pi$, k any integer. What does the graph of the tangent function look like near one of the values of x, say $\pi/2$, at which it is undefined? If $x < \pi/2$ but x is close to $\pi/2$, then b is close to 1 and a is positive and close to 0, so the ratio b/a is large and positive. Therefore, using the notation of Section 3-5,

$$\tan x \to \infty \qquad \text{as} \qquad x \to (\pi/2)^-$$

Similarly, if $x > \pi/2$ but x is close to $\pi/2$, then b is close to 1 and a is negative and close to 0, so the ratio b/a is large in absolute value and negative. Therefore,

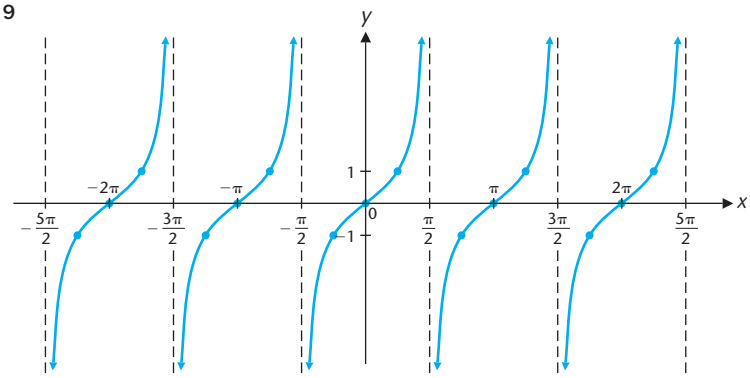
$$\tan x \to -\infty \qquad \text{as} \qquad x \to (\pi/2)^+$$

Therefore the line $x = \pi/2$ is a vertical asymptote for the tangent function and, by periodicity, so are the vertical lines $x = \pi/2 + k\pi$, k any integer. From our knowledge of key circular points, $\tan(\pi/4) = 1$ and $\tan(3\pi/4) = -1$. Figure 9 summarizes

these properties of the tangent function and shows its graph. The analogous properties of the cotangent function and its graph are shown in Figure 10.

GRAPH OF $y = \tan x$

Figure 9



Period: π

Domain: All real numbers except $\pi/2 + k\pi$, k an integer

Range: All real numbers

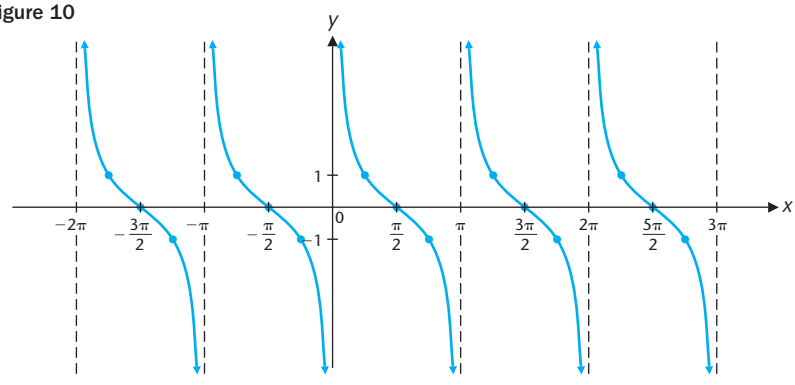
Symmetric with respect to the origin

Increasing function between consecutive asymptotes

Discontinuous at $x = \pi/2 + k\pi$, k an integer

GRAPH OF $y = \cot x$

Figure 10



Period: π

Domain: All real numbers except $k\pi$, k an integer

Range: All real numbers

Symmetric with respect to the origin

Decreasing function between consecutive asymptotes

Discontinuous at $x = k\pi$, k an integer

››› EXPLORE-DISCUSS 2

(A) Discuss how the graphs of the tangent and cotangent functions are related.

(B) How would you shift and/or reflect the tangent graph to obtain the cotangent graph?

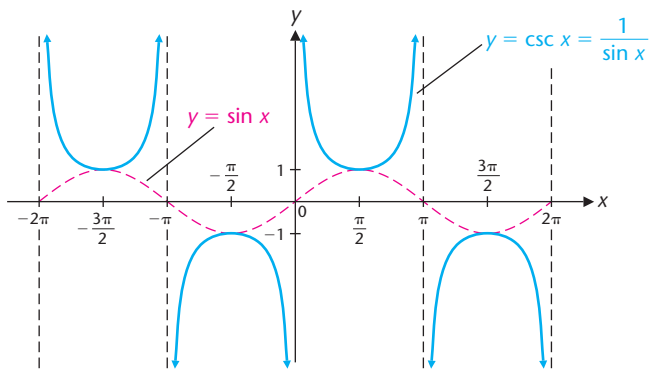
(C) Is either the graph of $y = \tan(x - \pi/2)$ or $y = -\tan(x - \pi/2)$ the same as the graph of $y = \cot x$? Explain in terms of shifts and/or reflections.

Note that for a particular value of x, the y value on the graph of $y = \cot x$ is the reciprocal of the y value on the graph of $y = \tan x$. The vertical asymptotes of $y = \cot x$ occur at the zeros of $y = \tan x$, and vice versa.

The graphs of $y = \csc x$ and $y = \sec x$ can be obtained by taking the reciprocals of the y values of the graphs of $y = \sin x$ and $y = \cos x$, respectively. Vertical asymptotes occur at the zeros of $y = \sin x$ or $y = \cos x$. Figures 11 and 12 summarize the properties and show the graphs of $y = \csc x$ and $y = \sec x$. To emphasize the reciprocal relationships, the graphs of $y = \sin x$ and $y = \cos x$ are indicated in broken lines.

GRAPH OF $y = \csc x$

Figure 11



Period: 2π

Domain: All real numbers except $k\pi$, k an integer

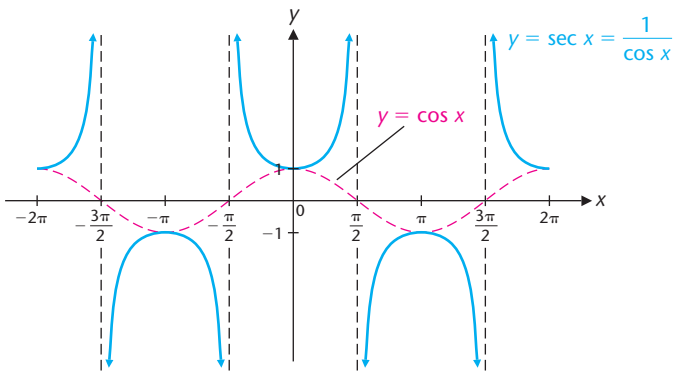
Range: All real numbers y such that $y \le -1$ or $y \ge 1$

Symmetric with respect to the origin

Discontinuous at $x = k\pi$, k an integer

GRAPH OF $y = \sec x$

Figure 12



Period: 2π

Domain: All real numbers except $\pi/2 + k\pi$, k an integer

Range: All real numbers y such that $y \le -1$ or $y \ge 1$

Symmetric with respect to the y axis

Discontinuous at $x = \pi/2 + k\pi$, k an integer

ANSWERS TO MATCHED PROBLEMS

1. $\sin x = -1/\sqrt{2}$, $\tan x = -1$, $\csc x = -\sqrt{2}$, $\sec x = \sqrt{2}$, $\cot x = -1$
2. $\sin \theta = -10/\sqrt{101}$, $\cos \theta = -1/\sqrt{101}$, $\csc \theta = -\sqrt{101}/10$, $\sec \theta = -\sqrt{101}$, $\cot \theta = 1/10$
3. $\sin \theta = -\frac{8}{17}$, $\cos \theta = \frac{15}{17}$, $\tan \theta = -\frac{8}{15}$, $\csc \theta = -\frac{17}{8}$, $\sec \theta = \frac{17}{15}$, $\cot \theta = -\frac{15}{8}$
4. Odd

5-4 Exercises

1. When is an equation an identity?

2. Explain the meaning of the expressions $\sin^2 x$, $\sin x^2$, and $(\sin x)^2$.

3. How would you use your calculator to evaluate the expressions in Problem 2 for $x = \frac{\pi}{6}$?

4. Explain how to form a reference triangle.

5. How can you tell from the graph of a function whether it is periodic?

6. Explain why a periodic function either has no zeros, or infinitely many zeros, and give an example of each case.

In Problems 7–16, try to answer the questions by referring to the figure, without looking back in the text or using a calculator.

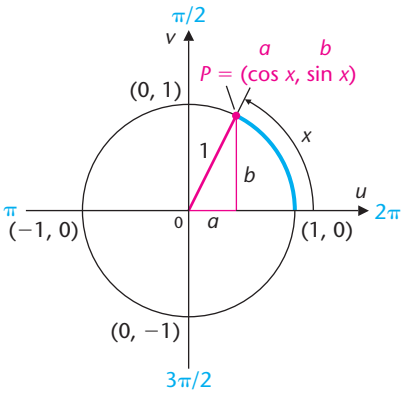


Figure for Problems 7–16.

7. What are the periods of the sine, cotangent, and cosecant functions?

8. What are the periods of the cosine, tangent, and secant functions?

9. How far does the graph of each function deviate from the x axis?
(A) $y = \cos x$ (B) $y = \tan x$ (C) $y = \csc x$

10. How far does the graph of each function deviate from the x axis?
(A) $y = \sin x$ (B) $y = \cot x$ (C) $y = \sec x$

11. What are the x intercepts for the graph of each function over the interval $-2\pi \le x \le 2\pi$?
(A) $y = \sin x$ (B) $y = \cot x$ (C) $y = \csc x$

12. What are the x intercepts for the graph of each function over the interval $-2\pi \le x \le 2\pi$?
(A) $y = \cos x$ (B) $y = \tan x$ (C) $y = \sec x$

13. For what values of x, $-2\pi \le x \le 2\pi$, are the following functions not defined?
(A) $y = \cos x$ (B) $y = \tan x$ (C) $y = \csc x$

14. For what values of x, $-2\pi \le x \le 2\pi$, are the following functions not defined?
(A) $y = \sin x$ (B) $y = \cot x$ (C) $y = \sec x$

15. At what values of x, $-2\pi \le x \le 2\pi$, do the vertical asymptotes for the following functions cross the x axis?
(A) $y = \cos x$ (B) $y = \tan x$ (C) $y = \csc x$

16. At what values of x, $-2\pi \le x \le 2\pi$, do the vertical asymptotes for the following functions cross the x axis?
(A) $y = \sin x$ (B) $y = \cot x$ (C) $y = \sec x$

17. (A) Describe a shift and/or reflection that will transform the graph of $y = \csc x$ into the graph of $y = \sec x$.
(B) Is either the graph of $y = -\csc(x + \pi/2)$ or $y = -\csc(x - \pi/2)$ the same as the graph of $y = \sec x$? Explain in terms of shifts and/or reflections.

18. (A) Describe a shift and/or reflection that will transform the graph of $y = \sec x$ into the graph of $y = \csc x$.
(B) Is either the graph of $y = -\sec(x - \pi/2)$ or $y = -\sec(x + \pi/2)$ the same as the graph of $y = \csc x$? Explain in terms of shifts and/or reflections.

Find the reference angle α for each angle θ in problems 19–26.

19. $\theta = 300°$ **20.** $\theta = 135°$ **21.** $\theta = \dfrac{7\pi}{6}$

22. $\theta = \dfrac{\pi}{4}$ **23.** $\theta = -\dfrac{5\pi}{3}$ **24.** $\theta = -\dfrac{5\pi}{4}$

25. $170°$ **26.** $280°$

Find the value of each of the other five trigonometric functions for an angle θ, without finding θ, given the information indicated in Problems 27–34. Sketching a reference triangle should be helpful.

27. $\sin\theta = \frac{3}{5}$ and $\cos\theta < 0$

28. $\tan\theta = -\frac{4}{3}$ and $\sin\theta < 0$

29. $\cos\theta = -\sqrt{5}/3$ and $\cot\theta > 0$

30. $\cos\theta = -\sqrt{5}/3$ and $\tan\theta > 0$

31. $\tan\theta = \frac{1}{2}$ and $\sec\theta < 0$

32. $\sin\theta = -\frac{1}{4}$ and $\cos\theta > 0$

33. $\sec\theta = 10$ and $\tan\theta < 0$

34. $\csc\theta = 5$ and $\cot\theta > 0$

Without finding θ, find the value of each of the six trigonometric functions for an angle θ whose terminal side contains the point indicated in Problems 35–42.

35. $(6, 8)$ **36.** $(-3, 4)$ **37.** $(-1, \sqrt{3})$

38. $(\sqrt{3}, 1)$ **39.** $(112, 15)$ **40.** $(17, 144)$

41. $(3, -2)$ **42.** $(-1, -4)$

In Problems 43–50, determine whether each function is even, odd, or neither.

43. $y = \dfrac{\tan x}{x}$ **44.** $y = \dfrac{\sec x}{x}$ **45.** $y = \dfrac{\csc x}{x}$

46. $y = \dfrac{\cot x}{x}$ **47.** $y = \sin x \cos x$ **48.** $y = x \sin x \cos x$

49. $y = x^2 \sin x$ **50.** $y = x^3 \sin x$

In Problems 51–56, without using a calculator, find the smallest positive θ in degree and radian measure for which

51. $\cos\theta = \dfrac{-1}{2}$ **52.** $\sin\theta = \dfrac{-\sqrt{3}}{2}$ **53.** $\sin\theta = \dfrac{-1}{2}$

54. $\tan\theta = -\sqrt{3}$ **55.** $\csc\theta = \dfrac{-2}{\sqrt{3}}$ **56.** $\sec\theta = -\sqrt{2}$

57. Which trigonometric functions are not defined when the terminal side of an angle lies along the vertical axis. Why?

58. Which trigonometric functions are not defined when the terminal side of an angle lies along the horizontal axis? Why?

59. Find exactly, all $\theta, 0° \le \theta < 360°$, for which $\cos\theta = -\sqrt{3}/2$.

60. Find exactly, all $\theta, 0° \le \theta < 360°$, for which $\cot\theta = -1/\sqrt{3}$.

61. Find exactly, all $\theta, 0 \le \theta < 2\pi$, for which $\tan\theta = 1$.

62. Find exactly, all $\theta, 0 \le \theta < 2\pi$, for which $\sec\theta = -\sqrt{2}$.

In Problems 63–72, determine whether the statement is true or false. Explain.

63. Each of the six trigonometric functions has infinitely many zeros.

64. Each of the six trigonometric functions has infinitely many turning points.

65. If a function f is periodic with period p, then $f(x) = f(x + 2p)$ for all x in the domain of f.

66. If a function f is periodic with period p, then $f(x) = f(x + p/2)$ for all x in the domain of f.

67. If the function f is not even, then it is odd.

68. The constant function with value 0 is both even and odd.

69. If f and g are each periodic with period p, then the function f/g is periodic with period p.

70. If f and g are each periodic with period p, then the function f/g is periodic.

71. If f and g are both odd, then the function fg is even.

72. If f and g are both even, then the function fg is odd.

73. Find all functions of the form $f(x) = ax + b$ that are periodic.

74. Find all functions of the form $f(x) = ax^2 + bx + c$ that are periodic.

75. Find all functions of the form $f(x) = ax + b$ that are even.

76. Find all functions of the form $f(x) = ax + b$ that are odd.

Problems 77–82 offer a preliminary investigation into the relationships of the graphs of $y = \sin x$ and $y = \cos x$ with the graphs of $y = A\sin x, y = A\cos x, y = \sin Bx, y = \cos Bx, y = \sin(x + C)$, and $y = \cos(x + C)$. This important topic is discussed in detail in the next section.

77. (A) Graph $y = A\cos x, (-2\pi \le x \le 2\pi, -3 \le y \le 3)$, for $A = 1, 2$, and -3, all in the same viewing window.
(B) Do the x intercepts change? If so, where?
(C) How far does each graph deviate from the x axis? (Experiment with additional values of A.)
(D) Describe how the graph of $y = \cos x$ is changed by changing the values of A in $y = A\cos x$.

78. (A) Graph $y = A\sin x, (-2\pi \le x \le 2\pi, -3 \le y \le 3)$, for $A = 1, 3$, and -2, all in the same viewing window.
(B) Do the x intercepts change? If so, where?
(C) How far does each graph deviate from the x axis? (Experiment with additional values of A.)
(D) Describe how the graph of $y = \sin x$ is changed by changing the values of A in $y = A\sin x$.

79. (A) Graph $y = \sin Bx\ (-\pi \le x \le \pi, -2 \le y \le 2)$, for $B = 1, 2$, and 3, all in the same viewing window.
(B) How many periods of each graph appear in this viewing rectangle? (Experiment with additional positive integer values of B.)
(C) Based on the observations in part B, how many periods of the graph of $y = \sin nx$, n a positive integer, would appear in this viewing window?

80. (A) Graph $y = \cos Bx\ (-\pi \le x \le \pi, -2 \le y \le 2)$, for $B = 1, 2$, and 3, all in the same viewing window.
(B) How many periods of each graph appear in this viewing rectangle? (Experiment with additional positive integer values of B.)
(C) Based on the observations in part B, how many periods of the graph of $y = \cos nx$, n a positive integer, would appear in this viewing window?

81. (A) Graph $y = \cos(x + C), -2\pi \le x \le 2\pi, -1.5 \le y \le 1.5$, for $C = 0, -\pi/2$, and $\pi/2$, all in the same viewing window. (Experiment with additional values of C.)
(B) Describe how the graph of $y = \cos x$ is changed by changing the values of C in $y = \cos(x + C)$.

82. (A) Graph $y = \sin(x + C), -2\pi \le x \le 2\pi, -1.5 \le y \le 1.5$, for $C = 0, -\pi/2$, and $\pi/2$, all in the same viewing window. (Experiment with additional values of C.)
(B) Describe how the graph of $y = \sin x$ is changed by changing the values of C in $y = \sin(x + C)$.

83. Try to calculate each of the following on your calculator. Explain the results.
(A) $\sec(\pi/2)$ (B) $\tan(-\pi/2)$ (C) $\cot(-\pi)$

84. Try to calculate each of the following on your calculator. Explain the results.
(A) $\csc\pi$ (B) $\tan(\pi/2)$ (C) $\cot 0$

85. Graph $f(x) = \sin x$ and $g(x) = x$ in the same viewing window $(-1 \le x \le 1, -1 \le y \le 1)$.
(A) What do you observe about the two graphs when x is close to 0, say $-0.5 \le x \le 0.5$?

(B) Complete the table to three decimal places (use the table feature on your graphing calculator if it has one):

x	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3
$\sin x$							

(In applied mathematics certain derivations, formulas, and calculations are simplified by replacing $\sin x$ with x for small values of $|x|$.)

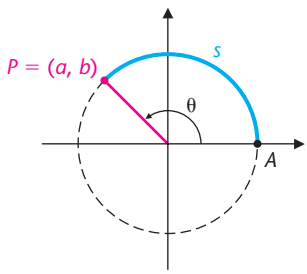
86. Graph $h(x) = \tan x$ and $g(x) = x$ in the same viewing window $(-1 \le x \le 1, -1 \le y \le 1)$.

(A) What do you observe about the two graphs when x is close to 0, say $-0.5 \le x \le 0.5$?

(B) Complete the table to three decimal places (use the table feature on your graphing calculator if it has one):

x	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3
$\tan x$							

(In applied mathematics certain derivations, formulas, and calculations are simplified by replacing $\tan x$ with x for small values of $|x|$.)



87. If the coordinates of A are (4, 0) and arc length s is 7 units, find

(A) The exact radian measure of θ

(B) The coordinates of P to three decimal places

88. If the coordinates of A are (2, 0) and arc length s is 8 units, find

(A) The exact radian measure of θ

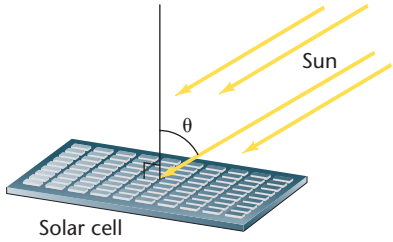
(B) The coordinates of P to three decimal places

89. In a rectangular coordinate system, a circle with center at the origin passes through the point $(6\sqrt{3}, 6)$. What is the length of the arc on the circle in quadrant I between the positive horizontal axis and the point $(6\sqrt{3}, 6)$?

90. In a rectangular coordinate system, a circle with center at the origin passes through the point $(2, 2\sqrt{3})$. What is the length of the arc on the circle in quadrant I between the positive horizontal axis and the point $(2, 2\sqrt{3})$?

APPLICATIONS

91. **SOLAR ENERGY** The intensity of light I on a solar cell changes with the angle of the sun and is given by the formula $I = k \cos \theta$, where k is a constant (see the figure). Find light intensity I in terms of k for $\theta = 0°$, $\theta = 30°$, and $\theta = 60°$.

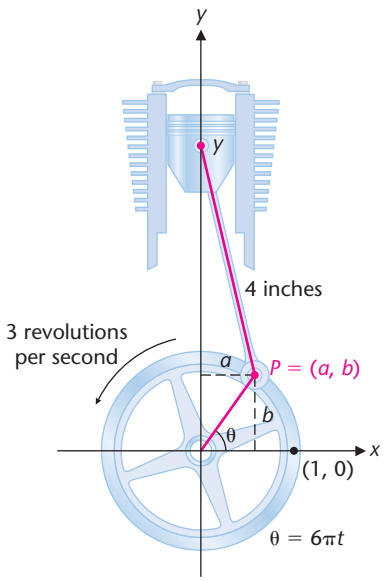


92. **SOLAR ENERGY** Refer to Problem 91. Find light intensity I in terms of k for $\theta = 20°$, $\theta = 50°$, and $\theta = 90°$.

93. **PHYSICS—ENGINEERING** The figure illustrates a piston connected to a wheel that turns 3 revolutions per second; hence, the angle θ is being generated at $3(2\pi) = 6\pi$ radians per second, or $\theta = 6\pi t$, where t is time in seconds. If P is at (1, 0) when $t = 0$, show that

$$y = b + \sqrt{4^2 - a^2}$$
$$= \sin 6\pi t + \sqrt{16 - (\cos 6\pi t)^2}$$

for $t \ge 0$.

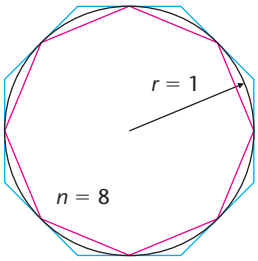


94. **PHYSICS—ENGINEERING** In Problem 93, find the position of the piston y when $t = 0.2$ second (to three significant digits).

95. **GEOMETRY** The area of a regular n-sided polygon circumscribed about a circle of radius 1 is given by

$$A = n \tan \frac{180°}{n}$$

(A) Find A for $n = 8, n = 100, n = 1{,}000$, and $n = 10{,}000$. Compute each to five decimal places.
(B) What number does A seem to approach as $n \to \infty$? (What is the area of a circle with radius 1?)



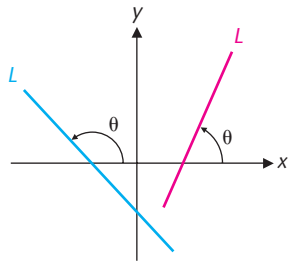
96. GEOMETRY The area of a regular n-sided polygon inscribed in a circle of radius 1 is given by

$$A = \frac{n}{2} \sin \frac{360^\circ}{n}$$

(A) Find A for $n = 8, n = 100, n = 1{,}000$, and $n = 10{,}000$. Compute each to five decimal places.
(B) What number does A seem to approach as $n \to \infty$? (What is the area of a circle with radius 1?)

97. ANGLE OF INCLINATION Recall (Section 2-3) the **slope** of a nonvertical line passing through points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is given by slope $= m = (y_2 - y_1)/(x_2 - x_1)$. The angle θ that the line L makes with the x axis, $0^\circ \le \theta < 180^\circ$, is called the **angle of inclination** of the line L (see figure). Therefore,

$$\text{Slope} = m = \tan \theta, \quad 0^\circ \le \theta < 180^\circ$$



(A) Compute the slopes to two decimal places of the lines with angles of inclination 88.7° and 162.3°.
(B) Find the equation of a line passing through $(-4, 5)$ with an angle of inclination 137°. Write the answer in the form $y = mx + b$, with m and b to two decimal places.

98. ANGLE OF INCLINATION Refer to Problem 97.
(A) Compute the slopes to two decimal places of the lines with angles of inclination 5.34° and 92.4°.
(B) Find the equation of a line passing through $(6, -4)$ with an angle of inclination 106°. Write the answer in the form $y = mx + b$, with m and b to two decimal places.

5-5 More General Trigonometric Functions and Models

- Graphs of $y = A \sin Bx$ and $y = A \cos Bx$
- Graphs of $y = A \sin (Bx + C)$ and $y = A \cos (Bx + C)$
- Finding an Equation from the Graph of a Simple Harmonic
- Mathematical Modeling and Data Analysis

Imagine a weight suspended from the ceiling by a spring. If the weight were pulled downward and released, then, assuming no air resistance or friction, it would move up and down with the same frequency and amplitude forever. This idealized motion is an example of **simple harmonic motion.** Simple harmonic motion can be described by functions of the form $y = A \sin (Bx + C)$ or $y = A \cos (Bx + C)$, called **simple harmonics.**

Simple harmonics are extremely important in both pure and applied mathematics. In applied mathematics they are used in the analysis of sound waves, radio waves, X-rays, gamma rays, visible light, infrared radiation, ultraviolet radiation, seismic waves,

ocean waves, electric circuits, electric generators, vibrations, bridge and building construction, spring–mass systems, bow waves of boats, sonic booms, and so on. Analysis involving simple harmonics is called *harmonic analysis*.

In this section we study properties, graphs, and applications of simple harmonics. A brief review of graph transformations (Section 1-4) should prove helpful.

Graphs of $y = A \sin Bx$ and $y = A \cos Bx$

We visualize the graphs of functions of the form $y = A \sin Bx$ or $y = A \cos Bx$, and determine their zeros and turning points, by understanding how each of the constants A and B transforms the graph of $y = \sin x$ or $y = \cos x$.

EXAMPLE 1 **Zeros and Turning Points**

Find the zeros and turning points of each function on the interval $[0, 2\pi]$.

(A) $y = \frac{1}{2} \sin x$ (B) $y = -2 \sin x$

SOLUTIONS

(A) The function $y = \frac{1}{2} \sin x$ is the vertical contraction of $y = \sin x$ that is obtained by multiplying each y value by $\frac{1}{2}$ (Fig. 1). Therefore its zeros on $[0, 2\pi]$ are identical to the zeros of $y = \sin x$, namely, $x = 0$, π, and 2π. Because the turning points of $y = \sin x$ are $(\pi/2, 1)$ and $(3\pi/2, -1)$, the turning points of $y = \frac{1}{2} \sin x$ are $(\pi/2, 1/2)$ and $(3\pi/2, -1/2)$.

(B) The function $y = -2 \sin x$ is the vertical expansion of $y = \sin x$ that is obtained by multiplying each y value by 2, followed by a reflection through the x axis (see Fig. 1). Therefore its zeros on $[0, 2\pi]$ are identical to the zeros of $y = \sin x$, namely $x = 0$, π, and 2π. Because the turning points of $y = \sin x$ are $(\pi/2, 1)$ and $(3\pi/2, -1)$, the turning points of $y = -2 \sin x$ are $(\pi/2, -2)$ and $(3\pi/2, 2)$.

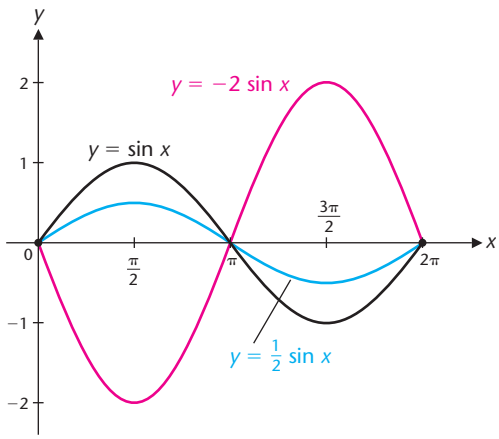


Figure 1

MATCHED PROBLEM 1

Find the zeros and turning points of each function on the interval $[\pi/2, 5\pi/2]$.

(A) $y = -5 \cos x$ (B) $y = \frac{1}{3} \cos x$

As Example 1 illustrates, the graph of $y = A \sin x$ can be obtained from the graph of $y = \sin x$ by multiplying each y value of $y = \sin x$ by the constant A. The graph of $y = A \sin x$ still crosses the x axis where the graph of $y = \sin x$ crosses the x axis, because $A \cdot 0 = 0$. Because the maximum value of $\sin x$ is 1, the maximum value of $A \sin x$ is $|A| \cdot 1 = |A|$. The constant $|A|$ is called the **amplitude** of the graph of $y = A \sin x$ and indicates the maximum deviation of the graph of $y = A \sin x$ from the x axis.

The period of $y = A \sin x$ (assuming $A \neq 0$) is the same as the period of $y = \sin x$, namely 2π, because $A \sin (x + 2\pi) = A \sin x$.

EXAMPLE 2 **Periods**

Find the period of each function.

(A) $y = \sin 2x$ (B) $y = \sin (x/2)$

SOLUTIONS

(A) Because the function $y = \sin x$ has period 2π, the function $y = \sin 2x$ completes one cycle as $2x$ varies from

$$2x = 0 \quad \text{to} \quad 2x = 2\pi$$

or as x varies from

$$x = 0 \quad \text{to} \quad x = \pi$$ Half the period for sin x

Therefore the period of $y = \sin 2x$ is π (Fig. 2).

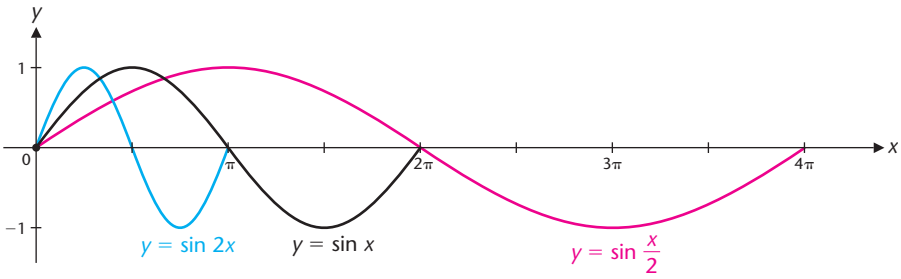
(B) Because the function $y = \sin x$ has period 2π, the function $y = \sin (x/2)$ completes one cycle as $x/2$ varies from

$$\frac{x}{2} = 0 \quad \text{to} \quad \frac{x}{2} = 2\pi$$

or as x varies from

$$x = 0 \quad \text{to} \quad x = 4\pi$$ Double the period for sin x

Therefore the period of $y = \sin (x/2)$ is 4π (see Fig. 2).



› Figure 2

MATCHED PROBLEM 2

Find the period of each function.

(A) $y = \cos(x/10)$ (B) $y = \cos(6\pi x)$

As Example 2 illustrates, the graph of $y = \sin Bx$, for a positive constant B, completes one cycle as Bx varies from

$$Bx = 0 \quad \text{to} \quad Bx = 2\pi$$

or as x varies from

$$x = 0 \quad \text{to} \quad x = \frac{2\pi}{B}$$

Therefore the period of $y = \sin Bx$ is $\frac{2\pi}{B}$. Note that the amplitude of $y = \sin Bx$ is 1, the same as the amplitude of $y = \sin x$. The effect of the constant B is to horizontally stretch or shrink the basic sine curve by changing the period of the function, but not its amplitude. A similar analysis applies to $y = \cos Bx$, for $B > 0$, where it can be shown that the period is also $\frac{2\pi}{B}$. We combine and summarize our results on period and amplitude as follows:

› **PERIOD AND AMPLITUDE**

For $y = A \sin Bx$ or $y = A \cos Bx$, $A \neq 0, B > 0$:

$$\text{Amplitude} = |A| \qquad \text{Period} = \frac{2\pi}{B}$$

If $0 < B < 1$, the basic sine or cosine curve is horizontally stretched.
If $B > 1$, the basic sine or cosine curve is horizontally shrunk.

You can either memorize the formula for the period, $\frac{2\pi}{B}$, or use the reasoning we used in deriving the formula. Recall, sin Bx or cos Bx completes one cycle as Bx varies from

$$Bx = 0 \quad \text{to} \quad Bx = 2\pi$$

that is, as x varies from

$$x = 0 \quad \text{to} \quad x = \frac{2\pi}{B}$$

Some prefer to memorize a formula, others to learn a process.

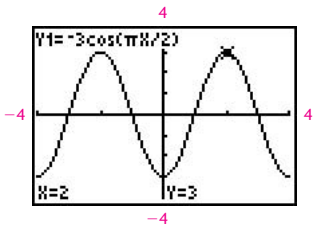
EXAMPLE 3 Amplitude, Period, and Turning Points

Find the amplitude, period, and turning points of $y = -3\cos(\pi x/2)$ on the interval $[-4, 4]$.

SOLUTION

$$\text{Amplitude} = |-3| = 3 \qquad \text{Period} = \frac{2\pi}{(\pi/2)} = 4$$

Because $y = \cos x$ has turning points at $x = 0$ and $x = \pm\pi$ (half of a complete cycle), $y = -3\cos(\pi x/2)$ has turning points at $x = 0$ and $x = \pm 2$. The turning points on the interval $[-4, 4]$ are thus $(-2, 3)$, $(0, -3)$, and $(2, 3)$. These results are confirmed by graphing $y = -3\cos(\pi x/2)$ (Fig. 3).



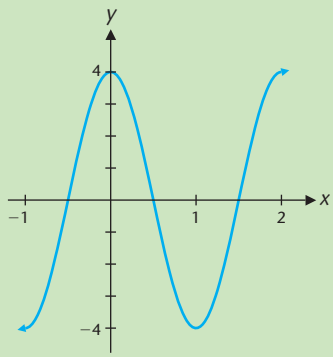
› Figure 3

MATCHED PROBLEM 3

Find the amplitude, period, and turning points of $y = \frac{1}{4}\sin(3\pi x)$ on the interval $[0, 1]$.

››› EXPLORE-DISCUSS 1

Find an equation of the form $y = A\cos Bx$ that produces the following graph. Check your equation with a graphing calculator.



Is it possible for an equation of the form $y = A\sin Bx$ to produce the same graph? Explain.

Graphs of $y = A \sin(Bx + C)$ and $y = A \cos(Bx + C)$

The graph of $y = A \sin(Bx + C)$ is a horizontal translation of the graph of the function $y = A \sin Bx$. In fact, because the period of the sine function is 2π, $y = A \sin(Bx + C)$ completes one cycle as $Bx + C$ varies from

$$Bx + C = 0 \quad \text{to} \quad Bx + C = 2\pi$$

or (solving for x in each equation) as x varies from

Phase shift Period

$$x = -\frac{C}{B} \quad \text{to} \quad x = -\frac{C}{B} + \frac{2\pi}{B}$$

We conclude that $y = A \sin(Bx + C)$ has a period of $2\pi/B$, and its graph is the graph of $y = A \sin Bx$ translated $|-C/B|$ units to the right if $-C/B$ is positive and $|-C/B|$ units to the left if $-C/B$ is negative. The number $-C/B$ is referred to as the **phase shift.**

EXAMPLE 4 Graphing $y = A \cos(Bx + C)$

Consider the equation $y = \frac{1}{2}\cos(4x - \pi)$.

(A) Find the amplitude, period, and phase shift.
(B) Sketch the graph for $-\pi \le x \le \pi$.
(C) Find all zeros.
(D) Find the smallest positive value of x for which $\frac{1}{2}\cos(4x - \pi) = -\frac{1}{2}$.

SOLUTIONS

(A) $\text{Amplitude} = |A| = \left|\frac{1}{2}\right| = \frac{1}{2}$

The graph completes one cycle as $4x - \pi$ varies from

$$4x - \pi = 0 \quad \text{to} \quad 4x - \pi = 2\pi$$

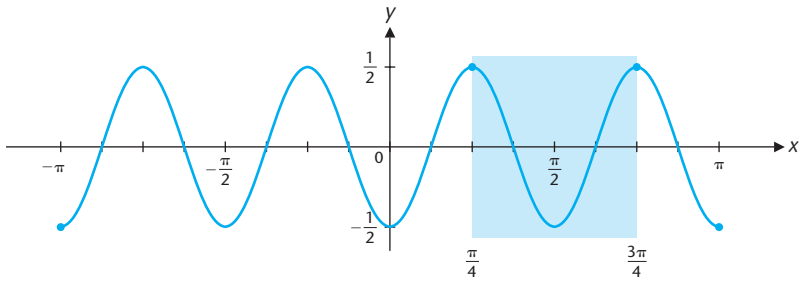
or as x varies from

$$x = \frac{\pi}{4} \quad \text{to} \quad x = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$$

Phase shift Period

$$\text{Phase shift} = \frac{\pi}{4} \qquad \text{Period} = \frac{\pi}{2}$$

(B) To sketch the graph, divide the interval $[\pi/4, 3\pi/4]$ into four equal parts and sketch one cycle of $y = \frac{1}{2}\cos(4x - \pi)$. Then extend the graph to cover $[-\pi, \pi]$ (Fig. 4).



› **Figure 4** $y = \frac{1}{2}\cos(4x - \pi)$, $-\pi \le x \le \pi$.

(C) The zeros of $y = \frac{1}{2}\cos(4x - \pi)$ are obtained by shifting the zeros of $y = \frac{1}{2}\cos(4x)$ to the right by $\pi/4$ units. Because $x = \pi/8$ and $x = 3\pi/8$ are zeros of $y = \frac{1}{2}\cos(4x)$, $x = \pi/8 + \pi/4 = 3\pi/8$ and $x = 3\pi/8 + \pi/4 = 5\pi/8$ are zeros of $y = \frac{1}{2}\cos(4x - \pi)$. By periodicity the zeros of $y = \frac{1}{2}\cos(4x - \pi)$ are $x = 3\pi/8 + k\pi/4$, k any integer, as confirmed by the graph (Fig. 4).

(D) Note that the equation is equivalent to $\cos(4x - \pi) = -1$. To find one solution, recall that $\cos\theta = -1$ at $\theta = \pi$, halfway through the cycle of $\cos\theta$ on $[0, 2\pi]$. Therefore, $\cos(4x - \pi) = -1$ halfway through its corresponding cycle, that is, at

$$x = \underbrace{\frac{\pi}{4}}_{\text{Phase shift}} + \frac{1}{2}\underbrace{\left(\frac{\pi}{2}\right)}_{\text{Period}} = \frac{\pi}{2}$$

Furthermore, $\pi/2$ is the smallest positive solution, because, by periodicity, all solutions are given by $\frac{\pi}{2} + \frac{k\pi}{2}$, k an integer, as confirmed by the graph (Fig. 4).

MATCHED PROBLEM 4

Consider the equation $y = \frac{3}{4}\sin(2x + \pi)$.

(A) Find the amplitude, period, and phase shift.

(B) Sketch the graph for $-\pi \le x \le \pi$.

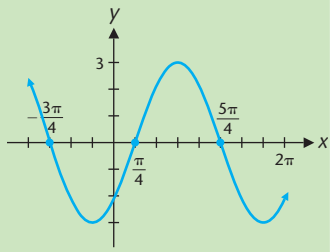
(C) Find all zeros.

(D) Find the smallest positive value of x for which $\frac{3}{4}\sin(2x + \pi) = -\frac{3}{4}$.

EXPLORE-DISCUSS 2

Find an equation of the form $y = A \sin(Bx + C)$ that produces the following graph.

Is it possible for an equation of the form $y = A \cos(Bx + C)$ to produce the same graph? Explain.



The graphs of $y = A \sin(Bx + C) + k$ and $y = A \cos(Bx + C) + k$ are vertical shifts (up k units if $k > 0$, down k units if $k < 0$) of the graphs of $y = A \sin(Bx + C)$ and $y = A \cos(Bx + C)$, respectively.

Because $y = \sec x$ and $y = \csc x$ are unbounded functions, amplitude is not defined for functions of the form $y = A \sec(Bx + C)$ and $y = A \csc(Bx + C)$. However, because both the secant and cosecant functions have period 2π, the functions $y = A \csc(Bx + C)$ and $y = A \sec(Bx + C)$ have period $2\pi/B$ and phase shift $-C/B$.

Because $y = \tan x$ and $y = \cot x$ are unbounded functions, amplitude is not defined for functions of the form $y = A \tan(Bx + C)$ or $y = A \cot(Bx + C)$. The tangent and cotangent functions both have period π, so the functions $y = A \tan(Bx + C)$ and $y = A \cot(Bx + C)$ have period π/B and phase shift $-C/B$.

Our results on amplitude, period, and phase shift are summarized in the following box.

AMPLITUDE, PERIOD, AND PHASE SHIFT

Let A, B, C be constants such that $A \neq 0$ and $B > 0$.

1. For $y = A \sin(Bx + C)$ and $y = A \cos(Bx + C)$,

$$\text{Amplitude} = |A| \qquad \text{Period} = \frac{2\pi}{B} \qquad \text{Phase shift} = \frac{-C}{B}$$

2. For $y = A \sec(Bx + C)$ and $y = A \csc(Bx + C)$,

$$\text{Period} = \frac{2\pi}{B} \qquad \text{Phase shift} = \frac{-C}{B}$$

3. For $y = A \tan(Bx + C)$ and $y = A \cot(Bx + C)$,

$$\text{Period} = \frac{\pi}{B} \qquad \text{Phase shift} = \frac{-C}{B}$$

Note: Amplitude is not defined for the secant, cosecant, tangent, and cotangent functions, all of which are unbounded.

EXAMPLE 5 **Amplitude, Period, and Phase Shift**

Find the amplitude (if applicable), period, and phase shift.

(A) $y = 4 \tan (6x + 9)$
(B) $y = -2 \sec (5\pi x - 2)$
(C) $y = -8 \sin (4x - \pi)$

SOLUTIONS

(A) Amplitude is not defined for the tangent function. Since $B = 6$ and $C = 9$,

$$\text{Period} = \frac{2\pi}{B} = \frac{2\pi}{6} = \frac{\pi}{3} \qquad \text{Phase shift} = \frac{-C}{B} = \frac{-9}{6} = -\frac{3}{2}$$

(B) Amplitude is not defined for the secant function. Since $B = 5\pi$ and $C = -2$,

$$\text{Period} = \frac{2\pi}{B} = \frac{2\pi}{5\pi} = \frac{2}{5} \qquad \text{Phase shift} = \frac{-C}{B} = \frac{2}{5\pi}$$

(C) Amplitude $= |A| = |-8| = 8$. Since $B = 4$ and $C = -\pi$,

$$\text{Period} = \frac{2\pi}{B} = \frac{2\pi}{4} = \frac{\pi}{2} \qquad \text{Phase shift} = \frac{-C}{B} = \frac{\pi}{4}$$

MATCHED PROBLEM 5

Find the amplitude (if applicable), period, and phase shift.

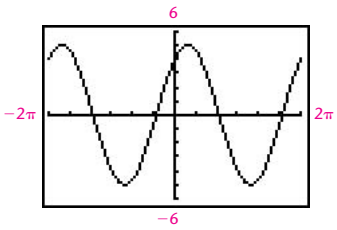
(A) $y = 0.15 \csc (9x - 3)$
(B) $y = -12 \cos (3\pi x + 1)$
(C) $y = 3 \cot (2x + 4\pi)$

› Finding an Equation from the Graph of a Simple Harmonic

Given the graph of a simple harmonic, we wish to find an equation of the form $y = A \sin (Bx + C)$ or $y = A \cos (Bx + C)$ that produces the graph. Example 6 illustrates the process.

EXAMPLE 6 **Finding an Equation of a Simple Harmonic Graph**

Graph $y_1 = 3 \sin x + 4 \cos x$ using a graphing calculator, and find an equation of the form $y_2 = A \sin (Bx + C)$ that has the same graph as y_1. Find A and B exactly and C to three decimal places.



› Figure 5
$y_1 = 3 \sin x + 4 \cos x$.

SOLUTION

The graph of y_1 is shown in Figure 5. The graph appears to be a sine curve shifted to the left. The amplitude and period appear to be 5 and 2π, respectively. (We will assume this for now and check it at the end.) Therefore, $A = 5$, and because the period P is $2\pi/B$, $B = 2\pi/P = 2\pi/2\pi = 1$. Using a graphing calculator, we find that the x intercept closest to the origin, to three decimal places, is -0.927. To find C, substitute $B = 1$ and $x = -0.927$ into the phase-shift formula $x = -C/B$ and solve for C:

$$x = -\frac{C}{B} \quad \text{Substitute } x = -0.927,\ B = 1.$$

$$-0.927 = -\frac{C}{1} \quad \text{Solve for } C.$$

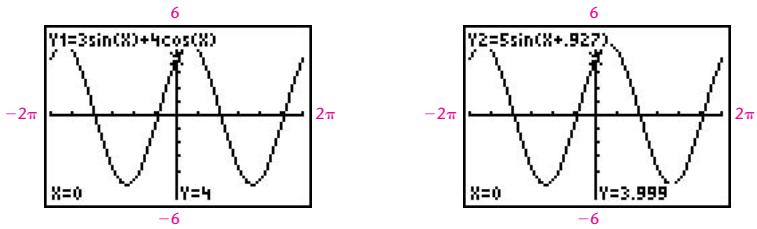
$$C = 0.927$$

We now have the equation we are looking for:

$$y_2 = 5 \sin (x + 0.927)$$

CHECK

Graph y_1 and y_2 in the same viewing window. If the graphs are the same, it appears that only one graph is drawn—the second graph is drawn over the first. To check further that the graphs are the same, use TRACE and switch back and forth between y_1 and y_2 at different values of x. Figure 6 shows a comparison at $x = 0$ (both graphs appear in the same viewing window).



› Figure 6

››› CAUTION ›››

There is more than one answer to Example 6. In fact, there are infinitely many equations of the form $y_2 = A \sin (Bx + C)$ that have the same graph as y_1. We found the one with the smallest phase shift by choosing the x intercept of y_1 that is closest to the origin.

If, however, we choose the x intercept that lies between 2 and 3, then we find that $y_3 = -5 \sin (x - 2.214)$ has the same graph as y_1. If we choose the x intercept that lies between 5 and 6, then we find that $y_4 = 5 \sin (x - 5.356)$ has the same graph as y_1.

MATCHED PROBLEM 6

Graph $y_1 = 4 \sin x - 3 \cos x$ using a graphing calculator, and find an equation of the form $y_2 = A \sin (Bx + C)$ that has the same graph as y_1. (Find the x intercept closest to the origin to three decimal places.)

› Mathematical Modeling and Data Analysis

The polynomial, exponential, and logarithmic functions studied in Chapters 3 and 4 are not suitable for modeling periodic phenomena. Instead, when given a data set that indicates periodic behavior, we use a technique called **sinusoidal regression** to model the data by a function of the form $f(x) = A \sin (Bx + C) + k$. [Note that $f(x)$ is a vertical translation of $y = A \sin (Bx + C)$].

EXAMPLE 7 Temperature Variation

The monthly average high temperatures in Fairbanks, Alaska, are given in Table 1. Use sinusoidal regression to find the function $y = A \sin (Bx + C) + k$ that best fits the data. Round the constants A, B, C, and k to three significant digits and use the sinusoidal regression function to estimate the average high temperature on April 1.

Table 1 Temperatures in Fairbanks, Alaska

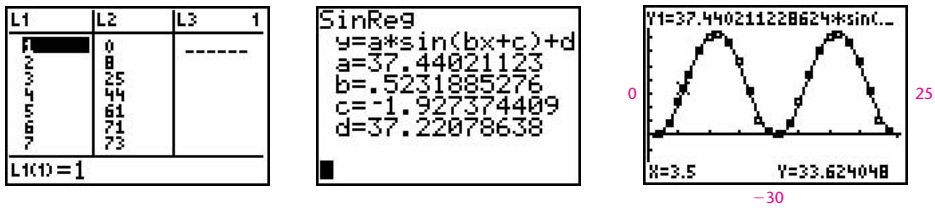
Month	1	2	3	4	5	6	7	8	9	10	11	12
Average High (°F)	0	8	25	44	61	71	73	66	54	31	11	3
Average Low (°F)	−19	−15	−3	20	37	49	52	46	35	16	−7	−15

SOLUTION

To observe the cyclical behavior of the data, we enter the average high temperatures for two consecutive years, from $x = 1$ to $x = 24$, where $x = 1$ represents January 15, $x = 2$ represents February 15, and so on. The data, the sinusoidal regression function, and a plot of the data and graph of the regression function are shown in Figure 7. Rounding constants to three significant digits, the sinusoidal regression function is

$$y = 37.4 \sin (0.523x - 1.93) + 37.2$$

To estimate the average high temperature on April 1 we substitute $x = 3.5$, obtaining a temperature of 33.5°. [Note the slight discrepancy, due to rounding, from the estimate shown in Figure 7(c).]



(a) (b) (c)

› Figure 7

MATCHED PROBLEM 7

The monthly average low temperatures in Fairbanks, Alaska, are given in Table 1. Use sinusoidal regression to find the function $y = A \sin(Bx + C) + k$ that best fits the data. Round the constants A, B, C, and k to three significant digits and use the sinusoidal regression function to estimate the average low temperature on April 1.

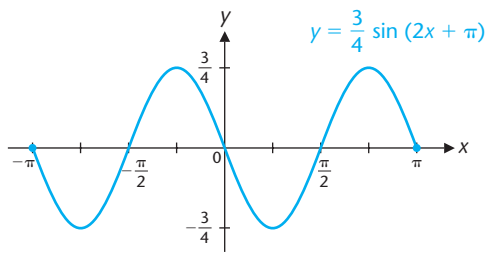
ANSWERS TO MATCHED PROBLEMS

1. (A) Zeros: $\pi/2, 3\pi/2, 5\pi/2$; turning points: $(\pi, 5), (2\pi, -5)$
(B) Zeros: $\pi/2, 3\pi/2, 5\pi/2$; turning points: $(\pi, -1/3), (2\pi, 1/3)$

2. (A) 20π (B) 1/3

3. Amplitude: 1/4; period: 2/3; turning points: (1/6, 1/4), (1/2, −1/4), (5/6, 1/4)

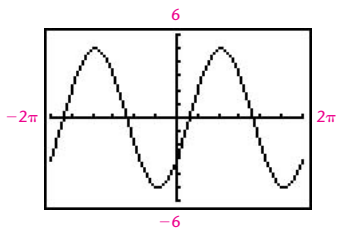
4. (A) Amplitude: 3/4; period: π; phase shift: $-\pi/2$
(B)



(C) $k\pi/2$, k any integer (D) $\pi/4$

5. (A) Period: $2\pi/9$; phase shift: 1/3
(B) Amplitude: 12; period: 2/3; phase shift: $-\frac{1}{3\pi}$ (C) Period: π; phase shift: -2π

6. $y_2 = 5 \sin(x - 0.644)$



7. 8.53°F

5-5 Exercises

1. What is simple harmonic motion?

2. Explain why amplitude is not applicable to all trigonometric functions.

3. In $y = A \sin Bx$, explain why the constant B determines the period.

4. Explain the connection between the graphs of $y = A \cos (Bx + C)$ and $y = A \cos Bx$.

5. Describe, in your own words, what the graph of a simple harmonic looks like.

6. Use a graphing calculator to graph $y = 3x \sin (2x + 1)$, and explain why it is not a simple harmonic.

In Problems 7–18, find the amplitude (if applicable) and period.

7. $y = 3 \sin x$

8. $y = \frac{1}{4} \cos x$

9. $y = -\frac{1}{2} \cos x$

10. $y = -2 \sin x$

11. $y = \sin 3x$

12. $y = \cos 2x$

13. $y = 2 \cot 4x$

14. $y = 3 \tan 2x$

15. $y = -\frac{1}{4} \tan 8\pi x$

16. $y = -\frac{1}{2} \cot 2\pi x$

17. $y = \csc (x/2)$

18. $y = \sec \pi x$

In Problems 19–22, find the amplitude (if applicable), the period, and all zeros in the given interval.

19. $y = \sin \pi x, -2 \le x \le 2$

20. $y = \cos \pi x, -2 \le x \le 2$

21. $y = \frac{1}{2} \cot (x/2), 0 < x < 4\pi$

22. $y = \frac{1}{2} \tan (x/2), -\pi < x < 3\pi$

In Problems 23–26, find the amplitude (if applicable), the period, and all turning points in the given interval.

23. $y = 3 \cos 2x, -\pi \le x \le \pi$

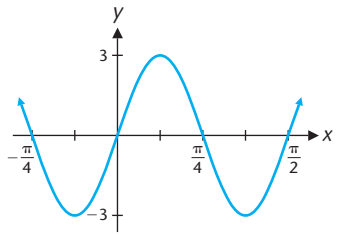
24. $y = 2 \sin 4x, -\pi \le x \le \pi$

25. $y = 2 \sec \pi x, -1 \le x \le 3$

26. $y = 2 \csc (x/2), 0 < x < 8\pi$

In Problems 27–30, find the equation of the form $y = A \sin Bx$ that produces the graph shown.

27.



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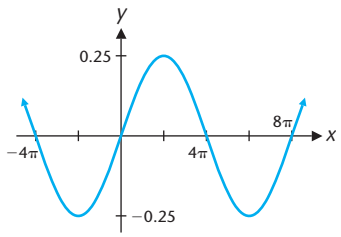

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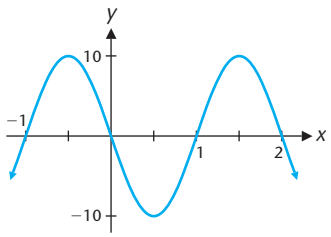

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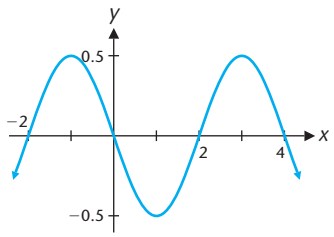
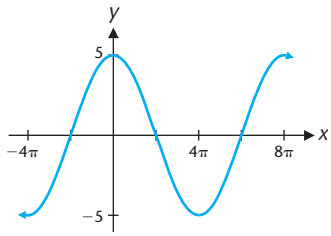
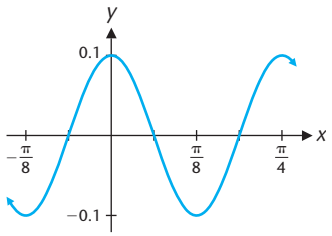


In Problems 31–34, find the equation of the form $y = A \cos Bx$ that produces the graph shown.

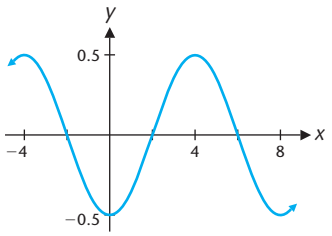
31.



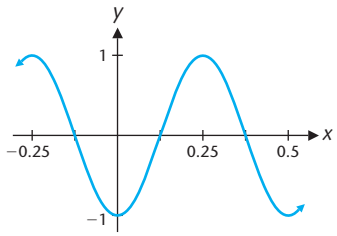
32.



33.



34.



In Problems 35–50, find the amplitude (if applicable), period, and phase shift, then sketch a graph of each function.

35. $y = 4 \cos x, 0 \le x \le 4\pi$

36. $y = 5 \sin x, 0 \le x \le 4\pi$

37. $y = \frac{1}{2} \sin (x + \pi/4), -2\pi \le x \le 2\pi$

38. $y = \frac{1}{3} \cos (x - \pi/4), -2\pi \le x \le 2\pi$

39. $y = \cot (x - \pi/6), -\pi \le x \le \pi$

40. $y = \tan (x + \pi/3), -\pi \le x \le \pi$

41. $y = 3 \tan 2x, 0 \le x \le 2\pi$

42. $y = -4 \cot 3x, -\pi/2 \le x \le \pi/2$

43. $y = 2\pi \sin (\pi x/2), 0 \le x \le 12$

44. $y = \pi \cos (\pi x/4), 0 \le x \le 12$

45. $y = -3 \sin [2\pi(x + \frac{1}{2})], -1 \le x \le 2$

46. $y = -2 \cos [\pi(x - 1)], -1 \le x \le 2$

47. $y = \sec (x + \pi), -\pi \le x \le \pi$

48. $y = \csc (x - \pi/2), -\pi \le x \le \pi$

49. $y = 10 \csc \pi x, 0 < x < 3$

50. $y = 8 \sec 2\pi x, 0 \le x \le 3$

In Problems 51–62, determine whether the statement is true or false. If true, explain why. If false, give a counterexample.

51. The graph of $y = A \sin Bx$ passes through the origin.

52. The graph of every simple harmonic passes through the origin.

53. Every simple harmonic is either even or odd.

54. The function $y = A \cos Bx$ is even.

55. Every simple harmonic is periodic.

56. Every simple harmonic is periodic with period 2π.

57. Every simple harmonic is bounded.

58. The amplitude of every simple harmonic is greater than its period.

59. If f is a simple harmonic, then the function g defined by $g(x) = f(3x)$ is a simple harmonic.

60. If f is a simple harmonic, then the function h defined by $h(x) = f(x + 3)$ is a simple harmonic.

61. If f is a simple harmonic, then the function j defined by $j(x) = 3 - f(x)$ is a simple harmonic.

62. If f is a simple harmonic, then the function k defined by $k(x) = 3f(x)$ is a simple harmonic.

Graph each function in Problems 63–66. (Select the dimensions of each viewing window so that at least two periods are visible.) Find an equation of the form $y = k + A \sin Bx$ or $y = k + A \cos Bx$ that has the same graph as the given equation. (These problems suggest the existence of further identities in addition to the basic identities discussed in Section 5-4.)

63. $y = \cos^2 x - \sin^2 x$

64. $y = \sin x \cos x$

65. $y = 2 \sin^2 x$

66. $y = 2 \cos^2 x$

In Problems 67–74, graph at least two cycles of the given equation in a graphing calculator, then find an equation of the form $y = A \tan Bx$, $y = A \cot Bx$, $y = A \sec Bx$, or $y = A \csc Bx$ that has the same graph. (These problems suggest additional identities beyond those discussed in Section 5-4. Additional identities are discussed in detail in Chapter 6.)

67. $y = \cot x - \tan x$

68. $y = \cot x + \tan x$

69. $y = \csc x + \cot x$

70. $y = \csc x - \cot x$

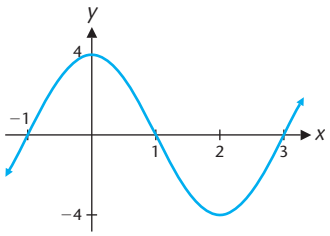
71. $y = \sin 3x + \cos 3x \cot 3x$

72. $y = \cos 2x + \sin 2x \tan 2x$

73. $y = \dfrac{\sin 4x}{1 + \cos 4x}$

74. $y = \dfrac{\sin 6x}{1 - \cos 6x}$

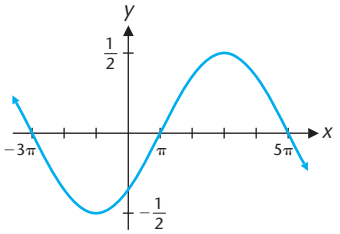
Problems 75 and 76 refer to the following graph:



75. If the graph is a graph of an equation of the form $y = A \sin(Bx + C)$, $0 < -C/B < 2$, find the equation.

76. If the graph is a graph of an equation of the form $y = A \sin(Bx + C)$, $-2 < -C/B < 0$, find the equation.

Problems 77 and 78 refer to the following graph:

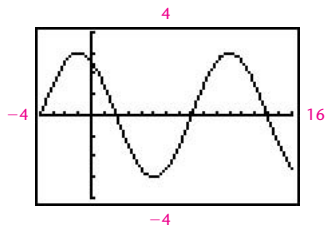


77. If the graph is a graph of an equation of the form $y = A \cos(Bx + C)$, $0 < -C/B < 4\pi$, find the equation.

78. If the graph is a graph of an equation of the form $y = A \cos(Bx + C)$, $-2\pi < -C/B < 0$, find the equation.

In Problems 79–82, the graph is the graph of a simple harmonic. Find the equation of the form $y = A \sin(Bx + C)$ or $y = A \cos(Bx + C)$ that produces the graph and has the smallest phase shift (in absolute value).

79.



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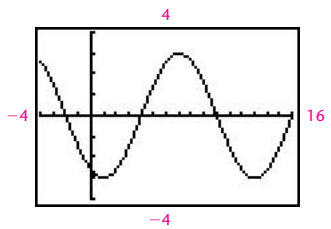

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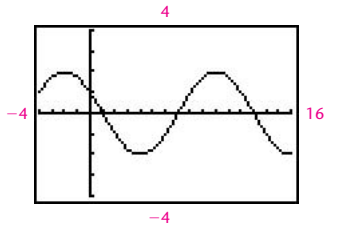
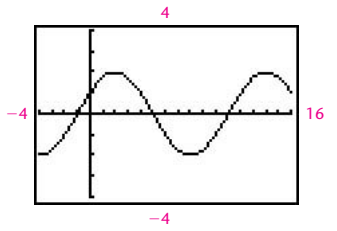


82.



In Problems 83–86, state the amplitude, period, and phase shift of each function and sketch a graph of the function with the aid of a graphing calculator.

83. $y = 3.5 \sin\left[\frac{\pi}{2}(t + 0.5)\right], 0 \le t \le 10$

84. $y = 5.4 \sin\left[\frac{\pi}{2.5}(t - 1)\right], 0 \le t \le 6$

85. $y = 50 \cos[2\pi(t - 0.25)], 0 \le t \le 2$

86. $y = 25 \cos[5\pi(t - 0.1)], 0 \le t \le 2$

In Problems 87–92, graph each equation. (Select the dimensions of each viewing window so that at least two periods are visible.) Find an equation of the form $y = A \sin(Bx + C)$ that has the same graph as the given equation. Find A and B exactly and C to three decimal places. Use the x intercept closest to the origin as the phase shift.

87. $y = \sqrt{2} \sin x + \sqrt{2} \cos x$

88. $y = \sqrt{2} \sin x - \sqrt{2} \cos x$

89. $y = \sqrt{3} \sin x - \cos x$

90. $y = \sin x + \sqrt{3} \cos x$

91. $y = 4.8 \sin 2x - 1.4 \cos 2x$

92. $y = 1.4 \sin 2x + 4.8 \cos 2x$

Problems 93–98 illustrate combinations of functions that occur in harmonic analysis applications. Graph parts A, B, and C of each problem in the same viewing window. In Problems 93–96, what is happening to the amplitude of the function in part C? Give an example of a physical phenomenon that might be modeled by a similar function.

93. $0 \le x \le 16$

(A) $y = \frac{1}{x}$ (B) $y = -\frac{1}{x}$ (C) $y = \frac{1}{x} \sin \frac{\pi}{2} x$

94. $0 \le x \le 10$

(A) $y = \dfrac{2}{x}$ (B) $y = -\dfrac{2}{x}$ (C) $y = \dfrac{2}{x}\cos \pi x$

95. $0 \le x \le 10$

(A) $y = x$ (B) $y = -x$ (C) $y = x \sin \dfrac{\pi}{2} x$

96. $0 \le x \le 10$

(A) $y = \dfrac{x}{2}$ (B) $y = -\dfrac{x}{2}$ (C) $y = \dfrac{x}{2}\cos \pi x$

97. $0 \le x \le 2\pi$

(A) $y = \sin x$ (B) $y = \sin x + \dfrac{\sin 3x}{3}$

(C) $y = \sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5}$

98. $0 \le x \le 4$

(A) $y = \sin \pi x$ (B) $y = \sin \pi x + \dfrac{\sin 2\pi x}{2}$

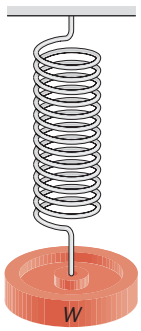
(C) $y = \sin \pi x + \dfrac{\sin 2\pi x}{2} + \dfrac{\sin 3\pi x}{3}$

APPLICATIONS

99. **SPRING-MASS SYSTEM** A 6-pound weight hanging from the end of a spring is pulled $\frac{1}{3}$ foot below the equilibrium position and then released (see figure). If air resistance and friction are neglected, the distance x that the weight is from the equilibrium position relative to time t (in seconds) is given by

$$x = \tfrac{1}{3}\cos 8t$$

State the period P and amplitude A of this function, and graph it for $0 \le t \le \pi$.



100. **ELECTRICAL CIRCUIT** An alternating current generator generates a current given by

$$I = 30 \sin 120t$$

where t is time in seconds. What are the amplitude A and period P of this function? What is the frequency of the current; that is, how many cycles (periods) will be completed in 1 second?

101. **SPRING-MASS SYSTEM** Assume the motion of the weight in Problem 99 has an amplitude of 8 inches and a period of 0.5 second, and that its position when $t = 0$ is 8 inches below its position at rest (displacement above rest position is positive and below is negative). Find an equation of the form $y = A \cos Bt$ that describes the motion at any time $t \ge 0$. (Neglect any damping forces—that is, friction and air resistance.)

102. **ELECTRICAL CIRCUIT** If the voltage E in an electrical circuit has an amplitude of 110 volts and a period of $\frac{1}{60}$ second, and if $E = 110$ volts when $t = 0$ seconds, find an equation of the form $E = A \cos Bt$ that gives the voltage at any time $t \ge 0$.

103. **POLLUTION** The amount of sulfur dioxide pollutant from heating fuels released in the atmosphere in a city varies seasonally. Suppose the number of tons of pollutant released into the atmosphere during the nth week after January 1 for a particular city is given by

$$A(n) = 1.5 + \cos \frac{n\pi}{26} \qquad 0 \le n \le 104$$

Graph the function over the indicated interval and describe what the graph shows.

104. **MEDICINE** A seated normal adult breathes in and exhales about 0.82 liter of air every 4.00 seconds. The volume of air in the lungs t seconds after exhaling is approximately

$$V(t) = 0.45 - 0.37 \cos \frac{\pi t}{2} \qquad 0 \le t \le 8$$

Graph the function over the indicated interval and describe what the graph shows.

105. **ELECTRICAL CIRCUIT** The current in an electrical circuit is given by $I = 15 \cos (120\pi t + \pi/2)$, $0 \le t \le \frac{2}{60}$, where I is measured in amperes and time t is in seconds.

(A) Find the amplitude A, period P, and phase shift.

(B) Graph the equation.

(C) Find the smallest positive value of t at which the current is -15 amperes.

106. **ELECTRICAL CIRCUIT** The current in an electrical circuit is given by $I = 30 \cos (120\pi t - \pi)$, $0 \le t \le \frac{3}{60}$, where I is measured in amperes and time t is in seconds.

(A) Find the amplitude A, period P, and phase shift.

(B) Graph the equation.

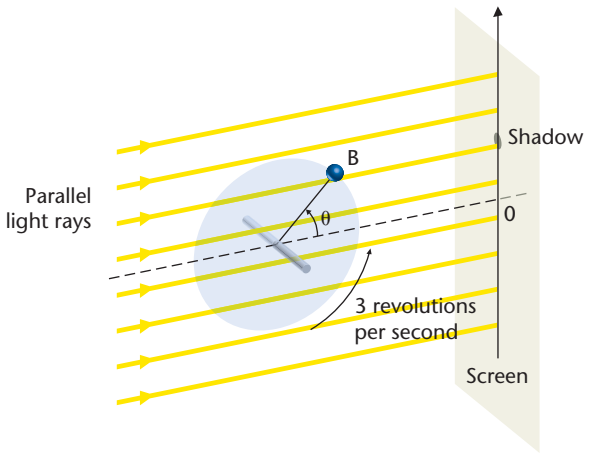
(C) Find the smallest positive value of t at which the current is -30 amperes.

107. **PHYSICS—ENGINEERING** The thin, plastic disk shown in the figure is rotated at 3 revolutions per second, starting at $\theta = 0$ (thus at the end of t seconds, $\theta = 6\pi t$—why?). If the disk has a

radius of 3, show that the position of the shadow on the y scale from the small steel ball B is given by

$$y = 3 \sin 6\pi t$$

Graph this equation for $0 \le t \le 1$.

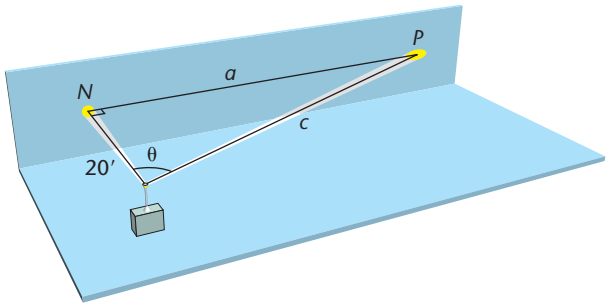


108. PHYSICS–ENGINEERING If in Problem 107 the disk started rotating at $\theta = \pi/2$, show that the position of the shadow at time t (in seconds) is given by

$$y = 3 \sin\left(6\pi t + \frac{\pi}{2}\right)$$

Graph this equation for $0 \le t \le 1$.

109. A beacon light 20 feet from a wall rotates clockwise at the rate of 1/4 revolution per second (rps) (see the figure), therefore, $\theta = \pi t/2$.
(A) Start counting time in seconds when the light spot is at N and write an equation for the length c of the light beam in terms of t.
(B) Graph the equation found in part A for the time interval $[0, 1]$.
(C) Describe what happens to the length c of the light beam as t goes from 0 to 1.



110. Refer to Problem 109.
(A) Write an equation for the distance a the light spot travels along the wall in terms of time t.
(B) Graph the equation found in part A for the time interval $[0, 1]$.
(C) Describe what happens to the distance a along the wall as t goes from 0 to 1.

111. MODELING SUNSET TIMES Sunset times for the fifth of each month over a period of 1 year were taken from a tide booklet for the San Francisco Bay to form Table 2. Daylight savings time was ignored and the times are for a 24-hour clock starting at midnight.

Table 2

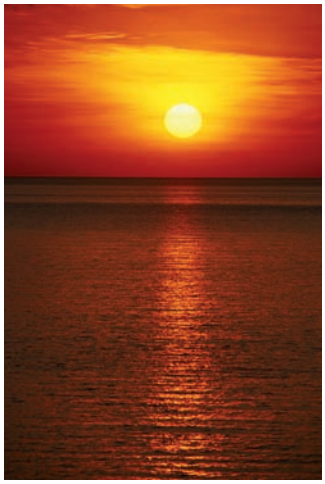
x (months)	y (sunset)*
1	17:05
2	17:38
3	18:07
4	18:36
5	19:04
6	19:29
7	19:35
8	19:15
9	18:34
10	17:47
11	17:07
12	16:51

*Time on a 24-hr clock, starting at midnight.

(A) Using 1 month as the basic unit of time, enter the data for a 2-year period in your graphing calculator and produce a scatter plot in the viewing window. Before entering Table 2 data into your graphing calculator, convert sunset times from hours and minutes to decimal hours rounded to two decimal places. Choose $15 \le y \le 20$ for the viewing window.
(B) It appears that a sine curve of the form

$$y = k + A \sin(Bx + C)$$

will closely model these data. The constants k, A, and B are easily determined from Table 2 as follows: $A = (\max y - \min y)/2$, $B = 2\pi/\text{Period}$, and $k = \min y + A$. To estimate C, visually estimate to one decimal place the smallest positive phase shift from the plot in part A. After determining A, B, k, and C, write the resulting equation. (Your value of C may differ slightly from the answer in the back of the book.)
(C) Plot the results of parts A and B in the same viewing window. (An improved fit may result by adjusting your value of C slightly.)
(D) If your graphing calculator has a sinusoidal regression feature, check your results from parts B and C by finding and plotting the regression equation.



112. MODELING TEMPERATURE VARIATION The 30-year average monthly temperature, °F, for each month of the year for Washington, D.C., is given in Table 3 (*World Almanac*).
(A) Using 1 month as the basic unit of time, enter the data for a 2-year period in your graphing calculator and produce a scatter plot in the viewing window. Choose $0 \le y \le 80$ for the viewing window.
(B) It appears that a sine curve of the form

$$y = k + A \sin (Bx + C)$$

will closely model these data. The constants k, A, and B are easily determined from Table 3 as follows: $A = (\max y - \min y)/2$, $B = 2\pi/\text{Period}$, and $k = \min y + A$. To estimate C, visually estimate to one decimal place the smallest positive phase shift from the plot in part A. After determining A, B, k, and C, write the resulting equation.
(C) Plot the results of parts A and B in the same viewing window. (An improved fit may result by adjusting your value of C slightly.)
(D) If your graphing calculator has a sinusoidal regression feature, check your results from parts B and C by finding and plotting the regression equation.

Table 3

x (months)	y (temp.)
1	31
2	34
3	43
4	53
5	62
6	71
7	76
8	74
9	67
10	55
11	45
12	35

5-6 Inverse Trigonometric Functions

- Inverse Sine Function
- Inverse Cosine Function
- Inverse Tangent Function
- Summary
- Inverse Cotangent, Secant, and Cosecant Functions (Optional)

In the process of solving right triangles in Section 5-3, we used the inverse trigonometric keys on a calculator to find the angle with a given trigonometric ratio. In this section we give detailed definitions of the inverse trigonometric functions and study their graphs. Inverse functions in general were discussed in Section 1-6, and some important facts are restated in the box.

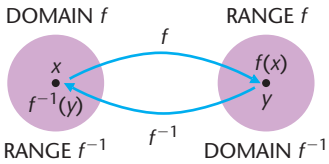
FACTS ABOUT INVERSE FUNCTIONS

For a one-to-one function f and its inverse f^{-1}:

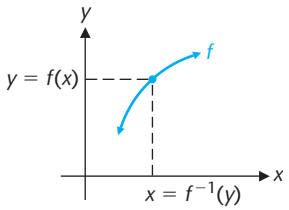
1. If (a, b) is an element of f, then (b, a) is an element of f^{-1}, and conversely.

2. Range of f = Domain of f^{-1}
Domain of f = Range of f^{-1}

3.



4. If $x = f^{-1}(y)$, then $y = f(x)$ for y in the domain of f^{-1} and x in the domain of f, and conversely.



5. $f(f^{-1}(y)) = y$ for y in the domain of f^{-1}
$f^{-1}(f(x)) = x$ for x in the domain of f

All trigonometric functions are periodic; therefore, each range value can be associated with infinitely many domain values (Fig. 1). As a result, no trigonometric function is one-to-one. Without restrictions, no trigonometric function has an inverse function. To resolve this problem, we restrict the domain of each function so that it is one-to-one over the restricted domain. For this restricted domain, an inverse function is guaranteed.

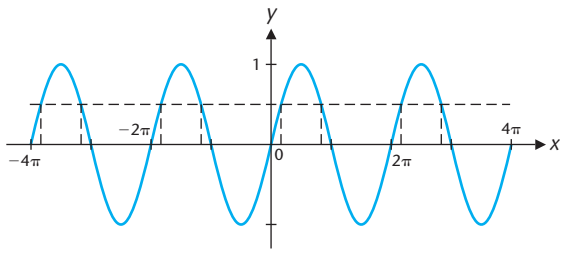


Figure 1 $y = \sin x$ is not one-to-one over $(-\infty, \infty)$.

Inverse trigonometric functions represent another group of basic functions that are added to our library of elementary functions. These functions are used in many

applications and mathematical developments, and will be particularly useful to us when we solve trigonometric equations in Section 6-5.

Inverse Sine Function

How can the domain of the sine function be restricted so that it is one-to-one? This can be done in infinitely many ways. A natural and generally accepted way is illustrated in Figure 2.

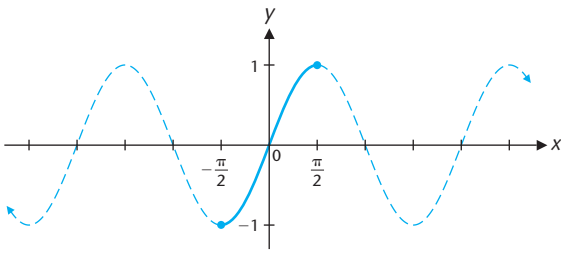


Figure 2 $y = \sin x$ is one-to-one over $[-\pi/2, \pi/2]$.

If the domain of the sine function is restricted to the interval $[-\pi/2, \pi/2]$, we see that the restricted function passes the horizontal line test (Section 1-6) and thus is one-to-one. Note that each range value from -1 to 1 is assumed exactly once as x moves from $-\pi/2$ to $\pi/2$. We use this restricted sine function to define the *inverse sine function.*

DEFINITION 1 Inverse Sine Function

The **inverse sine function,** denoted by $\sin^{-1}$ or arcsin, is defined as the inverse of the restricted sine function $y = \sin x$, $-\pi/2 \le x \le \pi/2$. Thus,

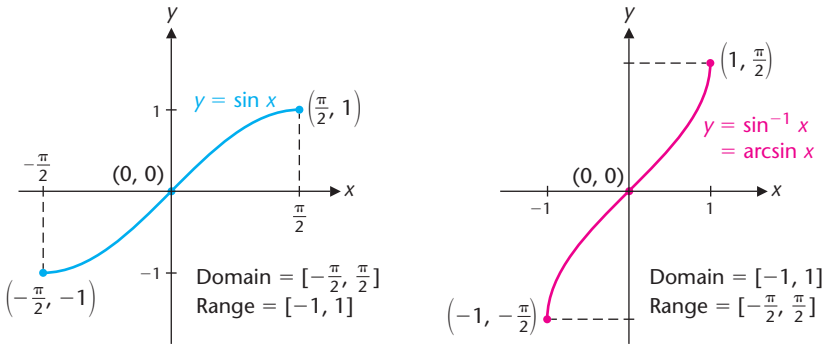
$$y = \sin^{-1} x \text{ and } y = \arcsin x$$

are equivalent to

$$\sin y = x \quad \text{where} \quad -\pi/2 \le y \le \pi/2, -1 \le x \le 1$$

In words, the inverse sine of x, or the arcsine of x, is the number or angle y, $-\pi/2 \le y \le \pi/2$, whose sine is x.

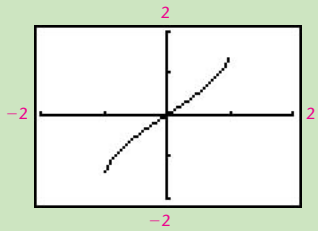
To graph $y = \sin^{-1} x$, take each point on the graph of the restricted sine function and reverse the order of the coordinates. For example, because $(-\pi/2, -1)$, $(0, 0)$, and $(\pi/2, 1)$ are on the graph of the restricted sine function [Fig. 3(a)], then $(-1, -\pi/2)$, $(0, 0)$, and $(1, \pi/2)$ are on the graph of the inverse sine function, as shown in Figure 3(b). Using these three points provides us with a quick way of sketching the graph of the inverse sine function. A more accurate graph can be obtained by using a calculator.



› **Figure 3** Inverse sine function.

››› EXPLORE-DISCUSS 1

A graphing calculator produced the graph in Figure 4 for $y_1 = \sin^{-1} x$, $-2 \le x \le 2$, and $-2 \le y \le 2$. (Try this on your own graphing calculator.) Explain why there are no parts of the graph on the intervals $[-2, -1)$ and $(1, 2]$.



› **Figure 4**

We state the important sine–inverse sine identities that follow from the general properties of inverse functions given in the box at the beginning of this section.

› SINE–INVERSE SINE IDENTITIES

$$\sin(\sin^{-1} x) = x \qquad -1 \le x \le 1 \qquad f(f^{-1}(x)) = x$$
$$\sin^{-1}(\sin x) = x \qquad -\pi/2 \le x \le \pi/2 \qquad f^{-1}(f(x)) = x$$

$\sin(\sin^{-1} 0.7) = 0.7$ $\quad$ $\sin(\sin^{-1} 1.3) \ne 1.3$

$\sin^{-1}[\sin(-1.2)] = -1.2$ $\quad$ $\sin^{-1}[\sin(-2)] \ne -2$

[*Note:* The number 1.3 is not in the domain of the inverse sine function, and -2 is not in the restricted domain of the sine function. Try calculating all these examples with your calculator and see what happens!]

EXAMPLE 1 Exact Values

Find exact values without using a calculator.

(A) $\arcsin(-\frac{1}{2})$ (B) $\sin^{-1}(\sin 1.2)$ (C) $\cos[\sin^{-1}(\frac{2}{3})]$

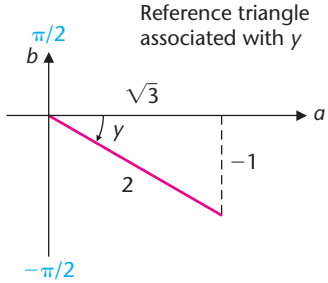
SOLUTIONS

(A) $y = \arcsin(-\frac{1}{2})$ is equivalent to

$$\sin y = -\tfrac{1}{2} \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$$y = -\frac{\pi}{6} = \arcsin(-\tfrac{1}{2})$$

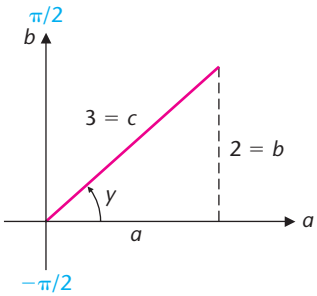
Reference triangle associated with y



[*Note:* $y \neq 11\pi/6$, even though $\sin(11\pi/6) = -\frac{1}{2}$, because y must be between $-\pi/2$ and $\pi/2$, inclusive.]

(B) $\sin^{-1}(\sin 1.2) = 1.2$ Sine–inverse sine identity, because $-\pi/2 \le 1.2 \le \pi/2$

(C) Let $y = \sin^{-1}(\frac{2}{3})$; then $\sin y = (\frac{2}{3})$, $-\pi/2 \le y \le \pi/2$. Draw the reference triangle associated with y. Then $\cos y = \cos[\sin^{-1}(\frac{2}{3})]$ can be determined directly from the triangle (after finding the third side) without actually finding y.



$$a^2 + b^2 = c^2 \qquad \text{Solve for } a.$$

$$a = \sqrt{3^2 - 2^2} \qquad \text{Because } a > 0 \text{ in Quadrant I}$$

$$= \sqrt{5}$$

Therefore, $\cos[\sin^{-1}(\frac{2}{3})] = \cos y = \frac{a}{c} = \sqrt{5}/3$.

MATCHED PROBLEM 1

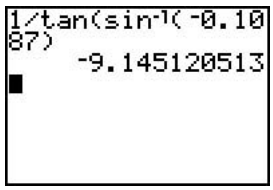
Find exact values without using a calculator.

(A) $\arcsin(\sqrt{2}/2)$ (B) $\sin[\sin^{-1}(-0.4)]$ (C) $\tan[\sin^{-1}(-1/\sqrt{5})]$

EXAMPLE 2 Calculator Values

Find to four significant digits using a calculator.

(A) $\arcsin(-0.3042)$ (B) $\sin^{-1} 1.357$ (C) $\cot[\sin^{-1}(-0.1087)]$



› Figure 5

SOLUTIONS

The function keys used to represent inverse trigonometric functions vary among different brands of calculators, so read the user's manual for your calculator. Set your calculator in radian mode:

(A) $\arcsin(-0.3042) = -0.3091$

(B) $\sin^{-1} 1.357 =$ Error 1.357 is not in the domain of $\sin^{-1}$.

(C) $\cot[\sin^{-1}(-0.1087)] = -9.145$ (See Fig. 5.)

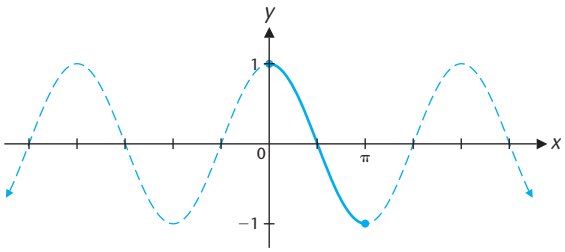
MATCHED PROBLEM 2

Find to four significant digits using a calculator.

(A) $\sin^{-1} 0.2903$ (B) $\arcsin(-2.305)$ (C) $\cot[\sin^{-1}(-0.3446)]$

› Inverse Cosine Function

To restrict the cosine function so that it becomes one-to-one, we choose the interval $[0, \pi]$. Over this interval the restricted function passes the horizontal line test, and each range value is assumed exactly once as x moves from 0 to π (Fig. 6). We use this restricted cosine function to define the *inverse cosine function.*



› Figure 6 $y = \cos x$ is one-to-one over $[0, \pi]$.

› DEFINITION 2 Inverse Cosine Function

The **inverse cosine function,** denoted by $\cos^{-1}$ or arccos, is defined as the inverse of the restricted cosine function $y = \cos x$, $0 \le x \le \pi$. Thus,

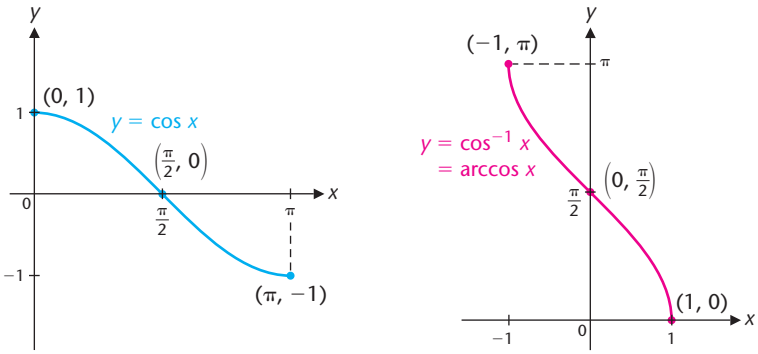
$$y = \cos^{-1} x \quad \text{and} \quad y = \arccos x$$

are equivalent to

$$\cos y = x \quad \text{where} \quad 0 \le y \le \pi, -1 \le x \le 1$$

In words, the inverse cosine of x, or the arccosine of x, is the number or angle y, $0 \le y \le \pi$, whose cosine is x.

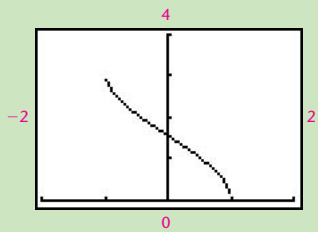
Figure 7 compares the graphs of the restricted cosine function and its inverse. Notice that $(0, 1)$, $(\pi/2, 0)$, and $(\pi, -1)$ are on the restricted cosine graph. Reversing the coordinates gives us three points on the graph of the inverse cosine function.



› **Figure 7** Inverse cosine function.

››› EXPLORE-DISCUSS 2

A graphing calculator produced the graph in Figure 8 for $y_1 = \cos^{-1} x$, $-2 \le x \le 2$, and $0 \le y \le 4$. (Try this on your own graphing calculator.) Explain why there are no parts of the graph on the intervals $[-2, -1)$ and $(1, 2]$.



› **Figure 8**

We complete the discussion by giving the cosine–inverse cosine identities:

› COSINE–INVERSE COSINE IDENTITIES

$$\cos(\cos^{-1} x) = x \qquad -1 \le x \le 1 \qquad f(f^{-1}(x)) = x$$

$$\cos^{-1}(\cos x) = x \qquad 0 \le x \le \pi \qquad f^{-1}(f(x)) = x$$

››› EXPLORE-DISCUSS 3

Evaluate each of the following with a calculator. Which illustrate a cosine–inverse cosine identity and which do not? Discuss why.

(A) $\cos(\cos^{-1} 0.2)$ **(B)** $\cos[\cos^{-1}(-2)]$

(C) $\cos^{-1}(\cos 2)$ **(D)** $\cos^{-1}[\cos(-3)]$

EXAMPLE 3 Exact Values

Find exact values without using a calculator.

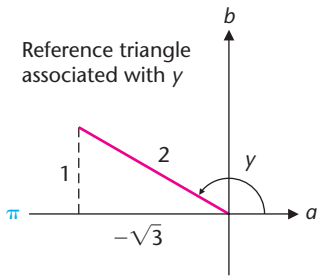
(A) $\arccos(-\sqrt{3}/2)$ (B) $\cos(\cos^{-1} 0.7)$ (C) $\sin[\cos^{-1}(-\frac{1}{3})]$

SOLUTIONS

(A) $y = \arccos(-\sqrt{3}/2)$ is equivalent to

$$\cos y = -\frac{\sqrt{3}}{2} \qquad 0 \le y \le \pi$$

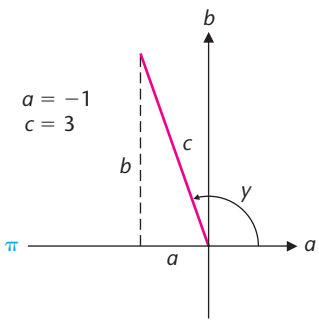
$$y = \frac{5\pi}{6} = \arccos\left(-\frac{\sqrt{3}}{2}\right)$$



[*Note:* $y \neq -5\pi/6$, even though $\cos(-5\pi/6) = -\sqrt{3}/2$ because y must be between 0 and π, inclusive.]

(B) $\cos(\cos^{-1} 0.7) = 0.7$ Cosine–inverse cosine identity, because $-1 \le 0.7 \le 1$

(C) Let $y = \cos^{-1}(-\frac{1}{3})$; then $\cos y = -\frac{1}{3}$, $0 \le y \le \pi$. Draw a reference triangle associated with y. Then $\sin y = \sin[\cos^{-1}(-\frac{1}{3})]$ can be determined directly from the triangle (after finding the third side) without actually finding y.



$$a^2 + b^2 = c^2 \qquad \text{Solve for } b.$$

$$b = \sqrt{3^2 - (-1^2)} \qquad \text{Because } b > 0 \text{ in Quadrant II}$$

$$= \sqrt{8} = 2\sqrt{2}$$

Therefore, $\sin[\cos^{-1}(-\frac{1}{3})] = \sin y = \frac{b}{c} = 2\sqrt{2}/3$.

MATCHED PROBLEM 3

Find exact values without using a calculator.

(A) $\arccos(\sqrt{2}/2)$ (B) $\cos^{-1}(\cos 3.05)$ (C) $\cot[\cos^{-1}(-1/\sqrt{5})]$

EXAMPLE 4 Calculator Values

Find to four significant digits using a calculator.

(A) $\arccos 0.4325$ (B) $\cos^{-1} 2.137$ (C) $\csc[\cos^{-1}(-0.0349)]$

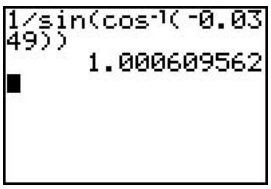
SOLUTIONS

Set your calculator in radian mode.

(A) $\arccos 0.4325 = 1.124$

(B) $\cos^{-1} 2.137 =$ Error 2.137 is not in the domain of $\cos^{-1}$.

(C) $\csc[\cos^{-1}(-0.0349)] = 1.001$ (See Fig. 9.)



› Figure 9

MATCHED PROBLEM 4

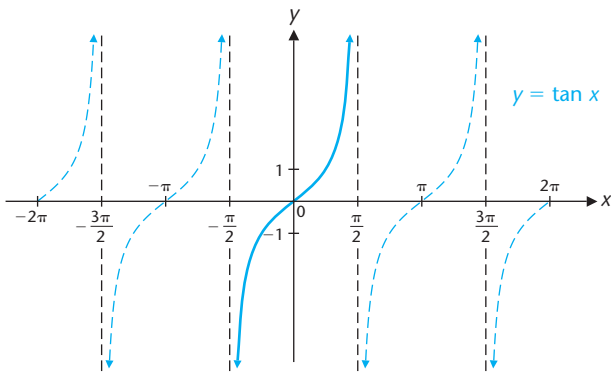
Find to four significant digits using a calculator.

(A) $\cos^{-1}(0.6773)$ (B) $\arccos(-1.003)$ (C) $\cot[\cos^{-1}(-0.5036)]$

› Inverse Tangent Function

To restrict the tangent function so that it becomes one-to-one, we choose the interval $(-\pi/2, \pi/2)$. Over this interval the restricted function passes the horizontal line test, and each range value is assumed exactly once as x moves across this restricted domain (Fig. 10). We use this restricted tangent function to define the *inverse tangent function.*

› Figure 10 $y = \tan x$ is one-to-one over $(-\pi/2, \pi/2)$.



› DEFINITION 3 Inverse Tangent Function

The **inverse tangent function,** denoted by $\tan^{-1}$ or arctan, is defined as the inverse of the restricted tangent function $y = \tan x$, $-\pi/2 < x < \pi/2$. Thus,

$$y = \tan^{-1} x \qquad \text{and} \qquad y = \arctan x$$

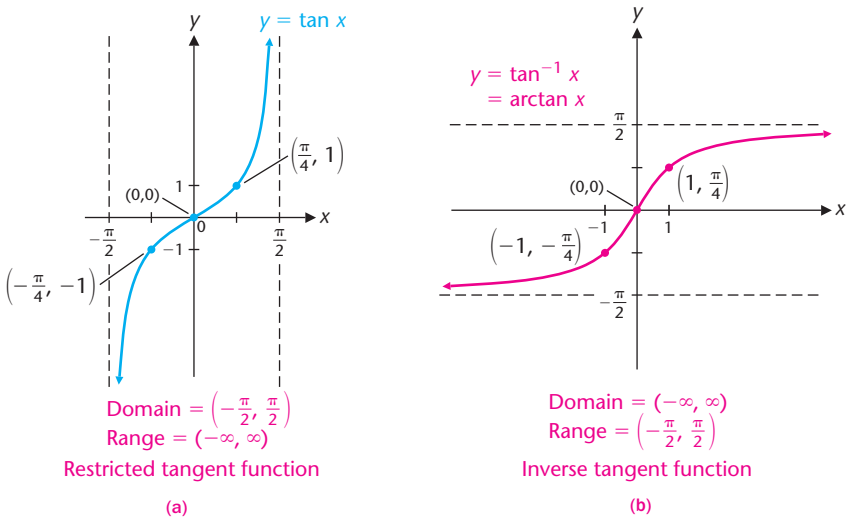
are equivalent to

$$\tan y = x \qquad \text{where} \qquad -\pi/2 < y < \pi/2 \text{ and } x \text{ is a real number}$$

In words, the inverse tangent of x, or the arctangent of x, is the number or angle y, $-\pi/2 < y < \pi/2$, whose tangent is x.

Figure 11 compares the graphs of the restricted tangent function and its inverse. Notice that $(-\pi/4, -1)$, $(0, 0)$, and $(\pi/4, 1)$ are on the restricted tangent graph. Reversing the coordinates gives us three points on the graph of the inverse tangent function. Also note that the vertical asymptotes become horizontal asymptotes for the inverse function.

› **Figure 11** Inverse tangent function.



We now state the tangent–inverse tangent identities.

› TANGENT–INVERSE TANGENT IDENTITIES

$$\tan(\tan^{-1} x) = x \qquad -\infty < x < \infty \qquad f(f^{-1}(x)) = x$$

$$\tan^{-1}(\tan x) = x \qquad -\pi/2 < x < \pi/2 \qquad f^{-1}(f(x)) = x$$

››› EXPLORE-DISCUSS 4

Evaluate each of the following with a calculator. Which illustrate a tangent–inverse tangent identity and which do not? Discuss why.

(A) $\tan(\tan^{-1} 30)$ **(B)** $\tan[\tan^{-1}(-455)]$

(C) $\tan^{-1}(\tan 1.4)$ **(D)** $\tan^{-1}[\tan(-3)]$

EXAMPLE 5 Exact Values

Find exact values without using a calculator.

(A) $\tan^{-1}(-1/\sqrt{3})$ (B) $\tan^{-1}(\tan 0.63)$

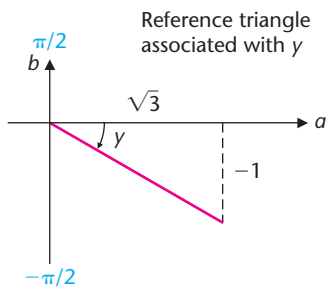
SOLUTIONS

(A) $y = \tan^{-1}(-1/\sqrt{3})$ is equivalent to

$$\tan y = -\frac{1}{\sqrt{3}} \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$y = -\frac{\pi}{6} = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

[*Note:* y cannot be $11\pi/6$ because y must be between $-\pi/2$ and $\pi/2$.]



(B) $\tan^{-1}(\tan 0.63) = 0.63$ Tangent–inverse tangent identity, because $-\pi/2 \le 0.63 \le \pi/2$

MATCHED PROBLEM 5

Find exact values without using a calculator.

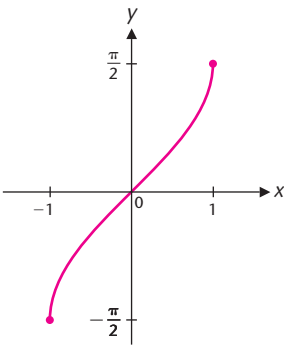
(A) $\arctan(-\sqrt{3})$ (B) $\tan(\tan^{-1} 43)$

› Summary

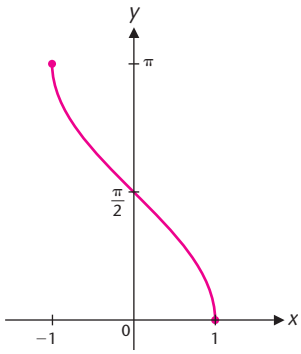
We summarize the definitions and graphs of the inverse trigonometric functions discussed so far for convenient reference.

SUMMARY OF $\sin^{-1}$, $\cos^{-1}$, AND $\tan^{-1}$

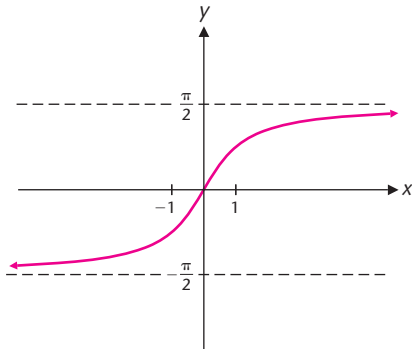
$y = \sin^{-1} x$	is equivalent to $x = \sin y$	where $-1 \le x \le 1, -\pi/2 \le y \le \pi/2$
$y = \cos^{-1} x$	is equivalent to $x = \cos y$	where $-1 \le x \le 1, 0 \le y \le \pi$
$y = \tan^{-1} x$	is equivalent to $x = \tan y$	where $-\infty < x < \infty, -\pi/2 < y < \pi/2$



$y = \sin^{-1} x$
Domain = $[-1, 1]$
Range = $[-\frac{\pi}{2}, \frac{\pi}{2}]$



$y = \cos^{-1} x$
Domain = $[-1, 1]$
Range = $[0, \pi]$



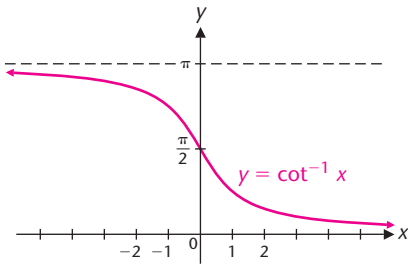
$y = \tan^{-1} x$
Domain = $(-\infty, \infty)$
Range = $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Inverse Cotangent, Secant, and Cosecant Functions

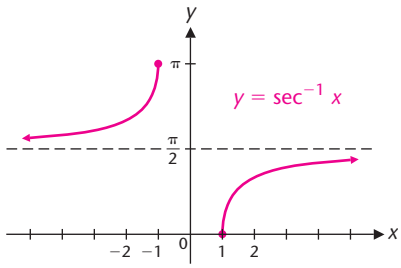
For completeness, we include the definitions and graphs of the inverse cotangent, secant, and cosecant functions.

DEFINITION 4 Inverse Cotangent, Secant, and Cosecant Functions

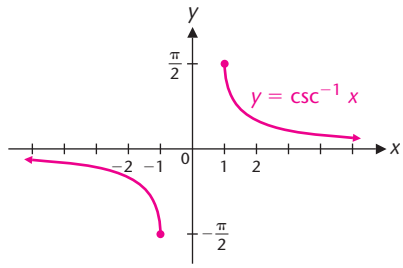
$y = \cot^{-1} x$	is equivalent to $x = \cot y$	where $0 < y < \pi, -\infty < x < \infty$
$y = \sec^{-1} x$	is equivalent to $x = \sec y$	where $0 \le y \le \pi, y \ne \pi/2, \lvert x\rvert \ge 1$
$y = \csc^{-1} x$	is equivalent to $x = \csc y$	where $-\pi/2 \le y \le \pi/2, y \ne 0, \lvert x\rvert \ge 1$



Domain: All real numbers
Range: $0 < y < \pi$



Domain: $x \le -1$ or $x \ge 1$
Range: $0 \le y \le \pi, y \ne \pi/2$



Domain: $x \le -1$ or $x \ge 1$
Range: $-\pi/2 \le y \le \pi/2, y \ne 0$

[*Note:* The definitions of $\sec^{-1}$ and $\csc^{-1}$ are not universally agreed upon.]

ANSWERS TO MATCHED PROBLEMS

1. (A) $\pi/4$ (B) -0.4 (C) $-1/2$
2. (A) 0.2945 (B) Not defined (C) -2.724
3. (A) $\pi/4$ (B) 3.05 (C) $-1/2$
4. (A) 0.8267 (B) Not defined (C) -0.5829
5. (A) $-\pi/3$ (B) 43

5-6 Exercises

Unless stated to the contrary, the inverse trigonometric functions are assumed to have real number ranges (use radian mode in calculator problems). A few problems involve ranges with angles in degree measure, and these are clearly indicated (use degree mode in calculator problems).

1. Explain why the function $f(x) = \sin x$, for $0 \le x \le \pi$, has no inverse.

2. Explain why the function $f(x) = \cos x$, for $-\pi/2 \le x \le \pi/2$, has no inverse.

3. Does $\tan(\tan^{-1} x) = x$ for all real x? Explain.

4. Does $\tan^{-1}(\tan x) = x$ for all real x? Explain.

5. If a function f has an inverse, how are the graphs of f and f^{-1} related?

6. If f is increasing, is f^{-1} also increasing? Explain.

In Problems 7–18, find exact values without using a calculator.

7. $\cos^{-1} 0$ **8.** $\sin^{-1} 0$ **9.** $\arcsin(\sqrt{3}/2)$

10. $\arccos(\sqrt{3}/2)$ **11.** $\arctan\sqrt{3}$ **12.** $\tan^{-1} 1$

13. $\sin^{-1}(\sqrt{2}/2)$ **14.** $\cos^{-1}(\frac{1}{2})$ **15.** $\arccos 1$

16. $\arctan(1/\sqrt{3})$ **17.** $\sin^{-1}(\frac{1}{2})$ **18.** $\tan^{-1} 0$

In Problems 19–24, evaluate to four significant digits using a calculator.

19. $\sin^{-1} 0.9103$ **20.** $\cos^{-1} 0.4038$ **21.** $\arctan 103.7$

22. $\tan^{-1} 43.09$ **23.** $\arccos 3.051$ **24.** $\arcsin 1.131$

In Problems 25–40, find exact values without using a calculator.

25. $\arcsin(-\sqrt{2}/2)$ **26.** $\arccos(-\frac{1}{2})$

27. $\tan^{-1}(-\sqrt{3})$ **28.** $\tan^{-1}(-1)$

29. $\tan(\tan^{-1} 25)$ **30.** $\sin[\sin^{-1}(-0.6)]$

31. $\cos^{-1}(\cos 2.3)$ **32.** $\tan^{-1}[\tan(-1.5)]$

33. $\sin[\cos^{-1}(\sqrt{3}/2)]$ **34.** $\tan[\cos^{-1}(\frac{1}{2})]$

35. $\csc[\tan^{-1}(-1)]$ **36.** $\cos[\sin^{-1}(-\sqrt{2}/2)]$

37. $\sin^{-1}[\sin\pi]$ **38.** $\cos^{-1}[\cos(-\pi/2)]$

39. $\cos^{-1}[\cos(4\pi/3)]$ **40.** $\sin^{-1}[\sin(5\pi/4)]$

In Problems 41–46, evaluate to four significant digits using a calculator.

41. $\arctan(-10.04)$ **42.** $\tan^{-1}(-4.038)$

43. $\cot[\cos^{-1}(-0.7003)]$ **44.** $\sec[\sin^{-1}(-0.0399)]$

45. $\sqrt{5 + \cos^{-1}(1 - \sqrt{2})}$ **46.** $\sqrt{2} + \tan^{-1}\sqrt[3]{5}$

In Problems 47–52, find the exact degree measure of each without the use of a calculator.

47. $\sin^{-1}(-\sqrt{2}/2)$ **48.** $\cos^{-1}(-\frac{1}{2})$ **49.** $\arctan(-\sqrt{3})$

50. $\arctan(-1)$ **51.** $\cos^{-1}(-1)$ **52.** $\sin^{-1}(-1)$

In Problems 53–58, find the degree measure of each to two decimal places using a calculator set in degree mode.

53. $\cos^{-1} 0.7253$ **54.** $\tan^{-1} 12.4304$

55. $\arcsin(-0.3662)$ **56.** $\arccos(-0.9206)$

57. $\tan^{-1}(-837)$ **58.** $\sin^{-1}(-0.7071)$

59. Evaluate $\sin^{-1}(\sin 2)$ with a calculator set in radian mode, and explain why this does or does not illustrate the inverse sine–sine identity.

60. Evaluate $\cos^{-1}[\cos(-0.5)]$ with a calculator set in radian mode, and explain why this does or does not illustrate the inverse cosine–cosine identity.

In Problems 61–68, determine whether the statement is true or false. Explain.

61. None of the six trigonometric functions is one-to-one.

62. Each of the six inverse trigonometric functions is one-to-one.

63. Each of the six inverse trigonometric functions is periodic.

64. Each of the six inverse trigonometric functions is bounded.

65. The function $y = \sin^{-1} x$ is odd.

66. The function $y = \cos^{-1} x$ is even.

67. None of the six inverse trigonometric functions has a turning point.

68. None of the six inverse trigonometric functions has a zero.

In Problems 69–76, sketch a graph of each function over the indicated interval.

69. $y = \sin^{-1} x, -1 \le x \le 1$

70. $y = \cos^{-1} x, -1 \le x \le 1$

71. $y = \cos^{-1} (x/3), -3 \le x \le 3$

72. $y = \sin^{-1} (x/2), -2 \le x \le 2$

73. $y = \sin^{-1} (x - 2), 1 \le x \le 3$

74. $y = \cos^{-1} (x + 1), -2 \le x \le 0$

75. $y = \tan^{-1} (2x - 4), -2 \le x \le 6$

76. $y = \tan^{-1} (2x + 3), -5 \le x \le 2$

77. The identity $\cos (\cos^{-1} x) = x$ is valid for $-1 \le x \le 1$.
(A) Graph $y = \cos (\cos^{-1} x)$ for $-1 \le x \le 1$.
(B) What happens if you graph $y = \cos (\cos^{-1} x)$ over a larger interval, say $-2 \le x \le 2$? Explain.

78. The identity $\sin (\sin^{-1} x) = x$ is valid for $-1 \le x \le 1$.
(A) Graph $y = \sin (\sin^{-1} x)$ for $-1 \le x \le 1$.
(B) What happens if you graph $y = \sin (\sin^{-1} x)$ over a larger interval, say $-2 \le x \le 2$? Explain.

In Problems 79–82, write each expression as an algebraic expression in x free of trigonometric or inverse trigonometric functions.

79. $\cos (\sin^{-1} x)$

80. $\sin (\cos^{-1} x)$

81. $\cos (\arctan x)$

82. $\tan (\arcsin x)$

In Problems 83–88, find $f^{-1}(x)$. How must x be restricted in $f^{-1}(x)$?

83. $f(x) = \sin (x + 2), -2 - \pi/2 \le x \le -2 + \pi/2$

84. $f(x) = \cos (x - 1), 1 \le x \le 1 + \pi$

85. $f(x) = 5 + 2 \cos x, 0 \le x \le \pi$

86. $f(x) = 3 - 4 \sin x, -\pi/2 \le x \le \pi/2$

87. $f(x) = 4 + 2 \cos (x - 3), 3 \le x \le (3 + \pi)$

88. $f(x) = 3 + 5 \sin (x - 1), (1 - \pi/2) \le x \le (1 + \pi/2)$

89. The identity $\cos^{-1} (\cos x) = x$ is valid for $0 \le x \le \pi$.
(A) Graph $y = \cos^{-1} (\cos x)$ for $0 \le x \le \pi$.
(B) What happens if you graph $y = \cos^{-1} (\cos x)$ over a larger interval, say $-2\pi \le x \le 2\pi$? Explain.

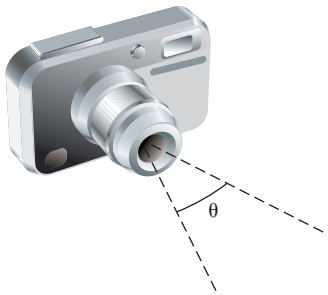
90. The identity $\sin^{-1} (\sin x) = x$ is valid for $-\pi/2 \le x \le \pi/2$.
(A) Graph $y = \sin^{-1} (\sin x)$ for $-\pi/2 \le x \le \pi/2$.
(B) What happens if you graph $y = \sin^{-1} (\sin x)$ over a larger interval, say $-2\pi \le x \le 2\pi$? Explain.

APPLICATIONS

91. **PHOTOGRAPHY** The viewing angle changes with the focal length of a camera lens. A 28-millimeter wide-angle lens has a wide viewing angle and a 300-millimeter telephoto lens has a narrow viewing angle. For a 35-millimeter format camera the viewing angle θ, in degrees, is given by

$$\theta = 2 \tan^{-1} \frac{21.634}{x}$$

where x is the focal length of the lens being used. What is the viewing angle (in decimal degrees to two decimal places) of a 28-millimeter lens? Of a 100-millimeter lens?



92. **PHOTOGRAPHY** Referring to Problem 91, what is the viewing angle (in decimal degrees to two decimal places) of a 17-millimeter lens? Of a 70-millimeter lens?

93. (A) Graph the function in Problem 91 in a graphing calculator using degree mode. The graph should cover lenses with focal lengths from 10 millimeters to 100 millimeters.
(B) What focal-length lens, to two decimal places, would have a viewing angle of 40°? Solve by graphing $\theta = 40$ and $\theta = 2 \tan^{-1} (21.634/x)$ in the same viewing window and using the INTERSECT command.

94. (A) Graph the function in Problem 91 in a graphing calculator, in degree mode, with the graph covering lenses with focal lengths from 100 millimeters to 1,000 millimeters.
(B) What focal length lens, to two decimal places, would have a viewing angle of 10°? Solve by graphing $\theta = 10$ and $\theta = \tan^{-1}(21.634/x)$ in the same viewing window and using the INTERSECT command.

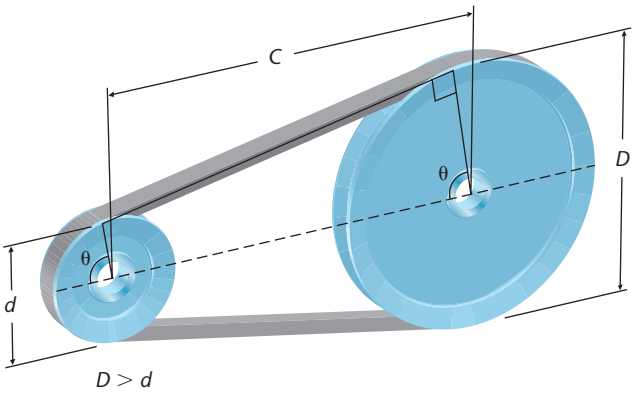
95. ENGINEERING The length of the belt around the two pulleys in the figure is given by

$$L = \pi D + (d - D)\theta + 2C \sin\theta$$

where θ (in radians) is given by

$$\theta = \cos^{-1}\frac{D - d}{2C}$$

Verify these formulas, and find the length of the belt to two decimal places if $D = 4$ inches, $d = 2$ inches, and $C = 6$ inches.



96. ENGINEERING For Problem 95, find the length of the belt if $D = 6$ inches, $d = 4$ inches, and $C = 10$ inches.

97. ENGINEERING The function

$$y_1 = 4\pi - 2\cos^{-1}\frac{1}{x} + 2x\sin\left(\cos^{-1}\frac{1}{x}\right)$$

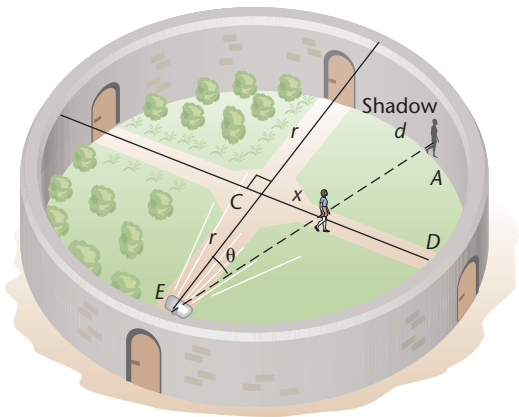
represents the length of the belt around the two pulleys in Problem 95 when the centers of the pulleys are x inches apart.
(A) Graph y_1 in a graphing calculator (in radian mode), with the graph covering pulleys with their centers from 3 to 10 inches apart.
(B) How far, to two decimal places, should the centers of the two pulleys be placed to use a belt 24 inches long? Solve by graphing y_1 and $y_2 = 24$ in the same viewing window and using the INTERSECT command.

98. ENGINEERING The function

$$y_1 = 6\pi - 2\cos^{-1}\frac{1}{x} + 2x\sin\left(\cos^{-1}\frac{1}{x}\right)$$

represents the length of the belt around the two pulleys in Problem 96 when the centers of the pulleys are x inches apart.
(A) Graph y_1 in a graphing calculator (in radian mode), with the graph covering pulleys with their centers from 3 to 20 inches apart.
(B) How far, to two decimal places, should the centers of the two pulleys be placed to use a belt 36 inches long? Solve by graphing y_1 and $y_2 = 36$ in the same viewing window and using the INTERSECT command.

99. MOTION The figure represents a circular courtyard surrounded by a high stone wall. A floodlight located at E shines into the courtyard.



(A) If a person walks x feet away from the center along DC, show that the person's shadow will move a distance given by

$$d = 2r\theta = 2r\tan^{-1}\frac{x}{r}$$

where θ is in radians. [*Hint:* Draw a line from A to C.]
(B) Find d to two decimal places if $r = 100$ feet and $x = 40$ feet.

100. MOTION In Problem 99, find d for $r = 50$ feet and $x = 25$ feet.

CHAPTER 5 Review

5-1 Angles and Their Measure

An **angle** is formed by rotating (in a plane) a ray m, called the **initial side** of the angle, around its endpoint until it coincides with a ray n, called the **terminal side** of the angle. The common endpoint of m and n is called the **vertex.** If the rotation is counterclockwise, the angle is **positive;** if clockwise, **negative.** Two angles are **coterminal** if they have the same initial and terminal sides.

An angle is in **standard position** in a rectangular coordinate system if its vertex is at the origin and its initial side is along the positive x axis. **Quadrantal angles** have their terminal sides on a coordinate axis. An angle of **1 degree** is $\frac{1}{360}$ of a complete rotation. An angle of **1 radian** is a central angle of a circle subtended by an arc having the same length as the radius. If θ is a positive angle formed by the central angle of a circle, then its **radian measure** is given by

$$\theta = \frac{s}{r} \text{ radians}$$

where s is the length of arc opposite angle θ and r is the radius of the circle.

Radian–degree conversion: $\dfrac{\theta_{\text{deg}}}{180^\circ} = \dfrac{\theta_{\text{rad}}}{\pi \text{ radians}}$

If a point P moves through an angle θ and arc length s, in time t, on the circumference of a circle of radius r, then the (average) **linear speed** of P is

$$v = \frac{s}{t}$$

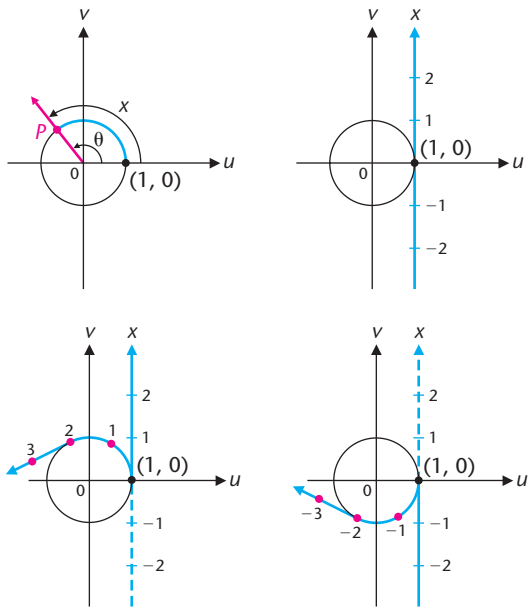
and the (average) **angular speed** is

$$\omega = \frac{\theta}{t}$$

Because $s = r\theta$ it follows that $v = r\omega$.

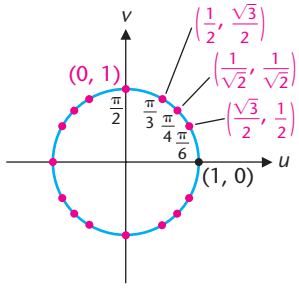
5-2 Trigonometric Functions: A Unit Circle Approach

If θ is a positive angle in standard position, and P is the point of intersection of the terminal side of θ with the unit circle, then the radian measure of θ equals the length x of the arc opposite θ; and if θ is negative, the radian measure of θ equals the negative of the length of the intercepted arc. The function W that associates with each real number x the point $W(x) = P$ is called the **wrapping function,** and the point P is called a **circular point.** The function $W(x)$ can be visualized as a wrapping of the real number line, with origin at (1, 0), around the unit circle—the positive real axis is wrapped counterclockwise and the negative real axis is wrapped clockwise—so that each real number is paired with a unique circular point. The function $W(x)$ is not one-to-one: for example, each of the real numbers $2\pi k$, k any integer, corresponds to the circular point (1, 0).



The coordinates of key circular points in the first quadrant can be found using simple geometric facts; the coordinates of the circular point associated with any multiple of $\pi/6$ or $\pi/4$ can then be determined by reflection through the x axis, y axis, or origin, as appropriate.

Coordinates of Key Circular Points

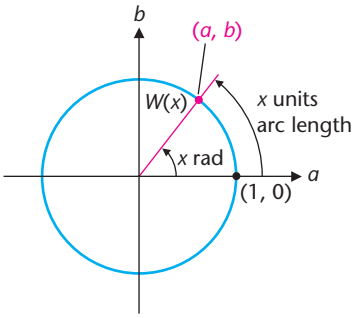


The six **trigonometric functions—sine, cosine, tangent, cotangent, secant,** and **cosecant**—are defined in terms of the coordinates (a, b) of the circular point $W(x)$ that lies on the terminal side of the angle with radian measure x:

$$\sin x = b \qquad \csc x = \frac{1}{b} \quad b \neq 0$$

$$\cos x = a \qquad \sec x = \frac{1}{a} \quad a \neq 0$$

$$\tan x = \frac{b}{a} \quad a \neq 0 \qquad \cot x = \frac{a}{b} \quad b \neq 0$$

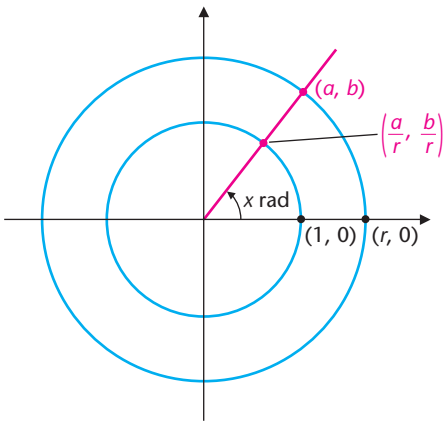


More generally, if (a, b) lies on the terminal side of the angle with radian measure x, and on the circle of radius $r > 0$, then:

$$\sin x = \frac{b}{r} \qquad \csc x = \frac{r}{b} \quad b \neq 0$$

$$\cos x = \frac{a}{r} \qquad \sec x = \frac{r}{a} \quad a \neq 0$$

$$\tan x = \frac{b}{a} \quad a \neq 0 \qquad \cot x = \frac{a}{b} \quad b \neq 0$$



The trigonometric functions of any multiple of $\pi/6$ or $\pi/4$ can be determined exactly from the coordinates of the circular point. A graphing calculator can be used to graph the trigonometric functions and approximate their values at arbitrary angles.

5-3 Solving Right Triangles

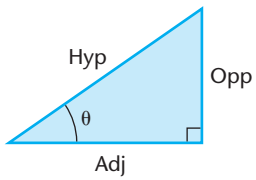
A **right triangle** is a triangle with one 90° angle. To **solve a right triangle** is to find all unknown angles and sides, given the measures of two sides or the measures of one side and an acute angle.

Trigonometric Functions of Acute Angles

$$\sin \theta = \frac{\text{Opp}}{\text{Hyp}} \qquad \csc \theta = \frac{\text{Hyp}}{\text{Opp}}$$

$$\cos \theta = \frac{\text{Adj}}{\text{Hyp}} \qquad \sec \theta = \frac{\text{Hyp}}{\text{Adj}}$$

$$\tan \theta = \frac{\text{Opp}}{\text{Adj}} \qquad \cot \theta = \frac{\text{Adj}}{\text{Opp}}$$



Computational Accuracy

Angle to nearest	Significant digits for side measure
1°	2
10′ or 0.1°	3
1′ or 0.01°	4
10″ or 0.001°	5

5-4 Properties of Trigonometric Functions

The definition of the trigonometric functions implies that the following **basic identities** hold true for all replacements of x by real numbers for which both sides of an equation are defined:

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Identities for Negatives

$$\sin(-x) = -\sin x \quad \cos(-x) = \cos x \quad \tan(-x) = -\tan x$$

Pythagorean identity

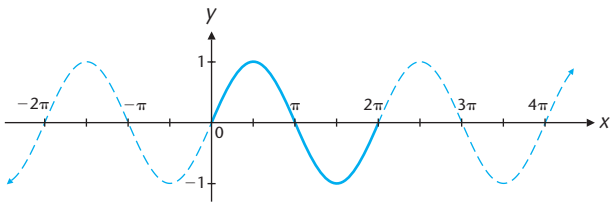
$$\sin^2 x + \cos^2 x = 1$$

A function f is **periodic** if there exists a positive real number p such that

$$f(x + p) = f(x)$$

for all x in the domain of f. The smallest such positive p, if it exists, is called the **fundamental period of f,** or often just the **period of f.** All the trigonometric functions are periodic.

Graph of $y = \sin x$:

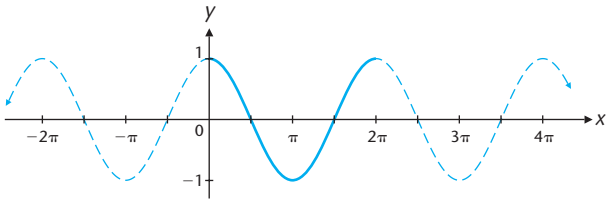


Period: 2π

Domain: All real numbers

Range: $[-1, 1]$

Graph of $y = \cos x$:

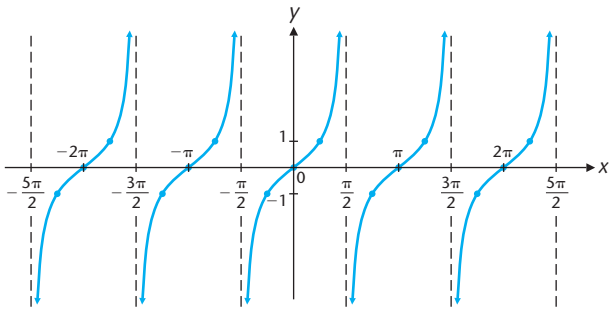


Period: 2π

Domain: All real numbers

Range: $[-1, 1]$

Graph of $y = \tan x$:

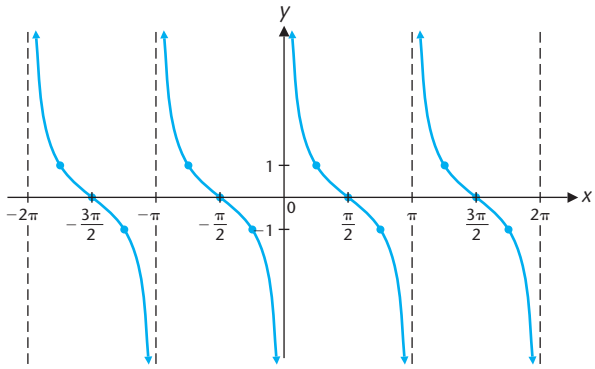


Period: π

Domain: All real numbers except $\pi/2 + k\pi$, k an integer

Range: All real numbers

Graph of $y = \cot x$:

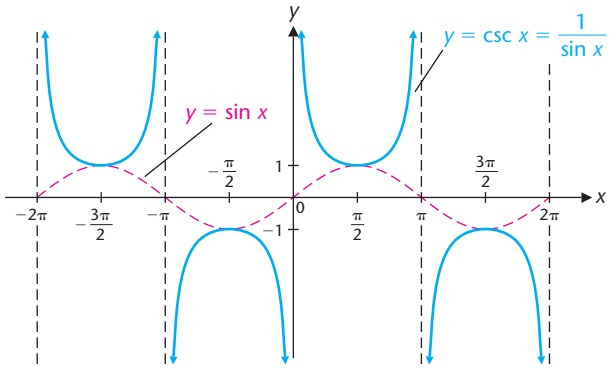


Period: π

Domain: All real numbers except $k\pi$, k an integer

Range: All real numbers

Graph of $y = \csc x$:

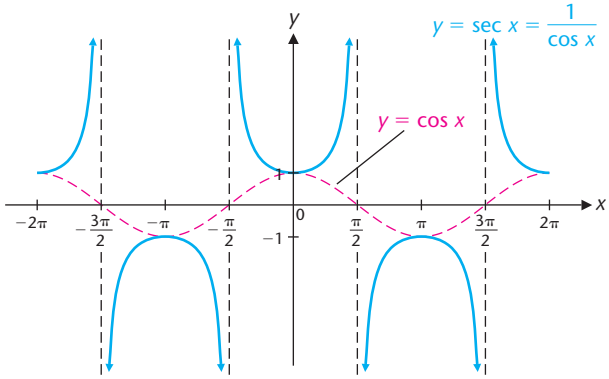


Period: 2π

Domain: All real numbers except $k\pi$, k an integer

Range: All real numbers y such that $y \le -1$ or $y \ge 1$

Graph of $y = \sec x$:

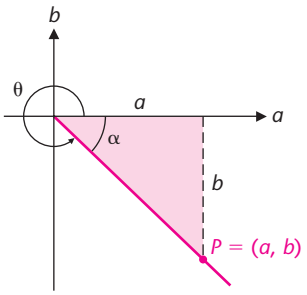


Period: 2π

Domain: All real numbers except $\pi/2 + k\pi$, k an integer

Range: All real numbers y such that $y \le -1$ or $y \ge 1$

Associated with each angle that does not terminate on a coordinate axis is a **reference triangle** for θ. The reference triangle is formed by drawing a perpendicular from point $P = (a, b)$ on the terminal side of θ to the horizontal axis. The **reference angle** α is the acute angle, always taken positive, between the terminal side of θ and the horizontal axis as indicated in the following figure.



Reference Triangle
$(a, b) \neq (0, 0)$
α is always positive.

5-5 More General Trigonometric Functions and Models

Let A, B, C be constants such that $A \neq 0$ and $B > 0$.
If $y = A \sin(Bx + C)$ or $y = A \cos(Bx + C)$:

$$\text{Amplitude} = |A| \qquad \text{Period} = \frac{2\pi}{B} \qquad \text{Phase shift} = \frac{-C}{B}$$

If $y = A \sec(Bx + C)$ or $y = \csc(Bx + C)$:

$$\text{Period} = \frac{2\pi}{B} \qquad \text{Phase shift} = \frac{-C}{B}$$

If $y = A \tan(Bx + C)$ or $y = A \cot(Bx + C)$:

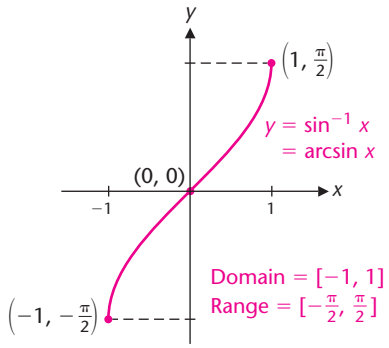
$$\text{Period} = \frac{\pi}{B} \qquad \text{Phase shift} = \frac{-C}{B}$$

(Amplitude is not defined for the secant, cosecant, tangent, and cotangent functions, all of which are unbounded.)

Sinusoidal regression is used to find the function of the form $y = A \sin(Bx + C) + k$ that best fits a set of data points.

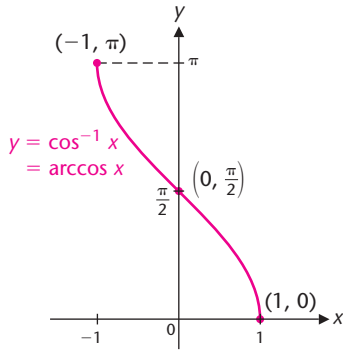
5-6 Inverse Trigonometric Functions

$y = \sin^{-1} x = \arcsin x$ if and only if $\sin y = x$, $-\pi/2 \leq y \leq \pi/2$ and $-1 \leq x \leq 1$.



Inverse sine function

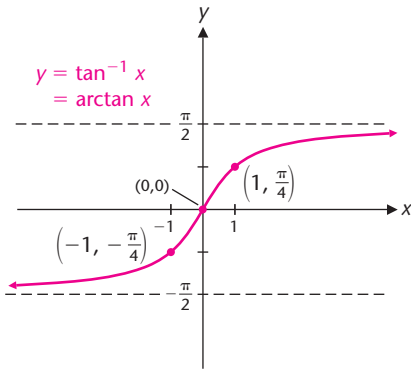
$y = \cos^{-1} x = \arccos x$ if and only if $\cos y = x$, $0 \leq y \leq \pi$ and $-1 \leq x \leq 1$.



Domain = [−1, 1]
Range = [0, π]

Inverse cosine function

$y = \tan^{-1} x = \arctan x$ if and only if $\tan y = x$, $-\pi/2 < y < \pi/2$ and x is any real number.



Domain = $(-\infty, \infty)$
Range = $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Inverse tangent function

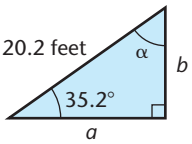
CHAPTER 5 Review Exercises

Work through all the problems in this chapter review and check answers in the back of the book. Answers to all review problems are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

1. Find the radian measure of a central angle opposite an arc 15 centimeters long on a circle of radius 6 centimeters.

2. In a circle of radius 3 centimeters, find the length of an arc opposite an angle of 2.5 radians.

3. Solve the triangle:



4. Find the reference angle associated with each angle θ.

(A) $\theta = \pi/3$ (B) $\theta = -120°$

(C) $\theta = -13\pi/6$ (D) $\theta = 210°$

5. In which quadrants is each negative?

(A) sin θ (B) cos θ (C) tan θ

6. If the circular point $W(x)$ has coordinates $(\frac{12}{13}, -\frac{5}{13})$, find

(A) cos x (B) csc x (C) cot x

7. If $(4, -3)$ is on the terminal side of angle θ, find

(A) sin θ (B) sec θ (C) cot θ

8. Complete Table 1 using exact values. Do not use a calculator.

Table 1

θ°	θ rad	sin θ	cos θ	tan θ	csc θ	sec θ	cot θ
0°					ND*		
30°							
45°	π/4		$1/\sqrt{2}$				
60°							
90°							
180°							
270°							
360°							

*ND = Not defined

9. What is the period of each of the following?

(A) $y = \cos x$ (B) $y = \csc x$ (C) $y = \tan x$

10. Indicate the domain and range of each.

(A) $y = \sin x$ (B) $y = \tan x$

11. Sketch the graph of $y = \sin x$, $-2\pi \le x \le 2\pi$.

12. Sketch the graph of $y = \cot x$, $-\pi < x < \pi$.

13. Verbally describe the meaning of a central angle in a circle with radian measure 0.5.

14. Describe the smallest shift of the graph of $y = \sin x$ that produces the graph of $y = \cos x$.

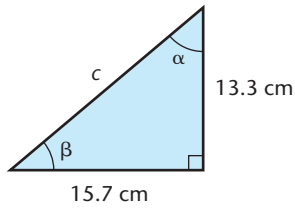
15. Convert 132°52′41″ to decimal degrees to three decimal places.

16. Convert 13.762 radians to degree-minute-second form.

17. Convert 64°28′14″ to radians to three decimal places.

18. Convert 1.37 radians to decimal degrees to two decimal places.

19. Solve the triangle:



20. Indicate whether the angle is a quadrant I, II, III, or IV angle or a quadrantal angle.

(A) −210° (B) $5\pi/2$ (C) 4.2 radians

21. Which of the following angles are coterminal with 120°?

(A) −240° (B) $-7\pi/6$ (C) 840°

22. Which of the following have the same value as cos 3?

(A) cos 3° (B) cos (3 radians) (C) cos $(3 + 2\pi)$

23 For which values of x, $0 \le x \le 2\pi$, is each of the following not defined?

(A) tan x (B) cot x (C) csc x

24. A circular point $P = (a, b)$ moves clockwise around the circumference of a unit circle starting at (1, 0) and stops after covering a distance of 8.305 units. Explain how you would find the coordinates of point P at its final position

and how you would determine which quadrant P is in. Find the coordinates of P to three decimal places and the quadrant for the final position of P.

25. Find the value of each of the other five trigonometric functions for an angle θ given that $\tan\theta = 4$ and $\cos\theta < 0$.

26. Find all zeros and turning points of $y = 3\cos x$ on $[-3\pi/2, 3\pi/2]$.

In Problems 27–42, evaluate exactly without the use of a calculator.

27. $\tan 0$

28. $\sec 90°$

29. $\cos^{-1} 1$

30. $\cos\left(-\frac{3\pi}{4}\right)$

31. $\sin^{-1}\frac{\sqrt{2}}{2}$

32. $\csc 300°$

33. $\arctan\sqrt{3}$

34. $\sin 570°$

35. $\tan^{-1}(-1)$

36. $\cot\left(-\frac{4\pi}{3}\right)$

37. $\arcsin\left(-\frac{1}{2}\right)$

38. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

39. $\cos(\cos^{-1} 0.33)$

40. $\csc[\tan^{-1}(-1)]$

41. $\sin\left[\arccos\left(-\frac{1}{2}\right)\right]$

42. $\tan\left(\sin^{-1}\frac{-4}{5}\right)$

Evaluate Problems 43–50 to four significant digits using a calculator.

43. $\cos 423.7°$

44. $\tan 93°46'17''$

45. $\sec(-2.073)$

46. $\sin^{-1}(-0.8277)$

47. $\arccos(-1.3281)$

48. $\tan^{-1} 75.14$

49. $\csc[\cos^{-1}(-0.4081)]$

50. $\sin^{-1}(\tan 1.345)$

51. Find the exact degree measure of each without a calculator.

(A) $\theta = \sin^{-1}(-\frac{1}{2})$ (B) $\theta = \arccos(-\frac{1}{2})$

52. Find the degree measure of each to two decimal places using a calculator.

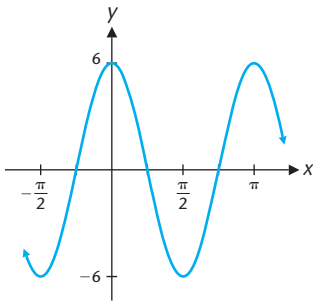
(A) $\theta = \cos^{-1}(-0.8763)$ (B) $\theta = \arctan 7.3771$

53. Evaluate $\cos^{-1}[\cos(-2)]$ with a calculator set in radian mode, and explain why this does or does not illustrate the inverse cosine–cosine identity.

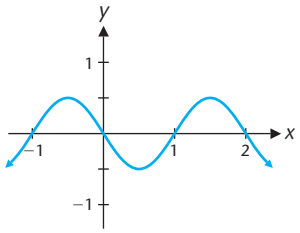
54. Sketch a graph of $y = -2\cos\pi x$, $-1 \le x \le 3$. Indicate amplitude A and period P.

55. Sketch a graph of $y = -2 + 3\sin(x/2)$, $-4\pi \le x \le 4\pi$.

56. Find the equation of the form $y = A\cos Bx$ that has the graph shown here.



57. Find the equation of the form $y = A\sin Bx$ that has the graph shown here.



58. Describe the smallest shift and/or reflection that transforms the graph of $y = \tan x$ into the graph of $y = \cot x$.

59. Simplify each of the following using appropriate basic identities:

(A) $\sin(-x)\cot(-x)$ (B) $\dfrac{\sin^2 x}{1 - \sin^2 x}$

60. Sketch a graph of $y = 3\sin[(x/2) + (\pi/2)]$ over the interval $-4\pi \le x \le 4\pi$.

61. Indicate the amplitude A, period P, and phase shift for the graph of $y = -2\cos[(\pi/2)x - (\pi/4)]$. Do not graph.

62. Sketch a graph of $y = \cos^{-1} x$, and indicate the domain and range.

63. Graph $y = 1/(1 + \tan^2 x)$ in a graphing calculator that displays at least two full periods of the graph. Find an equation of the form $y = k + A\sin Bx$ or $y = k + A\cos Bx$ that has the same graph.

64. Graph each equation in a graphing calculator and find an equation of the form $y = A\tan Bx$ or $y = A\cot Bx$ that has the same graph as the given equation. Select the dimensions of the viewing window so that at least two periods are visible.

(A) $y = \dfrac{2\sin^2 x}{\sin 2x}$ (B) $y = \dfrac{2\cos^2 x}{\sin 2x}$

65. Determine whether each function is even, odd, or neither.

(A) $f(x) = \dfrac{1}{1 + \tan^2 x}$ (B) $g(x) = \dfrac{1}{1 + \tan x}$

In Problems 66 and 67, determine whether the statement is true or false. If true, explain why. If false, give a counterexample.

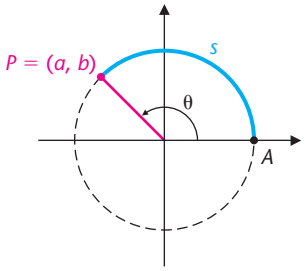
66. If α and β are the acute angles of a right triangle, then $\sin \alpha = \csc \beta$.

67. If α and β are the acute angles of a right triangle and $\alpha = \beta$, then all six trigonometric functions of α are greater than $\frac{1}{2}$ and less than $\frac{3}{2}$.

68. If in the figure the coordinates of A are (8, 0) and arc length s is 20 units, find:

(A) The exact radian measure of θ

(B) The coordinates of P to three significant digits



69. Find exactly the least positive real number for which

(A) $\cos x = -\frac{1}{2}$ (B) $\csc x = -\sqrt{2}$

70. Sketch a graph of $y = \sec x$, $-\pi/2 < x < 3\pi/2$.

71. Sketch a graph of $y = \tan^{-1} x$, and indicate the domain and range.

72. Indicate the period P and phase shift for the graph of $y = -5 \tan(\pi x + \pi/2)$. Do not graph.

73. Indicate the period and phase shift for the graph of $y = 3 \csc(x/2 - \pi/4)$. Do not graph.

74. Indicate whether each is symmetrical with respect to the x axis, y axis, or origin.

(A) Sine (B) Cosine (C) Tangent

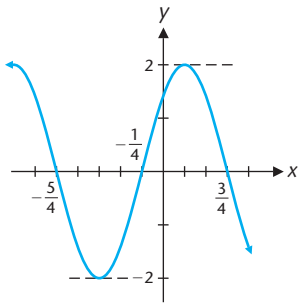
75. Write as an algebraic expression in x free of trigonometric or inverse trigonometric functions:

$$\sec(\sin^{-1} x)$$

76. Try to calculate each of the following on your calculator. Explain the results.

(A) $\csc(-\pi)$ (B) $\tan(-3\pi/2)$ (C) $\sin^{-1} 2$

77. The accompanying graph is a graph of an equation of the form $y = A \sin(Bx + C)$, $-1 < -C/B < 0$. Find the equation.



78. Graph $y = 1.2 \sin 2x + 1.6 \cos 2x$ in a graphing calculator. (Select the dimensions of the viewing window so that at least two periods are visible.) Find an equation of the form $y = A \sin(Bx + C)$ that has the same graph as the given equation. Find A and B exactly and C to three decimal places. Use the x intercept closest to the origin as the phase shift.

79. At what values of x, $-\pi \le x \le \pi$, do the vertical asymptotes of $y = 2 \tan 3x$ cross the x axis?

APPLICATIONS

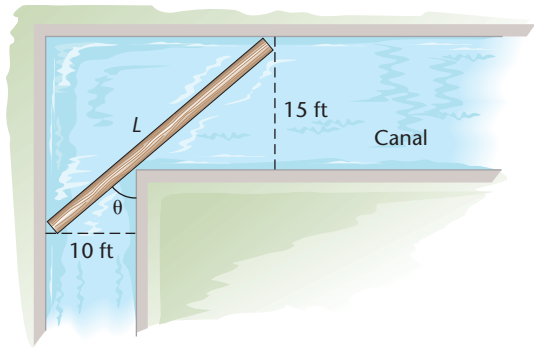
80. **ASTRONOMY** A line from the sun to the Earth sweeps out an angle of how many radians in 73 days? Express the answer in terms of π.

81. **GEOMETRY** Find the perimeter of a square inscribed in a circle of radius 5.00 centimeters.

82. **ANGULAR SPEED** A wind turbine of rotor diameter 40 feet makes 80 revolutions per minute. Find the angular speed (in radians per second) and the linear speed (in feet per second) of the rotor tip.

83. **ALTERNATING CURRENT** The current I in alternating electrical current has an amplitude of 30 amperes and a period of $\frac{1}{60}$ second. If $I = 30$ amperes when $t = 0$, find an equation of the form $I = A \cos Bt$ that gives the current at any time $t \ge 0$.

84. **RESTRICTED ACCESS** A 10-foot-wide canal makes a right turn into a 15-foot-wide canal. Long narrow logs are to be floated through the canal around the right angle turn (see the figure). We are interested in finding the longest log that will go around the corner, ignoring the log's diameter.



(A) Express the length L of the line that touches the two outer sides of the canal and the inside corner in terms of θ.

(B) Complete Table 2, each to one decimal place, and estimate from the table the longest log to the nearest foot that can make it around the corner. (The longest log is the shortest distance L.)

Table 2

θ (radians)	0.4	0.5	0.6	0.7	0.8	0.9	1.0
L (feet)	42.0						

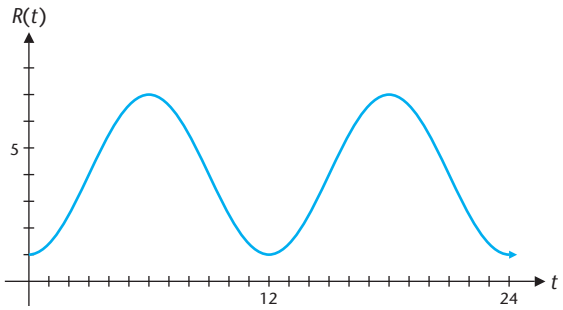
(C) Graph the function in part A in a graphing calculator and use the MINIMUM command to find the shortest distance L to one decimal place; that is, the length of the longest log that can make it around the corner.

(D) Explain what happens to the length L as θ approaches 0 or $\pi/2$.

85. MODELING SEASONAL BUSINESS CYCLES A soft drink company has revenues from sales over a 2-year period as shown by the accompanying graph, where $R(t)$ is revenue (in millions of dollars) for a month of sales t months after February 1.

(A) Find an equation of the form $R(t) = k + A \cos Bt$ that produces this graph, and check the result by graphing.

(B) Verbally interpret the graph



86. MODELING TEMPERATURE VARIATION The 30-year average monthly temperature, °F, for each month of the year for Los Angeles is given in Table 3 (*World Almanac*).

(A) Using 1 month as the basic unit of time, enter the data for a 2-year period in your graphing calculator and produce a scatter plot in the viewing window. Choose $40 \le y \le 90$ for the viewing window.

(B) It appears that a sine curve of the form

$$y = k + A \sin (Bx + C)$$

will closely model these data. The constants k, A, and B are easily determined from Table 3. To estimate C, visually estimate to one decimal place the smallest positive phase shift from the plot in part A. After determining A, B, k, and C, write the resulting equation. (Your value of C may differ slightly from the answer at the back of the book.)

(C) Plot the results of parts A and B in the same viewing window. (An improved fit may result by adjusting your value of C slightly.)

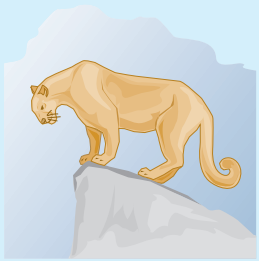
(D) If your graphing calculator has a sinusoidal regression feature, check your results from parts B and C by finding and plotting the regression equation.

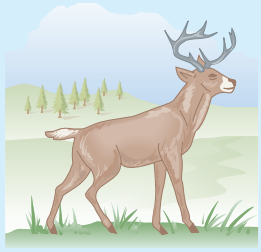
Table 3

x (months)	y (temperature)
1	58
2	60
3	61
4	63
5	66
6	70
7	74
8	75
9	74
10	70
11	63
12	58

CHAPTER 5

››› GROUP ACTIVITY A Predator–Prey Analysis Involving Mountain Lions and Deer





In some western state wilderness areas, deer and mountain lion populations are interrelated, because the mountain lions rely on the deer as a food source. The population of each species goes up and down in cycles, but out of phase with each other. A wildlife management research team estimated the respective populations in a particular region every 2 years over a 16-year period, with the results shown in Table 1.

Table 1 Mountain Lion/Deer Populations

Years	0	2	4	6	8	10	12	14	16
Deer	1,272	1,523	1,152	891	1,284	1,543	1,128	917	1,185
Mountain Lions	39	47	63	54	37	48	60	46	40

(A) Deer Population Analysis

1. Enter the data for the deer population for the time interval [0, 16] in a graphing calculator and produce a scatter plot of the data.
2. A function of the form $y = k + A \sin(Bx + C)$ can be used to model these data. Use the data in Table 1 to determine k, A, and B. Use the graph in part 1 to visually estimate C to one decimal place.
3. Plot the data from part 1 and the equation from part 2 in the same viewing window. If necessary, adjust the value of C for a better fit.
4. If your graphing calculator has a sinusoidal regression feature, check your results from parts 2 and 3 by finding and plotting the regression equation.
5. Write a summary of the results, describing fluctuations and cycles of the deer population.

(B) Mountain Lion Population Analysis

1. Enter the data for the mountain lion population for the time interval [0, 16] in a graphing calculator and produce a scatter plot of the data.
2. A function of the form $y = k + A \sin(Bx + C)$ can be used to model these data. Use the data in Table 1 to determine k, A, and B. Use the graph in part 1 to visually estimate C to one decimal place.

3. Plot the data from part 1 and the equation from part 2 in the same viewing window. If necessary, adjust the value of C for a better fit.
4. If your graphing calculator has a sinusoidal regression feature, check your results from parts 2 and 3 by finding and plotting the regression equation.
5. Write a summary of the results, describing fluctuations and cycles of the mountain lion population.

(C) Interrelationship of the Two Populations

1. Discuss the relationship of the maximum predator populations to the maximum prey populations relative to time.
2. Discuss the relationship of the minimum predator populations to the minimum prey populations relative to time.
3. Discuss the dynamics of the fluctuations of the two interdependent populations. What causes the two populations to rise and fall, and why are they out of phase with one another?

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Trigonometric Identities and Conditional Equations

TRIGONOMETRIC functions are widely used in solving real-world problems and in the development of mathematics. Whatever their use, it is often important to change a trigonometric expression from one form to an equivalent, more useful form. This involves the use of identities. An equation in one or more variables is said to be an **identity** if the left side is equal to the right side for all replacements of the variables for which both sides are defined.

For example, the equation $\sin^2 x + \cos^2 x = 1$ is an identity, but the equation $\sin x + \cos x = 1$ is not. The latter equation is called a **conditional equation,** because it holds for certain values of x (for example, $x = 0$ and $x = \pi/2$) but not for other values for which both sides are defined (for example, $x = \pi/4$). The first four sections of this chapter deal with trigonometric identities, and the last section with conditional trigonometric equations.

CHAPTER

6

OUTLINE

6-1 Basic Identities and Their Use

6-2 Sum, Difference, and Cofunction Identities

6-3 Double-Angle and Half-Angle Identities

6-4 Product–Sum and Sum–Product Identities

6-5 Trigonometric Equations

Chapter 6 Review

Chapter 6 Group Activity: From $M \sin Bt + N \cos Bt$ to $A \sin(Bt + C)$—A Harmonic Analysis Tool

6-1 Basic Identities and Their Use

- Basic Identities
- Establishing Other Identities

In this section, we review the basic identities introduced in Section 5-4 and show how they are used to establish other identities.

Basic Identities

In the box we list for convenient reference the basic identities introduced in Section 5-4. These identities will be used very frequently in the work that follows and should be memorized.

BASIC TRIGONOMETRIC IDENTITIES

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Identities for Negatives

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

All these identities, with the exception of the second and third Pythagorean identities, were established in Section 5-4. The two exceptions can be derived from the first Pythagorean identity (see Explore-Discuss 1 and Problems 109 and 110 in Exercises 6-1).

››› EXPLORE-DISCUSS 1

Discuss an easy way to recall the second and third Pythagorean identities from the first. [*Hint:* Divide both sides of the first Pythagorean identity by $\cos^2 x$ or $\sin^2 x$.]

› Establishing Other Identities

Identities are established to convert one form to an equivalent form that may be more useful. To *verify an identity* means to prove that both sides of an equation are equal for all replacements of the variables for which both sides are defined. Such a proof might use basic identities or other verified identities and algebraic operations such as multiplication, factoring, combining and reducing fractions, and so on. Examples 1 through 6 illustrate some of the techniques used to verify certain identities. The steps illustrated are not necessarily unique—often, there is more than one path to a desired goal. To become proficient in the use of identities, it is important that you work out many problems on your own.

EXAMPLE 1 Identity Verification

Verify the identity $\cos x \tan x = \sin x$.

VERIFICATION

Generally, we proceed by starting with the more complicated of the two sides, and transform that side into the other side in one or more steps using basic identities, algebra, or other established identities. Here we start with the left side and use a quotient identity to rewrite $\tan x$:

$$\cos x \tan x = \cos x \frac{\sin x}{\cos x} \qquad \text{Use algebra.}$$
$$= \sin x$$

MATCHED PROBLEM 1

Verify the identity $\sin x \cot x = \cos x$.

››› EXPLORE-DISCUSS 2

Graph the left and right sides of the identity in Example 1 in a graphing calculator by letting $y_1 = \cos x \tan x$ and $y_2 = \sin x$. Use TRACE, moving back and forth between the graphs of y_1 and y_2, to compare values of y for given values of x. What does this investigation illustrate?

EXAMPLE 2 Identity Verification

Verify the identity $\sec(-x) = \sec x$.

VERIFICATION

We start with the left side and use a reciprocal identity:

$$\begin{aligned} \sec(-x) &= \frac{1}{\cos(-x)} && \text{Use an identity for negatives.} \\ &= \frac{1}{\cos x} && \text{Use a reciprocal identity.} \\ &= \sec x \end{aligned}$$

MATCHED PROBLEM 2

Verify the identity $\csc(-x) = -\csc x$.

EXAMPLE 3 Identity Verification

Verify the identity $\cot x \cos x + \sin x = \csc x$.

VERIFICATION

We start with the left side and use a quotient identity to rewrite $\cot x$:

$$\begin{aligned} \cot x \cos x + \sin x &= \frac{\cos x}{\sin x}\cos x + \sin x && \text{Use algebra.} \\ &= \frac{\cos^2 x}{\sin x} + \sin x && \text{Write as a single fraction.} \\ &= \frac{\cos^2 x + \sin^2 x}{\sin x} && \text{Use a Pythagorean identity.} \\ &= \frac{1}{\sin x} && \text{Use a reciprocal identity.} \\ &= \csc x \end{aligned}$$

KEY ALGEBRAIC STEPS IN EXAMPLE 3

$$\frac{a}{b}a + b = \frac{a^2}{b} + b = \frac{a^2 + b^2}{b}$$

MATCHED PROBLEM 3

Verify the identity $\tan x \sin x + \cos x = \sec x$.

To verify an identity, proceed from one side to the other, making sure all steps are reversible. Do not use properties of equality to perform the same operation on both sides of the equation. Although there is no fixed method of verification that works for all identities, there are certain steps that help in many cases.

SUGGESTED STEPS IN VERIFYING IDENTITIES

1. Start with the more complicated side of the identity, and transform it into the simpler side.
2. Try algebraic operations such as multiplying, factoring, combining fractions, and splitting fractions.
3. If other steps fail, express each function in terms of sine and cosine functions, and then perform appropriate algebraic operations.
4. At each step, keep the other side of the identity in mind. This often reveals what you should do to get there.

EXAMPLE 4 Identity Verification

Verify the identity $\dfrac{1 + \sin x}{\cos x} + \dfrac{\cos x}{1 + \sin x} = 2 \sec x$.

VERIFICATION

We start with the left side and write it as a single fraction:

$$\frac{1 + \sin x}{\cos x} + \frac{\cos x}{1 + \sin x} = \frac{(1 + \sin x)^2 + \cos^2 x}{\cos x\,(1 + \sin x)} \qquad \text{Expand the numerator.}$$

$$= \frac{1 + 2 \sin x + \sin^2 x + \cos^2 x}{\cos x\,(1 + \sin x)} \qquad \text{Use a Pythagorean identity.}$$

$$= \frac{1 + 2 \sin x + 1}{\cos x\,(1 + \sin x)} \qquad \text{Simplify.}$$

$$= \frac{2 + 2 \sin x}{\cos x\,(1 + \sin x)} \qquad \text{Factor the numerator.}$$

$$= \frac{2\,(1 + \sin x)}{\cos x\,(1 + \sin x)} \qquad \text{Simplify.}$$

$$= \frac{2}{\cos x} \qquad \text{Use a reciprocal identity.}$$

$$= 2 \sec x$$

KEY ALGEBRAIC STEPS IN EXAMPLE 4

$$\frac{a}{b}+\frac{b}{a}=\frac{a^2+b^2}{ba} \qquad (1+c)^2=1+2c+c^2 \qquad \frac{m(a+b)}{n(a+b)}=\frac{m}{n}$$

MATCHED PROBLEM 4

Verify the identity $\dfrac{1+\cos x}{\sin x}+\dfrac{\sin x}{1+\cos x}=2\csc x$.

EXAMPLE 5 Identity Verification

Verify the identity $\dfrac{\sin^2 x+2\sin x+1}{\cos^2 x}=\dfrac{1+\sin x}{1-\sin x}$.

VERIFICATION

We start with the left side and factor its numerator:

$$\begin{aligned}
\frac{\sin^2 x+2\sin x+1}{\cos^2 x} &= \frac{(\sin x+1)^2}{\cos^2 x} && \text{Use a Pythagorean identity } (\cos^2 x = 1-\sin^2 x).\\
&= \frac{(\sin x+1)^2}{1-\sin^2 x} && \text{Factor the denominator.}\\
&= \frac{(1+\sin x)^2}{(1-\sin x)(1+\sin x)} && \text{Simplify.}\\
&= \frac{1+\sin x}{1-\sin x}
\end{aligned}$$

KEY ALGEBRAIC STEPS IN EXAMPLE 5

$$a^2+2a+1=(a+1)^2 \qquad 1-b^2=(1-b)(1+b)$$

MATCHED PROBLEM 5

Verify the identity $\sec^4 x-2\sec^2 x\tan^2 x+\tan^4 x=1$.

EXAMPLE 6 Identity Verification

Verify the identity $\dfrac{\tan x-\cot x}{\tan x+\cot x}=1-2\cos^2 x$.

VERIFICATION

We start with the left side and change to sines and cosines using quotient identities:

$$\frac{\tan x - \cot x}{\tan x + \cot x} = \frac{\dfrac{\sin x}{\cos x} - \dfrac{\cos x}{\sin x}}{\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}}$$

Multiply numerator and denominator by $(\sin x)(\cos x)$.

$$= \frac{(\sin x)(\cos x)\left(\dfrac{\sin x}{\cos x} - \dfrac{\cos x}{\sin x}\right)}{(\sin x)(\cos x)\left(\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}\right)}$$

Use algebra to transform the compound fraction into a simple fraction.

$$= \frac{\sin^2 x - \cos^2 x}{\sin^2 x + \cos^2 x}$$

Use Pythagorean identities.

$$= \frac{1 - \cos^2 x - \cos^2 x}{1}$$

Simplify.

$$= 1 - 2\cos^2 x$$

KEY ALGEBRAIC STEPS IN EXAMPLE 6

$$\frac{\dfrac{a}{b} - \dfrac{b}{a}}{\dfrac{a}{b} + \dfrac{b}{a}} = \frac{ab\left(\dfrac{a}{b} - \dfrac{b}{a}\right)}{ab\left(\dfrac{a}{b} + \dfrac{b}{a}\right)} = \frac{a^2 - b^2}{a^2 + b^2}$$

MATCHED PROBLEM 6

Verify the identity $\cot x - \tan x = \dfrac{2\cos^2 x - 1}{\sin x \cos x}$.

Just observing how others verify identities won't make you good at it. You must verify a large number on your own. With practice the process will seem less complicated.

EXAMPLE 7 Testing Identities Using a Graphing Calculator

Use a graphing calculator to test whether each of the following is an identity. If an equation appears to be an identity, verify it. If the equation does not appear to be an identity, find a value of x for which both sides are defined but are not equal.

(A) $\tan x + 1 = (\sec x)(\sin x - \cos x)$

(B) $\tan x - 1 = (\sec x)(\sin x - \cos x)$

SOLUTIONS

(A) Graph each side of the equation in the same viewing window (Fig. 1). The equation is not an identity, because the graphs do not match.

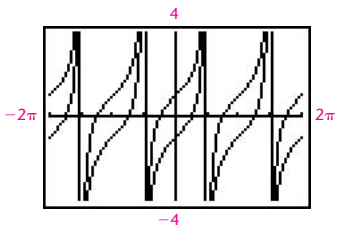
Try $x = 0$.

$$\text{Left side: } \tan 0 + 1 = 1$$
$$\text{Right side: } (\sec 0)(\sin 0 - \cos 0) = -1$$

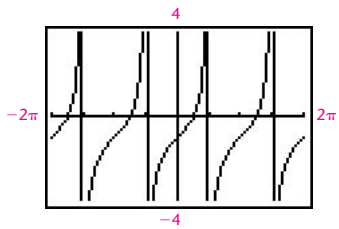
Finding one value of x for which both sides are defined, but are not equal, is enough to verify that the equation is not an identity.

(B) Graph each side of the equation in the same viewing window (Fig. 2). The equation appears to be an identity, which we now verify. We start with the right side and use a reciprocal identity to rewrite sec x:

$$\begin{aligned}(\sec x)(\sin x - \cos x) &= \left(\frac{1}{\cos x}\right)(\sin x - \cos x) && \text{Distribute.}\\ &= \left(\frac{\sin x}{\cos x} - \frac{\cos x}{\cos x}\right) && \text{Use a quotient identity.}\\ &= \tan x - 1\end{aligned}$$



› Figure 1



› Figure 2

MATCHED PROBLEM 7

Use a graphing calculator to test whether each of the following is an identity. If an equation appears to be an identity, verify it. If the equation does not appear to be an identity, find a value of x for which both sides are defined but are not equal. [Recall that $\cos^2 x$ should be entered in a graphing calculator as $(\cos(x))^2$ or $\cos(x)^2$, *not* as $\cos(x^2)$.]

(A) $\dfrac{\sin x}{1 - \cos^2 x} = \csc x$ (B) $\dfrac{\sin x}{1 - \cos^2 x} = \sec x$

ANSWERS TO MATCHED PROBLEMS

In the following identity verifications, other correct sequences of steps are possible—the process is not unique.

1. $\sin x \cot x = \sin x \dfrac{\cos x}{\sin x} = \cos x$

2. $\csc(-x) = \dfrac{1}{\sin(-x)} = \dfrac{1}{-\sin x} = -\csc x$

3. $\tan x \sin x + \cos x = \dfrac{\sin^2 x}{\cos x} + \cos x = \dfrac{\sin^2 x + \cos^2 x}{\cos x} = \dfrac{1}{\cos x} = \sec x$

4. $\dfrac{1+\cos x}{\sin x} + \dfrac{\sin x}{1+\cos x} = \dfrac{(1+\cos x)^2 + \sin^2 x}{\sin x(1+\cos x)}$

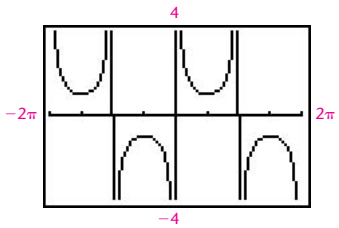
$= \dfrac{1 + 2\cos x + \cos^2 x + \sin^2 x}{\sin x(1+\cos x)} = \dfrac{2(1+\cos x)}{\sin x(1+\cos x)} = 2\csc x$

5. $\sec^4 x - 2\sec^2 x \tan^2 x + \tan^4 x = (\sec^2 x - \tan^2 x)^2 = 1^2 = 1$

6. $\cot x - \tan x = \dfrac{\cos x}{\sin x} - \dfrac{\sin x}{\cos x} = \dfrac{\cos^2 x - \sin^2 x}{\sin x \cos x}$

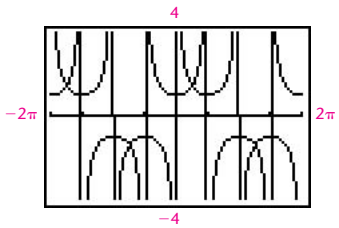
$= \dfrac{\cos^2 x - (1 - \cos^2 x)}{\sin x \cos x} = \dfrac{2\cos^2 x - 1}{\sin x \cos x}$

7. (A) An identity:



$\dfrac{\sin x}{1 - \cos^2 x} = \dfrac{\sin x}{\sin^2 x} = \dfrac{1}{\sin x} = \csc x$

(B) Not an identity: the left side is not equal to the right side for $x = \frac{\pi}{6}$, for example.



6-1 Exercises

1. What is an identity?

2. What is a conditional equation?

3. Can an equation be an identity if there is a value of x for which the left side is not equal to the right side? Explain.

4. Is $\sqrt{1-x} = \sqrt{x-1}$ an identity? Explain.

5. Use a graphing calculator to verify that the graphs of $y_1 = x$ and $y_2 = |x + 15| - 15$ are identical in the standard viewing window $-10 \le x \le 10$, $-10 \le y \le 10$. Is the equation $x = |x + 15| - 15$ an identity? Explain.

6. Explain why a graphing calculator can be used to prove that an equation is *not* an identity, but cannot be used to prove that an equation *is* an identity.

Verify that Problems 7–32 are identities.

7. $\sin\theta \sec\theta = \tan\theta$

8. $\cos\theta \csc\theta = \cot\theta$

9. $\cot u \sec u \sin u = 1$

10. $\tan\theta \csc\theta \cos\theta = 1$

11. $\dfrac{\sin(-x)}{\cos(-x)} = -\tan x$

12. $\cot(-x)\tan x = -1$

13. $\sin\alpha = \dfrac{\tan\alpha \cot\alpha}{\csc\alpha}$

14. $\tan\alpha = \dfrac{\cos\alpha \sec\alpha}{\cot\alpha}$

15. $\cot u + 1 = (\csc u)(\cos u + \sin u)$

16. $\tan u + 1 = (\sec u)(\sin u + \cos u)$

17. $\dfrac{\cos x - \sin x}{\sin x \cos x} = \csc x - \sec x$

18. $\dfrac{\cos^2 x - \sin^2 x}{\sin x \cos x} = \cot x - \tan x$

19. $\dfrac{\sin^2 t}{\cos t} + \cos t = \sec t$

20. $\dfrac{\cos^2 t}{\sin t} + \sin t = \csc t$

21. $\dfrac{\cos x}{1 - \sin^2 x} = \sec x$

22. $\dfrac{\sin u}{1 - \cos^2 u} = \csc u$

23. $(1 - \cos u)(1 + \cos u) = \sin^2 u$

24. $(1 - \sin t)(1 + \sin t) = \cos^2 t$

25. $\cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

26. $(\sin x + \cos x)^2 = 1 + 2\sin x \cos x$

27. $(\sec t + 1)(\sec t - 1) = \tan^2 t$

28. $(\csc t - 1)(\csc t + 1) = \cot^2 t$

29. $\csc^2 x - \cot^2 x = 1$

30. $\sec^2 u - \tan^2 u = 1$

31. $\cot x + \sec x = \dfrac{\cos x + \tan x}{\sin x}$

32. $\sin m\,(\csc m - \sin m) = \cos^2 m$

In Problems 33–40, show that the equation is not an identity by finding a value of x for which both sides are defined but are not equal.

33. $\sqrt{x^2} = x$

34. $\sqrt{x^2 + 4x + 4} = x + 2$

35. $x^4 + x^3 - x^2 - x = x^5 + x^4 - x^3 - x^2$

36. $x^5 - 3x^2 + 2x = x^6 - 3x^4 + 2x^2$

37. $\sin x \cos x = \tan x$

38. $\cot x \csc x = \sec x$

39. $\cos x = 1 - \sin x$

40. $1 + \sin x = \tan x + \sec x$

In Problems 41–44, graph all parts of each problem in the same viewing window in a graphing calculator.

41. $-\pi \le x \le \pi$
(A) $y = \sin^2 x$ (B) $y = \cos^2 x$
(C) $y = \sin^2 x + \cos^2 x$

42. $-\pi \le x \le \pi$
(A) $y = \sec^2 x$ (B) $y = \tan^2 x$
(C) $y = \sec^2 x - \tan^2 x$

43. $-\pi \le x \le \pi$
(A) $y = \dfrac{\cos x}{\cot x \sin x}$ (B) $y = 1$

44. $-\pi \le x \le \pi$
(A) $y = \dfrac{\sin x}{\cos x \tan x}$ (B) $y = 1$

In Problems 45–60, is the equation an identity? Explain.

45. $\dfrac{x^2 - 9}{x + 3} = x - 3$

46. $\dfrac{5x}{|x|} = 5$

47. $\sqrt{x^2 + 4x + 4} = x + 2$

48. $\dfrac{1}{\sqrt{x}} = \dfrac{\sqrt{x}}{|x|}$

49. $\sqrt{3 - x} + \sqrt{x - 3} = 0$

50. $\sqrt{5 - x} = \sqrt{x - 5}$

51. $\sqrt{4 - x} = \sqrt{x - 3}$

52. $x = |x + 100| - 100$

53. $\sin x - \cos x = 1$

54. $\sin x + \cos x = 1$

55. $\sin^2 x - \cos^2 x = 1$

56. $\sin^3 x + \cos^3 x = 1$

57. $\sin x = \dfrac{1}{\csc x}$

58. $\cos x = \dfrac{1}{\sec x}$

59. $\tan x = \dfrac{1}{\cot x}$

60. $\sin x = \dfrac{1}{\cos x}$

Verify that Problems 61–90 are identities.

61. $\dfrac{1 - (\sin x - \cos x)^2}{\sin x} = 2\cos x$

62. $\dfrac{1 - \cos^2 y}{(1 - \sin y)(1 + \sin y)} = \tan^2 y$

63. $\cos\theta + \sin\theta = \dfrac{\cot\theta + 1}{\csc\theta}$

64. $\sin\theta + \cos\theta = \dfrac{\tan\theta + 1}{\sec\theta}$

65. $\dfrac{1 + \cos y}{1 - \cos y} = \dfrac{\sin^2 y}{(1 - \cos y)^2}$

66. $1 - \sin y = \dfrac{\cos^2 y}{1 + \sin y}$

67. $\tan^2 x - \sin^2 x = \tan^2 x \sin^2 x$

68. $\sec^2 x + \csc^2 x = \sec^2 x \csc^2 x$

69. $\dfrac{\csc\theta}{\cot\theta + \tan\theta} = \cos\theta$

70. $\dfrac{1 + \sec\theta}{\sin\theta + \tan\theta} = \csc\theta$

71. $\ln(\tan x) = \ln(\sin x) - \ln(\cos x)$

72. $\ln(\cot x) = \ln(\cos x) - \ln(\sin x)$

73. $\ln(\cot x) = -\ln(\tan x)$

74. $\ln(\csc x) = -\ln(\sin x)$

75. $\dfrac{1 - \cos A}{1 + \cos A} = \dfrac{\sec A - 1}{\sec A + 1}$

76. $\dfrac{1 - \csc y}{1 + \csc y} = \dfrac{\sin y - 1}{\sin y + 1}$

77. $\sin^4 w - \cos^4 w = 1 - 2\cos^2 w$

78. $\sin^4 x + 2\sin^2 x \cos^2 x + \cos^4 x = 1$

79. $\sec x - \dfrac{\cos x}{1 + \sin x} = \tan x$

80. $\csc n - \dfrac{\sin n}{1 + \cos n} = \cot n$

81. $\dfrac{\cos^2 z - 3\cos z + 2}{\sin^2 z} = \dfrac{2 - \cos z}{1 + \cos z}$

82. $\dfrac{\sin^2 t + 4\sin t + 3}{\cos^2 t} = \dfrac{3 + \sin t}{1 - \sin t}$

83. $\dfrac{\cos^3 \theta - \sin^3 \theta}{\cos \theta - \sin \theta} = 1 + \sin \theta \cos \theta$

84. $\dfrac{\cos^3 u + \sin^3 u}{\cos u + \sin u} = 1 - \sin u \cos u$

85. $(\sec x - \tan x)^2 = \dfrac{1 - \sin x}{1 + \sin x}$

86. $(\cot u - \csc u)^2 = \dfrac{1 - \cos u}{1 + \cos u}$

87. $\dfrac{\csc^4 x - 1}{\cot^2 x} = 2 + \cot^2 x$

88. $\dfrac{\sec^4 x - 1}{\tan^2 x} = 2 + \tan^2 x$

89. $\dfrac{1 + \sin v}{\cos v} = \dfrac{\cos v}{1 - \sin v}$

90. $\dfrac{\sin x}{1 - \cos x} = \dfrac{1 + \cos x}{\sin x}$

In Problems 91–102, use a graphing calculator to test whether each of the following is an identity. If an equation appears to be an identity, verify it. If the equation does not appear to be an identity, find a value of x for which both sides are defined but are not equal.

91. $\dfrac{\sin(-x)}{\cos(-x)\tan(-x)} = -1$

92. $\dfrac{\cos(-x)}{\sin x \cot(-x)} = 1$

93. $\dfrac{\sin x}{\cos x \tan(-x)} = -1$

94. $\dfrac{\cos x}{\sin(-x)\cot(-x)} = 1$

95. $\sin x + \dfrac{\cos^2 x}{\sin x} = \sec x$

96. $\dfrac{1 - \tan^2 x}{1 - \cot^2 x} = \tan^2 x$

97. $\sin x + \dfrac{\cos^2 x}{\sin x} = \csc x$

98. $\dfrac{\tan^2 x - 1}{1 - \cot^2 x} = \tan^2 x$

99. $\dfrac{\tan x}{\sin x - 2\tan x} = \dfrac{1}{\cos x - 2}$

100. $\dfrac{\cos x}{1 - \sin x} + \dfrac{\cos x}{1 + \sin x} = 2\sec x$

101. $\dfrac{\tan x}{\sin x + 2\tan x} = \dfrac{1}{\cos x - 2}$

102. $\dfrac{\cos x}{\sin x + 1} - \dfrac{\cos x}{\sin x - 1} = 2\csc x$

Verify that Problems 103–108 are identities.

103. $\dfrac{2\sin^2 x + 3\cos x - 3}{\sin^2 x} = \dfrac{2\cos x - 1}{1 + \cos x}$

104. $\dfrac{3\cos^2 z + 5\sin z - 5}{\cos^2 z} = \dfrac{3\sin z - 2}{1 + \sin z}$

105. $\dfrac{\tan u + \sin u}{\tan u - \sin u} - \dfrac{\sec u + 1}{\sec u - 1} = 0$

106. $\dfrac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y} = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

107. $\tan \alpha + \cot \beta = \dfrac{\tan \beta + \cot \alpha}{\tan \beta \cot \alpha}$

108. $\dfrac{\cot \alpha + \cot \beta}{\cot \alpha \cot \beta - 1} = \dfrac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$

In Problems 109 and 110, fill in the blanks citing the appropriate basic trigonometric identity.

109.

Statement	Reason
$\cot^2 x + 1 = \left(\dfrac{\cos x}{\sin x}\right)^2 + 1$	(A) _____
$= \dfrac{\cos^2 x}{\sin^2 x} + 1$	Algebra
$= \dfrac{\cos^2 x + \sin^2 x}{\sin^2 x}$	Algebra
$= \dfrac{1}{\sin^2 x}$	(B) _____
$= \left(\dfrac{1}{\sin x}\right)^2$	Algebra
$= \csc^2 x$	(C) _____

110.

Statement	Reason
$\tan^2 x + 1 = \left(\dfrac{\sin x}{\cos x}\right)^2 + 1$	(A) _____
$= \dfrac{\sin^2 x}{\cos^2 x} + 1$	Algebra
$= \dfrac{\sin^2 x + \cos^2 x}{\cos^2 x}$	Algebra
$= \dfrac{1}{\cos^2 x}$	(B) _____
$= \left(\dfrac{1}{\cos x}\right)^2$	Algebra
$= \sec^2 x$	(C) _____

In Problems 111–116, examine the graph of f(x) in a graphing calculator to find a function of the form g(x) = k + AT(x) that has the same graph as f(x), where k and A are constants and T(x) is one of the six trigonometric functions. Verify the identity f(x) = g(x).

111. $f(x) = \dfrac{1 - \sin^2 x}{\tan x} + \sin x \cos x$

112. $f(x) = \dfrac{1 + \sin x}{2 \cos x} - \dfrac{\cos x}{2 + 2 \sin x}$

113. $f(x) = \dfrac{\cos^2 x}{1 + \sin x - \cos^2 x}$

114. $f(x) = \dfrac{\tan x \sin x}{1 - \cos x}$

115. $f(x) = \dfrac{1 + \cos x - 2 \cos^2 x}{1 - \cos x} - \dfrac{\sin^2 x}{1 + \cos x}$

116. $f(x) = \dfrac{3 \sin x - 2 \sin x \cos x}{1 - \cos x} - \dfrac{1 + \cos x}{\sin x}$

Each of the equations in Problems 117–124 is an identity in certain quadrants associated with x. Indicate which quadrants.

117. $\sqrt{1 - \cos^2 x} = -\sin x$ III, IV

118. $\sqrt{1 - \sin^2 x} = \cos x$ I, IV

119. $\sqrt{1 - \cos^2 x} = \sin x$ I, II

120. $\sqrt{1 - \sin^2 x} = -\cos x$ II, III

121. $\sqrt{1 - \sin^2 x} = |\cos x|$ I, II, III, IV

122. $\sqrt{1 - \cos^2 x} = |\sin x|$ I, II, III, IV

123. $\dfrac{\sin x}{\sqrt{1 - \sin^2 x}} = \tan x$ I, IV

124. $\dfrac{\sin x}{\sqrt{1 - \sin^2 x}} = -\tan x$ II, III

In calculus, trigonometric substitutions provide an effective way to rationalize the radical forms $\sqrt{a^2 - u^2}$ and $\sqrt{a^2 + u^2}$, which in turn leads to the solution to an important class of problems. Problems 125–128 involve such transformations. [Recall: $\sqrt{x^2} = |x|$ for all real numbers x.]

125. In the radical form $\sqrt{a^2 - u^2}$, $a > 0$, let $u = a \sin x$, $-\pi/2 < x < \pi/2$. Simplify, using a basic identity, and write the final form free of radicals. $a \cos x$

126. In the radical form $\sqrt{a^2 - u^2}$, $a > 0$, let $u = a \cos x$, $0 < x < \pi$. Simplify, using a basic identity, and write the final form free of radicals. $a \sin x$

127. In the radical form $\sqrt{a^2 + u^2}$, $a > 0$, let $u = a \tan x$, $0 < x < \pi/2$. Simplify, using a basic identity, and write the final form free of radicals. $a \sec x$

128. In the radical form $\sqrt{a^2 + u^2}$, $a > 0$, let $u = a \cot x$, $0 < x < \pi/2$. Simplify, using a basic identity, and write the final form free of radicals. $a \csc x$

6-2 Sum, Difference, and Cofunction Identities

- Sum and Difference Identities for Cosine
- Cofunction Identities
- Sum and Difference Identities for Sine and Tangent
- Summary and Use

The basic identities discussed in the last section involved only one variable. In this section, we consider identities that involve two variables.

Sum and Difference Identities for Cosine

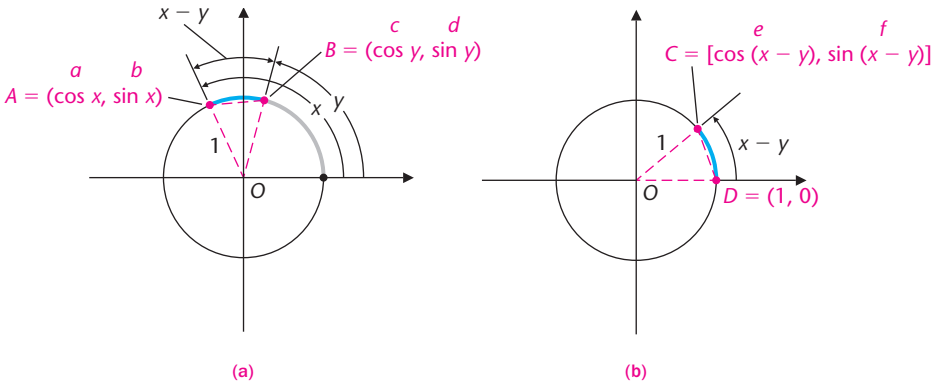
We start with the important **difference identity for cosine:**

$$\cos(x - y) = \cos x \cos y + \sin x \sin y \tag{1}$$

Many other useful identities can be readily verified from this particular one.

Here, we sketch a proof of equation (1) assuming x and y are in the interval $(0, 2\pi)$ and $x > y > 0$. It then follows easily, by periodicity and basic identities, that equation (1) holds for all real numbers x and y.

First, associate x and y with arcs and angles on the unit circle as indicated in Figure 1(a). Using the definitions of the trigonometric functions given in Section 5-2, label the terminal points of x and y as shown in Figure 1(a) [to simplify notation we let $a = \cos x$, $b = \sin x$, and so on, as indicated].



› **Figure 1** Obtaining the difference identity for cosine.

Now if you rotate the triangle AOB clockwise about the origin until the terminal point B coincides with $D = (1, 0)$, then terminal point A will be at C, as shown in Figure 1(b). Therefore, because rotation preserves lengths,

$$\begin{aligned}
d(A, B) &= d(C, D) && \text{Use distance formula.}\\
\sqrt{(c-a)^2 + (d-b)^2} &= \sqrt{(1-e)^2 + (0-f)^2} && \text{Square both sides.}\\
(c-a)^2 + (d-b)^2 &= (1-e)^2 + f^2 && \text{Expand.}\\
c^2 - 2ac + a^2 + d^2 - 2db + b^2 &= 1 - 2e + e^2 + f^2 && \text{Commute and associate.}\\
(c^2 + d^2) + (a^2 + b^2) - 2ac - 2db &= 1 - 2e + (e^2 + f^2) && (2)
\end{aligned}$$

Because points A, B, and C are on unit circles, $c^2 + d^2 = 1$, $a^2 + b^2 = 1$, and $e^2 + f^2 = 1$. We make these substitutions in equation (2) and simplify:

$$\begin{aligned}
1 + 1 - 2ac - 2db &= 1 - 2e + 1 && \text{Subtract 2 from both sides.}\\
-2ac - 2db &= -2e && \text{Multiply both sides by } -\tfrac{1}{2}.\\
ac + db &= e
\end{aligned}$$

Therefore, switching the sides of the equation,

$$e = ac + db \tag{3}$$

Replacing e, a, c, b, and d with $\cos(x - y)$, $\cos x$, $\cos y$, $\sin x$, and $\sin y$, respectively (see Fig. 1), we obtain

$$\begin{aligned}
\cos(x - y) &= \cos x \cos y + \sin y \sin x\\
&= \cos x \cos y + \sin x \sin y
\end{aligned} \tag{4}$$

This establishes the difference identity for cosine, assuming that x and y are in the interval $(0, 2\pi)$ and $x > y > 0$. It follows, using periodicity and basic identities, that equation (4) is true for all real numbers x and y.

If we replace y with $-y$ in equation (4) and use the identities for negatives (a good exercise for you), we obtain

$$\cos(x + y) = \cos x \cos y - \sin x \sin y \tag{5}$$

This is the **sum identity for cosine.**

››› EXPLORE-DISCUSS 1

(A) Verify the difference identity for cosine and the sum identity for cosine if $x = \pi/2$ and $y = \pi/3$.

(B) Discuss how you would show that the equations $\cos(x - y) = \cos x - \cos y$ and $\cos(x + y) = \cos x + \cos y$ are *not* identities.

› Cofunction Identities

To obtain sum and difference identities for the sine and tangent functions, we first derive *cofunction identities* directly from equation (1), the difference identity for cosine:

$$\begin{aligned} \cos(x - y) &= \cos x \cos y + \sin x \sin y && \text{Substitute } x = \tfrac{\pi}{2}. \\ \cos\left(\frac{\pi}{2} - y\right) &= \cos\frac{\pi}{2}\cos y + \sin\frac{\pi}{2}\sin y && \cos\tfrac{\pi}{2} = 0 \text{ and } \sin\tfrac{\pi}{2} = 1. \\ &= (0)(\cos y) + (1)(\sin y) && \text{Simplify.} \\ &= \sin y \end{aligned}$$

We have derived the **cofunction identity for cosine:**

$$\cos\left(\frac{\pi}{2} - y\right) = \sin y \tag{6}$$

for y any real number or angle in radian measure. If y is in degree measure, replace $\pi/2$ with $90°$.

Now, if we let $y = \pi/2 - x$ in equation (6), we have

$$\begin{aligned} \cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right] &= \sin\left(\frac{\pi}{2} - x\right) && \text{Simplify the left side.} \\ \cos x &= \sin\left(\frac{\pi}{2} - x\right) \end{aligned}$$

This is the **cofunction identity for sine;** that is,

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \tag{7}$$

where x is any real number or angle in radian measure. If x is in degree measure, replace $\pi/2$ with $90°$.

Finally, we state the **cofunction identity for tangent** (and leave its derivation to Problem 24 in Exercises 6-2):

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x \tag{8}$$

for x any real number or angle in radian measure. If x is in degree measure, replace $\pi/2$ with $90°$.

REMARK: If $0 < x < 90°$, then x and $90° - x$ are complementary angles. Originally, *cosine, cotangent,* and *cosecant* meant, respectively, "complement's sine," "complement's tangent," and "complement's secant." Now we simply refer to cosine, cotangent, and cosecant as **cofunctions** of sine, tangent, and secant, respectively.

› Sum and Difference Identities for Sine and Tangent

To derive a difference identity for sine, we first substitute $x - y$ for y in equation (6):

$$\begin{aligned}
\sin(x - y) &= \cos\left[\frac{\pi}{2} - (x - y)\right] && \text{Use algebra.}\\
&= \cos\left[\left(\frac{\pi}{2} - x\right) - (-y)\right] && \text{Use equation (1).}\\
&= \cos\left(\frac{\pi}{2} - x\right)\cos(-y) + \sin\left(\frac{\pi}{2} - x\right)\sin(-y) && \text{Use equations (6) and (7) and identities for negatives.}\\
&= \sin x \cos y - \cos x \sin y
\end{aligned}$$

The same result is obtained by replacing $\pi/2$ with $90°$. Therefore,

$$\sin(x - y) = \sin x \cos y - \cos x \sin y \tag{9}$$

is the **difference identity for sine.**

Now, if we replace y in equation (9) with $-y$ (a good exercise for you), we obtain

$$\sin(x + y) = \sin x \cos y + \cos x \sin y \tag{10}$$

the **sum identity for sine.**

It is not difficult to derive sum and difference identities for the tangent function. We first apply a quotient identity to $\tan(x - y)$:

$$\tan(x - y) = \frac{\sin(x - y)}{\cos(x - y)}$$ Use difference identities (1) and (9).

$$= \frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y + \sin x \sin y}$$ Divide the numerator and denominator by $\cos x \cos y$.

$$= \frac{\dfrac{\sin x \cos y}{\cos x \cos y} - \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} + \dfrac{\sin x \sin y}{\cos x \cos y}}$$ Simplify.

$$= \frac{\dfrac{\sin x}{\cos x} - \dfrac{\sin y}{\cos y}}{1 + \dfrac{\sin x \sin y}{\cos x \cos y}}$$ Use quotient identities.

$$= \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Therefore,

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} \tag{11}$$

for all angles or real numbers x and y for which both sides are defined. This is the **difference identity for tangent.**

If we replace y in equation (11) with $-y$ (another good exercise for you), we obtain

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \tag{12}$$

the **sum identity for tangent.**

››› EXPLORE-DISCUSS 2

(A) Verify the difference identity for tangent and the sum identity for tangent if $x = 5\pi/6$ and $y = \pi/6$.

(B) Discuss how you would show that the equations $\tan(x - y) = \tan x - \tan y$ and $\tan(x + y) = \tan x + \tan y$ are *not* identities.

› Summary and Use

Before proceeding with examples illustrating the use of these new identities, review the list given in the box.

SUMMARY OF IDENTITIES

Sum Identities

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Difference Identities

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$
$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Cofunction Identities

(Replace $\pi/2$ with $90°$ if x is in degrees.)

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x \qquad \sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \tan\left(\frac{\pi}{2} - x\right) = \cot x$$

EXAMPLE 1 Using a Difference Identity

Simplify $\cos(x - \pi)$ using a difference identity.

SOLUTION

First write the difference identity for cosine:

$$\begin{aligned} \cos(x - y) &= \cos x \cos y + \sin x \sin y && \text{Substitute } y = \pi. \\ \cos(x - \pi) &= \cos x \cos \pi + \sin x \sin \pi && \cos \pi = -1 \text{ and } \sin \pi = 0. \\ &= (\cos x)(-1) + (\sin x)(0) && \text{Simplify.} \\ &= -\cos x \end{aligned}$$

MATCHED PROBLEM 1

Simplify $\sin(x + 3\pi/2)$ using a sum identity.

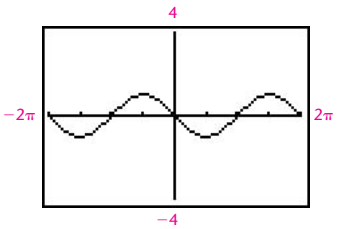
EXAMPLE 2 Checking the Use of Sum and Difference Identities on a Graphing Calculator

Simplify $\sin(x - \pi)$ using a difference identity. Enter the original form as y_1 and the converted form as y_2 in a graphing calculator, then graph both in the same viewing window.

SOLUTION

First write the difference identity for sine:

$$\begin{aligned} \sin(x - y) &= \sin x \cos y - \cos x \sin y && \text{Substitute } y = \pi. \\ \sin(x - \pi) &= \sin x \cos \pi - \cos x \sin \pi && \cos \pi = -1 \text{ and } \sin \pi = 0. \\ &= (\sin x)(-1) - (\cos x)(0) && \text{Simplify.} \\ &= -\sin x \end{aligned}$$



› Figure 2

Graph $y_1 = \sin(x - \pi)$ and $y_2 = -\sin x$ in the same viewing window (Fig. 2). Use TRACE and move back and forth between y_1 and y_2 for different values of x to see that the corresponding y values are the same, or nearly the same.

MATCHED PROBLEM 2

Simplify $\cos(x + 3\pi/2)$ using a sum identity. Enter the original form as y_1 and the converted form as y_2 in a graphing calculator, then graph both in the same viewing window.

EXAMPLE 3 Finding Exact Values

Find the exact value of $\tan 75°$ in radical form.

SOLUTION

Because we can write $75° = 45° + 30°$, the sum of two special angles, we can use the sum identity for tangent with $x = 45°$ and $y = 30°$:

$$\begin{aligned} \tan(x + y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} && \text{Substitute } x = 45°, y = 30°. \\ \tan(45° + 30°) &= \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°} && \text{Evaluate functions exactly.} \\ &= \frac{1 + (1/\sqrt{3})}{1 - 1(1/\sqrt{3})} && \text{Multiply numerator and denominator by } \sqrt{3} \text{ and simplify.} \\ &= \frac{\sqrt{3} + 1}{\sqrt{3} - 1} \end{aligned}$$

Therefore, $\tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1}$. Rationalizing the denominator and simplifying gives the alternative form $\tan 75° = 2 + \sqrt{3}$

MATCHED PROBLEM 3

Find the exact value of $\cos 15°$ in radical form.

EXAMPLE 4 Finding Exact Values

Find the exact value of $\sin(7\pi/12)$ in radical form.

SOLUTION

Because we can write $7\pi/12 = \pi/3 + \pi/4$, the sum of two special angles, we can use the sum identity for sine with $x = \pi/3$ and $y = \pi/4$:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y \qquad \text{Substitute } x = \frac{\pi}{3} \text{ and } y = \frac{\pi}{4}.$$

$$\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} + \cos\frac{\pi}{3}\sin\frac{\pi}{4} \qquad \text{Evaluate functions exactly.}$$

$$= \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} + \frac{1}{2}\frac{1}{\sqrt{2}} \qquad \text{Simplify.}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Therefore, $\sin(7\pi/12) = (\sqrt{3}+1)/2\sqrt{2}$. Rationalizing the denominator and simplifying gives the alternative form $\sin(7\pi/12) = (\sqrt{6}+\sqrt{2})/4$.

MATCHED PROBLEM 4

Find the exact value of $\tan(11\pi/12)$ in radical form.

EXAMPLE 5 Finding Exact Values

Find the exact value of $\cos(x + y)$, given $\sin x = \frac{3}{5}$, $\cos y = \frac{4}{5}$, x is an angle in Quadrant II, and y is an angle in Quadrant I. Do not use a calculator.

SOLUTION

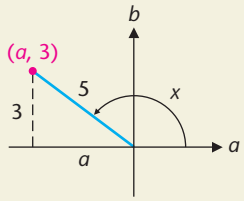
We start with the sum identity for cosine,

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

We know $\sin x$ and $\cos y$ but not $\cos x$ and $\sin y$. We find the latter two using two different methods as follows (use the method that is easiest for you).

Given $\sin x = \frac{3}{5}$ and x is an angle in Quadrant II, find $\cos x$:

Method I. Use a reference triangle:

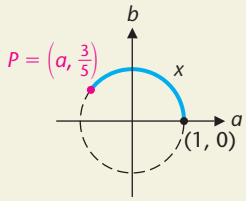


$$\cos x = \frac{a}{5}$$
$$a^2 + 3^2 = 5^2$$
$$a^2 = 16$$
$$a = \pm 4$$

In Quadrant II, $a = -4$

Therefore, $\cos x = -\frac{4}{5}$

Method II. Use a unit circle:



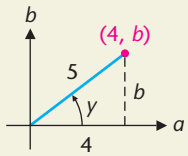
$$\cos x = a$$
$$a^2 + (\tfrac{3}{5})^2 = 1$$
$$a^2 = \tfrac{16}{25}$$
$$a = \pm\tfrac{4}{5}$$

In Quadrant II, $a = -\frac{4}{5}$

Therefore, $\cos x = -\frac{4}{5}$

Given $\cos y = \frac{4}{5}$ and y is an angle in Quadrant I, find $\sin y$:

Method I. Use a reference triangle:

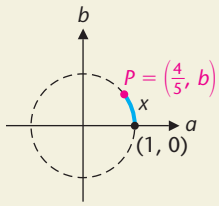


$$\sin y = \frac{b}{5}$$
$$4^2 + b^2 = 5^2$$
$$b^2 = 9$$
$$b = \pm 3$$

In Quadrant I, $b = 3$

Therefore, $\sin y = \frac{3}{5}$

Method II. Use a unit circle:



$$\sin y = b$$
$$(\tfrac{4}{5})^2 + b^2 = 1$$
$$b^2 = \tfrac{9}{25}$$
$$b = \pm\tfrac{3}{5}$$

In Quadrant I, $b = \frac{3}{5}$

Therefore, $\sin y = \frac{3}{5}$

We can now evaluate $\cos(x + y)$ without knowing x and y:

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$= (-\tfrac{4}{5})(\tfrac{4}{5}) - (\tfrac{3}{5})(\tfrac{3}{5}) = -\tfrac{25}{25} = -1$$

MATCHED PROBLEM 5

Find the exact value of $\sin(x - y)$, given $\sin x = -\frac{2}{3}$, $\cos y = \sqrt{5}/3$, x is an angle in Quadrant III, and y is an angle in Quadrant IV. Do not use a calculator.

EXAMPLE 6 Verifying an Identity

Verify the identity $\tan x + \cot y = \dfrac{\cos(x - y)}{\cos x \sin y}$.

VERIFICATION

We start with the right side and use the difference identity for cosine:

$$\frac{\cos(x - y)}{\cos x \sin y} = \frac{\cos x \cos y + \sin x \sin y}{\cos x \sin y}$$ Write as a sum of two fractions.

$$= \frac{\cancel{\cos x}\cos y}{\cancel{\cos x}\sin y} + \frac{\sin x \cancel{\sin y}}{\cos x \cancel{\sin y}}$$ Cancel and use quotient identities.

$$= \cot y + \tan x$$ Commute terms.

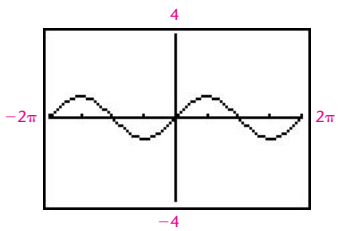
$$= \tan x + \cot y$$

MATCHED PROBLEM 6

Verify the identity $\cot y - \cot x = \dfrac{\sin(x - y)}{\sin x \sin y}$.

ANSWERS TO MATCHED PROBLEMS

1. $-\cos x$ **2.** $y_1 = \cos(x + 3\pi/2)$, $y_2 = \sin x$



3. $(1 + \sqrt{3})/2\sqrt{2}$ or $(\sqrt{6} + \sqrt{2})/4$ **4.** $\dfrac{1 - \sqrt{3}}{1 + \sqrt{3}}$ or $-2 + \sqrt{3}$ **5.** $-4\sqrt{5}/9$

6. $\dfrac{\sin(x - y)}{\sin x \sin y} = \dfrac{\sin x \cos y - \cos x \sin y}{\sin x \sin y} = \dfrac{\cancel{\sin x}\cos y}{\cancel{\sin x}\sin y} - \dfrac{\cos x \cancel{\sin y}}{\sin x \cancel{\sin y}} = \cot y - \cot x$

6-2 Exercises

1. What is a cofunction?

2. Explain how the cofunction identity $\cos(\pi/2 - x) = \sin x$ can be obtained from a difference identity.

3. Explain how each of the sum identities can be obtained from the corresponding difference identity.

4. In the sum identities, does it make a difference if x and y are given in degrees rather than in radians? Explain.

5. In the cofunction identities, does it make a difference if x and y are given in degrees rather than in radians? Explain.

6. Explain why you can't use the sum identity for tangent to obtain an identity with left side $\tan(\pi/2 + x)$. How could you obtain such an identity?

In Problems 7–14, show that the equation is not an identity by finding a value of x and a value of y for which both sides are defined but are not equal.

7. $(x + y)^2 = x^2 + y^2$ **8.** $(x - y)^3 = x^3 - y^3$

9. $x \sin y = \sin xy$ **10.** $x \tan y = \tan xy$

11. $\cos(x + y) = \cos x + \cos y$

12. $\tan(x + y) = \tan x + \tan y$

13. $\tan(x - y) = \tan x - \tan y$

14. $\sin(x - y) = \sin x - \sin y$

In Problems 15–22, is the equation an identity? Explain, making use of the sum or difference identities.

15. $\tan(x - \pi) = \tan x$ **16.** $\cos(x + \pi) = \cos x$

17. $\sin(x - \pi) = \sin x$ **18.** $\cot(x + \pi) = \cot x$

19. $\csc(2\pi - x) = \csc x$ **20.** $\sec(2\pi - x) = \sec x$

21. $\sin(x - \pi/2) = -\cos x$ **22.** $\cos(x - \pi/2) = -\sin x$

Verify each identity in Problems 23–26 using cofunction identities for sine and cosine and basic identities discussed in the last section.

23. $\cot\left(\frac{\pi}{2} - x\right) = \tan x$ **24.** $\tan\left(\frac{\pi}{2} - x\right) = \cot x$

25. $\csc\left(\frac{\pi}{2} - x\right) = \sec x$ **26.** $\sec\left(\frac{\pi}{2} - x\right) = \csc x$

Convert Problems 27–32 to forms involving sin x, cos x, and/or tan x using sum or difference identities.

27. $\sin(30° - x)$ **28.** $\sin(x - 45°)$

29. $\sin(180° - x)$ **30.** $\cos(x + 180°)$

31. $\tan\left(x + \frac{\pi}{3}\right)$ **32.** $\tan\left(\frac{\pi}{4} - x\right)$

Use appropriate identities to find exact values for Problems 33–44. Do not use a calculator.

33. $\sin 15°$ **34.** $\cos 105°$ **35.** $\cos 75°$

36. $\sin 165°$ **37.** $\tan \frac{7\pi}{12}$ **38.** $\tan \frac{\pi}{12}$

39. $\cos \frac{11\pi}{12}$ **40.** $\sin \frac{5\pi}{12}$

41. $\cos 74° \cos 44° + \sin 74° \sin 44°$

42. $\sin 22° \cos 38° + \cos 22° \sin 38°$

43. $\dfrac{\tan 27° + \tan 18°}{1 - \tan 27° \tan 18°}$ **44.** $\dfrac{\tan 110° - \tan 50°}{1 + \tan 110° \tan 50°}$

Find $\sin(x - y)$ and $\tan(x + y)$ exactly without a calculator using the information given in Problems 45–48.

45. $\sin x = -\frac{3}{5}$, $\sin y = \sqrt{8}/3$, x is a Quadrant IV angle, y is a Quadrant I angle.

46. $\sin x = \frac{2}{3}$, $\cos y = -\frac{1}{4}$, x is a Quadrant II angle, y is a Quadrant III angle.

47. $\tan x = \frac{3}{4}$, $\tan y = -\frac{1}{2}$, x is a Quadrant III angle, y is a Quadrant IV angle.

48. $\cos x = -\frac{1}{3}$, $\tan y = \frac{1}{2}$, x is a Quadrant II angle, y is a Quadrant III angle.

Verify each identity in Problems 49–62.

49. $\cos 2x = \cos^2 x - \sin^2 x$ **50.** $\sin 2x = 2 \sin x \cos x$

51. $\cot(x + y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}$

52. $\cot(x - y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}$

53. $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$ **54.** $\cot 2x = \dfrac{\cot^2 x - 1}{2 \cot x}$

55. $\dfrac{\sin(v+u)}{\sin(v-u)} = \dfrac{\cot u + \cot v}{\cot u - \cot v}$

56. $\dfrac{\sin(u+v)}{\sin(u-v)} = \dfrac{\tan u + \tan v}{\tan u - \tan v}$

57. $\cot x - \tan y = \dfrac{\cos(x+y)}{\sin x \cos y}$

58. $\tan x - \tan y = \dfrac{\sin(x-y)}{\cos x \cos y}$

59. $\tan(x-y) = \dfrac{\cot y - \cot x}{\cot x \cot y + 1}$

60. $\tan(x+y) = \dfrac{\cot x + \cot y}{\cot x \cot y - 1}$

61. $\dfrac{\cos(x+h) - \cos x}{h} = \cos x\left(\dfrac{\cos h - 1}{h}\right) - \sin x\left(\dfrac{\sin h}{h}\right)$

62. $\dfrac{\sin(x+h) - \sin x}{h} = \sin x\left(\dfrac{\cos h - 1}{h}\right) + \cos x\left(\dfrac{\sin h}{h}\right)$

Evaluate both sides of the difference identity for sine and the sum identity for tangent for the values of x and y indicated in Problems 63–66. Evaluate to four significant digits using a calculator.

63. $x = 5.288, y = 1.769$

64. $x = 3.042, y = 2.384$

65. $x = 42.08°, y = 68.37°$

66. $x = 128.3°, y = 25.62°$

67. Show that $\sec(x-y) = \sec x - \sec y$ is *not* identity.

68. Show that $\csc(x+y) = \csc x + \csc y$ is *not* an identity.

In Problems 69–74, use sum or difference identities to convert each equation to a form involving sin x, cos x, and/or tan x. Enter the original equation in a graphing calculator as y_1 and the converted form as y_2, then graph y_1 and y_2 in the same viewing window. Use TRACE to compare the two graphs.

69. $y = \sin(x + \pi/6)$

70. $y = \sin(x - \pi/3)$

71. $y = \cos(x - 3\pi/4)$

72. $y = \cos(x + 5\pi/6)$

73. $y = \tan(x + 2\pi/3)$

74. $y = \tan(x - \pi/4)$

In Problems 75–78, evaluate exactly as real numbers without the use of a calculator.

75. $\sin[\cos^{-1}(-\frac{4}{5}) + \sin^{-1}(-\frac{3}{5})]$

76. $\cos[\sin^{-1}(-\frac{3}{5}) + \cos^{-1}(\frac{4}{5})]$

77. $\sin[\arccos\frac{1}{2} + \arcsin(-1)]$

78. $\cos[\arccos(-\sqrt{3}/2) - \arcsin(-\frac{1}{2})]$

79. Express $\sin(\sin^{-1}x + \cos^{-1}y)$ in an equivalent form free of trigonometric and inverse trigonometric functions.

80. Express $\cos(\sin^{-1}x - \cos^{-1}y)$ in an equivalent form free of trigonometric and inverse trigonometric functions.

Verify the identities in Problems 81 and 82.

81. $\cos(x+y+z) = \cos x \cos y \cos z - \sin x \sin y \cos z - \sin x \cos y \sin z - \cos x \sin y \sin z$

82. $\sin(x+y+z) = \sin x \cos y \cos z + \cos x \sin y \cos z + \cos x \cos y \sin z - \sin x \sin y \sin z$

In Problems 83 and 84, write each equation in terms of a single trigonometric function. Enter the original equation in a graphing calculator as y_1 and the converted form as y_2, then graph y_1 and y_2 in the same viewing window. Use TRACE to compare the two graphs.

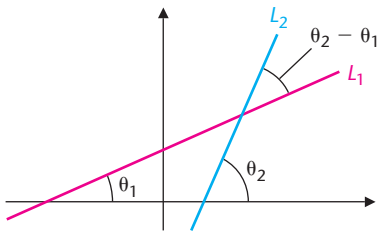
83. $y = \cos 1.2x \cos 0.8x - \sin 1.2x \sin 0.8x$

84. $y = \sin 0.8x \cos 0.3x - \cos 0.8x \sin 0.3x$

APPLICATIONS

85. ANALYTIC GEOMETRY Use the information in the figure to show that

$$\tan(\theta_2 - \theta_1) = \frac{m_2 - m_1}{1 + m_1 m_2}$$



$\tan\theta_1 =$ Slope of $L_1 = m_1$

$\tan\theta_2 =$ Slope of $L_2 = m_2$

86. ANALYTIC GEOMETRY Find the acute angle of intersection between the two lines $y = 3x + 1$ and $y = \frac{1}{2}x - 1$. (Use the results of Problem 85.)

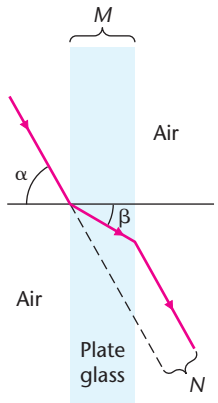
87. LIGHT REFRACTION Light rays passing through a plate glass window are refracted when they enter the glass and again when they leave to continue on a path parallel to the entering rays (see the figure on the next page). If the plate glass is M inches thick, the parallel displacement of the light rays is N inches, the angle of incidence is α, and the angle of refraction is β, show that

$$\tan\beta = \tan\alpha - \frac{N}{M}\sec\alpha$$

[*Hint:* First use geometric relationships to obtain

$$\frac{M}{\sec(90° - \beta)} = \frac{N}{\sin(\alpha - \beta)}$$

then use difference identities and fundamental identities to complete the derivation.]



88. LIGHT REFRACTION Use the results of Problem 87 to find β to the nearest degree if $\alpha = 43°$, $M = 0.25$ inch, and $N = 0.11$ inch.

89. SURVEYING El Capitan is a large monolithic granite peak that rises straight up from the floor of Yosemite Valley in Yosemite National Park. It attracts rock climbers worldwide. At certain times, the reflection of the peak can be seen in the Merced River that runs along the valley floor. How can the height H of El Capitan above the river be determined by using only a sextant h feet high to measure the angle of elevation, β, to the top of the peak, and the angle of depression, α, of the reflected peak top in the river? (See accompanying figure, which is not to scale.)

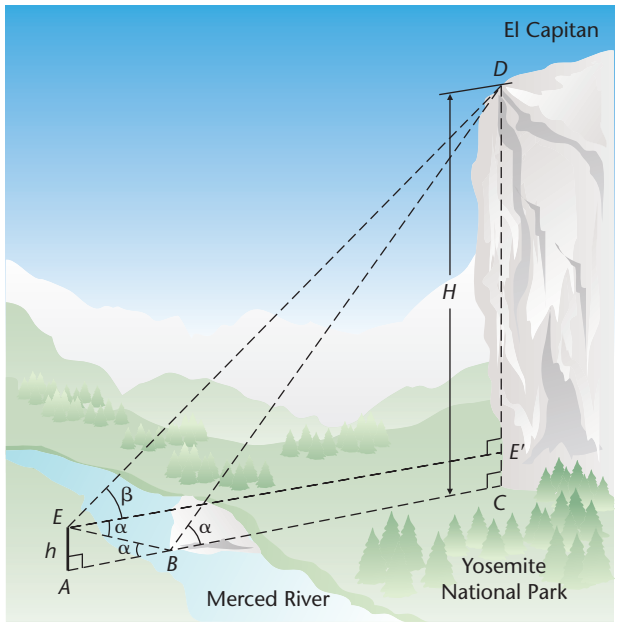
(A) Using right triangle relationships, show that

$$H = h\left(\frac{1 + \tan\beta \cot\alpha}{1 - \tan\beta \cot\alpha}\right)$$

(B) Using sum or difference identities, show that the result in part A can be written in the form

$$H = h\left[\frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)}\right]$$

(C) If a sextant of height 4.90 feet measures α to be 46.23° and β to be 46.15°, compute the height H of El Capitan above the Merced River to three significant digits.



6-3 Double-Angle and Half-Angle Identities

- Double-Angle Identities
- Half-Angle Identities

In this section, we develop another important set of identities called *double-angle* and *half-angle identities*. We can derive these identities directly from the sum and difference identities given in the last section. Although the names use the word *angle,* the new identities hold for real numbers as well.

Double-Angle Identities

Start with the sum identity for sine,

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

and replace y with x to obtain

$$\sin(x + x) = \sin x \cos x + \cos x \sin x$$

On simplification, this gives

$$\sin 2x = 2 \sin x \cos x \quad \text{Double-angle identity for sine} \tag{1}$$

If we start with the sum identity for cosine,

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

and replace y with x, we obtain

$$\cos(x + x) = \cos x \cos x - \sin x \sin x$$

On simplification, this gives

$$\cos 2x = \cos^2 x - \sin^2 x \quad \text{First double-angle identity for cosine} \tag{2}$$

Now, using the Pythagorean identity

$$\sin^2 x + \cos^2 x = 1 \tag{3}$$

in the form

$$\cos^2 x = 1 - \sin^2 x \tag{4}$$

and substituting it into equation (2), we get

$$\cos 2x = 1 - \sin^2 x - \sin^2 x$$

On simplification, this gives

$$\cos 2x = 1 - 2 \sin^2 x \quad \text{Second double-angle identity for cosine} \tag{5}$$

Or, if we use equation (3) in the form

$$\sin^2 x = 1 - \cos^2 x$$

and substitute it into equation (2), we get

$$\cos 2x = \cos^2 x - (1 - \cos^2 x)$$

On simplification, this gives

$$\cos 2x = 2\cos^2 x - 1 \qquad \text{Third double-angle identity for cosine} \tag{6}$$

Double-angle identities can be established for the tangent function in the same way by starting with the sum formula for tangent (a good exercise for you).

We list the double-angle identities below for convenient reference.

› DOUBLE-ANGLE IDENTITIES

$$\sin 2x = 2\sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} = \frac{2\cot x}{\cot^2 x - 1} = \frac{2}{\cot x - \tan x}$$

The identities in the second row can be solved for $\sin^2 x$ and $\cos^2 x$ to obtain the identities

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

These are useful in calculus to transform a power form to a nonpower form.

››› EXPLORE-DISCUSS 1

(A) Verify the double-angle identities if $x = \pi/6$.

(B) Discuss how you would show that the equations $\sin 2x = 2\sin x$, $\cos 2x = 2\cos x$, and $\tan 2x = 2\tan x$ are *not* identities.

EXAMPLE 1 Identity Verification

Verify the identity $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$.

VERIFICATION

We start with the right side and use quotient identities:

$$\frac{1-\tan^2 x}{1+\tan^2 x} = \frac{1-\dfrac{\sin^2 x}{\cos^2 x}}{1+\dfrac{\sin^2 x}{\cos^2 x}}$$ Multiply numerator and denominator by $\cos^2 x$.

$$= \frac{\cos^2 x\left(1-\dfrac{\sin^2 x}{\cos^2 x}\right)}{\cos^2 x\left(1+\dfrac{\sin^2 x}{\cos^2 x}\right)}$$ Use algebra to write the compound fraction as a simple fraction.

$$= \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x}$$ Use $\sin^2 x + \cos^2 x = 1$.

$$= \cos^2 x - \sin^2 x$$ Use a double-angle identity.

$$= \cos 2x$$ Double-angle identity

KEY ALGEBRAIC STEPS IN EXAMPLE 1

$$\frac{1-\dfrac{a^2}{b^2}}{1+\dfrac{a^2}{b^2}} = \frac{b^2\left(1-\dfrac{a^2}{b^2}\right)}{b^2\left(1+\dfrac{a^2}{b^2}\right)} = \frac{b^2-a^2}{b^2+a^2}$$

MATCHED PROBLEM 1

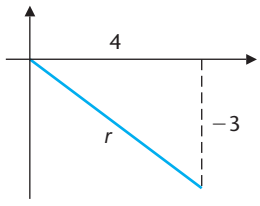
Verify the identity $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$.

EXAMPLE 2 Finding Exact Values

Find the exact values, without using a calculator, of $\sin 2x$ and $\cos 2x$ if $\tan x = -\frac{3}{4}$ and x is a Quadrant IV angle.

SOLUTION

First draw the reference triangle for x and find any unknown sides:



$$r = \sqrt{(-3)^2 + 4^2} = 5$$
$$\sin x = -\tfrac{3}{5}$$
$$\cos x = \tfrac{4}{5}$$

Now use double-angle identities for sine and cosine:

$$\sin 2x = 2 \sin x \cos x = 2(-\tfrac{3}{5})(\tfrac{4}{5}) = -\tfrac{24}{25}$$
$$\cos 2x = 2 \cos^2 x - 1 = 2(\tfrac{4}{5})^2 - 1 = \tfrac{7}{25}$$

MATCHED PROBLEM 2

Find the exact values, without using a calculator, of $\cos 2x$ and $\tan 2x$ if $\sin x = \frac{4}{5}$ and x is a Quadrant II angle.

Half-Angle Identities

Half-angle identities are simply double-angle identities stated in an alternate form. Let's start with the double-angle identity for cosine in the form

$$\cos 2m = 1 - 2 \sin^2 m$$

Now replace m with $x/2$ and solve for $\sin (x/2)$ [if $2m$ is twice m, then m is half of $2m$—think about this]:

$$\cos x = 1 - 2 \sin^2 \frac{x}{2}$$
$$\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}$$
$$\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}} \quad \text{Half-angle identity for sine} \tag{7}$$

where the choice of the sign is determined by the quadrant in which $x/2$ lies.

To obtain a half-angle identity for cosine, start with the double-angle identity for cosine in the form

$$\cos 2m = 2 \cos^2 m - 1$$

and let $m = x/2$ to obtain

$$\cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}} \quad \text{Half-angle identity for cosine} \tag{8}$$

where the sign is determined by the quadrant in which $x/2$ lies.

To obtain a *half-angle identity for tangent,* use the quotient identity and the half-angle formulas for sine and cosine:

$$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{\pm\sqrt{\frac{1-\cos x}{2}}}{\pm\sqrt{\frac{1+\cos x}{2}}} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$

Therefore,

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} \qquad \text{Half-angle identity for tangent} \tag{9}$$

where the sign is determined by the quadrant in which $x/2$ lies.

Simpler versions of equation (9) can be obtained as follows:

$$\left|\tan\frac{x}{2}\right| = \sqrt{\frac{1-\cos x}{1+\cos x}} \qquad \text{Multiply radicand by } \frac{1+\cos x}{1+\cos x}. \tag{10}$$

$$= \sqrt{\frac{1-\cos x}{1+\cos x}\cdot\frac{1+\cos x}{1+\cos x}} \qquad \text{Use algebra.}$$

$$= \sqrt{\frac{1-\cos^2 x}{(1+\cos x)^2}} \qquad \text{Use } \sin^2 x + \cos^2 x = 1.$$

$$= \sqrt{\frac{\sin^2 x}{(1+\cos x)^2}} \qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

$$= \frac{\sqrt{\sin^2 x}}{\sqrt{(1+\cos x)^2}} \qquad \sqrt{\sin^2 x} = |\sin x| \text{ and } \sqrt{(1+\cos x)^2} = 1+\cos x, \text{ because } 1+\cos x \text{ is never negative.}$$

$$= \frac{|\sin x|}{1+\cos x}$$

All absolute value signs can be dropped, because it can be shown that $\tan(x/2)$ and $\sin x$ always have the same sign (a good exercise for you). Therefore,

$$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x} \qquad \text{Half-angle identity for tangent} \tag{11}$$

By multiplying the numerator and the denominator in the radicand in equation (10) by $1-\cos x$ and reasoning as before, we also can obtain

$$\tan\frac{x}{2} = \frac{1-\cos x}{\sin x} \qquad \text{Half-angle identity for tangent} \tag{12}$$

We now list all the half-angle identities for convenient reference.

› HALF-ANGLE IDENTITIES

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$

where the sign is determined by the quadrant in which $x/2$ lies.

››› EXPLORE-DISCUSS 2

(A) Verify the half-angle identities if $x = \pi/2$.

(B) Discuss how you would show that the equations

$$\sin\frac{x}{2} = \frac{1}{2}\sin x, \qquad \cos\frac{x}{2} = \frac{1}{2}\cos x, \qquad \text{and} \qquad \tan\frac{x}{2} = \frac{1}{2}\tan x$$

are *not* identities.

EXAMPLE 3 Finding Exact Values

Compute the exact value of sin 165° without a calculator using a half-angle identity.

SOLUTION

$$\begin{aligned}\sin 165° &= \sin\frac{330°}{2} && \text{Use half-angle identity for sine with a positive radical, because 165° is in Quadrant II and sin 165° is positive.}\\ &= \sqrt{\frac{1-\cos 330°}{2}} && \cos 330° = \frac{\sqrt{3}}{2}\\ &= \sqrt{\frac{1-(\sqrt{3}/2)}{2}} && \text{Multiply numerator and denominator of radicand by 2 and simplify.}\\ &= \frac{\sqrt{2-\sqrt{3}}}{2}\end{aligned}$$

MATCHED PROBLEM 3

Compute the exact value of tan 105° without a calculator using a half-angle identity.

EXAMPLE 4 Finding Exact Values

Find the exact value of $\cos(5\pi/8)$ in radical form.

SOLUTION

$$\cos\frac{5\pi}{8} = \cos\frac{(5\pi/4)}{2}$$

Use half-angle identity for cosine with a negative radical, because $5\pi/8$ is in Quadrant II and $\cos(5\pi/8)$ is negative.

$$= -\sqrt{\frac{1+\cos(5\pi/4)}{2}}$$

$\cos(5\pi/4) = -\frac{1}{\sqrt{2}}$

$$= -\sqrt{\frac{1+(-1/\sqrt{2})}{2}}$$

Multiply numerator and denominator of radicand by 2.

$$= -\sqrt{\frac{2+(-2/\sqrt{2})}{4}}$$

Simplify; use $2/\sqrt{2} = \sqrt{2}$.

$$= \frac{-\sqrt{2-\sqrt{2}}}{2}$$

MATCHED PROBLEM 4

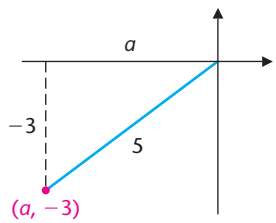
Find the exact value of $\sin(3\pi/8)$ in radical form.

EXAMPLE 5 Finding Exact Values

Find the exact values of $\cos(x/2)$ and $\cot(x/2)$ without using a calculator if $\sin x = -\frac{3}{5}$, $\pi < x < 3\pi/2$.

SOLUTION

Draw a reference triangle in the third quadrant, and find $\cos x$. Then use appropriate half-angle identities.



$$a = -\sqrt{5^2-(-3)^2} = -4$$
$$\cos x = -\tfrac{4}{5}$$

If $\pi < x < 3\pi/2$, then, dividing each member by 2, we obtain

$$\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$$

Therefore, $x/2$ is an angle in the second quadrant where cosine and cotangent are negative, and

$$\cos\frac{x}{2} = -\sqrt{\frac{1+\cos x}{2}} = -\sqrt{\frac{1+(-\frac{4}{5})}{2}} = -\sqrt{\frac{1}{10}} \text{ or } \frac{-\sqrt{10}}{10}$$

$$\cot\frac{x}{2} = \frac{1}{\tan(x/2)} = \frac{\sin x}{1-\cos x} = \frac{-\frac{3}{5}}{1-(-\frac{4}{5})} = -\frac{1}{3}$$

MATCHED PROBLEM 5

Find the exact values of $\sin(x/2)$ and $\tan(x/2)$ without using a calculator if $\cot x = -\frac{4}{3}$, $\pi/2 < x < \pi$.

EXAMPLE 6 Identity Verification

Verify the identity $\sin^2\frac{x}{2} = \frac{\tan x - \sin x}{2\tan x}$.

VERIFICATION

We start with the left side and use the half-angle identity for sine to rewrite $\sin(x/2)$:

$$\sin^2\frac{x}{2} = \left(\pm\sqrt{\frac{1-\cos x}{2}}\right)^2 \quad \text{Use algebra.}$$

$$= \frac{1-\cos x}{2} \quad \text{Multiply by } \frac{\tan x}{\tan x}.$$

$$= \frac{\tan x}{\tan x}\cdot\frac{1-\cos x}{2} \quad \text{Use algebra.}$$

$$= \frac{\tan x - \tan x\cos x}{2\tan x} \quad \text{Use a quotient identity.}$$

$$= \frac{\tan x - \left(\frac{\sin x}{\cos x}\right)\cos x}{2\tan x}^{*} \quad \text{Simplify.}$$

$$= \frac{\tan x - \sin x}{2\tan x}$$

MATCHED PROBLEM 6

Verify the identity $\cos^2\frac{x}{2} = \frac{\tan x + \sin x}{2\tan x}$.

*Throughout the book, dashed boxes—called think boxes—are used to represent steps that may be performed mentally.

ANSWERS TO MATCHED PROBLEMS

1. $$\frac{2\tan x}{1+\tan^2 x} = \frac{2\left(\frac{\sin x}{\cos x}\right)}{1+\frac{\sin^2 x}{\cos^2 x}} = \frac{\cos^2 x\left[2\left(\frac{\sin x}{\cos x}\right)\right]}{\cos^2 x\left(1+\frac{\sin^2 x}{\cos^2 x}\right)} = \frac{2\sin x\cos x}{\cos^2 x+\sin^2 x}$$
$$= 2\sin x\cos x = \sin 2x$$

2. $\cos 2x = -\frac{7}{25}$, $\tan 2x = \frac{24}{7}$ **3.** $-\sqrt{3}-2$ **4.** $\frac{\sqrt{2+\sqrt{2}}}{2}$

5. $\sin(x/2) = 3\sqrt{10}/10$, $\tan(x/2) = 3$

6. $$\cos^2\frac{x}{2} = \frac{1+\cos x}{2} = \frac{\tan x}{\tan x}\cdot\frac{1+\cos x}{2} = \frac{\tan x+\tan x\cos x}{2\tan x} = \frac{\tan x+\sin x}{2\tan x}$$

6-3 Exercises

1. Explain how the double-angle identity for sine can be obtained from the sum identity for sine.

2. Using the same technique as in Problem 1, find the identity that is obtained from the difference identity for sine.

3. Explain how the first double-angle identity for cosine can be obtained from the sum identity for cosine.

4. Using the same technique as in Problem 3, find the identity that is obtained from the difference identity for cosine.

5. Explain how the first double-angle identity for tangent can be obtained from the sum identity for tangent.

6. Using the same technique as in Problem 5, find the identity that is obtained from the difference identity for tangent.

In Problems 7–12, verify each identity for the values indicated.

7. $\cos 2x = \cos^2 x - \sin^2 x, x = 30°$

8. $\sin 2x = 2\sin x\cos x, x = 45°$

9. $\tan 2x = \dfrac{2}{\cot x - \tan x}, x = \dfrac{\pi}{3}$

10. $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}, x = \dfrac{\pi}{6}$

11. $\sin\dfrac{x}{2} = \pm\sqrt{\dfrac{1-\cos x}{2}}, x = \pi$
(Choose the correct sign.)

12. $\cos\dfrac{x}{2} = \pm\sqrt{\dfrac{1+\cos x}{2}}, x = \dfrac{\pi}{2}$
(Choose the correct sign.)

In Problems 13–24, find the exact value without a calculator using half-angle identities.

13. $\sin 22.5°$ **14.** $\tan 75°$ **15.** $\cos 67.5°$

16. $\tan 15°$ **17.** $\tan 105°$ **18.** $\sin 112.5°$

19. $\sin\dfrac{5\pi}{8}$ **20.** $\cos\dfrac{7\pi}{8}$ **21.** $\cos\dfrac{5\pi}{12}$

22. $\tan\dfrac{11\pi}{12}$ **23.** $\tan\dfrac{7\pi}{12}$ **24.** $\sin\dfrac{\pi}{12}$

In Problems 25–28, graph y_1 and y_2 in the same viewing window for $-2\pi \le x \le 2\pi$. Use TRACE to compare the two graphs.

25. $y_1 = \cos 2x, y_2 = \cos^2 x - \sin^2 x$

26. $y_1 = \sin 2x, y_2 = 2\sin x\cos x$

27. $y_1 = \tan\dfrac{x}{2}, y_2 = \dfrac{\sin x}{1+\cos x}$

28. $y_1 = \tan 2x, y_2 = \dfrac{2\tan x}{1-\tan^2 x}$

Verify the identities in Problems 29–46.

29. $(\sin x+\cos x)^2 = 1+\sin 2x$

30. $\sin 2x = (\tan x)(1+\cos 2x)$

31. $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ **32.** $\cos^2 x = \frac{1}{2}(\cos 2x+1)$

33. $1-\cos 2x = \tan x\sin 2x$

34. $1+\sin 2t = (\sin t+\cos t)^2$

35. $\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}$

36. $\cos^2 \frac{x}{2} = \frac{1 + \cos x}{2}$

37. $\cot 2x = \frac{1 - \tan^2 x}{2 \tan x}$

38. $\cot 2x = \frac{\cot x - \tan x}{2}$

39. $\cot \frac{\theta}{2} = \frac{\sin \theta}{1 - \cos \theta}$

40. $\cot \frac{\theta}{2} = \frac{1 + \cos \theta}{\sin \theta}$

41. $\cos 2u = \frac{1 - \tan^2 u}{1 + \tan^2 u}$

42. $\frac{\cos 2u}{1 - \sin 2u} = \frac{1 + \tan u}{1 - \tan u}$

43. $2 \csc 2x = \frac{1 + \tan^2 x}{\tan x}$

44. $\sec 2x = \frac{\sec^2 x}{2 - \sec^2 x}$

45. $\cos \alpha = \frac{1 - \tan^2 (\alpha/2)}{1 + \tan^2 (\alpha/2)}$

46. $\cos 2\alpha = \frac{\cot \alpha - \tan \alpha}{\cot \alpha + \tan \alpha}$

In Problems 47–54, show that the equation is not an identity by finding a value of x for which both sides are defined but are not equal.

47. $\tan 2x = 2 \tan x$

48. $\cos 2x = 2 \cos x$

49. $\sin \frac{x}{2} = \frac{1}{2} \sin x$

50. $\tan \frac{x}{2} = \frac{1}{2} \tan x$

51. $\cos \frac{x}{2} = \sqrt{\frac{1 + \cos x}{2}}$

52. $\sin \frac{x}{2} = \sqrt{\frac{1 - \cos x}{2}}$

53. $\tan \frac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}}$

54. $\tan 2x = \frac{2 \cot x}{1 - \cot^2 x}$

In Problems 55–60, is the equation an identity? Explain.

55. $\sin 4x = 4 \sin x \cos x$

56. $\csc 2x = 2 \csc x \sec x$

57. $\cot 2x = \frac{\tan x (\cot^2 x - 1)}{2}$

58. $\tan 4x = 4 \tan x$

59. $\cos 2x = 1 - 2 \cos^2 x$

60. $\tan 2x = \frac{2}{\tan x - \cot x}$

Compute the exact values of sin 2x, cos 2x, and tan 2x using the information given in Problems 61–68 and appropriate identities. Do not use a calculator.

61. $\sin x = \frac{3}{5}, \pi/2 < x < \pi$

62. $\cos x = -\frac{4}{5}, \pi/2 < x < \pi$

63. $\tan x = -\frac{5}{12}, -\pi/2 < x < 0$

64. $\cot x = -\frac{5}{12}, -\pi/2 < x < 0$

65. $\cos x = \frac{2}{3}, \tan x < 0$

66. $\sin x = \frac{12}{37}, \cot x < 0$

67. $\tan x = 2, \sin x < 0$

68. $\tan x = \frac{1}{2}, \cos x > 0$

In Problems 69–76, compute the exact values of sin (x/2), cos (x/2), and tan (x/2) using the information given and appropriate identities. Do not use a calculator.

69. $\sin x = -\frac{1}{3}, \pi < x < 3\pi/2$

70. $\cos x = -\frac{1}{4}, \pi < x < 3\pi/2$

71. $\cot x = \frac{3}{4}, -\pi < x < -\pi/2$

72. $\tan x = \frac{3}{4}, -\pi < x < -\pi/2$

73. $\cos x = -\frac{8}{17}, \sin x < 0$

74. $\sin x = \frac{12}{37}, \cot x < 0$

75. $\tan x = \frac{1}{3}, \cos x > 0$

76. $\cot x = -4, \csc x < 0$

Suppose you are tutoring a student who is having difficulties in finding the exact values of sin θ and cos θ from the information given in Problems 77 and 78. Assuming you have worked through each problem and have identified the key steps in the solution process, proceed with your tutoring by guiding the student through the solution process using the following questions. Record the expected correct responses from the student.

(A) The angle 2θ is in what quadrant and how do you know?

(B) How can you find sin 2θ and cos 2θ? Find each.

(C) What identities relate sin θ and cos θ with either sin 2θ or cos 2θ?

(D) How would you use the identities in part C to find sin θ and cos θ exactly, including the correct sign?

(E) What are the exact values for sin θ and cos θ?

77. Find the exact values of sin θ and cos θ, given $\tan 2\theta = -\frac{4}{3}$, $0° < \theta < 90°$.

78. Find the exact values of sin θ and cos θ, given $\sec 2\theta = -\frac{5}{4}$, $0° < \theta < 90°$.

Verify each of the following identities for the value of x indicated in Problems 79–82. Compute values to five significant digits using a calculator.

(A) $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$ *(B)* $\cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}}$

(Choose the correct sign.)

79. $x = 252.06°$

80. $x = 72.358°$

81. $x = 0.934\,57$

82. $x = 4$

In Problems 83–86, graph y_1 and y_2 in the same viewing window for $-2\pi \le x \le 2\pi$, and state the intervals for which the equation $y_1 = y_2$ is an identity.

83. $y_1 = \cos(x/2), y_2 = \sqrt{\dfrac{1 + \cos x}{2}}$

84. $y_1 = \cos(x/2), y_2 = -\sqrt{\dfrac{1 + \cos x}{2}}$

85. $y_1 = \sin(x/2), y_2 = -\sqrt{\dfrac{1 - \cos x}{2}}$

86. $y_1 = \sin(x/2), y_2 = \sqrt{\dfrac{1 - \cos x}{2}}$

Verify the identities in Problems 87–90.

87. $\cos 3x = 4\cos^3 x - 3\cos x$

88. $\sin 3x = 3\sin x - 4\sin^3 x$

89. $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$

90. $\sin 4x = (\cos x)(4\sin x - 8\sin^3 x)$

In Problems 91–96, find the exact value of each without using a calculator.

91. $\cos[2\cos^{-1}(\frac{3}{5})]$

92. $\sin[2\cos^{-1}(\frac{3}{5})]$

93. $\tan[2\cos^{-1}(-\frac{4}{5})]$

94. $\tan[2\tan^{-1}(-\frac{3}{4})]$

95. $\cos[\frac{1}{2}\cos^{-1}(-\frac{3}{5})]$

96. $\sin[\frac{1}{2}\tan^{-1}(-\frac{4}{3})]$

In Problems 97–102, graph f(x) in a graphing calculator, find a simpler function g(x) that has the same graph as f(x), and verify the identity $f(x) = g(x)$. [Assume $g(x) = k + A\,T(Bx)$ where k, A, and B are constants and T(x) is one of the six trigonometric functions.]

97. $f(x) = \csc x - \cot x$

98. $f(x) = \csc x + \cot x$

99. $f(x) = \dfrac{1 - 2\cos 2x}{2\sin x - 1}$

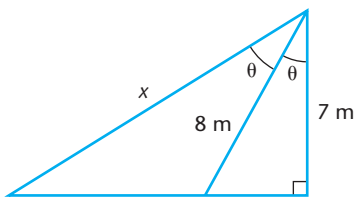
100. $f(x) = \dfrac{1 + 2\cos 2x}{1 + 2\cos x}$

101. $f(x) = \dfrac{1}{\cot x \sin 2x - 1}$

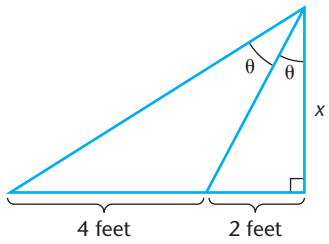
102. $f(x) = \dfrac{\cot x}{1 + \cos 2x}$

APPLICATIONS

103. INDIRECT MEASUREMENT Find the exact value of x in the figure; then find x and θ (in degrees) to three decimal places. [*Hint:* Use $\cos 2\theta = 2\cos^2\theta - 1$.]



104. INDIRECT MEASUREMENT Find the exact value of x in the figure; then find x and θ (in degrees) to three decimal places. [*Hint:* Use $\tan 2\theta = (2\tan\theta)/(1 - \tan^2\theta)$.]



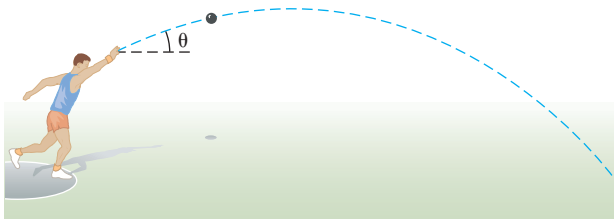
105. SPORTS—PHYSICS The theoretical distance d that a shotputter, discus thrower, or javelin thrower can achieve on a given throw is found in physics to be given approximately by

$$d = \frac{2v_0^2 \sin\theta\cos\theta}{32 \text{ feet per second per second}}$$

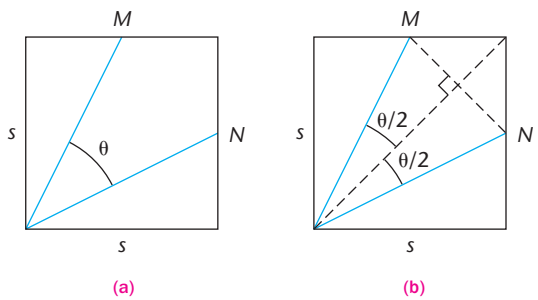
where v_0 is the initial speed of the object thrown (in feet per second) and θ is the angle above the horizontal at which the object leaves the hand (see the figure).

(A) Write the formula in terms of $\sin 2\theta$ by using a suitable identity.

(B) Using the resulting equation in part A, determine the angle θ (in degrees) that will produce the maximum distance d for a given initial speed v_0. This result is an important consideration for shotputters, javelin throwers, and discus throwers.



106. GEOMETRY In part (a) of the figure, M and N are the midpoints of the sides of a square. Find the exact value of $\cos\theta$. [*Hint:* The solution uses the Pythagorean theorem, the definition of sine and cosine, a half-angle identity, and some auxiliary lines as drawn in part (b) of the figure.]



107. AREA An n-sided regular polygon is inscribed in a circle of radius R.

(A) Show that the area of the n-sided polygon is given by

$$A_n = \frac{1}{2} nR^2 \sin \frac{2\pi}{n}$$

[*Hint:* (Area of a triangle) $= (\frac{1}{2})$(base)(altitude). Also, a double-angle identity is useful.]

(B) For a circle of radius 1, complete Table 1, to five decimal places, using the formula in part A:

Table 1

n	10	100	1,000	10,000
A_n				

(C) What number does A_n seem to approach as n increases without bound? (What is the area of a circle of radius 1?)

(D) Will A_n exactly equal the area of the circumscribed circle for some sufficiently large n? How close can A_n be to the area of the circumscribed circle? [In calculus, the area of the circumscribed circle is called the *limit* of A_n as n increases without bound. In symbols, for a circle of radius 1, we would write $\lim_{n\to\infty} A_n = \pi$. The limit concept is the cornerstone on which calculus is constructed.]

6-4 Product–Sum and Sum–Product Identities

- Product–Sum Identities
- Sum–Product Identities

Our work with identities is concluded by developing the *product–sum* and *sum–product identities,* which are easily derived from the sum and difference identities developed in the second section of this chapter. These identities are used in calculus to convert product forms to more convenient sum forms. They also are used in the study of sound waves in music to convert sum forms to more convenient product forms.

Product–Sum Identities

First, add left side to left side and right side to right side, the sum and difference identities for sine:

$$\begin{aligned} \sin(x + y) &= \sin x \cos y + \cos x \sin y \\ \sin(x - y) &= \sin x \cos y - \cos x \sin y \\ \hline \sin(x + y) + \sin(x - y) &= 2 \sin x \cos y \end{aligned}$$

or

$$\sin x \cos y = \tfrac{1}{2}[\sin(x + y) + \sin(x - y)]$$

Similarly, by adding or subtracting the appropriate sum and difference identities, we can obtain three other **product–sum identities.** These are listed in the box for convenient reference.

PRODUCT–SUM IDENTITIES

$$\sin x \cos y = \tfrac{1}{2}[\sin(x + y) + \sin(x - y)]$$
$$\cos x \sin y = \tfrac{1}{2}[\sin(x + y) - \sin(x - y)]$$
$$\sin x \sin y = \tfrac{1}{2}[\cos(x - y) - \cos(x + y)]$$
$$\cos x \cos y = \tfrac{1}{2}[\cos(x + y) + \cos(x - y)]$$

EXAMPLE 1 A Product as a Difference

Write the product $\cos 3t \sin t$ as a sum or difference.

SOLUTION

$$\cos x \sin y = \tfrac{1}{2}[\sin(x + y) - \sin(x - y)] \quad \text{Let } x = 3t \text{ and } y = t.$$
$$\cos 3t \sin t = \tfrac{1}{2}[\sin(3t + t) - \sin(3t - t)] \quad \text{Simplify.}$$
$$= \tfrac{1}{2}\sin 4t - \tfrac{1}{2}\sin 2t$$

MATCHED PROBLEM 1

Write the product $\cos 5\theta \cos 2\theta$ as a sum or difference.

EXAMPLE 2 Finding Exact Values

Evaluate $\sin 105° \sin 15°$ exactly using an appropriate product–sum identity.

SOLUTION

$$\sin x \sin y = \tfrac{1}{2}[\cos(x - y) - \cos(x + y)] \quad \text{Let } x = 105° \text{ and } y = 15°.$$
$$\sin 105° \sin 15° = \tfrac{1}{2}[\cos(105° - 15°) - \cos(105° + 15°)] \quad \text{Simplify.}$$
$$= \tfrac{1}{2}[\cos 90° - \cos 120°] \quad \cos 90° = 0,\ \cos 120° = -\tfrac{1}{2}$$
$$= \tfrac{1}{2}[0 - (-\tfrac{1}{2})] = \tfrac{1}{4} \text{ or } 0.25$$

MATCHED PROBLEM 2

Evaluate $\cos 165° \sin 75°$ exactly using an appropriate product–sum identity.

EXAMPLE 3 **Finding Exact Values**

Evaluate $\sin(19\pi/24)\cos(13\pi/24)$ exactly using an appropriate product–sum identity.

SOLUTION

$$\sin x \cos y = \tfrac{1}{2}[\sin(x+y) + \sin(x-y)] \qquad \text{Let } x = \frac{19\pi}{24} \text{ and } y = \frac{13\pi}{24}.$$

$$\sin\frac{19\pi}{24}\cos\frac{13\pi}{24} = \frac{1}{2}\left[\sin\left(\frac{19\pi}{24}+\frac{13\pi}{24}\right) + \sin\left(\frac{19\pi}{24}-\frac{13\pi}{24}\right)\right] \qquad \text{Add fractions.}$$

$$= \frac{1}{2}\left(\sin\frac{32\pi}{24} + \sin\frac{6\pi}{24}\right) \qquad \text{Reduce fractions.}$$

$$= \frac{1}{2}\left(\sin\frac{4\pi}{3} + \sin\frac{\pi}{4}\right) \qquad \sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2},\ \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2}\left(-\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\right) \qquad \text{Simplify.}$$

$$= \frac{\sqrt{2}-\sqrt{3}}{4}$$

MATCHED PROBLEM 3

Evaluate $\cos(23\pi/24)\cos(17\pi/24)$ exactly using an appropriate product–sum identity.

› Sum–Product Identities

The product–sum identities can be transformed into equivalent forms called **sum–product identities.** These identities are used to express sums and differences involving sines and cosines as products involving sines and cosines. We illustrate the transformation for one identity. The other three identities can be obtained by following similar procedures.

We start with a product–sum identity:

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \tag{1}$$

We would like

$$\alpha + \beta = x$$
$$\alpha - \beta = y$$

Solving this system, we have

$$\alpha = \frac{x+y}{2} \qquad \beta = \frac{x-y}{2} \tag{2}$$

Substituting equation (2) into equation (1) and simplifying, we obtain

$$\sin x + \sin y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2}$$

All four sum–product identities are listed next for convenient reference.

SUM–PRODUCT IDENTITIES

$$\sin x + \sin y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2}$$

$$\sin x - \sin y = 2 \cos \frac{x + y}{2} \sin \frac{x - y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x + y}{2} \cos \frac{x - y}{2}$$

$$\cos x - \cos y = -2 \sin \frac{x + y}{2} \sin \frac{x - y}{2}$$

EXAMPLE 4 A Difference as a Product

Write the difference $\sin 7\theta - \sin 3\theta$ as a product.

SOLUTION

$$\sin x - \sin y = 2 \cos \frac{x + y}{2} \sin \frac{x - y}{2} \qquad \text{Let } x = 7\theta \text{ and } y = 3\theta.$$

$$\sin 7\theta - \sin 3\theta = 2 \cos \frac{7\theta + 3\theta}{2} \sin \frac{7\theta - 3\theta}{2} \qquad \text{Simplify.}$$

$$= 2 \cos 5\theta \sin 2\theta$$

MATCHED PROBLEM 4

Write the sum $\cos 3t + \cos t$ as a product.

EXAMPLE 5 Finding Exact Values

Find the exact value of $\sin 105° - \sin 15°$ using an appropriate sum–product identity.

SOLUTION

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$ Let $x = 105°$ and $y = 15°$.

$$\sin 105° - \sin 15° = 2\cos\frac{105° + 15°}{2}\sin\frac{105° - 15°}{2}$$ Simplify.

$$= 2\cos 60°\sin 45°$$ $\cos 60° = \frac{1}{2}$, $\sin 45° = \frac{\sqrt{2}}{2}$

$$= 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$$

MATCHED PROBLEM 5

Find the exact value of $\cos 165° - \cos 75°$ using an appropriate sum–product identity.

EXAMPLE 6 Finding Exact Values

Find the exact value of $\cos(19\pi/12) + \cos(\pi/12)$ using an appropriate sum–product identity.

SOLUTION

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$ Let $x = \frac{19\pi}{12}$ and $y = \frac{\pi}{12}$.

$$\cos\frac{19\pi}{12} + \cos\frac{\pi}{12} = 2\cos\frac{19\pi/12 + \pi/12}{2}\cos\frac{19\pi/12 - \pi/12}{2}$$ Simplify the compound fractions.

$$= 2\cos\frac{20\pi}{24}\cos\frac{18\pi}{24}$$ Reduce fractions.

$$= 2\cos\frac{5\pi}{6}\cos\frac{3\pi}{4}$$ $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$, $\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$

$$= 2\left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{\sqrt{2}}\right)$$ Simplify.

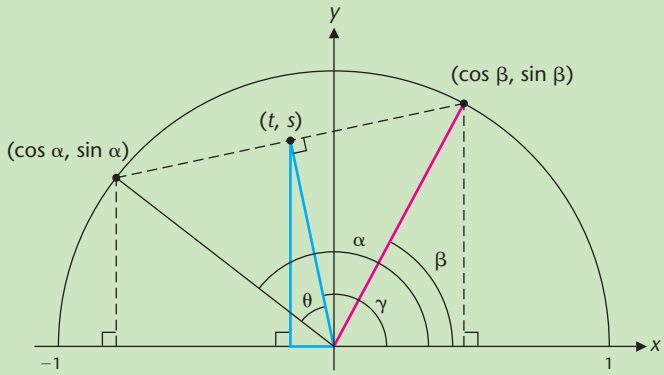
$$= \frac{\sqrt{6}}{2}$$

MATCHED PROBLEM 6

Find the exact value of $\sin(17\pi/12) + \sin(11\pi/12)$ using an appropriate sum–product identity.

››› EXPLORE-DISCUSS 1

The following "proof without words" of two of the sum–product identities is based on a similar "proof" by Sidney H. Kung, Jacksonville University, that was printed in the October 1996 issue of *Mathematics Magazine*. Discuss how the relationships below the figure are verified from the figure.



$$\theta = \frac{\alpha - \beta}{2} \qquad \gamma = \frac{\alpha + \beta}{2}$$

$$\frac{\sin\alpha + \sin\beta}{2} = s = \cos\frac{\alpha - \beta}{2}\sin\frac{\alpha + \beta}{2}$$

$$\frac{\cos\alpha + \cos\beta}{2} = t = \cos\frac{\alpha - \beta}{2}\cos\frac{\alpha + \beta}{2}$$

ANSWERS TO MATCHED PROBLEMS

1. $\frac{1}{2}\cos 7\theta + \frac{1}{2}\cos 3\theta$ **2.** $(-\sqrt{3} - 2)/4$ **3.** $(1 + \sqrt{2})/4$
4. $2\cos 2t\cos t$ **5.** $-\sqrt{6}/2$ **6.** $-\sqrt{2}/2$

6-4 Exercises

1. What is a product–sum identity?

2. What is a sum–product identity?

3. Which identity is obtained if you add, left side to left side and right side to right side, the first two product–sum identities?

4. Which identity is obtained if you add, left side to left side and right side to right side, the last two product–sum identities?

5. Which identities are obtained if you substitute x for y in the product–sum identities?

6. Which identities are obtained if you substitute x for y in the sum–product identities?

In Problems 7–10, write each product as a sum or difference involving sine and cosine.

7. $\sin 3m \cos m$ **8.** $\cos 7A \cos 5A$

9. $\sin u \sin 3u$ **10.** $\cos 2\theta \sin 3\theta$

In Problems 11–14, write each difference or sum as a product involving sines and cosines.

11. $\sin 3t + \sin t$ **12.** $\cos 7\theta + \cos 5\theta$

13. $\cos 5w - \cos 9w$ **14.** $\sin u - \sin 5u$

Evaluate Problems 15–26 exactly using an appropriate identity.

15. $\sin 195^\circ \cos 75^\circ$ **16.** $\cos 75^\circ \sin 15^\circ$

17. $\cos 157.5^\circ \cos 67.5^\circ$ **18.** $\sin 112.5^\circ \sin 22.5^\circ$

19. $\cos 37.5^\circ \sin 7.5^\circ$ **20.** $\sin 262.5^\circ \cos 52.5^\circ$

21. $\sin \frac{5\pi}{8} \sin \frac{\pi}{8}$ **22.** $\cos \frac{3\pi}{8} \cos \frac{7\pi}{8}$

23. $\cos \frac{11\pi}{12} \cos \frac{\pi}{12}$ **24.** $\cos \frac{7\pi}{12} \sin \frac{5\pi}{12}$

25. $\sin \frac{13\pi}{24} \cos \frac{5\pi}{24}$ **26.** $\sin \frac{17\pi}{24} \sin \frac{\pi}{24}$

Evaluate Problems 27–34 exactly using an appropriate identity.

27. $\sin 195^\circ - \sin 105^\circ$ **28.** $\cos 105^\circ + \cos 15^\circ$

29. $\cos 75^\circ - \cos 15^\circ$ **30.** $\sin 165^\circ + \sin 105^\circ$

31. $\cos \frac{17\pi}{12} + \cos \frac{\pi}{12}$ **32.** $\cos \frac{13\pi}{12} - \cos \frac{5\pi}{12}$

33. $\sin \frac{\pi}{12} + \sin \frac{5\pi}{12}$ **34.** $\sin \frac{11\pi}{12} - \sin \frac{7\pi}{12}$

Use sum and difference identities to verify the identities in Problems 35 and 36.

35. $\cos x \cos y = \frac{1}{2}[\cos (x + y) + \cos (x - y)]$

36. $\sin x \sin y = \frac{1}{2}[\cos (x - y) - \cos (x + y)]$

37. Explain how you can transform the product–sum identity

$$\sin u \sin v = \tfrac{1}{2}[\cos (u - v) - \cos (u + v)]$$

into the sum–product identity

$$\cos x - \cos y = -2 \sin \frac{x + y}{2} \sin \frac{x - y}{2}$$

by a suitable substitution.

38. Explain how you can transform the product–sum identity

$$\cos u \cos v = \tfrac{1}{2}[\cos (u + v) + \cos (u - v)]$$

into the sum–product identity

$$\cos x + \cos y = 2 \cos \frac{x + y}{2} \cos \frac{x - y}{2}$$

by a suitable substitution.

Verify each identity in Problems 39–46.

39. $\dfrac{\sin 2t + \sin 4t}{\cos 2t - \cos 4t} = \cot t$ **40.** $\dfrac{\cos t - \cos 3t}{\sin t + \sin 3t} = \tan t$

41. $\dfrac{\sin x - \sin y}{\cos x - \cos y} = -\cot \dfrac{x + y}{2}$

42. $\dfrac{\sin x + \sin y}{\cos x + \cos y} = \tan \dfrac{x + y}{2}$

43. $\dfrac{\cos x + \cos y}{\sin x - \sin y} = \cot \dfrac{x - y}{2}$

44. $\dfrac{\cos x - \cos y}{\sin x + \sin y} = -\tan \dfrac{x - y}{2}$

45. $\dfrac{\cos x + \cos y}{\cos x - \cos y} = -\cot \dfrac{x + y}{2} \cot \dfrac{x - y}{2}$

46. $\dfrac{\sin x + \sin y}{\sin x - \sin y} = \dfrac{\tan [\frac{1}{2}(x + y)]}{\tan [\frac{1}{2}(x - y)]}$

In Problems 47–54, show that the equation is not an identity by finding a value of x and a value of y for which both sides are defined but are not equal.

47. $\sin x \cos y = \sin x + \cos y$

48. $\cos x \sin y = \cos x - \sin y$

49. $\sin x \sin y = \sin (x + y)$ **50.** $\cos x \cos y = \cos (x + y)$

51. $\cos x + \cos y = (\cos x)(\cos y)$

52. $\sin x + \sin y = (\sin x)(\sin y)$

53. $\sin x - \sin y = \cos \dfrac{x + y}{2} \sin \dfrac{x - y}{2}$

54. $\cos x - \cos y = 2 \sin \dfrac{x + y}{2} \sin \dfrac{x - y}{2}$

In Problems 55–60, is the equation an identity? Explain.

55. $\sin 3x - \sin x = 2 \cos 2x \sin x$

56. $2 \sin x \cos 2x = \sin x + \sin 3x$

57. $\cos 3x - \cos x = 2 \sin 2x \sin x$

58. $2 \cos 3x \cos 5x = \cos 8x + \cos 2x$

59. $\cos x + \cos 5x = 2 \cos 2x \cos 3x$

60. $2 \sin 4x \cos 2x = \sin 8x + \sin 2x$

Verify each of the following identities for the values of x and y indicated in Problems 61–64. Evaluate each side to five significant digits.

(A) $\cos x \sin y = \frac{1}{2}[\sin (x + y) - \sin (x - y)]$

(B) $\cos x + \cos y = 2 \cos \dfrac{x + y}{2} \cos \dfrac{x - y}{2}$

61. $x = 172.63°, y = 20.177°$

62. $x = 50.137°, y = 18.044°$

63. $x = 1.1255, y = 3.6014$

64. $x = 0.039\,17, y = 0.610\,52$

In Problems 65–72, write each as a product if y is a sum or difference, or as a sum or difference if y is a product. Enter the original equation in a graphing calculator as y_1, the converted form as y_2, and graph y_1 and y_2 in the same viewing window. Use TRACE to compare the two graphs.

65. $y = \sin 2x + \sin x$

66. $y = \cos 3x + \cos x$

67. $y = \cos 1.7x - \cos 0.3x$

68. $y = \sin 2.1x - \sin 0.5x$

69. $y = \sin 3x \cos x$

70. $y = \cos 5x \cos 3x$

71. $y = \sin 2.3x \sin 0.7x$

72. $y = \cos 1.9x \sin 0.5x$

Verify each identity in Problems 73 and 74.

73. $\cos x \cos y \cos z = \frac{1}{4}[\cos (x + y - z) + \cos (y + z - x) + \cos (z + x - y) + \cos (x + y + z)]$

74. $\sin x \sin y \sin z = \frac{1}{4}[\sin (x + y - z) + \sin (y + z - x) + \sin (z + x - y) - \sin (x + y + z)]$

In Problems 75–78,
(A) Graph y_1, y_2, and y_3 in a graphing calculator for $0 \le x \le 1$ and $-2 \le y \le 2$.
(B) Convert y_1 to a sum or difference and repeat part A.

75. $y_1 = 2 \cos (28\pi x) \cos (2\pi x)$
$y_2 = 2 \cos (2\pi x)$
$y_3 = -2 \cos (2\pi x)$

76. $y_1 = 2 \sin (24\pi x) \sin (2\pi x)$
$y_2 = 2 \sin (2\pi x)$
$y_3 = -2 \sin (2\pi x)$

77. $y_1 = 2 \sin (20\pi x) \cos (2\pi x)$
$y_2 = 2 \cos (2\pi x)$
$y_3 = -2 \cos (2\pi x)$

78. $y_1 = 2 \cos (16\pi x) \sin (2\pi x)$
$y_2 = 2 \sin (2\pi x)$
$y_3 = -2 \sin (2\pi x)$

APPLICATIONS

Problems 79 and 80 involve the phenomenon of sound called beats. If two tones having the same loudness and close together in pitch (frequency) are sounded, one following the other, most people have difficulty in differentiating the two tones. However, if the tones are sounded simultaneously, they will interact with each other, producing a low warbling sound called a ***beat.*** *Musicians, when tuning an instrument with other instruments or a tuning fork, listen for these lower beat frequencies and try to eliminate them by adjusting their instruments. Problems 79 and 80 provide a visual illustration of the beat phenomenon.*

79. MUSIC—BEAT FREQUENCIES The equations $y = 0.5 \cos 128\pi t$ and $y = -0.5 \cos 144\pi t$ model sound waves with frequencies 64 and 72 hertz, respectively. If both sounds are emitted simultaneously, a *beat* frequency results.

(A) Show that

$$0.5 \cos 128\pi t - 0.5 \cos 144\pi t = \sin 8\pi t \sin 136\pi t$$

(The product form is more useful to sound engineers.)
(B) Graph each equation in a different viewing window for $0 \le t \le 0.25$:

$$y = 0.5 \cos 128\pi t$$
$$y = -0.5 \cos 144\pi t$$
$$y = 0.5 \cos 128\pi t - 0.5 \cos 144\pi t$$
$$y = \sin 8\pi t \sin 136\pi t$$

80. MUSIC—BEAT FREQUENCIES The equations $y = 0.25 \cos 256\pi t$ and $y = -0.25 \cos 288\pi t$ model sound waves with frequencies 128 and 144 hertz, respectively. If both sounds are emitted simultaneously, a *beat* frequency results.
(A) Show that

$$0.25 \cos 256\pi t - 0.25 \cos 288\pi t = 0.5 \sin 16\pi t \sin 272\pi t$$

(The product form is more useful to sound engineers.)
(B) Graph each equation in a different viewing window for $0 \le t \le 0.125$:

$$y = 0.25 \cos 256\pi t$$
$$y = -0.25 \cos 288\pi t$$
$$y = 0.25 \cos 256\pi t - 0.25 \cos 288\pi t$$
$$y = 0.5 \sin 16\pi t \sin 272\pi t$$

6-5 Trigonometric Equations

- Solving Trigonometric Equations Using an Algebraic Approach
- Solving Trigonometric Equations Using a Graphing Calculator

The first four sections of this chapter consider trigonometric equations called *identities*. These are equations that are true for all replacements of the variable(s) for which both sides are defined. We now consider another class of trigonometric equations, called **conditional equations,** which may be true for some replacements of the variable but false for others. For example,

$$\cos x = \sin x$$

is a conditional equation, because it is true for some values of x, for example, $x = \pi/4$, and false for others, such as $x = 0$. (Check both values.)

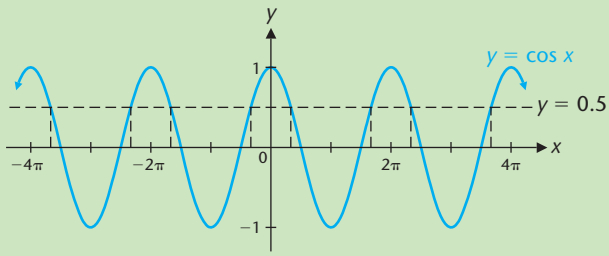
In this section, we present two approaches for solving conditional trigonometric equations: an algebraic approach and a graphing calculator approach. Solving trigonometric equations using an algebraic approach often requires the use of algebraic manipulation, identities, and ingenuity. In some cases algebraic methods lead to exact solutions, which are very useful in certain contexts. Graphing calculator methods can be used to approximate solutions to a greater variety of trigonometric equations, but usually do not produce exact solutions. Each method has its strengths.

››› EXPLORE-DISCUSS 1

We are interested in solutions to the equation

$$\cos x = 0.5$$

The figure shows a partial graph of the left and right sides of the equation.



(A) How many solutions does the equation have on the interval $[0, 2\pi)$? What are they?

(B) How many solutions does the equation have on the interval $(-\infty, \infty)$? Discuss a method of writing all solutions to the equation.

Solving Trigonometric Equations Using an Algebraic Approach

You might find the following suggestions for solving trigonometric equations using an algebraic approach useful:

SUGGESTIONS FOR SOLVING TRIGONOMETRIC EQUATIONS ALGEBRAICALLY

1. Regard one particular trigonometric function as a variable, and solve for it.
   (A) Consider using algebraic manipulation such as factoring, combining or separating fractions, and so on.
   (B) Consider using identities.
2. After solving for a trigonometric function, solve for the variable.

Examples 1–6 should help make the algebraic approach clear.

EXAMPLE 1 Finding Exact Solutions

Find all solutions exactly for $5 + 5 \tan \theta = 0$, θ in degrees.

SOLUTION

Step 1. *Solve for* $\tan \theta$.

$$\begin{aligned} 5 + 5 \tan \theta &= 0 && \text{Subtract 5 from both sides.} \\ 5 \tan \theta &= -5 && \text{Divide both sides by 5.} \\ \tan \theta &= -1 \end{aligned}$$

Step 2. *Solve the equation over one period* $(-90°, 90°)$. Recall that the period of the tangent function is 180°. Sketch a graph of $y = \tan \theta$ and $y = -1$ in the same coordinate system to provide an aid to writing all solutions over one period (Fig. 1)

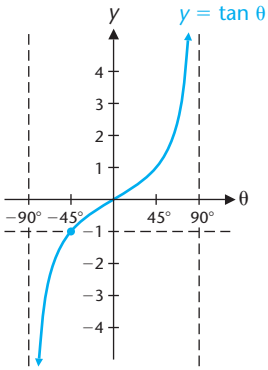


Figure 1

$$\begin{aligned} \tan \theta &= -1 \\ \theta &= -45° \end{aligned}$$

Step 3. *Write an expression for all solutions*. Because the tangent function is periodic with period 180°, all solutions are given by

$$\theta = -45° + k(180°), \; k \text{ any integer}$$

(In other words, all solutions are obtained by adding a multiple of 180° to −45°; for example, for $k = -2, -1, 0, 1$, and 2, the corresponding solutions are −405°, −225°, −45°, 135°, and 315°).

MATCHED PROBLEM 1

Find all solutions exactly for $3 - 3\tan\theta = 0$, θ in degrees.

››› CAUTION ›››

There is only one solution set for an equation, but there may be more than one way to specify that solution set. Consider the sets S and T:

$$S = \{-45° + k(180°) \mid k \text{ any integer}\}$$
$$T = \{-45° + m(360°), 135° + m(360°) \mid m \text{ any integer}\}$$

If k is even, then $-45° + k(180°)$ can be written in the form $-45° + m(360°)$ (for $m = k/2$); and if k is odd, then $-45° + k(180°)$ can be written in the form $135° + m(360°)$ [for $m = (k - 1)/2$]. Therefore, $S = T$ (that is, S and T are the same set). Although S and T are both correct answers to Example 1, we prefer S because of its shorter description.

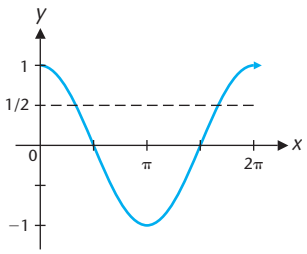
EXAMPLE 2 Exact Solutions Using Factoring

Find all solutions exactly for $2\cos^2 x - \cos x = 0$.

SOLUTION

Step 1. *Solve for cos x.*

$$2\cos^2 x - \cos x = 0 \qquad \text{Factor: } 2a^2 - a = a(2a - 1)$$
$$\cos x(2\cos x - 1) = 0 \qquad ab = 0 \text{ only if } a = 0 \text{ or } b = 0$$
$$\cos x = 0 \quad \text{or} \quad 2\cos x - 1 = 0$$
$$\cos x = \tfrac{1}{2}$$



› Figure 2

Step 2. *Solve each equation over one period [0, 2π).* Sketch a graph of $y = \cos x$, $y = 0$, and $y = \frac{1}{2}$ in the same coordinate system to provide an aid to writing all solutions over one period (Fig. 2).

$$\cos x = 0 \qquad \cos x = \tfrac{1}{2}$$
$$x = \pi/2, 3\pi/2 \qquad x = \pi/3, 5\pi/3$$

Step 3. *Write an expression for all solutions.* Because the cosine function is periodic with period 2π, all solutions are given by:

$$x = \begin{cases} \pi/3 + 2k\pi \\ \pi/2 + 2k\pi \\ 3\pi/2 + 2k\pi \\ 5\pi/3 + 2k\pi \end{cases} \qquad k \text{ any integer}$$

MATCHED PROBLEM 2

Find all solutions exactly for $2\sin^2 x + \sin x = 0$.

EXAMPLE 3 **Approximate Solutions Using Identities and Factoring**

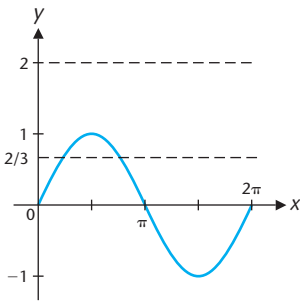
Find all real solutions for $3\cos^2 x + 8\sin x = 7$. Compute all inverse functions to four decimal places.

SOLUTION

Step 1. *Solve for sin x and/or cos x.* Move all nonzero terms to the left of the equal sign and express the left side in terms of $\sin x$:

$$\begin{aligned} 3\cos^2 x + 8\sin x &= 7 && \text{Subtract 7 from both sides.} \\ 3\cos^2 x + 8\sin x - 7 &= 0 && \cos^2 x = 1 - \sin^2 x \\ 3(1 - \sin^2 x) + 8\sin x - 7 &= 0 && \text{Multiply both sides by } -1 \text{ and simplify.} \\ 3\sin^2 x - 8\sin x + 4 &= 0 && \text{Factor: } 3u^2 - 8u + 4 = (u-2)(3u-2) \\ (\sin x - 2)(3\sin x - 2) &= 0 && ab = 0 \text{ only if } a = 0 \text{ or } b = 0 \\ \sin x - 2 &= 0 \quad \text{or} \quad 3\sin x - 2 = 0 \\ \sin x &= 2 \qquad\quad \sin x = \tfrac{2}{3} \end{aligned}$$

Step 2. *Solve each equation over one period [0, 2π):* Sketch a graph of $y = \sin x$, $y = 2$, and $y = \frac{2}{3}$ in the same coordinate system to provide an aid to writing all solutions over one period (Fig. 3).



› Figure 3

Solve the first equation:

$$\sin x = 2 \qquad \text{No solution, because } -1 \le \sin x \le 1.$$

Solve the second equation:

$$\begin{aligned} &\sin x = \tfrac{2}{3} && \text{From the graph we see there are solutions in the first and second quadrants.} \\ &x = \sin^{-1}\tfrac{2}{3} = 0.7297 && \text{First-quadrant solution} \\ &x = \pi - 0.7297 = 2.4119 && \text{Second-quadrant solution} \end{aligned}$$

CHECK

$$\sin 0.7297 = 0.6667; \sin 2.4119 = 0.6666$$

(Checks may not be exact because of roundoff errors.)

Step 3. *Write an expression for all solutions.* Because the sine function is periodic with period 2π, all solutions are given by

$$x = \begin{cases} 0.7297 + 2k\pi \\ 2.4119 + 2k\pi \end{cases} \quad k \text{ any integer}$$

MATCHED PROBLEM 3

Find all real solutions to $8 \sin^2 x = 5 - 10 \cos x$. Compute all inverse functions to four decimal places.

EXAMPLE 4 **Approximate Solutions Using Substitution**

Find θ in degree measure to three decimal places so that $5 \sin (2\theta - 5) = -3.045$, $0° \le 2\theta - 5 \le 360°$.

SOLUTION

Step 1. *Make a substitution.* Let $u = 2\theta - 5$ to obtain

$$5 \sin u = -3.045, \ 0° \le u \le 360°$$

Step 2. *Solve for sin u.*

$$\sin u = \frac{-3.045}{5} = -0.609$$

Step 3. *Solve for u over $0° \le u \le 360°$.* Sketch a graph of $y = \sin u$ and $y = -0.609$ in the same coordinate system to provide an aid to writing all solutions over $0° \le u \le 360°$ (Fig. 4).

Solutions are in the third and fourth quadrants. If the reference angle is α, then $u = 180° + \alpha$ or $u = 360° - \alpha$.

$$\begin{aligned} \alpha = \sin^{-1} 0.609 &= 37.517° && \text{Reference angle} \\ u &= 180° + 37.517° \\ &= 217.517° && \text{Third quadrant solution} \\ u &= 360° - 37.517° \\ &= 322.483° && \text{Fourth quadrant solution} \end{aligned}$$

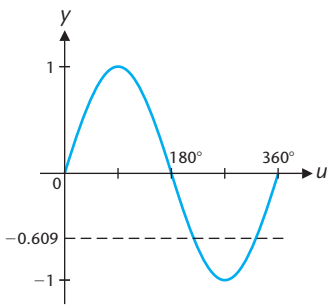


Figure 4

CHECK

$$\sin 217.517^\circ = -0.609;\ \sin 322.483^\circ = -0.609$$

Step 4. *Now solve for* θ*:*

$$\begin{aligned} u &= 217.517^\circ & u &= 322.483^\circ \\ 2\theta - 5 &= 217.517^\circ & 2\theta - 5 &= 322.483^\circ \\ \theta &= 111.259^\circ & \theta &= 163.742^\circ \end{aligned}$$

A final check in the original equation is left to the reader.

MATCHED PROBLEM 4

Find θ in degree measure to three decimal places so that $8 \tan(6\theta + 15) = -64.328$, $-90^\circ < 6\theta + 15 < 90^\circ$.

EXAMPLE 5 Exact Solutions Using Identities and Factoring

Find exact solutions for $\sin^2 x = \frac{1}{2}\sin 2x$, $0 \le x \le 2\pi$.

SOLUTION

The following solution includes only the key steps. Sketch graphs as appropriate on scratch paper.

$$\begin{aligned} \sin^2 x &= \tfrac{1}{2}\sin 2x && \text{Use double-angle identity.} \\ &= \tfrac{1}{2}(2\sin x \cos x) && \text{Subtract } \sin x \cos x \text{ from both sides.} \\ \sin^2 x - \sin x \cos x &= 0 && \text{Factor: } a^2 - ab = a(a - b) \\ \sin x(\sin x - \cos x) &= 0 && a(a - b) = 0 \text{ only if } a = 0 \text{ or } a - b = 0 \\ \sin x &= 0 \quad \text{or} \quad \sin x - \cos x = 0 \\ x &= 0, \pi \qquad\qquad \sin x = \cos x \\ & \qquad\qquad \frac{\sin x}{\cos x} = 1 \\ & \qquad\qquad \tan x = 1 \\ & \qquad\qquad x = \pi/4, 5\pi/4 \end{aligned}$$

Combining the solutions from both equations, we have the complete set of solutions:

$$x = 0, \pi/4, \pi, 5\pi/4$$

MATCHED PROBLEM 5

Find exact solutions for $\sin 2x = \sin x$, $0 \le x \le 2\pi$.

EXAMPLE 6 **Approximate Solutions Using Identities and the Quadratic Formula**

Solve $\cos 2x = 4 \cos x - 2$ for all real x. Compute inverse functions to four decimal places.

SOLUTION

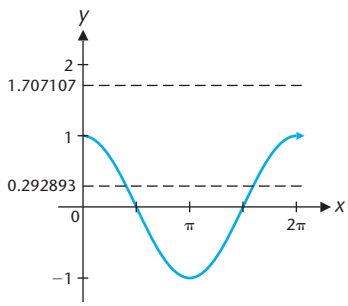
Step 1. *Solve for cos x.*

$$\cos 2x = 4 \cos x - 2 \qquad \text{Use double-angle identity.}$$
$$2 \cos^2 x - 1 = 4 \cos x - 2 \qquad \text{Subtract 4 cos } x \text{ and add 2 to both sides.}$$
$$2 \cos^2 x - 4 \cos x + 1 = 0 \qquad \text{Quadratic in cos } x\text{. Left side does not factor using integer coefficients. Solve using quadratic formula.}$$

$$\cos x = \frac{4 \pm \sqrt{16 - 4(2)(1)}}{2(2)}$$
$$= 1.707107 \text{ or } 0.292893$$

Step 2. *Solve each equation over one period [0, 2π):* Sketch a graph of $y = \cos x$, $y = 1.707107$, and $y = 0.292893$ in the same coordinate system to provide an aid to writing all solutions over one period (Fig. 5).

› Figure 5



Solve the first equation:

$$\cos x = 1.707107 \qquad \text{No solution, because } -1 \le \cos x \le 1$$

Solve the second equation:

$$\cos x = 0.292893$$

Figure 5 indicates a first quadrant solution and a fourth quadrant solution. If the reference angle is α, then $x = \alpha$ or $x = 2\pi - \alpha$.

$$\alpha = \cos^{-1} 0.292893 = 1.2735$$
$$2\pi - \alpha = 2\pi - 1.2735 = 5.0096$$

CHECK

$$\cos 1.2735 = 0.292936; \cos 5.0096 = 0.292854$$

Step 3. *Write an expression for all solutions.* Because the cosine function is periodic with period 2π, all solutions are given by:

$$x = \begin{cases} 1.2735 + 2k\pi \\ 5.0096 + 2k\pi \end{cases} \quad k \text{ any integer}$$

MATCHED PROBLEM 6

Solve $\cos 2x = 2(\sin x - 1)$ for all real x. Compute inverse functions to four decimal places.

Solving Trigonometric Equations Using a Graphing Calculator

All the trigonometric equations that were solved earlier with algebraic methods can also be solved, though usually not exactly, with graphing calculator methods. In addition, there are many trigonometric equations that can be solved approximately using graphing calculator methods, but cannot be solved exactly in a finite sequence of steps using algebraic methods. Examples 7–9 are examples of such equations.

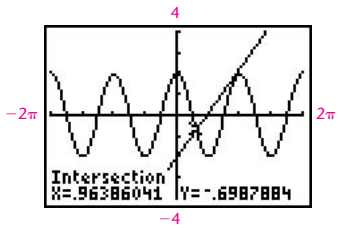
EXAMPLE 7 Solution Using a Graphing Calculator

Find all real solutions to four decimal places for $2\cos 2x = 1.35x - 2$.

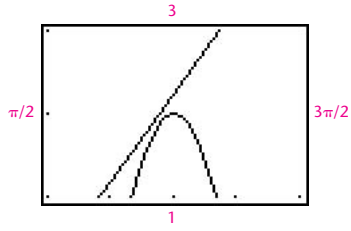
SOLUTION

This relatively simple trigonometric equation cannot be solved using a finite number of algebraic steps (try it!). However, it can be solved rather easily to the accuracy desired using a graphing calculator. Graph $y_1 = 2\cos 2x$ and $y_2 = 1.35x - 2$ in the same viewing window, and find any points of intersection using the INTERSECT command. The first point of intersection is shown in Figure 6. It appears there may be more than one point of intersection, but zooming in on the portion of the graph in question shows that the two graphs do not intersect in that region (Fig. 7). The only solution is

$$x = 0.9639$$



› Figure 6



› Figure 7

CHECK

Left side: $2 \cos 2(0.9639) = -0.6989$

Right side: $1.35(0.9639) - 2 = -0.6987$

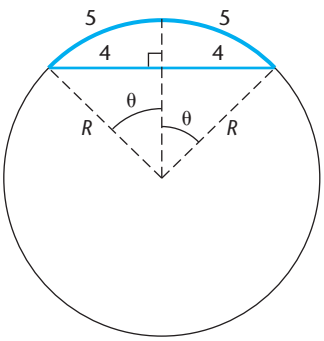
MATCHED PROBLEM 7

Find all real solutions to four decimal places for $\sin x/2 = 0.2x - 0.5$.

EXAMPLE 8 Geometric Application

A 10-centimeter arc on a circle has an 8-centimeter chord. What is the radius of the circle to four decimal places? What is the radian measure of the central angle, to four decimal places, subtended by the arc?

SOLUTION



› Figure 8

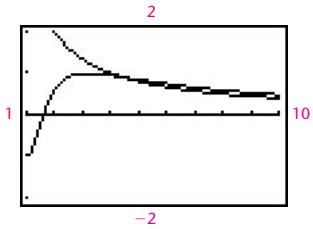
Sketch a figure with auxiliary lines (Fig. 8). From the figure, θ in radians is

$$\theta = \frac{5}{R} \quad \text{and} \quad \sin\theta = \frac{4}{R}$$

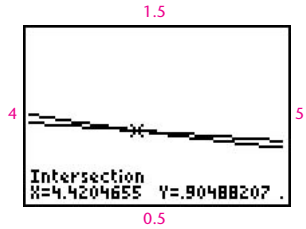
Therefore,

$$\sin\frac{5}{R} = \frac{4}{R}$$

and our problem is to solve this trigonometric equation for R. Algebraic methods will not isolate R, so turn to the use of a graphing calculator. Start by graphing $y_1 = \sin 5/x$ and $y_2 = 4/x$ in the same viewing window for $1 \le x \le 10$ and $-2 \le y \le 2$ (Fig. 9). It appears that the graphs intersect for x between 4 and 5. To get a clearer look at the intersection point we change the window dimensions to $4 \le x \le 5$ and $0.5 \le y \le 1.5$, and use the INTERSECT command to find the point of intersection (Fig. 10).



› Figure 9



› Figure 10

From Figure 10, we see that

$$R = 4.4205 \text{ centimeters}$$

CHECK

$\sin 5/R = \sin(5/4.4205) = 0.9049$ and $4/R = 4/4.4205 = 0.9049$

Having R, we can compute the radian measure of the central angle subtended by the 10-centimeter arc:

$$\text{Central angle} = \frac{10}{R} = \frac{10}{4.4205} = 2.2622 \text{ radians}$$

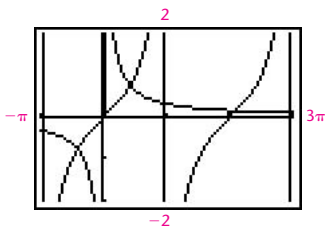
MATCHED PROBLEM 8

An 8.2456-inch arc on a circle has a 6.0344-inch chord. What is the radius of the circle to four decimal places? What is the measure of the central angle, to four decimal places, subtended by the arc?

EXAMPLE 9 Solution Using a Graphing Calculator

Find all real solutions, to four decimal places, for $\tan(x/2) = 1/x$, $-\pi < x \leq 3\pi$.

SOLUTION



› Figure 11

Graph $y_1 = \tan(x/2)$ and $y_2 = 1/x$ in the same viewing window for $-\pi < x < 3\pi$ (Fig. 11). Solutions are at points of intersection.

Using the INTERSECT command, the three solutions are found to be

$$x = -1.3065, 1.3065, 6.5846$$

Checking these solutions is left to the reader.

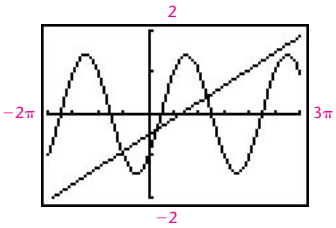
MATCHED PROBLEM 9

Find all real solutions, to four decimal places, for $0.25 \tan(x/2) = \ln x$, $0 < x < 4\pi$.

Solving trigonometric inequalities using a graphing calculator is as easy as solving trigonometric equations using a graphing calculator. Example 10 illustrates the process.

EXAMPLE 10 Solving a Trigonometric Inequality

Solve $\sin x - \cos x < 0.25x - 0.5$, using two-decimal-place accuracy.



› Figure 12

SOLUTION

Graph $y_1 = \sin x - \cos x$ and $y_2 = 0.25x - 0.5$ in the same viewing window (Fig. 12).

Finding the three points of intersection by the INTERSECT command, we see that the graph of y_1 is below the graph of y_2 on the following two intervals: $(-1.65, 0.52)$ and $(3.63, \infty)$. Therefore, the solution set to the inequality is $(-1.65, 0.52) \cup (3.63, \infty)$.

MATCHED PROBLEM 10

Solve $\cos x - \sin x > 0.4 - 0.3x$, using two-decimal-place accuracy.

››› EXPLORE-DISCUSS 2

How many solutions does the following equation have?

$$\sin(1/x) = 0 \tag{1}$$

Graph $y_1 = \sin(1/x)$ and $y_2 = 0$ for each of the indicated intervals in parts A–G. From each graph estimate the number of solutions that equation (1) appears to have. What final conjecture would you be willing to make regarding the number of solutions to equation (1)? Explain.

(A) $[-20, 20]$; Can 0 be a solution? Explain.

(B) $[-2, 2]$ **(C)** $[-1, 1]$ **(D)** $[-0.1, 0.1]$

(E) $[-0.01, 0.01]$ **(F)** $[-0.001, 0.001]$ **(G)** $[-0.0001, 0.0001]$

ANSWERS TO MATCHED PROBLEMS

1. $\theta = 45° + k(180°)$, k any integer

2. $x = \begin{cases} 0 + 2k\pi \\ \pi + 2k\pi \\ 7\pi/6 + 2k\pi \\ 11\pi/6 + 2k\pi \end{cases}$ k any integer **3.** $x = \begin{cases} 1.8235 + 2k\pi \\ 4.4597 + 2k\pi \end{cases}$ k any integer

4. $-16.318°$ **5.** $x = 0, \pi/3, \pi, 5\pi/3$ **6.** $x = \begin{cases} 0.9665 + 2k\pi \\ 2.1751 + 2k\pi \end{cases}$ k any integer

7. $x = 5.1609$ **8.** $R = 3.1103$ inches; central angle $= 2.6511$ radians

9. $x = 1.1828, 2.6369, 9.2004$ **10.** $(-1.67, 0.64) \cup (3.46, \infty)$

6-5 Exercises

1. What is a conditional equation?

2. Suppose that f is a periodic function and that you have found all solutions to the equation $f(x) = 0$ over one period of f. Explain how you can write down all solutions to $f(x) = 0$.

In Problems 3–20, find exact solutions over the indicated intervals (x a real number, θ in degrees).

3. $\sin x + 1 = 0, 0 \le x < 2\pi$

4. $\cos x - 1 = 0, 0 \le x < 2\pi$

5. $\sin x + 1 = 0$, all real x

6. $\cos x - 1 = 0$, all real x

7. $\tan \theta - 1 = 0, 0° \le \theta < 360°$

8. $\tan \theta + 1 = 0, 0° \le \theta < 360°$

9. $2 \sin x - 1 = 0, 0 \le x < 2\pi$

10. $2 \cos x - 1 = 0, 0 \le x < 2\pi$

11. $2 \sin x - 1 = 0$, all real x

12. $2 \cos x - 1 = 0$, all real x

13. $2 \sin \theta + \sqrt{3} = 0, 0° \le \theta < 360°$

14. $\sqrt{2} \cos \theta - 1 = 0, 0° \le \theta < 360°$

15. $2 \sin \theta + \sqrt{3} = 0$, all θ

16. $\sqrt{2} \cos \theta - 1 = 0$, all θ

17. $\tan x - \sqrt{3} = 0, 0 \le x < 2\pi$

18. $\sqrt{3} \tan x - 1 = 0, 0 \le x < 2\pi$

19. $\tan x - \sqrt{3} = 0$, all real x

20. $\sqrt{3} \tan x - 1 = 0$, all real x

Solve Problems 21–26 to four decimal places (θ in degrees, x real).

21. $7 \cos x - 3 = 0, 0 \le x < 2\pi$

22. $5 \cos x - 2 = 0, 0 \le x < 2\pi$

23. $2 \tan \theta - 7 = 0, 0° \le \theta < 180°$

24. $4 \tan \theta + 15 = 0, 0° \le \theta < 180°$

25. $1.3224 \sin x + 0.4732 = 0$, all real x

26. $5.0118 \sin x - 3.1105 = 0$, all real x

Solve Problems 27–30 to four decimal places using a graphing calculator.

27. $1 - x = 2 \sin x$, all real x

28. $2x - \cos x = 0$, all real x

29. $\tan (x/2) = 8 - x, 0 \le x < \pi$

30. $\tan 2x = 1 + 3x, 0 \le x < \pi/4$

In Problems 31–46, find exact solutions for x real and θ in degrees.

31. $2 \sin^2 \theta + \sin 2\theta = 0$, all θ

32. $\cos^2 \theta = \frac{1}{2} \sin 2\theta$, all θ

33. $\tan x = -2 \sin x, 0 \le x < 2\pi$

34. $\cos x = \cot x, 0 \le x < 2\pi$

35. $\sec (x/2) + 2 = 0, 0 \le x < 2\pi$

36. $\tan (x/2) - 1 = 0, 0 \le x < 2\pi$

37. $2 \cos^2 \theta + 3 \sin \theta = 0, 0° \le \theta < 360°$

38. $\sin^2 \theta + 2 \cos \theta = -2, 0° \le \theta < 360°$

39. $\cos 2\theta + \cos \theta = 0, 0° \le \theta < 360°$

40. $\cos 2\theta + \sin^2 \theta = 0, 0° \le \theta < 360°$

41. $2 \sin^2 (x/2) - 3 \sin (x/2) + 1 = 0, 0 \le x \le 2\pi$

42. $4 \cos^2 2x - 4 \cos 2x + 1 = 0, 0 \le x \le 2\pi$

43. $\cos^2 x + \sin^2 x = 1, 0 \le x < 2\pi$

44. $\sin x + \cos x = 3, 0 \le x < 2\pi$

45. $2 \sin \theta = 5 + \cos \theta, 0° \le \theta < 360°$

46. $2 \cos^2 \theta = 1 + \cos 2\theta, 0° \le \theta < 360°$

Solve Problems 47–52 (x real and θ in degrees). Compute inverse functions to four significant digits.

47. $6 \sin^2 \theta + 5 \sin \theta = 6, 0° \le \theta \le 90°$

48. $4 \cos^2 \theta = 7 \cos \theta + 2, 0° \le \theta \le 180°$

49. $3 \cos^2 x - 8 \cos x = 3, 0 \le x \le \pi$

50. $8 \sin^2 x + 10 \sin x = 3, 0 \le x \le \pi/2$

51. $2 \sin x = \cos 2x, 0 \le x < 2\pi$

52. $\cos 2x + 10 \cos x = 5, 0 \le x < 2\pi$

Solve Problems 53 and 54 for all real number solutions. Compute inverse functions to four significant digits.

53. $2 \sin^2 x = 1 - 2 \sin x$

54. $\cos^2 x = 3 - 5 \cos x$

Solve Problems 55–64 to four decimal places using a graphing calculator.

55. $2 \sin x = \cos 2x, 0 \le x < 2\pi$

56. $\cos 2x + 10 \cos x = 5, 0 \le x < 2\pi$

57. $2 \sin^2 x = 1 - 2 \sin x$, all real x

58. $\cos^2 x = 3 - 5 \cos x$, all real x

59. $\cos 2x > x^2 - 2$, all real x

60. $2 \sin (x - 2) < 3 - x^2$, all real x

61. $\cos (2x + 1) \le 0.5x - 2$, all real x

62. $\sin (3 - 2x) \ge 1 - 0.4x$, all real x

63. $e^{\sin x} = 2x - 1$, all real x

64. $e^{-\sin x} = 3 - x$, all real x

65. Explain the difference between evaluating $\tan^{-1}(-5.377)$ and solving the equation $\tan x = -5.377$.

66. Explain the difference between evaluating $\cos^{-1}(-0.7334)$ and solving the equation $\cos x = -0.7334$.

Find exact solutions to Problems 67–70. [Hint: Square both sides at an appropriate point, solve, then eliminate extraneous solutions at the end.]

67. $\cos x - \sin x = 1, 0 \le x < 2\pi$

68. $\sin x + \cos x = 1, 0 \le x < 2\pi$

69. $\tan x - \sec x = 1, 0 \le x < 2\pi$

70. $\sec x + \tan x = 1, 0 \le x < 2\pi$

Solve Problems 71–72 to four significant digits using a graphing calculator.

71. $\sin (1/x) = 1.5 - 5x, 0.04 \le x \le 0.2$

72. $2 \cos (1/x) = 950x - 4, 0.006 < x < 0.007$

73. We are interested in the zeros of $f(x) = \sin (1/x)$ for $x > 0$.

(A) Explore the graph of f over different intervals $[0.1, b]$ for various values of b, $b > 0.1$. Does the function f have a largest zero? If so, what is it (to four decimal places)? Explain what happens to the graph of f as x increases without bound. Does the graph have an asymptote? If so, what is its equation?

(B) Explore the graph of f over different intervals $(0, b]$ for various values of b, $0 < b \le 0.1$. How many zeros exist between 0 and b, for any $b > 0$, however small? Explain why this happens. Does f have a smallest positive zero? Explain.

74. We are interested in the zeros of $g(x) = \cos (1/x)$ for $x > 0$.

(A) Explore the graph of g over different intervals $[0.1, b]$ for various values of b, $b > 0.1$. Does the function g have a largest zero? If so, what is it (to four decimal places)? Explain what happens to the graph of g as x increases without bound. Does the graph have an asymptote? If so, what is its equation?

(B) Explore the graph of g over different intervals $(0, b]$ for various values of b, $0 < b \le 0.1$. How many zeros exist between 0 and b, for any $b > 0$, however small? Explain why this happens. Does g have a smallest positive zero? Explain.

APPLICATIONS

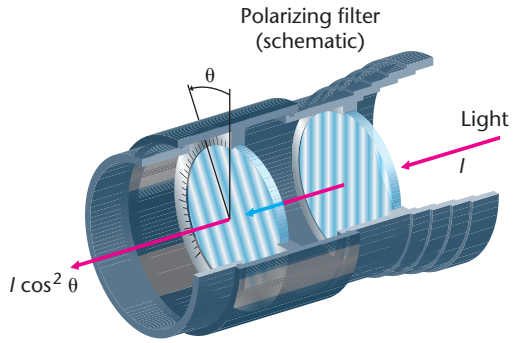
75. ELECTRIC CURRENT An alternating current generator produces a current given by the equation

$$I = 30 \sin 120\pi t$$

where t is time in seconds and I is current in amperes. Find the smallest positive t (to four significant digits) such that $I = -10$ amperes.

76. ELECTRIC CURRENT Refer to Problem 75. Find the smallest positive t (to four significant digits) such that $I = 25$ amperes.

77. OPTICS A polarizing filter for a camera contains two parallel plates of polarizing glass, one fixed and the other able to rotate. If θ is the angle of rotation from the position of maximum light transmission, then the intensity of light leaving the filter is $\cos^2 \theta$ times the intensity I of light entering the filter (see the figure).



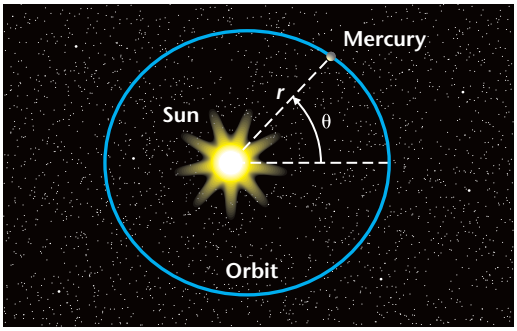
Find the smallest positive θ (in decimal degrees to two decimal places) so that the intensity of light leaving the filter is 40% of that entering.

78. OPTICS Refer to Problem 77. Find the smallest positive θ so that the light leaving the filter is 70% of that entering.

79. **ASTRONOMY** The planet Mercury travels around the sun in an elliptical orbit given approximately by

$$r = \frac{3.44 \times 10^7}{1 - 0.206 \cos \theta}$$

(see the figure). Find the smallest positive θ (in decimal degrees to three significant digits) such that Mercury is 3.09×10^7 miles from the sun.

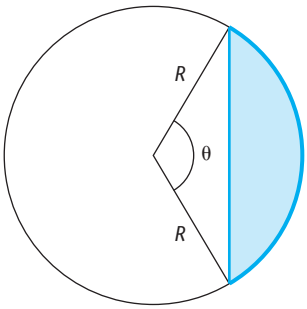


80. **ASTRONOMY** Refer to Problem 79. Find the smallest positive θ (in decimal degrees to three significant digits) such that Mercury is 3.78×10^7 miles from the sun.

81. **GEOMETRY** The area of the segment of a circle in the figure is given by

$$A = \tfrac{1}{2}R^2(\theta - \sin \theta)$$

where θ is in radian measure. Use a graphing calculator to find the radian measure, to three decimal places, of angle θ, if the radius is 8 inches and the area of the segment is 48 square inches.



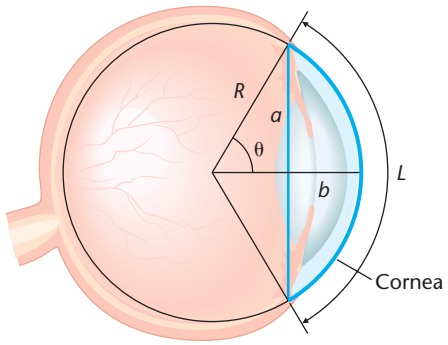
82. **GEOMETRY** Repeat Problem 81, if the radius is 10 centimeters and the area of the segment is 40 square centimeters.

83. **EYE SURGERY** A surgical technique for correcting an astigmatism involves removing small pieces of tissue to change the curvature of the cornea.* In the cross section of a cornea shown in the figure, the circular arc, with radius R and central angle 2θ, represents a cross section of the surface of the cornea.

(A) If $a = 5.5$ millimeters and $b = 2.5$ millimeters, find L correct to four decimal places.

(B) Reducing the chord length $2a$ without changing the length L of the arc has the effect of pushing the cornea outward and giving it a rounder, yet still a circular, shape. With the aid of a graphing calculator in part of the solution, approximate b to four decimal places if a is reduced to 5.4 millimeters and L remains the same as it was in part A.

*Based on the article "The Surgical Correction of Astigmatism" by Sheldon Rothman and Helen Strassberg in the *UMAP Journal,* Vol. v, no. 2, 1984.



84. **EYE SURGERY** Refer to Problem 83.

(A) If in the figure $a = 5.4$ millimeters and $b = 2.4$ millimeters, find L correct to four decimal places.

(B) Increasing the chord length without changing the arc length L has the effect of pulling the cornea inward and giving it a flatter, yet still circular, shape. With the aid of a graphing calculator in part of the solution, approximate b to four decimal places if a is increased to 5.5 millimeters and L remains the same as it was in part A.

ANALYTIC GEOMETRY *Find simultaneous solutions for each system of equations in Problems 85 and 86 ($0° \le \theta \le 360°$). These are polar equations, which will be discussed in Chapter 7.*

85. $r = 2 \sin \theta$
$r = \sin 2\theta$

86. $r = 2 \sin \theta$
$r = 2(1 - \sin \theta)$

Problems 87 and 88 are related to rotation of axes in analytic geometry.

87. **ANALYTIC GEOMETRY** Given the equation $2xy = 1$, replace x and y with

$$x = u \cos \theta - v \sin \theta$$
$$y = u \sin \theta + v \cos \theta$$

and simplify the left side of the resulting equation. Find the smallest positive θ in degree measure so that the coefficient of the uv term is 0.

88. **ANALYTIC GEOMETRY** Repeat Problem 87 for $xy = -2$.

CHAPTER 6 Review

6-1 Basic Identities and Their Use

The following 11 identities are basic to the process of changing trigonometric expressions to equivalent but more useful forms:

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Identities for Negatives

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

Although there is no fixed method of verification that works for all identities, the following suggested steps are helpful in many cases:

Suggested Steps in Verifying Identities

1. Start with the more complicated side of the identity, and transform it into the simpler side.
2. Try algebraic operations such as multiplying, factoring, combining fractions, and splitting fractions.
3. If other steps fail, express each function in terms of sine and cosine functions, and then perform appropriate algebraic operations.
4. At each step, keep the other side of the identity in mind. This often reveals what you should do to get there.

6-2 Sum, Difference, and Cofunction Identities

Sum Identities

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Difference Identities

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$
$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Cofunction Identities

(Replace $\pi/2$ with $90°$ if x is in degrees.)

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$
$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$
$$\tan\left(\frac{\pi}{2} - x\right) = \cot x$$

6-3 Double-Angle and Half-Angle Identities

Double-Angle Identities

$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} = \frac{2\cot x}{\cot^2 x - 1} = \frac{2}{\cot x - \tan x}$$

Half-Angle Identities

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}}$$
$$\cos\frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}}$$
$$\tan\frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$$

6-4 Product–Sum and Sum–Product Identities

Product–Sum Identities

$$\sin x \cos y = \tfrac{1}{2}[\sin(x + y) + \sin(x - y)]$$
$$\cos x \sin y = \tfrac{1}{2}[\sin(x + y) - \sin(x - y)]$$
$$\sin x \sin y = \tfrac{1}{2}[\cos(x - y) - \cos(x + y)]$$
$$\cos x \cos y = \tfrac{1}{2}[\cos(x + y) + \cos(x - y)]$$

Sum–Product Identities

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

6-5 Trigonometric Equations

Sections 6-1 through 6-4 of the chapter considered trigonometric equations called **identities.** Identities are true for all replacements of the variable(s) for which both sides are defined. Section 6-5 considered **conditional equations.** Conditional equations may be true for some variable replacements, but are false for other variable replacements for which both sides are defined. The equation $\sin x = \cos x$ is a conditional equation.

In solving a trigonometric equation using an algebraic approach, no particular rule will always lead to all solutions of every trigonometric equation you are likely to encounter. Solving trigonometric equations algebraically often requires the use of algebraic manipulation, identities, and ingenuity.

Suggestions for Solving Trigonometric Equations Algebraically

1. Regard one particular trigonometric function as a variable, and solve for it.

(A) Consider using algebraic manipulation such as factoring, combining or separating fractions, and so on.

(B) Consider using identities.

2. After solving for a trigonometric function, solve for the variable.

In *solving a trigonometric equation using a graphing calculator approach* you can solve a larger variety of problems than with the algebraic approach. The solutions are generally approximations.

CHAPTER 6

Review Exercises

Work through all the problems in this chapter review and check answers in the back of the book. Answers to all review problems, except verifications, are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

Verify each identity in Problems 1–4.

1. $\tan x + \cot x = \sec x \csc x$

2. $\sec^4 x - 2 \sec^2 x \tan^2 x + \tan^4 x = 1$

3. $\dfrac{1}{1 - \sin x} + \dfrac{1}{1 + \sin x} = 2 \sec^2 x$

4. $\cos\left(x - \dfrac{3\pi}{2}\right) = -\sin x$

5. Write as a sum: $\sin 5\alpha \cos 3\alpha$.

6. Write as a product: $\cos 7x - \cos 5x$.

7. Simplify: $\sin\left(x + \dfrac{9\pi}{2}\right)$.

Solve Problems 8 and 9 exactly (θ in degrees, x real).

8. $\sqrt{2} \cos \theta + 1 = 0$, all θ

9. $\sin x \tan x - \sin x = 0$, all real x

Solve Problems 10–13 to four decimal places (θ in degrees and x real).

10. $\sin x = 0.7088$, all real x

11. $\cos \theta = 0.2557$, all θ

12. $\cot x = -0.1692$, $-\pi/2 < x < \pi/2$

13. $3 \tan (11 - 3x) = 23.46$, $-\pi/2 < 11 - 3x < \pi/2$

14. Use a graphing calculator to test whether each of the following is an identity. If an equation appears to be an identity, verify it. If the equation does not appear to be an identity, find a value of x for which both sides are defined but are not equal.

(A) $(\sin x + \cos x)^2 = 1 - 2 \sin x \cos x$

(B) $\cos^2 x - \sin^2 x = 1 - 2 \sin^2 x$

Verify each identity in Problems 15–23.

15. $\dfrac{1 - 2\cos x - 3\cos^2 x}{\sin^2 x} = \dfrac{1 - 3\cos x}{1 - \cos x}$

16. $(1 - \cos x)(\csc x + \cot x) = \sin x$

17. $\dfrac{1 + \sin x}{\cos x} = \dfrac{\cos x}{1 - \sin x}$

18. $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$

19. $\cot \dfrac{x}{2} = \dfrac{\sin x}{1 - \cos x}$

20. $\cot x - \tan x = \dfrac{4\cos^2 x - 2}{\sin^2 x}$

21. $\left(\dfrac{1 - \cot x}{\csc x}\right)^2 = 1 - \sin 2x$

22. $\tan m + \tan n = \dfrac{\sin (m + n)}{\cos m \cos n}$

23. $\tan (x + y) = \dfrac{\cot x + \cot y}{\cot x \cot y - 1}$

24. Use a sum identity to find the exact value of tan 75°.

25. Use a difference identity to find the exact value of cos $(\pi/12)$.

26. Use a half-angle identity to find the exact value of sin 105°.

27. Use a half-angle identity to find the exact value of cos $(7\pi/8)$.

Evaluate Problems 28–31 exactly using appropriate sum–product or product–sum identities.

28. cos 195° sin 75°

29. cos 195° + cos 105°

30. $\sin \dfrac{11\pi}{24} \sin \dfrac{5\pi}{24}$

31. $\sin \dfrac{5\pi}{12} - \sin \dfrac{\pi}{12}$

In Problems 32–35, is the equation an identity? Explain.

32. $\cot^2 x = \csc^2 x + 1$

33. $\cos 3x = \cos x (\cos 2x - 2\sin^2 x)$

34. $\sin (x + 3\pi/2) = \cos x$

35. $\cos (x - 3\pi/2) = \sin x$

Solve Problems 36–40 exactly (θ in degrees, x real).

36. $4\sin^2 x - 3 = 0, 0 \le x < 2\pi$

37. $2\sin^2 \theta + \cos \theta = 1, 0° \le \theta \le 180°$

38. $2\sin^2 x - \sin x = 0$, all real x

39. $\sin 2x = \sqrt{3} \sin x$, all real x

40. $2\sin^2 \theta + 5\cos \theta + 1 = 0$, all θ

Solve Problems 41–43 to four significant digits (θ in degrees, x real).

41. $\tan \theta = 0.2557$, all θ

42. $\sin^2 x + 2 = 4\sin x$, all real x

43. $\tan^2 x = 2\tan x + 1, 0 \le x < \pi$

In Problems 44–47, solve the equation or inequality to four decimal places.

44. $3\sin 2x = 2x - 2.5$, all real x

45. $3\sin 2x > 2x - 2.5$, all real x

46. $2\sin^2 x - \cos 2x = 1 - x^2$, all real x

47. $2\sin^2 x - \cos 2x \le 1 - x^2$, all real x

48. Given the equation $\tan (x + y) = \tan x + \tan y$:

(A) Is $x = 0$ and $y = \pi/4$ a solution?

(B) Is the equation an identity or a conditional equation? Explain.

49. Explain the difference in evaluating $\sin^{-1} 0.3351$ and solving the equation $\sin x = 0.3351$.

50. Use a graphing calculator to test whether each of the following is an identity. If an equation appears to be an identity, verify it. If the equation does not appear to be an identity, find a value of x for which both sides are defined but are not equal.

(A) $\dfrac{\tan x}{\sin x + 2\tan x} = \dfrac{1}{\cos x - 2}$

(B) $\dfrac{\tan x}{\sin x - 2\tan x} = \dfrac{1}{\cos x - 2}$

51. Use a sum or difference identity to convert $y = \cos (x - \pi/3)$ to a form involving $\sin x$ and/or $\cos x$. Enter the original equation in a graphing calculator as y_1, the converted form as y_2, and graph y_1 and y_2 in the same viewing window. Use TRACE to compare the two graphs.

52. (A) Solve $\tan (x/2) = 2\sin x$ exactly, $0 \le x < 2\pi$, using algebraic calculator.

(B) Solve $\tan (x/2) = 2\sin x, 0 \le x < 2\pi$, to four decimal places using a graphing calculator.

53. Solve $3\cos (x - 1) = 2 - x^2$ for all real x, to three decimal places using a graphing calculator.

In Problems 54–56, find exact values without the use of a calculator.

54. Given $\tan x = -\frac{3}{4}, \pi/2 \le x \le \pi$, find

(A) $\sin (x/2)$

(B) $\cos 2x$

55. $\sin [2\tan^{-1} (-\frac{3}{4})]$

56. $\sin [\sin^{-1} (\frac{3}{5}) + \cos^{-1} (\frac{4}{5})]$

57. (A) Solve $\cos^2 2x = \cos 2x + \sin^2 2x$, $0 \le x < \pi$, exactly using algebraic methods.

(B) Solve $\cos^2 2x = \cos 2x + \sin^2 2x$, $0 \le x < \pi$, to four decimal places using a graphing calculator.

58. We are interested in the zeros of $f(x) = \sin \dfrac{1}{x-1}$ for $x > 0$.

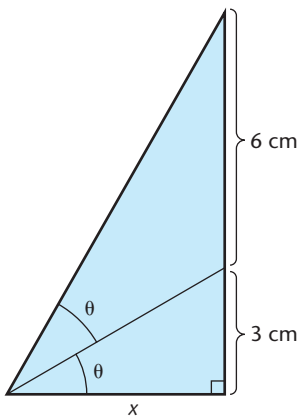
(A) Explore the graph of f over different intervals $[a, b]$ for various values of a and b, $0 < a < b$. Does the function f have a smallest zero? If so, what is it (to four decimal places)? Does the function have a largest zero? If so, what is it (to four decimal places)?

(B) Explain what happens to the graph as x increases without bound. Does the graph have an asymptote? If so, what is its equation?

(C) Explore the graph of f over smaller and smaller intervals containing $x = 1$. How many zeros exist on any interval containing $x = 1$? Is $x = 1$ a zero? Explain.

APPLICATIONS

59. INDIRECT MEASUREMENT Find the exact value of x in the figure, then find x and θ to three decimal places. [*Hint:* Use a suitable identity involving $\tan 2\theta$.]



60. ELECTRIC CURRENT An alternating current generator produces a current given by the equation

$$I = 50 \sin 120\pi(t - 0.001)$$

where t is time in seconds and I is current in amperes. Find the smallest positive t, to three significant digits, such that $I = 40$ amperes.

61. MUSIC—BEAT FREQUENCIES The equations

$$y = 0.6 \cos 184\pi t \text{ and } y = -0.6 \cos 208\pi t$$

model sound waves with frequencies 92 and 104 hertz, respectively. If both sounds are emitted simultaneously, a beat frequency results.

(A) Show that

$$0.6 \cos 184\pi t - 0.6 \cos 208\pi t = 1.2 \sin 12\pi t \sin 196\pi t$$

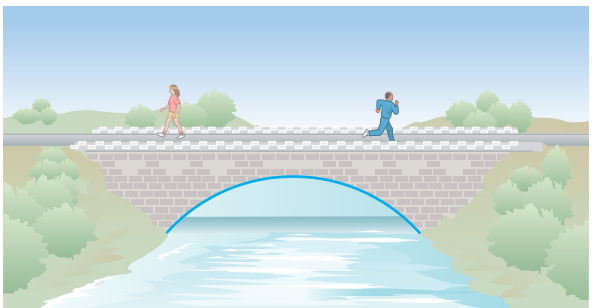
(B) Graph each of the following equations in a different viewing window for $0 \le t \le 0.2$.

$$y = 0.6 \cos 184\pi t$$
$$y = -0.6 \cos 208\pi t$$
$$y = 0.6 \cos 184\pi t - 0.6 \cos 208\pi t$$
$$y = 1.2 \sin 12\pi t \sin 196\pi t$$

62. ENGINEERING The circular arch of a bridge has an arc length of 36 feet and spans a 32-foot canal (see the figure). Determine the height of the circular arch above the water at the center of the bridge, and the radius of the circular arch, both to three decimal places. Start by drawing auxiliary lines in the figure, labeling appropriate parts, then explain how the trigonometric equation

$$\sin \theta = \frac{8}{9}\theta$$

is related to the problem. After solving the trigonometric equation for θ, the radius is easy to find and the height of the arch above the water can be found with a little ingenuity.



CHAPTER 6

››› GROUP ACTIVITY From $M \sin Bt + N \cos Bt$ to $A \sin (Bt + C)$—A Harmonic Analysis Tool

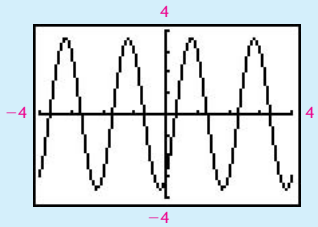
In solving certain kinds of more advanced applied mathematical problems—problems dealing with electrical circuits, spring-mass systems, heat flow, and so on—the solution process leads naturally to a function of the form

$$y = M \sin Bt + N \cos Bt \tag{1}$$

(A) *Graphing Calculator Exploration.* Use a graphing calculator to explore the nature of the graph of equation (1) for various values of M, N, and B. Does the graph appear to be simple harmonic; that is, does it appear to be a graph of an equation of the form $y = A \sin (Bt + C)$?

The graph of $y = 2 \sin (\pi t) - 3 \cos (\pi t)$, which is typical of the various graphs from equation (1), is shown in Figure 1. It turns out that the graph in Figure 1 can also be obtained from an equation of the form

$$y = A \sin (Bt + C) \tag{2}$$



› **Figure 1** $y = 2 \sin (\pi t) - 3 \cos (\pi t)$.

for suitable values of A, B, and C.

The problem now is: given M, N, and B in equation (1), find A, B, and C in equation (2) so that equation (2) produces the same graph as equation (1). The form of equation (2) is often preferred over (1), because from (2) you can easily read amplitude, period, and phase shift and recognize a phenomenon as simple harmonic.

The process of finding A, B, and C, given M, N, and B, requires a little ingenuity and the use of the sum identity

$$\sin (x + y) = \sin x \cos y + \cos x \sin y \tag{3}$$

How do we proceed? We start by trying to get the right side of equation (1) to look like the right side of identity (3). Then we use equation (3), from right to left, to obtain equation (2).

(B) *Establishing a Transformation Identity.* Show that

$$y = M \sin Bt + N \cos Bt = \sqrt{M^2 + N^2} \sin (Bt + C) \tag{4}$$

where C is any angle (in radians if t is real) having (M, N) on its terminal side. [*Hint:* A first step is the following:

$$M \sin Bt + N \cos Bt = \frac{\sqrt{M^2 + N^2}}{\sqrt{M^2 + N^2}} (M \sin Bt + N \cos Bt)]$$

(C) *Use of Transformation Identity.* Use equation (4) to transform

$$y_1 = -4 \sin (t/2) + 3 \cos (t/2)$$

into the form $y_2 = A \sin (Bt + C)$, where C is chosen so that $|C|$ is minimum. Compute C to three decimal places. From the new equation, determine the amplitude, period, and phase shift.

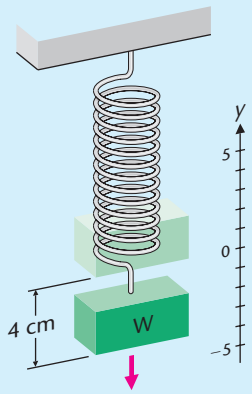
(D) *Graphing Calculator Visualization and Verification.* Graph y_1 and y_2 from part C in the same viewing window.

(E) *Physics Application.* A weight suspended from a spring, with spring constant 64, is pulled 4 centimeters below its equilibrium position and is then given a downward thrust to produce an initial downward velocity of 24 centimeters per second. In more advanced mathematics (differential equations) the equation of motion (neglecting air resistance and friction) is found to be given approximately by

$$y_1 = -3 \sin 8t - 4 \cos 8t$$

where y_1 is the coordinate of the bottom of the weight in Figure 2 at time t (y is in centimeters and t is in seconds). Transform the equation into the form

$$y_2 = A \sin (Bt + C)$$



› **Figure 2** Spring-mass system.

and indicate the amplitude, period, and phase shift of the motion. Choose the least positive C and keep A positive.

(F) *Graphing Calculator Visualization and Verification.* Graph y_1 and y_2 from part E in the same viewing window of a graphing calculator, $0 \le t \le 6$. How many times will the bottom of the weight pass $y = 2$ in the first 6 seconds?

(G) *Solving a Trigonometric Equation.* How long, to three decimal places, will it take the bottom of the weight to reach $y = 2$ for the first time?

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Additional Topics in Trigonometry

IN this chapter, we return to the problem of solving triangles—not just right triangles, as in Section 5-2, but any triangles. We use our knowledge of trigonometry to develop the important concept of vector, and we apply vectors to solve problems in navigation, physics, and engineering. We also introduce the polar coordinate system, probably the most important coordinate system after the rectangular coordinate system. The polar form of certain equations is simpler than the rectangular form, and the simpler form makes it easier to analyze their graphs. Similarly, we gain insight into the multiplication and division of complex numbers by representing them in polar form.

CHAPTER

7

OUTLINE

7-1 Law of Sines

7-2 Law of Cosines

7-3 Vectors in the Plane

7-4 Polar Coordinates and Graphs

7-5 Complex Numbers and De Moivre's Theorem

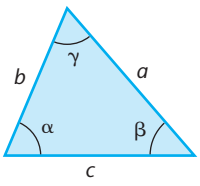
Chapter 7 Review

Chapter 7 Group Activity: Conic Sections and Planetary Orbits

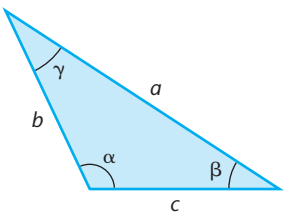
Cumulative Review Chapters 5–7

7-1 Law of Sines

- Deriving the Law of Sines
- Solving the ASA and AAS Cases
- Solving the SSA Case—Including the Ambiguous Case



Acute triangle
(a)



Obtuse triangle
(b)

› **Figure 1** Oblique triangles.

In Chapter 5 we used trigonometric functions to solve problems concerning right triangles. We now consider analogous problems for **oblique triangles**—triangles without a right angle.

Every oblique triangle is either **acute** (all angles between 0° and 90°) or **obtuse** (one angle between 90° and 180°). Figure 1 illustrates both types of triangles.

Note how the sides and angles of the oblique triangles in Figure 1 have been labeled: Side a is opposite angle α, side b is opposite angle β, and side c is opposite angle γ. Also note that the largest side of a triangle is opposite the largest angle. Given any three of the six quantities indicated in Figure 1, we are interested in finding the remaining three, if possible. This process is called *solving the triangle.*

If only the three angles of a triangle are known, it is impossible to solve for the sides. (Why?) But if we are given two angles and a side, or two sides and an angle, or all three sides, then it is possible to determine whether a triangle having the given quantities exists, and, if so, to solve for the remaining quantities.

The basic tools for solving oblique triangles are the *law of sines,* developed in this section, and the *law of cosines,* developed in the next section.

Before proceeding with specific examples, it is important to recall the rules in Table 1 regarding accuracy of angle and side measure. Table 1 is repeated inside the cover of the text for easy reference.

Table 1 Triangles and Significant Digits

Angle to nearest	Significant digits for side measure
1°	2
10′ or 0.1°	3
1′ or 0.01°	4
10″ or 0.001°	5

› CALCULATOR CALCULATIONS

When solving for a particular side or angle, carry out all operations within the calculator and then round to the appropriate number of significant digits (as specified in Table 1) at the end of the calculation. Your answers may still differ slightly from those in the book, depending on the order in which you solve for the sides and angles.

Deriving the Law of Sines

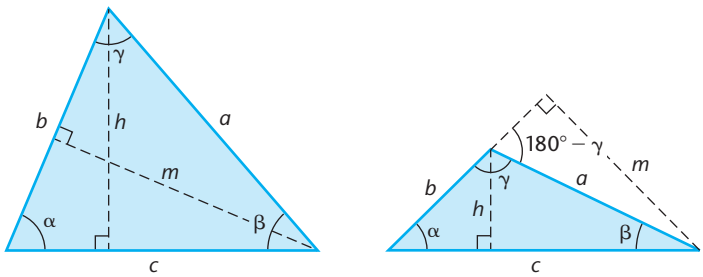
The law of sines is relatively easy to prove using the right triangle properties studied in Chapter 5. We will also use the fact that

$$\sin(180^\circ - x) = \sin x$$

which is readily obtained using a difference identity (a good exercise for you). Referring to Figure 2, we proceed as follows: Angles α and β in Figure 2(a), and also in Figure 2(b), satisfy

$$\sin\alpha = \frac{h}{b} \qquad \text{and} \qquad \sin\beta = \frac{h}{a}$$

Figure 2



Acute triangle (a) Obtuse triangle (b)

Solving each equation for h, we obtain

$$h = b\sin\alpha \qquad \text{and} \qquad h = a\sin\beta$$

Therefore,

$$\begin{aligned} b\sin\alpha &= a\sin\beta \\ \frac{\sin\alpha}{a} &= \frac{\sin\beta}{b} \end{aligned} \tag{1}$$

Similarly, angles α and γ in Figure 2(a), and also in Figure 2(b), satisfy

$$\sin\alpha = \frac{m}{c} \qquad \text{and} \qquad \sin\gamma = \sin(180^\circ - \gamma) = \frac{m}{a}$$

Solving each equation for m, we obtain

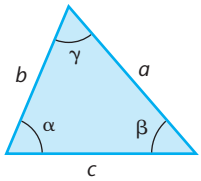
$$m = c\sin\alpha \qquad \text{and} \qquad m = a\sin\gamma$$

Therefore,

$$\begin{aligned} c\sin\alpha &= a\sin\gamma \\ \frac{\sin\alpha}{a} &= \frac{\sin\gamma}{c} \end{aligned} \tag{2}$$

If we combine equations (1) and (2), we obtain the **law of sines.**

› THEOREM 1 Law of Sines



$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$

In words, the ratio of the sine of an angle to its opposite side is the same as the ratio of the sine of either of the other angles to its opposite side.

Suppose that an angle of a triangle and its opposite side are known. Then the ratio in Theorem 1 can be calculated. So if one additional part of the triangle, either of the other angles or either of the other sides, is known, then the law of sines can be used to solve the triangle.

Therefore, the law of sines is used to solve triangles, given

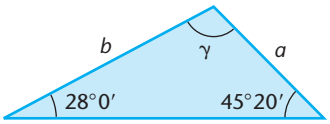
1. Two sides and an angle opposite one of them (SSA), or
2. Two angles and any side (ASA or AAS)

If the given information for a triangle consists of two sides and the included angle (SAS) or three sides (SSS), then the law of sines cannot be applied. The key to handling these two cases, the law of cosines, is developed in the next section.

We will apply the law of sines to the easier ASA and AAS cases first, and then will turn to the more challenging SSA case.

› Solving the ASA and AAS Cases

EXAMPLE 1 **Solving the ASA Case**



› Figure 3

Solve the triangle in Figure 3.

SOLUTION

We are given two angles and the included side, which is the ASA case. Find the third angle, then solve for the other two sides using the law of sines.

We solve for γ:

$$\begin{aligned} \alpha + \beta + \gamma &= 180^\circ \\ \gamma &= 180^\circ - (\alpha + \beta) \\ &= 180^\circ - (28^\circ 0' + 45^\circ 20') \\ &= 106^\circ 40' \end{aligned}$$

We solve for a:

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \qquad \text{Law of sines}$$

$$a = \frac{c\sin\alpha}{\sin\gamma}$$

$$= \frac{120\sin 28°0'}{\sin 106°40'}$$

$$= 58.8 \text{ meters}$$

We solve for b:

$$\frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \qquad \text{Law of sines}$$

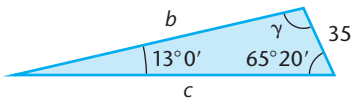
$$b = \frac{c\sin\beta}{\sin\gamma}$$

$$= \frac{120\sin 45°20'}{\sin 106°40'}$$

$$= 89.1 \text{ meters}$$

MATCHED PROBLEM 1

Solve the triangle in Figure 4.



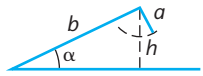
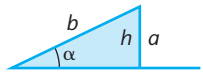
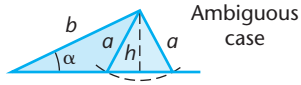
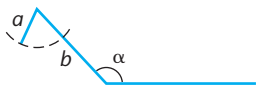
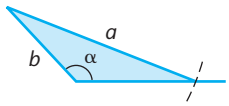
› Figure 4

Note that the AAS case can always be converted to the ASA case by first solving for the third angle. For the ASA or AAS case to determine a unique triangle, the sum of the two angles must be between 0° and 180°, because the sum of all three angles in a triangle is 180° and no angle can be zero or negative.

› Solving the SSA Case—Including the Ambiguous Case

We now look at the case where we are given two sides and an angle opposite one of the sides—the SSA case. This case has several possible outcomes, depending on the measures of the two sides and the angle. Table 2 illustrates the various possibilities.

Table 2 SSA Variations

α	a $[h = b \sin \alpha]$	Number of triangles	Figure	Case
Acute	$0 < a < h$	0		(a)
Acute	$a = h$	1		(b)
Acute	$h < a < b$	2		(c)
Acute	$a \geq b$	1		(d)
Obtuse	$0 < a \leq b$	0		(e)
Obtuse	$a > b$	1		(f)

It is unnecessary to memorize Table 2 to solve triangles in the SSA case. Instead, given sides a, b, and angle α, we use the law of sines to solve for the angle β opposite side b. The number of triangles is equal to the number of solutions β, $0° < \beta < 180°$, of the law of sines equation

$$\frac{\sin \beta}{b} = \frac{\sin \alpha}{a} \tag{3}$$

that satisfy

$$\alpha + \beta < 180° \tag{4}$$

In practice, we check each solution of equation (3) to determine whether inequality (4) is satisfied. If it is, we can easily solve for the remaining parts of the triangle. Examples 2–4 will make the procedure clear.

EXAMPLE 2 The SSA Case: One Triangle

Solve the triangle(s) with $a = 47$ centimeters, $b = 23$ centimeters, and $\alpha = 123°$.

SOLUTION

We solve for β:

$$\frac{\sin\beta}{b} = \frac{\sin\alpha}{a} \qquad \text{Law of sines}$$

$$\sin\beta = \frac{b\sin\alpha}{a} = \frac{23\sin 123^\circ}{47}$$

This equation has two solutions between 0° and 180°:

$$\beta = \sin^{-1}\left(\frac{23\sin 123^\circ}{47}\right) = 24^\circ$$

$$\beta' = 180^\circ - 24^\circ = 156^\circ$$

Because

$$\alpha + \beta = 123^\circ + 24^\circ = 147^\circ < 180^\circ$$

$$\alpha + \beta' = 123^\circ + 156^\circ = 279^\circ \geq 180^\circ$$

there is only one triangle. [Note that this conclusion is consistent with Table 2. Because α is obtuse and $a > b$, we are in Case (f).]

We solve for γ:

$$\alpha + \beta + \gamma = 180^\circ$$

$$\gamma = 180^\circ - 123^\circ - 24^\circ = 33^\circ$$

We solve for c:

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c}$$

$$c = \frac{a\sin\gamma}{\sin\alpha} = \frac{47\sin 33^\circ}{\sin 123^\circ} = 31 \text{ centimeters}$$

MATCHED PROBLEM 2

Solve the triangle(s) with $a = 88$ meters, $b = 62$ meters, and $\alpha = 81^\circ$.

EXAMPLE 3 The SSA Case: No Triangle

Solve the triangle(s) with $a = 27$ inches, $b = 28$ inches, and $\alpha = 110^\circ$.

SOLUTION

We solve for β:

$$\frac{\sin\beta}{b} = \frac{\sin\alpha}{a} \qquad \text{Law of sines}$$

$$\sin\beta = \frac{b\sin\alpha}{a} = \frac{28\sin 110^\circ}{27}$$

This equation has two solutions between 0° and 180°:

$$\beta = \sin^{-1}\left(\frac{28\sin 110^\circ}{27}\right) = 77^\circ$$

$$\beta' = 180^\circ - 77^\circ = 103^\circ$$

Because

$$\alpha + \beta = 110^\circ + 77^\circ = 187^\circ \geq 180^\circ$$

$$\alpha + \beta' = 110^\circ + 103^\circ = 213^\circ \geq 180^\circ$$

there is no triangle. [Note that this conclusion is consistent with Table 2. Because α is obtuse and $a \leq b$, we are in Case (e).]

MATCHED PROBLEM 3

Solve the triangle(s) with $a = 64$ feet, $b = 79$ feet, and $\alpha = 57^\circ$.

EXAMPLE 4 The SSA Case: Two Triangles

Solve the triangle(s) with $a = 1.0$ meters, $b = 1.8$ meters, and $\alpha = 26^\circ$.

SOLUTION

We solve for β:

$$\frac{\sin\beta}{b} = \frac{\sin\alpha}{a} \qquad \text{Law of sines}$$

$$\sin\beta = \frac{b\sin\alpha}{a} = \frac{1.8\sin 26^\circ}{1.0}$$

This equation has two solutions between 0° and 180°:

$$\beta = \sin^{-1}\left(\frac{1.8\sin 26^\circ}{1.0}\right) = 52^\circ$$

$$\beta' = 180^\circ - 52^\circ = 128^\circ$$

Because

$$\alpha + \beta = 26° + 52° = 78° < 180°$$
$$\alpha + \beta' = 26° + 128° = 154° < 180°$$

there are two triangles. [Note that this conclusion is consistent with Table 2. Because α is acute and $h = b \sin \alpha < a < b$, we are in Case (c), the ambiguous case.]

We solve for γ and γ':

$$\gamma = 180° - 26° - 52° = 102°$$
$$\gamma' = 180° - 26° - 128° = 26°$$

We solve for c and c':

$$c = \frac{a \sin \gamma}{\sin \alpha} = \frac{1.0 \sin 102°}{\sin 26°} = 2.2 \text{ meters}$$
$$c' = \frac{a \sin \gamma'}{\sin \alpha} = \frac{1.0 \sin 26°}{\sin 26°} = 1.0 \text{ meters}$$

In summary:

Triangle I:	$\beta = 52°$	$\gamma = 102°$	$c = 2.2$ meters
Triangle II:	$\beta' = 128°$	$\gamma' = 26°$	$c' = 1.0$ meters

MATCHED PROBLEM 4

Solve the triangle(s) with $a = 8$ kilometers, $b = 10$ kilometers, and $\alpha = 35°$.

››› EXPLORE-DISCUSS 1

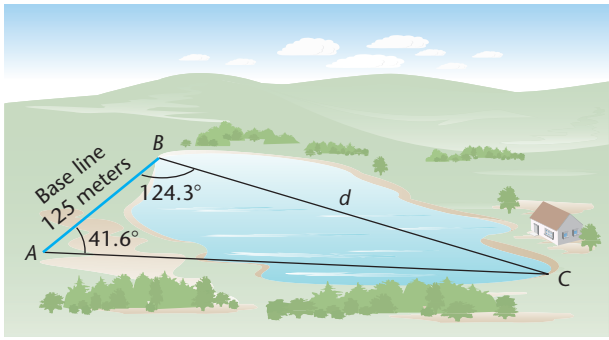
Sides a and b and acute angle α of a triangle are given. Explain which case(s) of Table 2 could apply if, in solving the triangle, it is found that

(A) $\sin \beta > 1$ **(B)** $\sin \beta = 1$ **(C)** $\sin \beta < 1$

The law of sines is useful in many applications, as can be seen in Examples 5 and 6 and the applications in the exercises.

EXAMPLE 5 Surveying

To measure the length d of a lake (Fig. 5), a base line AB is established and measured to be 125 meters. Angles A and B are measured to be 41.6° and 124.3°, respectively. How long is the lake?



› Figure 5

SOLUTION

Find angle C and use the law of sines.

$$\text{Angle } C = 180° - (124.3° + 41.6°) = 14.1°$$

$$\frac{\sin 14.1°}{125} = \frac{\sin 41.6°}{d}$$

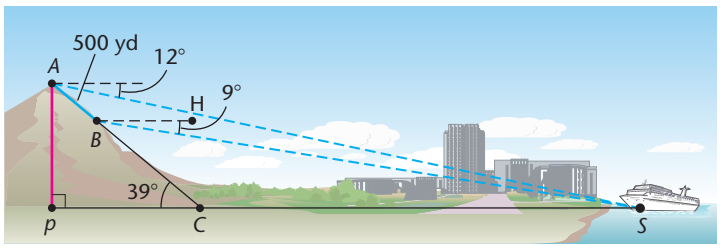
$$d = 125\left(\frac{\sin 41.6°}{\sin 14.1°}\right) = 341 \text{ meters}$$

MATCHED PROBLEM 5

In Example 5, find the distance AC.

EXAMPLE 6 **Estimating Elevation**

A hill slopes 39° relative to the horizontal. From the top of the hill, a cruise ship anchored in port can be seen at an angle of depression of 12°. Five hundred yards down the slope, the angle of depression of the cruise ship is 9° (Fig. 6). Find the elevation (in feet above sea level, to two significant digits) at the top of the hill.



› Figure 6

SOLUTION

If a woman climbing the hill looks down toward the cruise ship, the acute angle between the horizontal and her line of sight, below the horizontal, is called the *angle of depression.* If a man aboard the cruise ship looks up toward the hill, the acute angle between the horizontal and his line of sight, above the horizontal, is called the *angle of elevation.* If the two persons look toward each other, then, by alternate interior angles, the angle of depression for the woman on the hill is equal to the angle of elevation for the man on the cruise ship. Therefore, $\angle ASC = 12°$ and $\angle BSC = 9°$, so $\angle ASB = 3°$. By alternate interior angles, $\angle HBC = 39°$, so $\angle SBC = 39° - 9° = 30°$, and its supplement $\angle ABS = 180° - 30° = 150°$.

We convert 500 yards to 1,500 feet and use the law of sines to find an expression for side AS of triangle ABS:

$$\frac{\sin \angle ABS}{AS} = \frac{\sin \angle ASB}{1{,}500} \qquad \angle ABS = 150°,\ \angle ASB = 3°$$

$$\frac{\sin 150°}{AS} = \frac{\sin 3°}{1{,}500} \qquad \text{Solve for } AS.$$

$$AS = 1{,}500 \frac{\sin 150°}{\sin 3°}$$

Finally, we use the right triangle APS to find the elevation AP at the top of the hill:

$$\frac{AP}{AS} = \sin 12° \qquad \text{Solve for } AP.$$

$$AP = AS \sin 12° \qquad \text{Substitute } AS = 1{,}500 \frac{\sin 150°}{\sin 3°}.$$

$$= 1{,}500 \frac{\sin 150°}{\sin 3°} \sin 12°$$

$$= 3{,}000 \text{ feet} \qquad \text{To two significant digits}$$

MATCHED PROBLEM 6

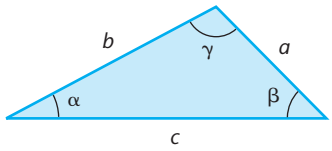
If the hill in Example 6 has a slope of 30° with respect to the horizontal, find the elevation (in feet above sea level, to two significant digits) at the top of the hill, assuming all other information remains the same.

ANSWERS TO MATCHED PROBLEMS

1. $\gamma = 101°40'$, $b = 141$, $c = 152$
2. $\beta = 44°$, $\gamma = 55°$, $c = 73$ meters
3. No solution
4. Triangle I: $\beta = 46°$, $\gamma = 99°$, $c = 14$ kilometers;
   Triangle II: $\beta' = 134°$, $\gamma' = 11°$, $c' = 2.7$ kilometers
5. 424 meters
6. 2,100 feet

7-1 Exercises

The labeling in the figure below is the convention we will follow in this exercise set. Your answers to some problems may differ slightly from those in the book, depending on the order in which you solve for the sides and angles of a given triangle.



1. What is an oblique triangle?

2. What is an acute triangle?

3. What is an obtuse triangle?

4. Explain why it is impossible to solve a triangle if only its three angles are known.

5. Explain what the abbreviations SSA, ASA, AAS, SAS, and SSS mean in the context of solving triangles.

6. Explain how the AAS case can always be reduced to ASA.

7. Explain why the law of sines cannot be applied to the SAS or SSS cases.

8. Explain why one of the SSA variations is called the "ambiguous case."

Solve each triangle in Problems 9–16.

9. $\alpha = 73°, \beta = 28°, c = 42$ feet

10. $\alpha = 41°, \beta = 33°, c = 21$ centimeters

11. $\alpha = 122°, \gamma = 18°, b = 12$ kilometers

12. $\beta = 43°, \gamma = 36°, a = 92$ millimeters

13. $\beta = 112°, \gamma = 19°, c = 23$ yards

14. $\alpha = 52°, \gamma = 105°, c = 47$ meters

15. $\alpha = 52°, \gamma = 47°, a = 13$ centimeters

16. $\beta = 83°, \gamma = 77°, c = 25$ miles

In Problems 17–24, determine whether the information in each problem enables you to construct zero, one, or two triangles. Do not solve the triangle. Explain which case in Table 2 applies.

17. $a = 2$ inches, $b = 4$ inches, $\alpha = 30°$

18. $a = 3$ feet, $b = 6$ feet, $\alpha = 30°$

19. $a = 6$ inches, $b = 4$ inches, $\alpha = 30°$

20. $a = 8$ feet, $b = 6$ feet, $\alpha = 30°$

21. $a = 1$ inch, $b = 4$ inches, $\alpha = 30°$

22. $a = 2$ feet, $b = 6$ feet, $\alpha = 30°$

23. $a = 3$ inches, $b = 4$ inches, $\alpha = 30°$

24. $a = 5$ feet, $b = 6$ feet, $\alpha = 30°$

Solve each triangle in Problems 25–40. If a problem has no solution, say so.

25. $\alpha = 118.3°, \gamma = 12.2°, b = 17.3$ feet

26. $\beta = 27.5°, \gamma = 54.5°, a = 9.27$ inches

27. $\alpha = 67.7°, \beta = 54.2°, b = 123$ meters

28. $\alpha = 122.7°, \beta = 34.4°, b = 18.3$ kilometers

29. $\alpha = 46.5°, a = 7.9$ millimeters, $b = 13.1$ millimeters

30. $\alpha = 26.3°, a = 14.7$ inches, $b = 35.2$ inches

31. $\alpha = 15.9°, a = 22.4$ inches, $b = 29.6$ inches

32. $\alpha = 43.5°, a = 138$ centimeters, $b = 172$ centimeters

33. $\beta = 38.9°, a = 42.7$ inches, $b = 30.0$ inches

34. $\beta = 27.3°, a = 244$ centimeters, $b = 135$ centimeters

35. $\alpha = 123.2°, a = 101$ yards, $b = 152$ yards

36. $\alpha = 137.3°, a = 13.9$ meters, $b = 19.1$ meters

37. $\beta = 29°30', a = 43.2$ millimeters, $b = 56.5$ millimeters

38. $\beta = 33°50', a = 673$ meters, $b = 1{,}240$ meters

39. $\alpha = 30°, a = 29$ feet, $b = 58$ feet

40. $\beta = 30°, a = 92$ inches, $b = 46$ inches

41. Let $\alpha = 42.3°$ and $b = 25.2$ centimeters. Determine a value k so that if $0 < a < k$, there is no solution; if $a = k$, there is one solution; and if $k < a < b$, there are two solutions.

42. Let $\alpha = 37.3°$ and $b = 42.8$ centimeters. Determine a value k so that if $0 < a < k$, there is no solution; if $a = k$, there is one solution; and if $k < a < b$, there are two solutions.

43. Mollweide's equation,

$$(a - b)\cos\frac{\gamma}{2} = c\sin\frac{\alpha - \beta}{2}$$

is often used to check the final solution of a triangle, because all six parts of a triangle are involved in the equation. If the left side does not equal the right side after substitution, then an error has been made in solving a triangle. Use this equation to check Problem 9. (Because of rounding errors, both sides may not be exactly the same.)

44. (A) Use the law of sines and suitable identities to show that for any triangle

$$\frac{a - b}{a + b} = \frac{\tan\dfrac{\alpha - \beta}{2}}{\tan\dfrac{\alpha + \beta}{2}}$$

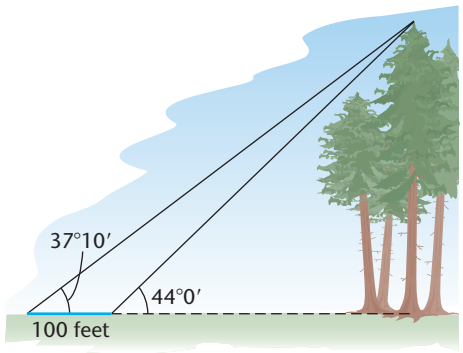
(B) Verify the formula with values from Problem 9.

APPLICATIONS

45. COAST GUARD Two lookout posts, A and B (10.0 miles apart), are established along a coast to watch for illegal ships coming within the 3-mile limit. If post A reports a ship S at angle $BAS = 37°30'$ and post B reports the same ship at angle $ABS = 20°0'$, how far is the ship from post A? How far is the ship from the shore (assuming the shore is along the line joining the two observation posts)?

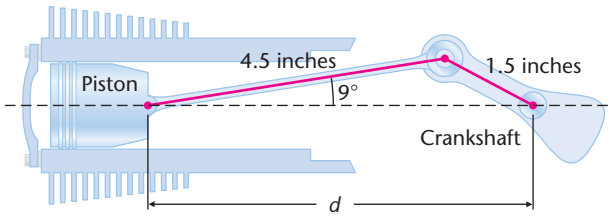
46. FIRE LOOKOUT A fire at F is spotted from two fire lookout stations, A and B, which are 10.0 miles apart. If station B reports the fire at angle $ABF = 53°0'$ and station A reports the fire at angle $BAF = 28°30'$, how far is the fire from station A? From station B?

47. NATURAL SCIENCE The tallest trees in the world grow in Redwood National Park in California; they are taller than a football field is long. Find the height of one of these trees, given the information in the figure. (The 100-foot measurement is accurate to three significant digits.)



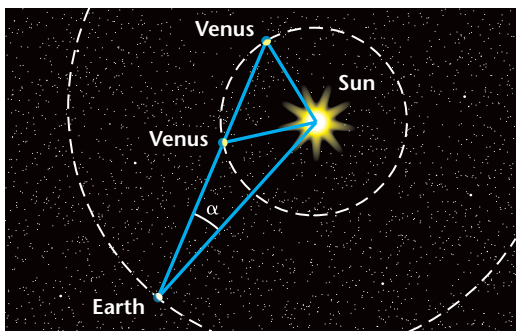
48. SURVEYING To measure the height of Mt. Whitney in California, surveyors used a scheme like the one shown in the figure in Problem 47. They set up a horizontal base line 2,000 feet long at the foot of the mountain and found the angle nearest the mountain to be 43°5′; the angle farthest from the mountain was found to be 38°0′. If the base line was 5,000 feet above sea level, how high is Mt. Whitney above sea level?

49. ENGINEERING A 4.5-inch piston rod joins a piston to a 1.5-inch crankshaft (see the figure). How far is the base of the piston from the center of the crankshaft (distance d) when the rod makes an angle of 9° with the centerline? There are two answers to the problem.



50. ENGINEERING Repeat Problem 49 if the piston rod is 6.3 inches, the crankshaft is 1.7 inches, and the angle is 11°.

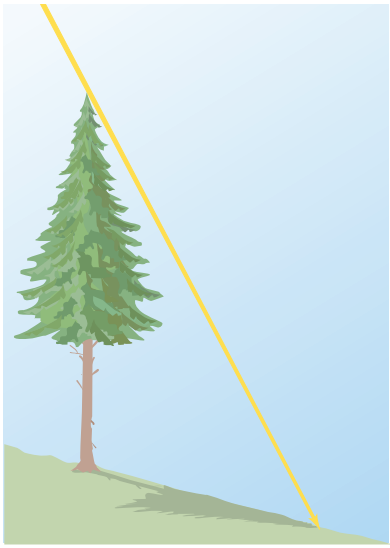
51. ASTRONOMY The orbits of the Earth and Venus are approximately circular, with the sun at the center. A sighting of Venus is made from Earth, and the angle α is found to be 18°40′. If the radius of the orbit of the Earth is 1.495×10^8 kilometers and the radius of the orbit of Venus is 1.085×10^8 kilometers, what are the possible distances from the Earth to Venus? (See the figure.)



52. **ASTRONOMY** In Problem 51, find the maximum angle α. [*Hint:* The angle is maximum when a straight line joining the Earth and Venus is tangent to Venus' orbit.]

53. **SURVEYING** A tree growing on a hillside casts a 102-foot shadow straight down the hill (see the figure). Find the vertical height of the tree if, relative to the horizontal, the hill slopes 15.0° and the angle of elevation of the sun is 62.0°.

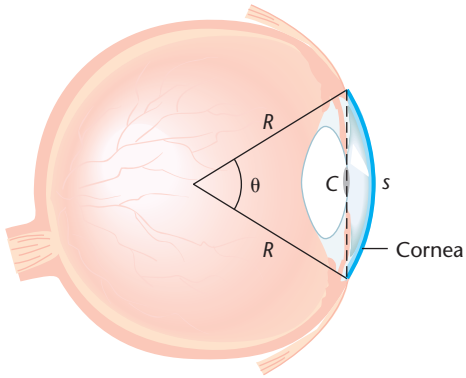
54. **SURVEYING** Find the height of the tree in Problem 53 if the shadow length is 157 feet and, relative to the horizontal, the hill slopes 11.0° and the angle of elevation of the sun is 42.0°.



55. **BALLOONING** A woman on the ground, looking due west, observes a hot air balloon at an angle of elevation of 75°. At the same time, $\frac{1}{2}$ mile east of her, a friend observes the same balloon at an angle of elevation of 20°. Find the altitude of the balloon, in feet, to two significant digits.

56. **BALLOONING** A man in a hot air balloon, looking due west at the moment the balloon passes over a point 1 mile east of the launch site, observes a plane at an angle of elevation of 20°, and the launch site at an angle of depression of 35°. At the same time, his friend at the launch site observes the same plane due west at an angle of elevation of 65°. Find the altitude of the balloon and the altitude of the plane.

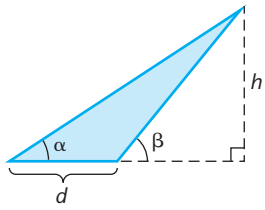
57. **LIFE SCIENCE** A cross-section of the cornea of an eye, a circular arc, is shown in the figure. Find the arc radius R and the arc length s, given the chord length $C = 11.8$ millimeters and the central angle $\theta = 98.9°$.



58. **LIFE SCIENCE** Referring to the preceding figure, find the arc radius R and the arc length s, given the chord length $C = 10.2$ millimeters and the central angle $\theta = 63.2°$.

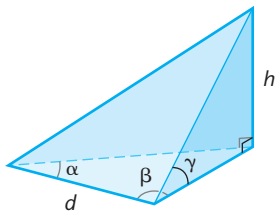
59. **SURVEYING** The procedure illustrated in Problems 47 and 48 is used to determine an inaccessible height h when a base line d on a line perpendicular to h can be established (see the figure) and the angles α and β can be measured. Show that

$$h = d\left[\frac{\sin\alpha \sin\beta}{\sin(\beta - \alpha)}\right]$$



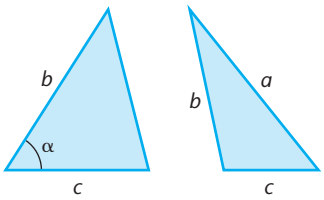
60. **SURVEYING** The layout in the figure is used to determine an inaccessible height h when a base line d in a plane perpendicular to h can be established and the angles α, β, and γ can be measured. Show that

$$h = d \sin\alpha \csc(\alpha + \beta) \tan\gamma$$



7-2 Law of Cosines

- Deriving the Law of Cosines
- Solving the SAS Case
- Solving the SSS Case



(a) SAS case (b) SSS case

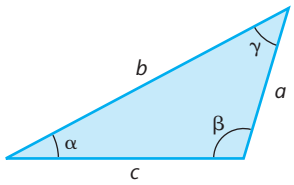
› Figure 1

If in a triangle two sides and the included angle are given (SAS), or three sides are given (SSS), the law of sines cannot be used to solve the triangle—neither case involves an angle and its opposite side (Fig. 1). Both cases can be solved starting with the *law of cosines,* which we discuss in this section.

› Deriving the Law of Cosines

Theorem 1 states the **law of cosines.**

› THEOREM 1 Law of Cosines



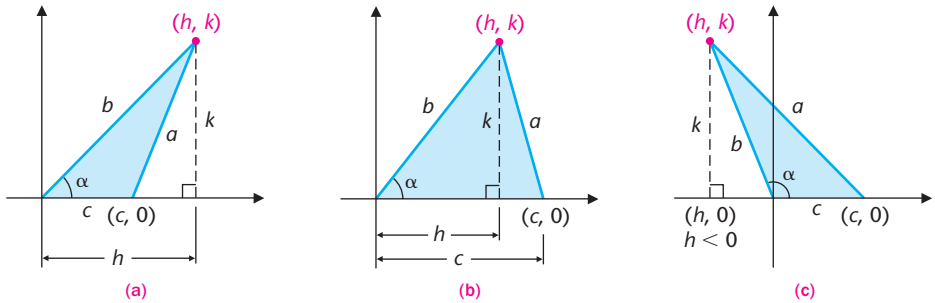
$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

All three equations say essentially the same thing.

The law of cosines is used to solve triangles, given:

1. Two sides and the included angle (SAS), or
2. Three sides (SSS)

We will establish the first equation in Theorem 1. The other two equations then can be obtained from this one simply by relabeling the figure. We start by locating a triangle in a rectangular coordinate system. Figure 2 on the next page shows three typical triangles.



› **Figure 2** Three representative triangles.

For an arbitrary triangle located as in Figure 2, the distance-between-two-points formula is used to obtain

$$\begin{aligned} a &= \sqrt{(h-c)^2 + (k-0)^2} && \text{Square both sides.} \\ a^2 &= (h-c)^2 + k^2 && \text{Expand.} \\ &= h^2 - 2hc + c^2 + k^2 \end{aligned} \tag{1}$$

From Figure 2, we note that

$$b^2 = h^2 + k^2$$

Substituting b^2 for $h^2 + k^2$ in equation (1), we obtain

$$a^2 = b^2 + c^2 - 2hc \tag{2}$$

But

$$\cos \alpha = \frac{h}{b}$$
$$h = b \cos \alpha$$

By replacing h in equation (2) with $b \cos \alpha$, we reach our objective:

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

[*Note:* If α is acute, then $\cos \alpha$ is positive; if α is obtuse, then $\cos \alpha$ is negative.]

› Solving the SAS Case

For the SAS case, start by using the law of cosines to find the side opposite the given angle. Then use either the law of cosines or the law of sines to find a second angle. Because of the simpler computations, the law of sines will generally be used to find the second angle.

››› EXPLORE-DISCUSS 1

After using the law of cosines to find the side opposite the angle for an SAS case, the law of sines is used to find one of the two remaining angles.

(A) If the given angle is obtuse, can either of the remaining angles be obtuse? Explain.

(B) If the given angle is acute, then one of the remaining angles may or may not be obtuse. Explain why choosing the angle opposite the shorter side guarantees the selection of an acute angle.

(C) Starting with $(\sin \alpha)/a = (\sin \beta)/b$, show that

$$\alpha = \sin^{-1}\left(\frac{a \sin \beta}{b}\right) \tag{3}$$

(D) Explain why equation (3) gives us the correct angle α only if α is acute.

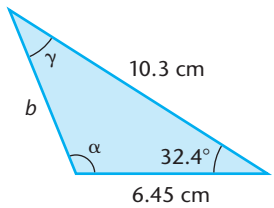
The preceding discussion leads to the following strategy for solving the SAS case:

› STRATEGY FOR SOLVING THE SAS CASE

Step	Find	Method
1	Side opposite given angle	Law of cosines
2	Second angle (Find the angle opposite the shorter of the two given sides—this angle will always be acute.)	Law of sines
3	Third angle	Subtract the sum of the measures of the given angle and the angle found in step 2 from 180°.

EXAMPLE 1 Solving the SAS Case

Solve the triangle in Figure 3.



› Figure 3

SOLUTION

We solve for b:

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac \cos \beta && \text{Law of cosines} \\ b &= \sqrt{a^2 + c^2 - 2ac \cos \beta} \\ &= \sqrt{(10.3)^2 + (6.45)^2 - 2(10.3)(6.45) \cos 32.4^\circ} \\ &= 5.96 \text{ cm} \end{aligned}$$

We solve for γ (the angle opposite the shorter side):

$$\frac{\sin\gamma}{c} = \frac{\sin\beta}{b} \qquad \text{Law of sines}$$

$$\sin\gamma = \frac{c\sin\beta}{b} \qquad \text{Solve for } \gamma.$$

$$\gamma = \sin^{-1}\left(\frac{c\sin\beta}{b}\right) \qquad \text{Because } \gamma \text{ is acute, the inverse sine function gives us } \gamma \text{ directly.}$$

$$= \sin^{-1}\left(\frac{6.45\sin 32.4^\circ}{5.96}\right)$$

$$= 35.4^\circ$$

We solve for α:

$$\alpha = 180^\circ - (\beta + \gamma)$$
$$= 180^\circ - (32.4^\circ + 35.4^\circ) = 112.2^\circ$$

MATCHED PROBLEM 1

Solve the triangle with $\alpha = 77.5^\circ$, $b = 10.4$ feet, and $c = 17.7$ feet.

Solving the SSS Case

Starting with three sides of a triangle, the problem is to find the three angles. Subsequent calculations are simplified if we solve for the obtuse angle first, if present. The law of cosines is used for this purpose. A second angle, which must be acute, can be found using either law, although computations are usually simpler with the law of sines.

EXPLORE-DISCUSS 2

(A) Starting with $a^2 = b^2 + c^2 - 2bc\cos\alpha$, show that

$$\alpha = \cos^{-1}\left(\frac{a^2 - b^2 - c^2}{-2bc}\right) \tag{4}$$

(B) Does equation (4) give us the correct angle α irrespective of whether α is acute or obtuse? Explain.

The preceding discussion leads to the following strategy for solving the SSS case:

› STRATEGY FOR SOLVING THE SSS CASE

Step	Find	Method
1	Angle opposite longest side—this will take care of an obtuse angle, if present.	Law of cosines
2	Either of the remaining angles, which will be acute. (Why?)	Law of sines
3	Third angle	Subtract the sum of the measures of the angles found in steps 1 and 2 from 180°.

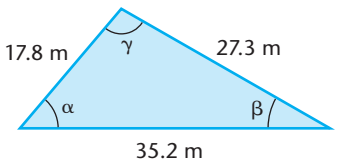
EXAMPLE 2 Solving the SSS Case

Solve the triangle with $a = 27.3$ meters, $b = 17.8$ meters, and $c = 35.2$ meters.

SOLUTION

Three sides of the triangle are given and we are to find the three angles. This is the SSS case.

Sketch the triangle (Fig. 4) and use the law of cosines to find the largest angle, then use the law of sines to find one of the two remaining acute angles.



› Figure 4

We solve for γ:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \qquad \text{Law of cosines}$$

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab} \qquad \text{Solve for } \gamma.$$

$$\gamma = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$$

$$= \cos^{-1}\left[\frac{(27.3)^2 + (17.8)^2 - (35.2)^2}{2(27.3)(17.8)}\right]$$

$$= 100.5°$$

We solve for α:

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \qquad \text{Law of sines}$$

$$\sin\alpha = \frac{a\sin\gamma}{c} = \frac{27.3\sin 100.5^\circ}{35.2} \qquad \text{Solve for } \alpha.$$

$$\alpha = \sin^{-1}\left(\frac{27.3\sin 100.5^\circ}{35.2}\right) \qquad \text{Calculate.}$$

$$= 49.7^\circ \qquad \alpha \text{ is acute.}$$

We solve for β:

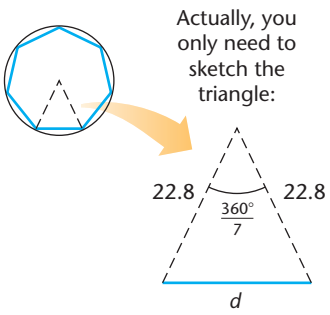
$$\alpha + \beta + \gamma = 180^\circ$$
$$\beta = 180^\circ - (\alpha + \gamma)$$
$$= 180^\circ - (49.7^\circ + 100.5^\circ)$$
$$= 29.8^\circ$$

MATCHED PROBLEM 2

Solve the triangle with $a = 1.25$ yards, $b = 2.05$ yards, and $c = 1.52$ yards.

EXAMPLE 3 **Finding the Side of a Regular Polygon**

If a seven-sided regular polygon is inscribed in a circle of radius 22.8 centimeters, find the length of one side of the polygon.



› Figure 5

SOLUTION

Sketch a figure (Fig. 5) and use the law of cosines.

$$d^2 = 22.8^2 + 22.8^2 - 2(22.8)(22.8)\cos\frac{360^\circ}{7}$$

$$d = \sqrt{2(22.8)^2 - 2(22.8)^2\cos\frac{360^\circ}{7}}$$

$$= 19.8 \text{ centimeters}$$

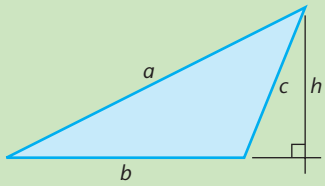
MATCHED PROBLEM 3

If an 11-sided regular polygon is inscribed in a circle with radius 4.63 inches, find the length of one side of the polygon.

››› EXPLORE-DISCUSS 3

(A) The area of a rectangle is its length times its width. Use this fact to explain why the area of a right triangle is one-half the product of its legs (the legs are the sides adjacent to the right angle).

(B) Explain why the area A of any triangle is given by $A = \frac{1}{2}bh$ [*Hint:* Use right triangles.]



(C) The law of cosines can be used to derive the formula given by Heron of Alexandria (A.D. 75) for the area of any triangle in terms of the lengths a, b, and c of its sides:

$$A = \sqrt{s(s-a)(s-b)(s-c)} \qquad \text{where} \qquad s = \frac{a+b+c}{2}$$

is the **semiperimeter** of the triangle. Verify that Heron's formula gives the correct area for any equilateral triangle.

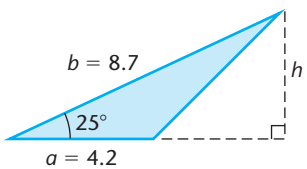
EXAMPLE 4 Finding the Area of a Triangle

Find the area of each triangle (to the same number of significant digits as the side with the least number of significant digits):

(A) $a = 4.2$ inches, $b = 8.7$ inches, $\gamma = 25°$

(B) $a = 3.52$ inches, $b = 2.91$ inches, $c = 4.67$ inches

SOLUTIONS



› Figure 6

(A) We are given two sides and the included angle. Either of the known sides can be used as the base; we will use side a. The height h is the perpendicular distance to the base (extend the base if necessary), from the vertex that is not on the base (Fig. 6).

Therefore,

$$\frac{h}{8.7} = \sin 25° \qquad \text{Multiply both sides by 8.7.}$$

$$h = 8.7 \sin 25°$$

The area of any triangle (see Explore-Discuss 3) is given by

$$\begin{aligned} A &= \tfrac{1}{2}(\text{base})(\text{height}) && \text{Base = 4.2, height = 8.7 sin 25°} \\ &= \tfrac{1}{2}(4.2)(8.7 \sin 25°) \\ &= 7.7 \text{ square inches} && \text{To two significant digits} \end{aligned}$$

(B) Because we are given the lengths of all three sides, we can calculate the semiperimeter s and then use Heron's formula (see Explore-Discuss 3):

$$\begin{aligned} s &= \frac{a + b + c}{2} \\ &= \frac{3.52 + 2.91 + 4.67}{2} \\ &= 5.55 \\ A &= \sqrt{s(s - a)(s - b)(s - c)} && \text{Substitute values of } s, a, b, \text{ and } c. \\ &= \sqrt{5.55(5.55 - 3.52)(5.55 - 2.91)(5.55 - 4.67)} && \text{Calculate.} \\ &= 5.12 \text{ square inches} && \text{To three significant digits} \end{aligned}$$

MATCHED PROBLEM 4

Find the area of each triangle (to the same number of significant digits as the side with the least number of significant digits):

(A) $a = 38$ meters, $b = 25$ meters, $\gamma = 74°$

(B) $a = 135$ yards, $b = 94$ yards, $c = 172$ yards

››› CAUTION ›››

The formula for the area of a triangle,

$$A = \tfrac{1}{2}(\text{base})(\text{height}), \text{ is often written as } A = \tfrac{1}{2}bh.$$

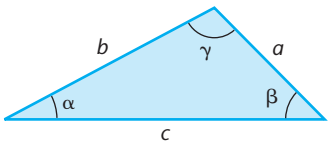
Don't confuse the letter b in the formula, which stands for base, with the letters a, b, and c that we use to label the sides of a triangle. Remember that any known side of a triangle can serve as the base, provided the height h is calculated with respect to that base (see Example 4A).

ANSWERS TO MATCHED PROBLEMS

1. $a = 18.5$ feet, $\beta = 33.3°$, $\gamma = 69.2°$ **2.** $\alpha = 37.4°$, $\beta = 95.0°$, $\gamma = 47.6°$
3. 2.61 inches **4.** (A) 460 square meters (B) 6,300 square yards

7-2 Exercises

The labeling in the figure shown here is the convention we will follow in this exercise set. Your answers to some problems may differ slightly from those in the book, depending on the order in which you solve for the sides and angles of a given triangle.



1. Explain how to solve a triangle given two of its sides and the included angle.

2. Explain how to solve a triangle given all three of its sides.

3. Explain how the law of cosines simplifies if $\gamma = 90°$.

4. Explain how the law of cosines simplifies if $a = b = c$.

5. In Euclidean geometry, the SAS congruence theorem says that if two sides and the included angle of one triangle are congruent, respectively, to two sides and the included angle of a second triangle, then the two triangles are congruent. Is there an ASA congruence theorem? An AAA congruence theorem? Explain.

6. See Problem 5. Is there an SSA congruence theorem? An SSS congruence theorem? Explain.

7. Referring to the figure, if $\alpha = 47.3°$, $b = 11.7$ centimeters, and $c = 6.04$ centimeters, which of the two angles, β or γ, can you say for certain is acute and why?

8. Referring to the figure, if $\alpha = 93.5°$, $b = 5.34$ inches, and $c = 8.77$ inches, which of the two angles, β or γ, can you say for certain is acute and why?

Solve each triangle in Problems 9–12.

9. $\alpha = 71.2°$, $b = 5.32$ yards, $c = 5.03$ yards

10. $\beta = 57.3°$, $a = 6.08$ centimeters, $c = 5.25$ centimeters

11. $\gamma = 120°20'$, $a = 5.73$ millimeters, $b = 10.2$ millimeters

12. $\alpha = 135°50'$, $b = 8.44$ inches, $c = 20.3$ inches

13. Referring to the figure at the beginning of the exercise set, if $a = 13.5$ feet, $b = 20.8$ feet, and $c = 8.09$ feet, then, if the triangle has an obtuse angle, which angle must it be and why?

14. Suppose you are told that a triangle has sides $a = 12.5$ centimeters, $b = 25.3$ centimeters, and $c = 10.7$ centimeters. Explain why the triangle has no solution.

Solve each triangle in Problems 15–18 if the triangle has a solution. Use decimal degrees for angle measure.

15. $a = 4.00$ meters, $b = 10.2$ meters, $c = 9.05$ meters

16. $a = 10.5$ miles, $b = 20.7$ miles, $c = 12.2$ miles

17. $a = 6.00$ kilometers, $b = 5.30$ kilometers, $c = 5.52$ kilometers

18. $a = 31.5$ meters, $b = 29.4$ meters, $c = 33.7$ meters

Problems 19–34 represent a variety of problems involving both the law of sines and the law of cosines. Solve each triangle. If a problem does not have a solution, say so.

19. $\alpha = 94.5°$, $\gamma = 88.3°$, $b = 23.7$ centimeters

20. $\beta = 85.6°$, $\gamma = 97.3°$, $a = 14.3$ millimeters

21. $\beta = 104.5°$, $a = 17.2$ inches, $c = 11.7$ inches

22. $\beta = 27.3°$, $a = 13.7$ yards, $c = 20.1$ yards

23. $\alpha = 57.2°$, $\gamma = 112.0°$, $c = 24.8$ meters

24. $\beta = 132.4°$, $\gamma = 17.3°$, $b = 67.6$ kilometers

25. $\beta = 38.4°$, $a = 11.5$ inches, $b = 14.0$ inches

26. $\gamma = 66.4°$, $b = 25.5$ meters, $c = 25.5$ meters

27. $a = 32.9$ meters, $b = 42.4$ meters, $c = 20.4$ meters

28. $a = 10.5$ centimeters, $b = 5.23$ centimeters, $c = 8.66$ centimeters

29. $\gamma = 58.4°$, $b = 7.23$ meters, $c = 6.54$ meters

30. $\alpha = 46.7°$, $a = 18.1$ meters, $b = 22.6$ meters

31. $\beta = 39.8°$, $a = 12.5$ inches, $b = 7.31$ inches

32. $\gamma = 47.9°$, $b = 35.2$ inches, $c = 25.5$ inches

33. $\beta = 13.6°$, $b = 21.6$ meters, $c = 58.4$ meters

34. $\beta = 25.1°$, $b = 53.7$ meters, $c = 98.5$ meters

In Problems 35–52, find the area of each triangle (to the same number of significant digits as the side with the least number of significant digits).

35. $a = 13.4$ feet, $b = 25.1$ feet, $\gamma = 90°$

36. $a = 67$ feet, $b = 92$ feet, $\gamma = 90°$

37. $a = 33$ yards, $b = 28$ yards, $\alpha = 90°$

38. $a = 542$ yards, $b = 167$ yards, $\alpha = 90°$

39. $a = 75$ meters, $b = 14$ meters, $\gamma = 37°$

40. $a = 183$ meters, $b = 10.1$ meters, $\gamma = 49.3°$

41. $a = 12$ inches, $b = 5.0$ inches, $c = 13$ inches

42. $a = 35$ inches, $b = 12$ inches, $c = 37$ inches

43. $a = 152$ feet, $b = 363$ feet, $\gamma = 112.5°$

44. $a = 42$ feet, $b = 210$ feet, $\gamma = 139°$

45. $a = 6.5$ kilometers, $b = 3.9$ kilometers, $c = 4.8$ kilometers

46. $a = 35.8$ kilometers, $b = 42.3$ kilometers, $c = 56.9$ kilometers

47. $\alpha = 72°$, $\beta = 48°$, $c = 2.6$ meters

48. $\alpha = 41°$, $\beta = 113°$, $c = 9.5$ meters

49. $a = 237$ yards, $b = 513$ yards, $c = 455$ yards

50. $a = 95$ yards, $b = 19$ yards, $c = 104$ yards

51. $\alpha = 82°$, $\beta = 61°$, $a = 16$ inches

52. $\alpha = 136°$, $\beta = 28°$, $a = 52$ inches

In Problems 53–56, determine whether the statement is true or false. If true, explain why. If false, give a counterexample.

53. If two triangles have the same perimeter, then they have the same area.

54. If two squares have the same perimeter, then they have the same area.

55. If two squares have the same area, then they have the same perimeter.

56. If two triangles have the same area, then they have the same perimeter.

57. Show, using the law of cosines, that if $\gamma = 90°$, then $c^2 = a^2 + b^2$ (the Pythagorean theorem).

58. Show, using the law of cosines, that if $c^2 = a^2 + b^2$, then $\gamma = 90°$.

59. Show that for any triangle,

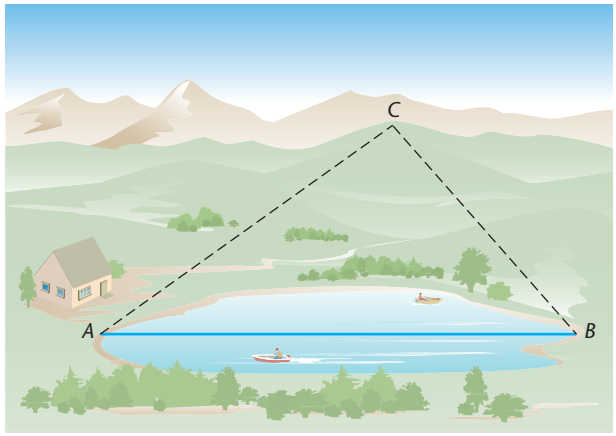
$$\frac{a^2 + b^2 + c^2}{2abc} = \frac{\cos\alpha}{a} + \frac{\cos\beta}{b} + \frac{\cos\gamma}{c}$$

60. Show that for any triangle,

$$a = b\cos\gamma + c\cos\beta$$

APPLICATIONS

61. **SURVEYING** To find the length AB of a small lake, a surveyor measured angle ACB to be 96°, AC to be 91 yards, and BC to be 71 yards. What is the approximate length of the lake?



62. **SURVEYING** Refer to Problem 61. If a surveyor finds $\angle ACB = 110°$, $AC = 85$ meters, and $BC = 73$ meters, what is the approximate length of the lake?

63. **GEOMETRY** Find the measure in decimal degrees of a central angle subtended by a chord of length 112 millimeters in a circle of radius 72.8 millimeters.

64. **GEOMETRY** Find the measure in decimal degrees of a central angle subtended by a chord of length 13.8 feet in a circle of radius 8.26 feet.

65. **GEOMETRY** Two adjacent sides of a parallelogram meet at an angle of 35°10′ and have lengths of 3 and 8 feet. What is the length of the shorter diagonal of the parallelogram (to three significant digits)?

66. **GEOMETRY** What is the length of the longer diagonal of the parallelogram in Problem 65 (to three significant digits)?

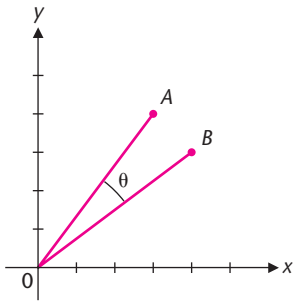
67. **NAVIGATION** Los Angeles and Las Vegas are approximately 200 miles apart. A pilot 80 miles from Los Angeles finds that she is 6°20′ off course relative to her start in Los Angeles. How far is she from Las Vegas at this time? (Compute the answer to three significant digits.)

68. **SEARCH AND RESCUE** At noon, two search planes set out from San Francisco to find a downed plane in the ocean. Plane A travels due west at 400 miles per hour, and plane B flies northwest at 500 miles per hour. At 2 P.M. plane A spots the survivors of the downed plane and radios plane B to come and assist in the rescue. How far is plane B from plane A at this time (to three significant digits)?

69. **GEOMETRY** Find the perimeter of a pentagon inscribed in a circle of radius 12.6 meters.

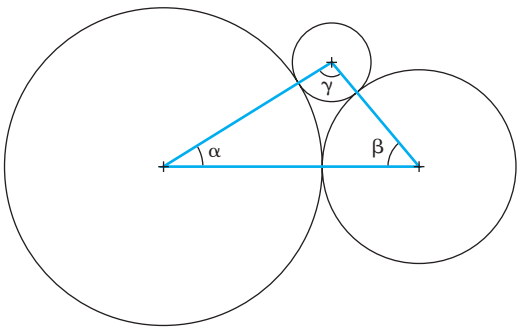
70. **GEOMETRY** Find the perimeter of a nine-sided regular polygon inscribed in a circle of radius 7.09 centimeters.

71. **ANALYTIC GEOMETRY** If point A in the figure has coordinates (3, 4) and point B has coordinates (4, 3), find the radian measure of angle θ to three decimal places.



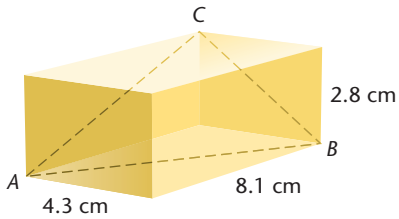
72. **ANALYTIC GEOMETRY** If point A in the figure above has coordinates (4, 3) and point B has coordinates (5, 1), find the radian measure of $\angle AOB$ to three decimal places.

73. **ENGINEERING** Three circles of radius 2.03, 5.00, and 8.20 centimeters are tangent to one another (see the figure). Find the three angles formed by the lines joining their centers (to the nearest 10′).



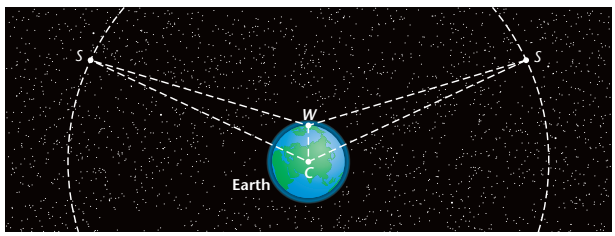
74. **ENGINEERING** Three circles of radius 2.00, 5.00, and 8.00 inches are tangent to each other (see the figure). Find the three angles formed by the lines joining their centers (to the nearest 10′).

75. **GEOMETRY** A rectangular solid has sides as indicated in the figure. Find $\angle CAB$ to the nearest degree.

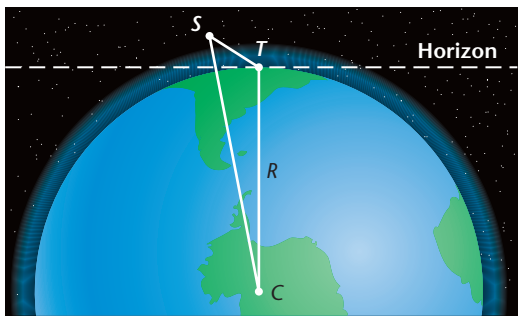


76. **GEOMETRY** Referring to Problem 75, find $\angle ACB$ to the nearest degree.

77. **SPACE SCIENCE** For communications between a space shuttle and the White Sands tracking station in southern New Mexico, two satellites are placed in geostationary orbit, 130° apart relative to the center of the Earth, and 22,300 miles above the surface of the Earth (see the figure). (A satellite in **geostationary** orbit remains stationary above a fixed point on the surface of the Earth.) Radio signals are sent from the tracking station by way of the satellites to the shuttle, and vice versa. How far to the nearest 100 miles is one of the geostationary satellites from the White Sands tracking station, W? The radius of the Earth is 3,964 miles.



78. **SPACE SCIENCE** A satellite S, in circular orbit around the Earth, is sighted by a tracking station T (see the figure). The distance TS is determined by radar to be 1,034 miles, and the angle of elevation above the horizon is 32.4°. How high is the satellite above the Earth at the time of the sighting? The radius of the Earth is 3,964 miles.



7-3 Vectors in the Plane

- Vectors
- Vector Addition and Scalar Multiplication
- Unit Vectors
- Velocity Vectors
- Force Vectors

Many physical quantities such as length, area, or volume can be completely specified by a single real number. They are called *scalar quantities*. Other quantities such as directed distances, velocities, and forces require both a magnitude and direction. They are *vector quantities*. Vector quantities have wide application in many areas of science and engineering.

In this section we introduce the concept of a vector and study its applications. Although we restrict our discussion to vectors in the plane, the methods we introduce can be readily generalized to vectors in three-dimensional or higher-dimensional spaces.

Vectors

A **vector v** is a quantity that has both magnitude and direction. We picture a vector as an arrow from an initial point O to a terminal point P with this provision: Arrows that have the same length (magnitude) and direction represent the same vector (Fig. 1).

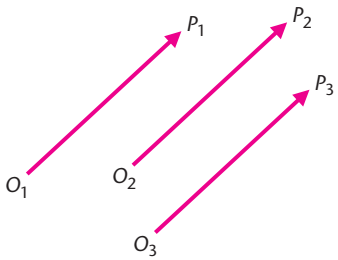
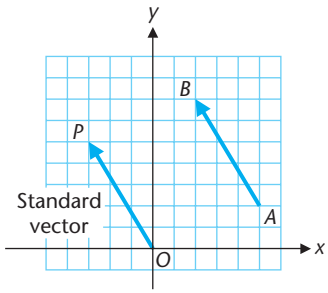


Figure 1 All three arrows represent the same vector **v**.

The vector **v** of Figure 1 is also denoted by $\overrightarrow{O_1P_1}$ (or $\overrightarrow{O_2P_2}$ or $\overrightarrow{O_3P_3}$). We use boldface letters such as **v** to denote vectors. But because it is difficult to write boldface on paper, we suggest that you use $\vec{v}$ as a substitute for **v** when you want to denote a vector by a single letter.

The **magnitude** of the vector $\mathbf{v} = \overrightarrow{OP}$, denoted by $|\mathbf{v}|$, $|\vec{v}|$, or $|\overrightarrow{OP}|$, is the length of the line segment OP. Two vectors have the **same direction** if they are parallel and point in the same direction. Two vectors have **opposite directions** if they are parallel and point in opposite directions. The **zero vector,** denoted by **0** or $\vec{0}$, has magnitude 0 and arbitrary direction. Two vectors are **equal** if they have the same magnitude and



› **Figure 2** $\overrightarrow{OP}$ is the standard vector for $\overrightarrow{AB}$.

direction. Therefore, a vector can be translated from one location to another as long as the magnitude and direction do not change.

Any vector $\overrightarrow{AB}$ in a rectangular coordinate system can be translated so that its initial point is the origin O. The vector $\overrightarrow{OP}$ such that $\overrightarrow{OP} = \overrightarrow{AB}$ is said to be the **standard vector** for $\overrightarrow{AB}$ (Fig. 2).

››› EXPLORE-DISCUSS 1

If the tail of a vector is at point $A = (-3, 2)$ and its tip is at $B = (6, 4)$, discuss how you would find the coordinates of P so that $\overrightarrow{OP}$ is the standard vector for $\overrightarrow{AB}$.

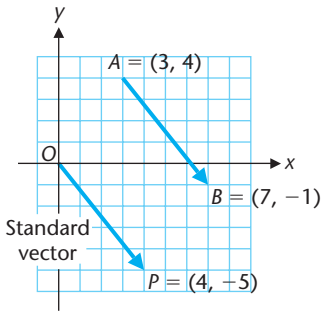
Given the coordinates of the endpoints of vector $\overrightarrow{AB}$, how do we find its corresponding standard vector $\overrightarrow{OP}$? The coordinates of the origin O, the initial point of OP, are always (0, 0). The coordinates of P, the terminal point of $\overrightarrow{OP}$, are given by

$$(x_p, y_p) = (x_b - x_a, y_b - y_a)$$

where $A = (x_a, y_a)$ and $B = (x_b, y_b)$.

EXAMPLE 1 Finding a Standard Vector for a Given Vector

Given the vector $\overrightarrow{AB}$ with initial point $A = (3, 4)$ and terminal point $B = (7, -1)$, find the coordinates of the point P such that $\overrightarrow{OP} = \overrightarrow{AB}$.



› **Figure 3**

SOLUTION

The coordinates of P are given by

$$\begin{aligned}(x_p, y_p) &= (x_b - x_a, y_b - y_a)\\ &= (7 - 3, -1 - 4)\\ &= (4, -5)\end{aligned}$$

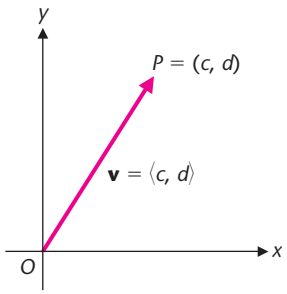
Note in Figure 3 that if we start at A, then move to the right four units and down five units, we will be at B. If we start at the origin, then move to the right four units and down five units, we will be at P.

MATCHED PROBLEM 1

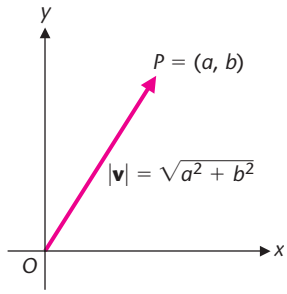
Given the vector $\overrightarrow{AB}$ with initial point $A = (8, -3)$ and terminal point $B = (4, 5)$, find the standard vector $\overrightarrow{OP}$ for $\overrightarrow{AB}$.

Example 1 suggests that there is a one-to-one correspondence between vectors in a rectangular coordinate system and points in the system. Any vector $\overrightarrow{AB}$ is completely specified by the point $P = (x_p, y_p)$ such that $\overrightarrow{OP} = \overrightarrow{AB}$ (we are not concerned that $\overrightarrow{OP}$ has a different position than $\overrightarrow{AB}$; we are free to translate a vector anywhere we please). Conversely, any point P of the system corresponds to the vector $\overrightarrow{OP}$.

A vector can thus be denoted by an ordered pair of real numbers. To avoid confusion, we use $\langle c, d\rangle$ to denote the vector $\overrightarrow{OP}$ with initial point (0, 0) and terminal point (c, d) (Fig. 4). The real numbers c and d are called the **scalar components** of the vector $\langle c, d\rangle$. Two vectors $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$ are equal if their corresponding components are equal, that is, if $a = c$ and $b = d$. The zero vector is $\mathbf{0} = \langle 0, 0\rangle$. The magnitude of the vector $\langle a, b\rangle$ is the length of the line segment from (0, 0) to (a, b) [Fig. 5].



› **Figure 4** Vector $\mathbf{v} = \langle c, d\rangle$.



› **Figure 5** Magnitude of $\mathbf{v} = \langle a, b\rangle$.

› MAGNITUDE OF A VECTOR

The **magnitude,** or **norm,** of a vector $\mathbf{v} = \langle a, b\rangle$ is denoted by $|\mathbf{v}|$ and given by

$$|\mathbf{v}| = \sqrt{a^2 + b^2}$$

EXAMPLE 2 Finding the Magnitude of a Vector

Find the magnitude of the vector $\mathbf{v} = \langle 3, -5\rangle$.

SOLUTION

$$|\mathbf{v}| = \sqrt{3^2 + (-5)^2} = \sqrt{34}$$

MATCHED PROBLEM 2

Find the magnitude of the vector $\mathbf{v} = \langle -2, 4\rangle$.

Vector Addition and Scalar Multiplication

The sum $\mathbf{u} + \mathbf{v}$ of two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by the **tail-to-tip rule:** Translate $\mathbf{v}$ so that its tail (initial point) is at the tip (terminal point) of $\mathbf{u}$. Then, the vector from the tail of $\mathbf{u}$ to the tip of $\mathbf{v}$ is the sum, denoted $\mathbf{u} + \mathbf{v}$, of the vectors $\mathbf{u}$ and $\mathbf{v}$ (Fig. 6).

If $\mathbf{u}$ and $\mathbf{v}$ are not parallel, the **parallelogram rule** gives an alternative description of $\mathbf{u} + \mathbf{v}$: The sum of two nonparallel vectors $\mathbf{u}$ and $\mathbf{v}$ is the diagonal of the parallelogram formed using $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides (Fig. 7).

The vector $\mathbf{u} + \mathbf{v}$ is also called the **resultant** of the two vectors $\mathbf{u}$ and $\mathbf{v}$, and $\mathbf{u}$ and $\mathbf{v}$ are called **vector components** of $\mathbf{u} + \mathbf{v}$.

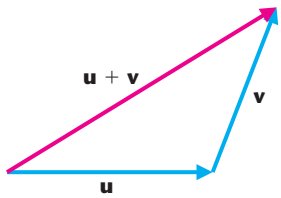


Figure 6 Vector addition: tail-to-tip rule.

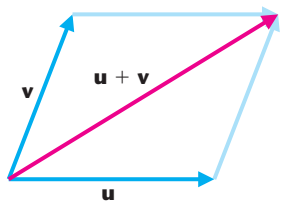


Figure 7 Vector addition: parallelogram rule.

EXPLORE-DISCUSS 2

Use the tail-to-tip rule and/or the parallelogram rule to explain why vector addition is *commutative,* that is, for any vectors $\mathbf{u}$ and $\mathbf{v}$,

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

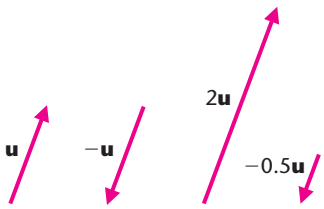


Figure 8 Scalar multiplication.

The **scalar product** $k\mathbf{u}$ of a scalar (real number) k and a vector $\mathbf{u}$ is the vector with magnitude $|k|\|\mathbf{u}\|$ that has the same direction as $\mathbf{u}$ if k is positive and the opposite direction if k is negative. For example, $2\mathbf{u}$ has twice the magnitude of $\mathbf{u}$ and the same direction. Similarly, $-0.5\mathbf{u}$ has half the magnitude of $\mathbf{u}$ and the opposite direction (Fig. 8).

Both the sum $\mathbf{u} + \mathbf{v}$ and the scalar product $k\mathbf{u}$ are easy to calculate if the scalar components of $\mathbf{u}$ and $\mathbf{v}$ are given: for the sum, just add corresponding components; for the scalar product, multiply each component by the scalar.

VECTOR ADDITION AND SCALAR MULTIPLICATION

If $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$ are vectors and k is a scalar (real number), then

$$\mathbf{u} + \mathbf{v} = \langle a + c, b + d \rangle \quad \textbf{Vector addition}$$
$$k\mathbf{u} = \langle ka, kb \rangle \quad \textbf{Scalar multiplication}$$

EXAMPLE 3 Vector Addition and Scalar Multiplication

Let $\mathbf{u} = \langle 4, -3\rangle$, $\mathbf{v} = \langle 2, 3\rangle$, and $\mathbf{w} = \langle 0, -5\rangle$. Find

(A) $\mathbf{u} + \mathbf{v}$ (B) $-2\mathbf{u}$ (C) $2\mathbf{u} - 3\mathbf{v}$ (D) $3\mathbf{u} + 2\mathbf{v} - \mathbf{w}$

SOLUTIONS

(A) $\mathbf{u} + \mathbf{v} = \langle 4, -3\rangle + \langle 2, 3\rangle = \langle 6, 0\rangle$

(B) $-2\mathbf{u} = -2\langle 4, -3\rangle = \langle -8, 6\rangle$

(C) $2\mathbf{u} - 3\mathbf{v} = 2\langle 4, -3\rangle - 3\langle 2, 3\rangle$
$= \langle 8, -6\rangle + \langle -6, -9\rangle = \langle 2, -15\rangle$

(D) $3\mathbf{u} + 2\mathbf{v} - \mathbf{w} = 3\langle 4, -3\rangle + 2\langle 2, 3\rangle - \langle 0, -5\rangle$
$= \langle 12, -9\rangle + \langle 4, 6\rangle + \langle 0, 5\rangle$
$= \langle 16, 2\rangle$

MATCHED PROBLEM 3

Let $\mathbf{u} = \langle -5, 3\rangle$, $\mathbf{v} = \langle 4, -6\rangle$, and $\mathbf{w} = \langle -2, 0\rangle$. Find

(A) $\mathbf{u} + \mathbf{v}$ (B) $-3\mathbf{u}$ (C) $3\mathbf{u} - 2\mathbf{v}$ (D) $2\mathbf{u} - \mathbf{v} + 3\mathbf{w}$

Vector addition and scalar multiplication possess algebraic properties similar to the real numbers. These properties enable us to manipulate symbols representing vectors and scalars in much the same way we manipulate symbols that represent real numbers in algebra. These properties are listed here for convenient reference.

› ALGEBRAIC PROPERTIES OF VECTORS

A. Addition Properties. For all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$,

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	**Commutative Property**
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	**Associative Property**
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$	**Additive Identity**
4. $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$	**Additive Inverse**

B. Scalar Multiplication Properties. For all vectors $\mathbf{u}$ and $\mathbf{v}$ and all scalars m and n:

1. $m(n\mathbf{u}) = (mn)\mathbf{u}$	**Associative Property**
2. $m(\mathbf{u} + \mathbf{v}) = m\mathbf{u} + m\mathbf{v}$	**Distributive Property**
3. $(m + n)\mathbf{u} = m\mathbf{u} + n\mathbf{u}$	**Distributive Property**
4. $1\mathbf{u} = \mathbf{u}$	**Multiplicative Identity**

Unit Vectors

Any vector that has magnitude 1 is called a **unit vector.** If $\mathbf{v}$ is an arbitrary nonzero vector and k is a scalar, then the scalar product $k\mathbf{v}$ has magnitude $|k||\mathbf{v}|$. Therefore, by choosing k to be $1/|\mathbf{v}|$, the scalar product $k\mathbf{v}$ will be a unit vector with the same direction as $\mathbf{v}$.

A UNIT VECTOR WITH THE SAME DIRECTION AS v

If $\mathbf{v}$ is a nonzero vector, then

$$\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v}$$

is a unit vector with the same direction as $\mathbf{v}$.

EXAMPLE 4 Finding a Unit Vector with the Same Direction as a Given Vector

Given a vector $\mathbf{v} = \langle 1, -2 \rangle$, find a unit vector $\mathbf{u}$ with the same direction as $\mathbf{v}$.

SOLUTION

$$|\mathbf{v}| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$$
$$\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v} = \frac{1}{\sqrt{5}}\langle 1, -2 \rangle$$
$$= \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle$$

CHECK

$$|\mathbf{u}| = \sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{-2}{\sqrt{5}}\right)^2} = \sqrt{\frac{1}{5} + \frac{4}{5}} = \sqrt{1} = 1$$

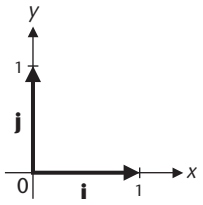
And we see that $\mathbf{u}$ is a unit vector with the same direction as $\mathbf{v}$.

MATCHED PROBLEM 4

Given a vector $\mathbf{v} = \langle 3, 1 \rangle$, find a unit vector $\mathbf{u}$ with the same direction as $\mathbf{v}$.

The unit vectors in the directions of the positive x axis and the positive y axis are denoted by $\mathbf{i}$ and $\mathbf{j}$, respectively.

› THE i AND j UNIT VECTORS



$$\mathbf{i} = \langle 1, 0 \rangle$$
$$\mathbf{j} = \langle 0, 1 \rangle$$

Why are the **i** and **j** unit vectors important? Any vector $\mathbf{v} = \langle a, b \rangle$ can be expressed as a linear combination of these two vectors; that is, as $a\mathbf{i} + b\mathbf{j}$.

$$\begin{aligned} \mathbf{v} &= \langle a, b \rangle = \langle a, 0 \rangle + \langle 0, b \rangle \\ &= a\langle 1, 0 \rangle + b\langle 0, 1 \rangle = a\mathbf{i} + b\mathbf{j} \end{aligned}$$

EXAMPLE 5 Expressing a Vector in Terms of the i and j Vectors

Express each vector as a linear combination of the **i** and **j** unit vectors.

(A) $\langle -2, 4 \rangle$ (B) $\langle 2, 0 \rangle$ (C) $\langle 0, -7 \rangle$

SOLUTIONS

(A) $\langle -2, 4 \rangle = -2\mathbf{i} + 4\mathbf{j}$
(B) $\langle 2, 0 \rangle = 2\mathbf{i} + 0\mathbf{j} = 2\mathbf{i}$
(C) $\langle 0, -7 \rangle = 0\mathbf{i} - 7\mathbf{j} = -7\mathbf{j}$

MATCHED PROBLEM 5

Express each vector as a linear combination of the **i** and **j** unit vectors.

(A) $\langle 5, -3 \rangle$ (B) $\langle -9, 0 \rangle$ (C) $\langle 0, 6 \rangle$

EXAMPLE 6 Algebraic Operations on Vectors Expressed in Terms of the i and j Vectors

For $\mathbf{u} = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j}$, compute each of the following:

(A) $\mathbf{u} + \mathbf{v}$ (B) $\mathbf{u} - \mathbf{v}$ (C) $2\mathbf{u} - 3\mathbf{v}$

SOLUTIONS

(A) $\mathbf{u} + \mathbf{v} = (\mathbf{i} - 2\mathbf{j}) + (5\mathbf{i} + 2\mathbf{j}) = \mathbf{i} - 2\mathbf{j} + 5\mathbf{i} + 2\mathbf{j} = 6\mathbf{i} + 0\mathbf{j} = 6\mathbf{i}$

(B) $\mathbf{u} - \mathbf{v} = (\mathbf{i} - 2\mathbf{j}) - (5\mathbf{i} + 2\mathbf{j}) = \mathbf{i} - 2\mathbf{j} - 5\mathbf{i} - 2\mathbf{j} = -4\mathbf{i} - 4\mathbf{j}$

(C) $2\mathbf{u} + 3\mathbf{v} = 2(\mathbf{i} - 2\mathbf{j}) + 3(5\mathbf{i} + 2\mathbf{j}) = 2\mathbf{i} - 4\mathbf{j} + 15\mathbf{i} + 6\mathbf{j} = 17\mathbf{i} + 2\mathbf{j}$

MATCHED PROBLEM 6

For $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{v} = 4\mathbf{i} + 5\mathbf{j}$, compute each of the following:

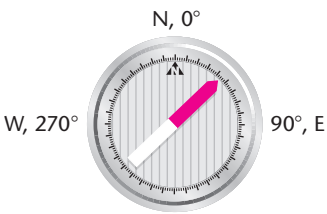
(A) $\mathbf{u} + \mathbf{v}$ (B) $\mathbf{u} - \mathbf{v}$ (C) $3\mathbf{u} - 2\mathbf{v}$

›››› CAUTION ››››

Don't confuse the unit vector $\mathbf{i}$ with the complex number i. The unit vector $\mathbf{i}$ is denoted by boldface; the complex number i, the imaginary unit, is denoted by italic. We have not defined the product of two vectors, so $\mathbf{i}^2$ is undefined. On the other hand, $i^2 = -1$ (see Section 2-4).

Velocity Vectors

A vector that represents the direction and speed of an object in motion is called a **velocity vector.** Problems involving objects in motion often can be analyzed using vector methods. Many of these problems involve the use of a **navigational compass,** which is marked clockwise in degrees starting at north as indicated in Figure 9.



Navigational compass

Figure 9

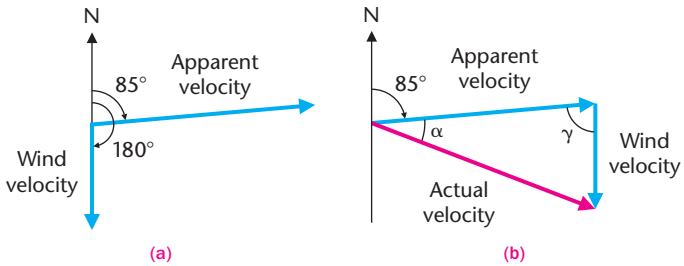
EXAMPLE 7 **Apparent and Actual Velocity**

An airplane has a compass heading (the direction the plane is pointing) of 85° and an airspeed (relative to the air) of 140 miles per hour. The wind is blowing from north to south at 66 miles per hour. The velocity of a plane relative to the air is called **apparent velocity,** and the velocity relative to the ground is called **resultant,** or

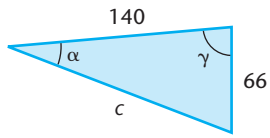
actual, velocity. The resultant velocity is the vector sum of the apparent velocity and the wind velocity. Find the resultant velocity; that is, find the actual speed and direction of the airplane relative to the ground. Directions are given to the nearest degree and magnitudes to two significant digits.

SOLUTION

Vectors [Fig. 10(a)] are used to represent the apparent velocity and the wind velocity. Add the two vectors using the tail-to-tip method to obtain the resultant (actual) velocity vector [Fig. 10(b)]. From the vector diagram [Fig. 10(b)], we obtain the triangle in Figure 11 and solve for γ, c, and α.



› Figure 10



› Figure 11

SOLVE FOR γ

Because the wind velocity vector is parallel to the north–south line, $\gamma = 85°$ [alternate interior angles of two parallel lines cut by a transversal are equal—see Fig. 10(b)].

SOLVE FOR c

Use the law of cosines:

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos\gamma \\ c &= \sqrt{a^2 + b^2 - 2ab\cos\gamma} \\ &= \sqrt{66^2 + 140^2 - 2(66)(140)\cos 85°} \\ &= 150 \text{ miles per hour} \end{aligned}$$

Speed relative to the ground

SOLVE FOR α

Use the law of sines:

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c}$$

$$\alpha = \sin^{-1}\left(\frac{a\sin\gamma}{c}\right)$$

$$= \sin^{-1}\left(\frac{66\sin 85}{150}\right) = 26°$$

Actual heading $= 85° + \alpha = 85° + 26° = 111°$. Therefore, the magnitude and direction of the resultant velocity vector are 150 miles per hour and 111°, respectively. That is, the plane, relative to the ground, is traveling at 150 miles per hour in a direction of 111°.

MATCHED PROBLEM 7

A river is flowing southwest (225°) at 3.0 miles per hour. A boat crosses the river with a compass heading of 90°. If the speedometer on the boat reads 5.0 miles per hour (the boat's speed relative to the water), what is the resultant velocity? That is, what is the boat's actual speed and direction relative to the ground? Directions are to the nearest degree, and magnitudes are to two significant digits.

Force Vectors

A vector that represents the direction and magnitude of an applied force is called a **force vector.** If an object is subjected to two forces, then the sum of these two forces, the **resultant force,** is a single force. If the resultant force replaced the original two forces, it would act on the object in the same way as the two original forces taken together. In physics it is shown that the resultant force vector can be obtained using vector addition to add the two individual force vectors. It seems natural to use the parallelogram rule for adding force vectors, as is illustrated in Example 8.

EXAMPLE 8 **Finding the Resultant Force**

Two forces of 30 and 70 pounds act on a point in a plane. If the angle between the force vectors is 40°, what are the magnitude and direction (relative to the 70-pound force) of the resultant force? The magnitudes of the forces are to two significant digits and the angles to the nearest degree.

SOLUTION

We start with a diagram (Fig. 12), letting vectors represent the various forces. Because adjacent angles in a parallelogram are supplementary, the measure of angle $OCB = 180° - 40° = 140°$. We can now find the magnitude of the resultant vector **R** using the law of cosines (Fig. 13).

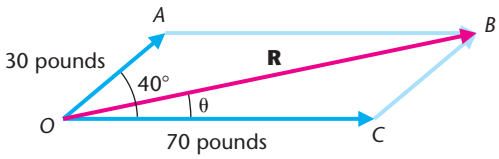
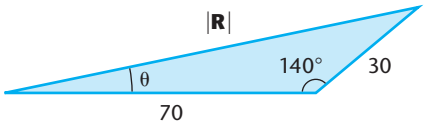


Figure 12

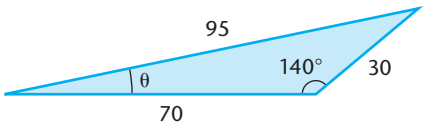
$$|\mathbf{R}|^2 = 30^2 + 70^2 - 2(30)(70)\cos 140^\circ$$
$$|\mathbf{R}| = \sqrt{30^2 + 70^2 - 2(30)(70)\cos 140^\circ}$$
$$= 95 \text{ pounds}$$



› Figure 13

To find θ, the direction of **R,** we use the law of sines (Fig. 14).

$$\frac{\sin\theta}{30} = \frac{\sin 140^\circ}{95}$$
$$\sin\theta = \frac{30\sin 140^\circ}{95}$$
$$\theta = \sin^{-1}\left(\frac{30\sin 140^\circ}{95}\right) = 12^\circ$$



› Figure 14

Therefore, the two given forces are equivalent to a single force of 95 pounds in the direction of 12° (relative to the 70-pound force).

MATCHED PROBLEM 8

Repeat Example 8 using an angle of 100° between the two forces.

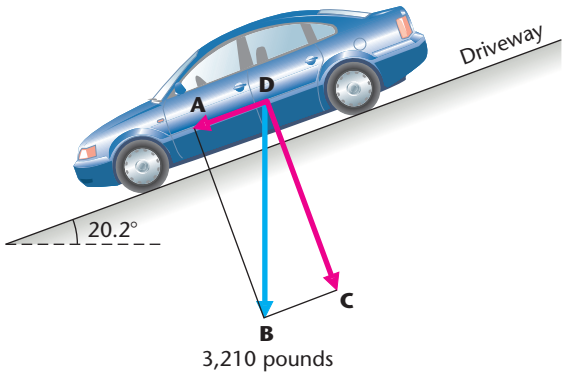
Instead of adding vectors, many problems require the resolution of vectors into components. As we indicated earlier, whenever a vector is expressed as the sum or resultant of two vectors, the two vectors are called *vector components* of the given vector. Example 9 illustrates an application of the process of resolving a vector into vector components.

EXAMPLE 9 Resolving a Force Vector into Components

A car weighing 3,210 pounds is on a driveway inclined 20.2° to the horizontal. Neglecting friction, find the magnitude of the force parallel to the driveway that will keep the car from rolling down the hill.

SOLUTION

We start by drawing a vector diagram (Fig. 15).



› Figure 15

The force vector $\overrightarrow{DB}$ acts in a downward direction and represents the weight of the car. Note that $\overrightarrow{DB} = \overrightarrow{DC} + \overrightarrow{DA}$, where $\overrightarrow{DC}$ is the perpendicular component of $\overrightarrow{DB}$ relative to the driveway and $\overrightarrow{DA}$ is the parallel component of $\overrightarrow{DB}$ relative to the driveway.

To keep the car at D from rolling down the hill, we need a force with the magnitude of $\overrightarrow{DA}$ but oppositely directed. To find $|\overrightarrow{DA}|$, we note that $\angle ABD = 20.2°$. This is true because $\angle ABD$ and the driveway angle have the same complement, $\angle ADB$.

$$\sin 20.2° = \frac{|\overrightarrow{DA}|}{3{,}210}$$

$$|\overrightarrow{DA}| = 3{,}210 \sin 20.2°$$

$$= 1{,}110 \text{ pounds}$$

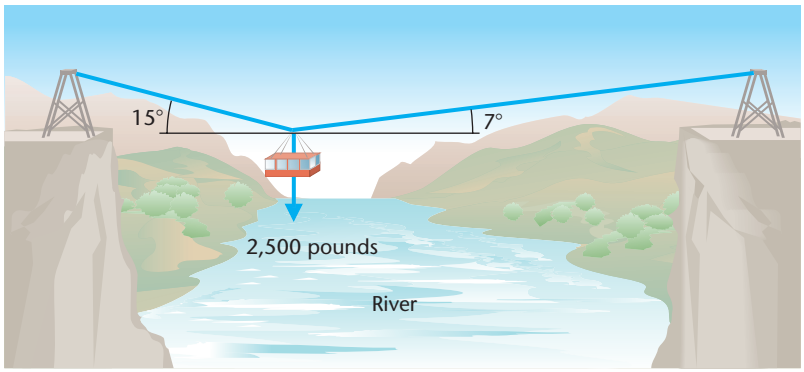
MATCHED PROBLEM 9

Find the magnitude of the perpendicular component of $\overrightarrow{DB}$ in Example 9.

An object at rest is said to be in **static equilibrium.** Example 10 illustrates how important physics and engineering problems can be solved using the condition for static equilibrium: For an object to remain in static equilibrium, the sum of all the force vectors acting on the object must be the zero vector.

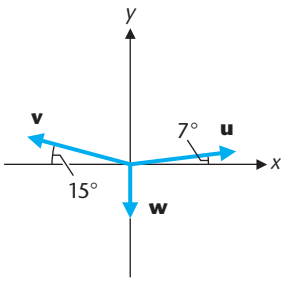
EXAMPLE 10 Tension in Cables

A cable car, used to ferry people and supplies across a river, weighs 2,500 pounds fully loaded. The car stops when partway across and deflects the cable relative to the horizontal, as indicated in Figure 16. What is the tension in each part of the cable running to each tower? (The *tension* in a cable is the magnitude of the force it exerts in the direction parallel to the cable.)



› Figure 16

SOLUTION



$|\mathbf{w}| = 2{,}500$ pounds

› Figure 17

Step 1. Draw a force diagram with all force vectors in standard position at the origin (Fig. 17). The objective is to find $|\mathbf{u}|$ and $|\mathbf{v}|$.

Step 2. Write each force vector in terms of the **i** and **j** unit vectors:

$$\mathbf{u} = |\mathbf{u}|(\cos 7°)\mathbf{i} + |\mathbf{u}|(\sin 7°)\mathbf{j}$$
$$\mathbf{v} = |\mathbf{v}|(-\cos 15°)\mathbf{i} + |\mathbf{v}|(\sin 15°)\mathbf{j}$$
$$\mathbf{w} = -2{,}500\mathbf{j}$$

Step 3. For the system to be in static equilibrium, the sum of the force vectors must be the zero vector. That is,

$$\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$$

Replacing vectors **u**, **v**, and **w** from step 2, we obtain

$$[|\mathbf{u}|(\cos 7°)\mathbf{i} + |\mathbf{u}|(\sin 7°)\mathbf{j}] + [|\mathbf{v}|(-\cos 15°)\mathbf{i} + |\mathbf{v}|(\sin 15°)\mathbf{j}] - 2{,}500\mathbf{j} = 0\mathbf{i} + 0\mathbf{j}$$

which, upon combining **i** and **j** vectors, becomes

$$[|\mathbf{u}|(\cos 7°) + |\mathbf{v}|(-\cos 15°)]\mathbf{i} + [|\mathbf{u}|(\sin 7°) + |\mathbf{v}|(\sin 15°) - 2{,}500]\mathbf{j} = 0\mathbf{i} + 0\mathbf{j}$$

Because two vectors are equal if and only if their corresponding components are equal, we are led to the following system of two equations in the two variables $|\mathbf{u}|$ and $|\mathbf{v}|$:

$$\begin{aligned}(\cos 7°)|\mathbf{u}| + (-\cos 15°)|\mathbf{v}| &= 0\\ (\sin 7°)|\mathbf{u}| + (\sin 15°)|\mathbf{v}| - 2{,}500 &= 0\end{aligned}$$

Solving this system by standard methods, we find that

$$|\mathbf{u}| = 6{,}400 \text{ pounds} \quad \text{and} \quad |\mathbf{v}| = 6{,}600 \text{ pounds}$$

Did you expect that the tension in each part of the cable is more than the weight hanging from the cable?

MATCHED PROBLEM 10

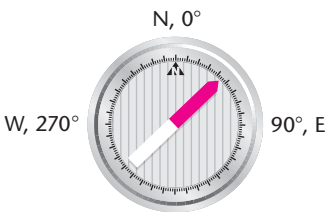
Repeat Example 10 with 15° replaced with 13°, 7° replaced with 9°, and the 2,500 pounds replaced with 1,900 pounds.

ANSWERS TO MATCHED PROBLEMS

1. $\overrightarrow{OP} = \langle -4, 8 \rangle$ 2. $2\sqrt{5}$
3. (A) $\langle -1, -3 \rangle$ (B) $\langle 15, -9 \rangle$ (C) $\langle -23, 21 \rangle$ (D) $\langle -20, 12 \rangle$
4. $\mathbf{u} = \langle 3/\sqrt{10}, 1/\sqrt{10} \rangle$
5. (A) $5\mathbf{i} - 3\mathbf{j}$ (B) $-9\mathbf{i}$ (C) $6\mathbf{j}$
6. (A) $6\mathbf{i} + 4\mathbf{j}$ (B) $-2\mathbf{i} - 6\mathbf{j}$ (C) $-2\mathbf{i} - 13\mathbf{j}$
7. Resultant velocity: magnitude = 3.6 miles per hour, direction = 126°
8. $|\mathbf{R}| = 71$ pounds, $\theta = 25°$ 9. $|\overrightarrow{DC}| = 3{,}010$ pounds
10. $|\mathbf{u}| = 4{,}900$ pounds, $|\mathbf{v}| = 5{,}000$ pounds

7-3 Exercises

Express all angle measures in decimal degrees. In navigation problems, refer to the figure of a navigational compass.



Navigational compass

1. What is a vector?

2. Explain the parallelogram rule.

3. What is the difference between a vector and a scalar?

4. Explain how each of the scalar products $3\mathbf{v}$ and $-3\mathbf{v}$ is related to the vector $\mathbf{v}$.

5. What is a unit vector?

6. What is the difference between apparent velocity and actual velocity?

In Problems 7–14, find the standard vector $\overrightarrow{OP}$ for each vector $\overrightarrow{AB}$.

7. $A = (4, 6); B = (10, 11)$ **8.** $A = (2, 7); B = (3, 15)$

9. $A = (3, -9); B = (-4, 5)$

10. $A = (-5, 2); B = (8, -1)$ **11.** $A = (0, 0); B = (-6, 7)$

12. $A = (9, -7); B = (0, 0)$ **13.** $A = (5, 8); B = (0, 0)$

14. $A = (0, 0); B = (7, 1)$

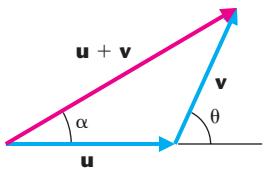
In Problems 15–22, find the magnitude of each vector.

15. $\langle -10, 0\rangle$ **16.** $\langle 0, 8\rangle$ **17.** $\langle 5, -12\rangle$

18. $\langle -4, 3\rangle$ **19.** $\langle -24, 7\rangle$ **20.** $\langle 10, -10\rangle$

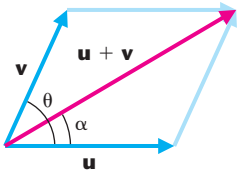
21. $\mathbf{i} + \mathbf{j}$ **22.** $2\mathbf{i} - \mathbf{j}$

Problems 23–30 refer to figures (a) and (b) showing vector addition for vectors **u** *and* **v.**



Tail-to-tip rule

(a)



Parallelogram rule

(b)

In Problems 23–30, find $|\mathbf{u} + \mathbf{v}|$ *and* α *given* $|\mathbf{u}|$, $|\mathbf{v}|$, *and* θ *in figures (a) and (b).*

23. $|\mathbf{u}| = 66$ grams, $|\mathbf{v}| = 22$ grams, $\theta = 68°$

24. $|\mathbf{u}| = 120$ grams, $|\mathbf{v}| = 84$ grams, $\theta = 44°$

25. $|\mathbf{u}| = 21$ knots, $|\mathbf{v}| = 3.2$ knots, $\theta = 53°$

26. $|\mathbf{u}| = 8.0$ knots, $|\mathbf{v}| = 2.0$ knots, $\theta = 64°$

In Problems 27–30, find $|\mathbf{u}|$ *and* $|\mathbf{v}|$, *given* $|\mathbf{u} + \mathbf{v}|$, α *and* θ *in figures (a) and (b).*

27. $|\mathbf{u} + \mathbf{v}| = 14$ kilograms, $\alpha = 25°$, $\theta = 79°$

28. $|\mathbf{u} + \mathbf{v}| = 33$ kilograms, $\alpha = 17°$, $\theta = 43°$

29. $|\mathbf{u} + \mathbf{v}| = 223$ miles per hour, $\alpha = 42.3°$, $\theta = 69.4°$

30. $|\mathbf{u} + \mathbf{v}| = 437$ miles per hour, $\alpha = 17.8°$, $\theta = 50.5°$

In Problems 31–34, find:
(A) $\mathbf{u} + \mathbf{v}$ *(B)* $\mathbf{u} - \mathbf{v}$ *(C)* $2\mathbf{u} - \mathbf{v} + 3\mathbf{w}$

31. $\mathbf{u} = \langle 2, 1\rangle$, $\mathbf{v} = \langle -1, 3\rangle$, $\mathbf{w} = \langle 3, 0\rangle$

32. $\mathbf{u} = \langle -1, 2\rangle$, $\mathbf{v} = \langle 3, -2\rangle$, $\mathbf{w} = \langle 0, -2\rangle$

33. $\mathbf{u} = \langle -4, -1\rangle$, $\mathbf{v} = \langle 2, 2\rangle$ $\mathbf{w} = \langle 0, 1\rangle$

34. $\mathbf{u} = \langle -3, 2\rangle$, $\mathbf{v} = \langle -2, 2\rangle$, $\mathbf{w} = \langle -3, 0\rangle$

In Problems 35–40, express **v** *in terms of the* **i** *and* **j** *unit vectors.*

35. $\mathbf{v} = \langle -3, 4\rangle$ **36.** $\mathbf{v} = \langle 2, -5\rangle$

37. $\mathbf{v} = \langle 3, 0\rangle$ **38.** $\mathbf{v} = \langle 0, -27\rangle$

39. $\mathbf{v} = \overrightarrow{AB}$, where $A = (2, 3)$ and $B = (-3, 1)$

40. $\mathbf{v} = \overrightarrow{AB}$, where $A = (-2, -1)$ and $B = (0, 2)$

In Problems 41–46, let $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j}$, $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j}$, *and* $\mathbf{w} = 2\mathbf{i}$, *and perform the indicated operations.*

41. $\mathbf{u} + \mathbf{v}$ **42.** $\mathbf{u} - \mathbf{v}$

43. $2\mathbf{u} - 3\mathbf{v}$ **44.** $3\mathbf{u} + 2\mathbf{v}$

45. $2\mathbf{u} - \mathbf{v} - 2\mathbf{w}$ **46.** $\mathbf{u} - 3\mathbf{v} + 2\mathbf{w}$

In Problems 47–54, find a unit vector with the same direction as **v**.

47. $\mathbf{v} = \langle 4, 3\rangle$ **48.** $\mathbf{v} = \langle 5, 12\rangle$

49. $\mathbf{v} = \langle -1, 1\rangle$ **50.** $\mathbf{v} = \langle 2, -3\rangle$

51. $\mathbf{v} = \langle -8, 0\rangle$ **52.** $\mathbf{v} = \langle 0, -17\rangle$

53. $\mathbf{v} = 5\mathbf{i} + \sqrt{11}\mathbf{j}$ **54.** $\mathbf{v} = -\sqrt{2}\mathbf{i} - \sqrt{7}\mathbf{j}$

In Problems 55–62, determine whether the statement is true or false. If true, explain why. If false, give a counterexample.

55. If two vectors have the same magnitude, then they are equal.

56. Every vector has the same magnitude as its standard vector.

57. If a vector has the same initial and terminal points, then it is the zero vector.

58. The only unit vectors in the plane are **i** and **j.**

59. Every vector **v** has the same magnitude as $\mathbf{v} + \mathbf{v}$.

60. Every vector **v** has the same direction as $\mathbf{v} + \mathbf{v}$.

61. The magnitude of every vector is a positive real number.

62. If **u** and **v** are unit vectors, then $\mathbf{u} + \mathbf{v}$ is a unit vector.

63. If **v** is a vector and k is a scalar, then $k\mathbf{v}$ has the same direction as **v.**

64. If **v** is a vector, k is a scalar, and $k\mathbf{v}$ is a unit vector, then $|k| = 1$.

In Problems 65–72, let $\mathbf{u} = \langle a, b\rangle$, $\mathbf{v} = \langle c, d\rangle$, *and* $\mathbf{w} = \langle e, f\rangle$ *be vectors and* m *and* n *be scalars. Prove each of the following vector properties using appropriate properties of real numbers and the definitions of vector addition and scalar multiplication.*

65. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

66. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ **67.** $\mathbf{u} + 0 = \mathbf{u}$

68. $\mathbf{u} + (-\mathbf{u}) = 0$ **69.** $(m + n)\mathbf{u} = m\mathbf{u} + n\mathbf{u}$

70. $m(\mathbf{u} + \mathbf{v}) = m\mathbf{u} + m\mathbf{v}$ **71.** $m(n\mathbf{u}) = (mn)\mathbf{u}$

72. $1\mathbf{u} = \mathbf{u}$

APPLICATIONS

In Problems 73–76, assume the north, east, south, and west directions are exact.

73. NAVIGATION An airplane is flying with a compass heading of 285° and an airspeed of 230 miles per hour. A steady wind of 35 miles per hour is blowing in the direction of 260°. What is the plane's actual velocity; that is, what is its speed and direction relative to the ground?

74. NAVIGATION A power boat crossing a wide river has a compass heading of 25° and speed relative to the water of 15 miles per hour. The river is flowing in the direction of 135° at 3.9 miles per hour. What is the boat's actual velocity; that is, what is its speed and direction relative to the ground?

75. NAVIGATION Two docks are directly opposite each other on a southward-flowing river. A boat pilot wishes to go in a straight line from the east dock to the west dock in a ferryboat with a cruising speed in still water of 8.0 knots. If the river's current is 2.5 knots, what compass heading should be maintained while crossing the river? What is the actual speed of the boat relative to the land?

76. NAVIGATION An airplane can cruise at 255 miles per hour in still air. If a steady wind of 46.0 miles per hour is blowing from the west, what compass heading should the pilot fly for the course of the plane relative to the ground to be north (0°)? Compute the ground speed for this course.

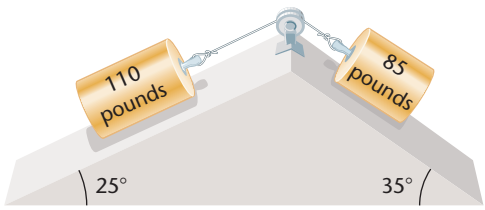
77. RESULTANT FORCE A large ship has gone aground in a harbor and two tugs, with cables attached, attempt to pull it free. If one tug pulls with a compass course of 52° and a force of 2,300 pounds and a second tug pulls with a compass course of 97° and a force of 1,900 pounds, what is the compass direction and the magnitude of the resultant force?

78. RESULTANT FORCE Repeat Problem 77 if one tug pulls with a compass direction of 161° and a force of 2,900 kilograms and a second tug pulls with a compass direction of 192° and a force of 3,600 kilograms.

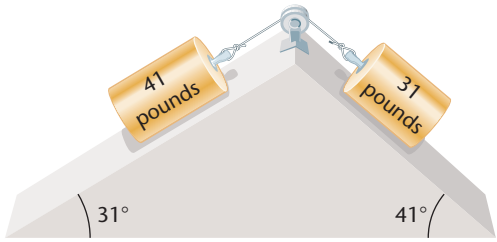
79. RESOLUTION OF FORCES An automobile weighing 4,050 pounds is standing on a driveway inclined 5.5° with the horizontal.
(A) Find the magnitude of the force parallel to the driveway necessary to keep the car from rolling down the hill.
(B) Find the magnitude of the force perpendicular to the driveway.

80. RESOLUTION OF FORCES Repeat Problem 79 for a car weighing 2,500 pounds parked on a hill inclined at 15° to the horizontal.

81. RESOLUTION OF FORCES If two weights are fastened together and placed on inclined planes as shown in the figure, neglecting friction, which way will they slide?

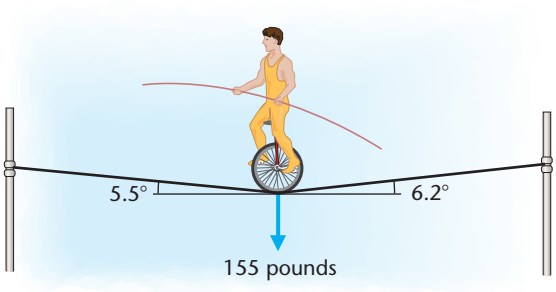


82. RESOLUTION OF FORCES If two weights are fastened together and placed on inclined planes as indicated in the figure, neglecting friction, which way will they slide?



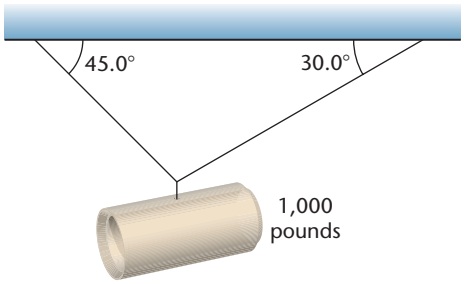
In Problems 83–90, compute all answers to three significant digits.

83. STATIC EQUILIBRIUM A unicyclist at a certain point on a tightrope deflects the rope as indicated in the figure. If the total weight of the cyclist and the unicycle is 155 pounds, how much tension is in each part of the cable?

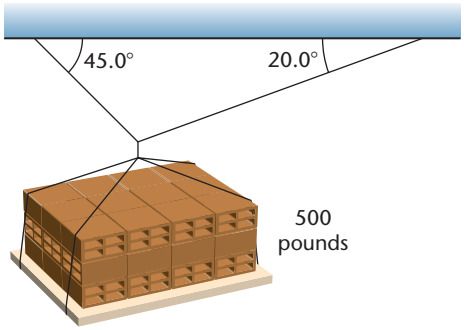


84. STATIC EQUILIBRIUM Repeat Problem 83 with the left angle 4.2°, the right angle 5.3°, and the total weight 112 pounds.

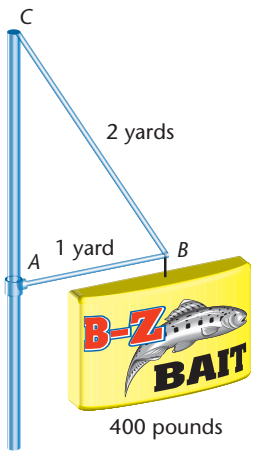
85. STATIC EQUILIBRIUM A weight of 1,000 pounds is suspended from two cables as shown in the figure. What is the tension in each cable?



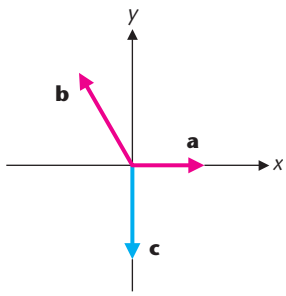
86. **STATIC EQUILIBRIUM** A weight of 500 pounds is supported by two cables as illustrated. What is the tension in each cable?



87. **STATIC EQUILIBRIUM** A 400-pound sign is suspended as shown in figure (a). The corresponding force diagram (b) is formed by observing the following: Member AB is "pushing" at B and is under compression. This "pushing" force also can be thought of as the force vector **a** "pulling" to the right at B. The force vector **b** reflects the fact that member CB is under tension—that is, it is "pulling" at B. The force vector **c** corresponds to the weight of the sign "pulling" down at B. Find the magnitudes of the forces in the rigid supporting members; that is, find $|\mathbf{a}|$ and $|\mathbf{b}|$ in the force diagram (b).

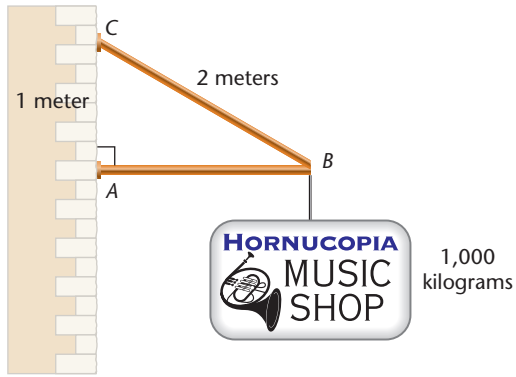


(a)

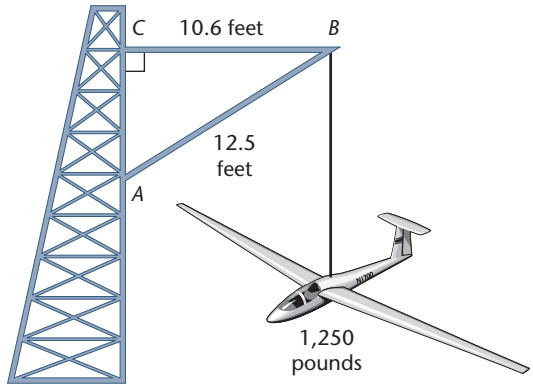


(b)

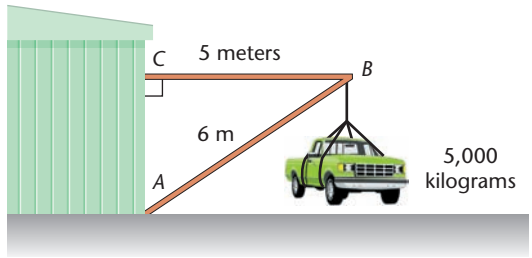
88. **STATIC EQUILIBRIUM** A weight of 1,000 kilograms is supported as shown in the figure. What are the magnitudes of the forces on the members AB and BC? (Refer to Problem 87.)



89. **STATIC EQUILIBRIUM** A 1,250-pound weight is hanging from a hoist as indicated in the figure. What are the magnitudes of the forces on the members AB and BC? (Refer to Problem 87.)



90. **STATIC EQUILIBRIUM** A weight of 5,000 kilograms is supported as shown in the figure. What are the magnitudes of the forces on the members AB and BC? (Refer to Problem 87.)



7-4 Polar Coordinates and Graphs

- Polar Coordinate System
- Converting from Polar to Rectangular Form, and Vice Versa
- Graphing Polar Equations
- Some Standard Polar Curves
- Application

Until now we have used only the rectangular coordinate system. Other coordinate systems have particular advantages in certain situations. Of the many that are possible, the *polar coordinate system* ranks second in importance to the rectangular coordinate system and is the subject matter of this section.

Polar Coordinate System

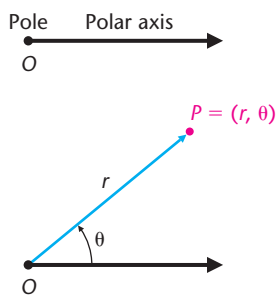


Figure 1 Polar coordinate system.

To form a **polar coordinate system** in a plane (Fig. 1), start with a fixed point O and call it the **pole,** or **origin.** From this point draw a half-line, or ray (usually horizontal and to the right), and call this line the **polar axis.**

If P is an arbitrary point in a plane, then associate polar coordinates (r, θ) (Fig. 1) with it as follows: Starting with the polar axis as the initial side of an angle, rotate the terminal side until it, or the extension of it through the pole, passes through the point. The θ coordinate in (r, θ) is this angle, in degree or radian measure. The angle θ is positive if the rotation is counterclockwise and negative if the rotation is clockwise. The r coordinate in (r, θ) is the directed distance from the pole to the point P. It is positive if measured from the pole along the terminal side of θ and negative if measured along the terminal side extended through the pole.

Figure 2 illustrates a point P with three different sets of polar coordinates. Study this figure carefully. The pole has polar coordinates $(0, \theta)$ for arbitrary θ. For example, $(0, 0°)$, $(0, \pi/3)$, and $(0, -371°)$ are all coordinates of the pole.

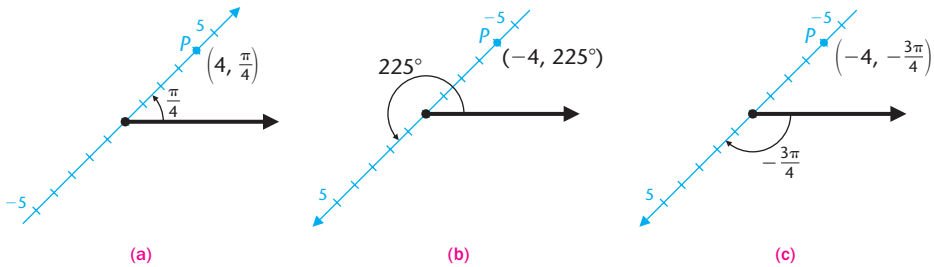


Figure 2 Polar coordinates of a point.

We now see a distinct difference between rectangular and polar coordinates for the given point. For a given point in a rectangular coordinate system, there exists exactly one set of rectangular coordinates. On the other hand, in a polar coordinate system, a point has infinitely many sets of polar coordinates.

Just as graph paper with a rectangular grid is readily available for plotting rectangular coordinates, polar graph paper is available for plotting polar coordinates.

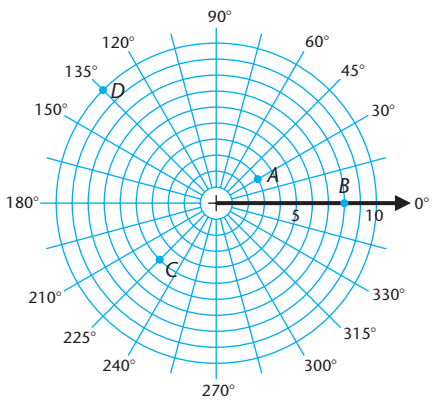
EXAMPLE 1 Plotting Points in a Polar Coordinate System

Plot the following points in a polar coordinate system:

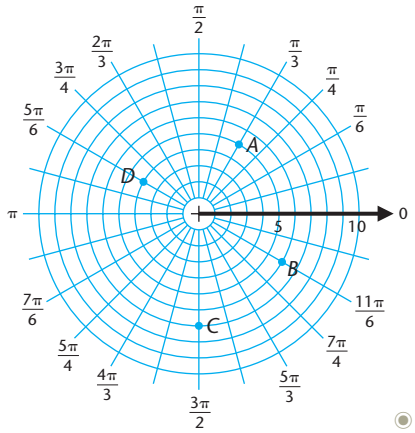
(A) $A = (3, 30°)$, $B = (-8, 180°)$, $C = (5, -135°)$, $D = (-10, -45°)$
(B) $A = (5, \pi/3)$, $B = (-6, 5\pi/6)$, $C = (7, -\pi/2)$, $D = (-4, -\pi/6)$

SOLUTIONS

(A)



(B)



MATCHED PROBLEM 1

Plot the following points in a polar coordinate system:

(A) $A = (8, 45°)$, $B = (-5, 150°)$, $C = (4, -210°)$, $D = (-6, -90°)$
(B) $A = (9, \pi/6)$, $B = (-3, -\pi)$, $C = (-7, 7\pi/4)$, $D = (5, -5\pi/6)$

››› EXPLORE-DISCUSS 1

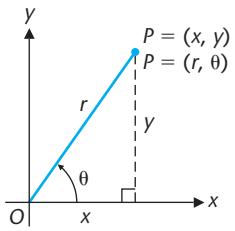
A point in a polar coordinate system has coordinates (5, 30°). How many other polar coordinates does the point have for θ restricted to $-360° \le \theta \le 360°$? Find the other coordinates of the point and explain how they are found.

Converting from Polar to Rectangular Form, and Vice Versa

Often, it is necessary to transform coordinates or equations in rectangular form into polar form, or vice versa. The following polar–rectangular relationships are useful in this regard:

POLAR–RECTANGULAR RELATIONSHIPS

We have the following relationships between rectangular coordinates (x, y) and polar coordinates (r, θ):



$$r^2 = x^2 + y^2$$

$$\sin\theta = \frac{y}{r} \quad \text{or} \quad y = r\sin\theta$$

$$\cos\theta = \frac{x}{r} \quad \text{or} \quad x = r\cos\theta$$

$$\tan\theta = \frac{y}{x}$$

[*Note:* The signs of x and y determine the quadrant for θ. The angle θ is chosen so that $-\pi < \theta \le \pi$ or $-180° < \theta \le 180°$, unless directed otherwise.]

Many calculators can automatically convert rectangular coordinates to polar form, and vice versa. (Read the manual for your particular calculator.) Example 2 illustrates calculator conversions in both directions.

EXAMPLE 2 Converting from Polar to Rectangular Form, and Vice Versa

(A) Convert the polar coordinates $(-4, 1.077)$ to rectangular coordinates to three decimal places.

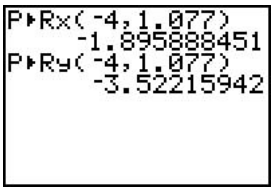
(B) Convert the rectangular coordinates $(-3.207, -5.719)$ to polar coordinates with θ in degree measure, $-180° < \theta \le 180°$ and $r \ge 0$.

SOLUTIONS

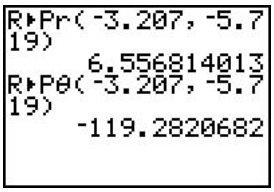
(A) Use a calculator set in radian mode.

$$(r, \theta) = (-4, 1.077)$$
$$x = r\cos\theta = (-4)\cos 1.077 = -1.896$$
$$y = r\sin\theta = (-4)\sin 1.077 = -3.522$$

Rectangular coordinates are $(-1.896, -3.522)$.



› Figure 3



› Figure 4

Figure 3 shows the same conversion done in a graphing calculator with a built-in conversion routine.

(B) Use a calculator set in degree mode.

$$(x, y) = (-3.207, -5.719)$$
$$r = \sqrt{x^2 + y^2} = \sqrt{(-3.207)^2 + (-5.719)^2} = 6.557$$
$$\tan \theta = \frac{y}{x} = \frac{-5.719}{-3.207}$$

θ is a third-quadrant angle and is to be chosen so that $-180° < \theta \leq 180°$.

$$\theta = -180° + \tan^{-1} \frac{-5.719}{-3.207} = -119.28°$$

Polar coordinates are $(6.557, -119.28°)$.

Figure 4 shows the same conversion done in a graphing calculator with a built-in conversion routine.

MATCHED PROBLEM 2

(A) Convert the polar coordinates $(8.677, -1.385)$ to rectangular coordinates to three decimal places.

(B) Convert the rectangular coordinates $(-6.434, 4.023)$ to polar coordinates with θ in degree measure, $-180° < \theta \leq 180°$ and $r \geq 0$.

Generally, a more important use of the polar–rectangular relationships is in the conversion of equations in rectangular form to polar form, and vice versa.

EXAMPLE 3 Converting an Equation from Rectangular Form to Polar Form

Change $x^2 + y^2 - 4y = 0$ to polar form.

SOLUTION

Use $r^2 = x^2 + y^2$ and $y = r \sin \theta$.

$$x^2 + y^2 - 4y = 0$$
$$r^2 - 4r \sin \theta = 0$$
$$r(r - 4 \sin \theta) = 0$$
$$r = 0 \quad \text{or} \quad r - 4 \sin \theta = 0$$

The graph of $r = 0$ is the pole. Because the pole is included in the graph of $r - 4 \sin \theta = 0$ (let $\theta = 0$), we can discard $r = 0$ and keep only

$$r - 4 \sin \theta = 0$$

or

$$r = 4 \sin \theta \qquad \text{The polar form of } x^2 + y^2 - 4y = 0$$

MATCHED PROBLEM 3

Change $x^2 + y^2 - 6x = 0$ to polar form.

EXAMPLE 4 Converting an Equation from Polar Form to Rectangular Form

Change $r = -3 \cos \theta$ to rectangular form.

SOLUTION

The transformation of this equation as it stands into rectangular form is fairly difficult. With a little trick, however, it becomes easy. We multiply both sides by r, which simply adds the pole to the graph. But the pole is already part of the graph of $r = -3 \cos \theta$ (let $\theta = \pi/2$), so we haven't actually changed anything.

$$\begin{aligned} r &= -3 \cos \theta && \text{Multiply both sides by } r. \\ r^2 &= -3r \cos \theta && r^2 = x^2 + y^2 \text{ and } r \cos \theta = x \\ x^2 + y^2 &= -3x \\ x^2 + y^2 + 3x &= 0 \end{aligned}$$

MATCHED PROBLEM 4

Change $r + 2 \sin \theta = 0$ to rectangular form.

Graphing Polar Equations

We now turn to graphing polar equations. The **graph** of a polar equation, such as $r = 3\theta$ or $r = 6 \cos \theta$, in a polar coordinate system is the set of all points having coordinates that satisfy the polar equation. Certain curves have simpler representations in polar coordinates, and other curves have simpler representations in rectangular coordinates.

To establish fundamentals in graphing polar equations, we start with a point-by-point graph. We then consider a more rapid way of making rough sketches of certain polar curves. And, finally, we show how polar curves are graphed in a graphing calculator.

To plot a polar equation using **point-by-point plotting,** just as in rectangular coordinates, make a table of values that satisfy the equation, plot these points, then join them with a smooth curve. Example 5 illustrates the process.

EXAMPLE 5 Point-by-Point Plotting

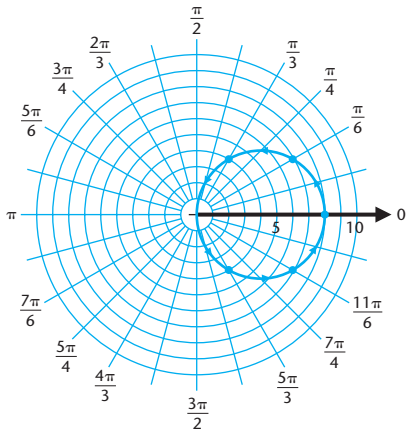
(A) Graph $r = 8 \cos \theta$ with θ in radians.

(B) Convert the polar equation in part A to rectangular form, and identify the graph.

SOLUTIONS

(A) We construct a table using multiples of $\pi/6$, plot these points, then join the points with a smooth curve (Fig. 5).

θ	r
0	8.0
$\pi/6$	6.9
$\pi/3$	4.0
$\pi/2$	0.0
$2\pi/3$	−4.0
$5\pi/6$	−6.9
π	−8.0
Graph repeats	



› Figure 5

(B)

$$r = 8 \cos \theta \qquad \text{Multiply both sides by } r.$$
$$r^2 = 8r \cos \theta \qquad \text{Change to rectangular form.}$$
$$x^2 + y^2 = 8x$$
$$x^2 - 8x + y^2 = 0 \qquad \text{Complete the square on the left side.}$$
$$x^2 - 8x + 16 + y^2 = 16 \qquad \text{Write in standard form.}$$
$$(x - 4)^2 + y^2 = 4^2 \qquad \text{Standard equation of a circle}$$

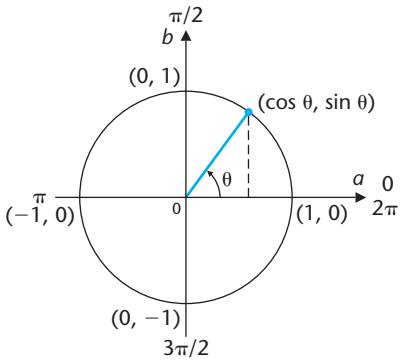
The graph in part A is a circle with center at (4, 0) and radius 4 (see Section 2-2).

MATCHED PROBLEM 5

(A) Graph $r = 8 \sin \theta$ with θ in degrees.

(B) Convert the polar equation in part A to rectangular form, and identify the graph.

If only a rough sketch of a polar equation involving $\sin\theta$ or $\cos\theta$ is desired, you can speed up the point-by-point graphing process by taking advantage of the uniform variation of $\sin\theta$ and $\cos\theta$ as θ moves around a unit circle. This process is referred to as **rapid polar sketching.** It is convenient to visualize Figure 6 in the process. With a little practice most of the table work in rapid sketching can be done mentally and a rough sketch can be made directly from the equation.



› Figure 6

EXAMPLE 6 Rapid Polar Sketching

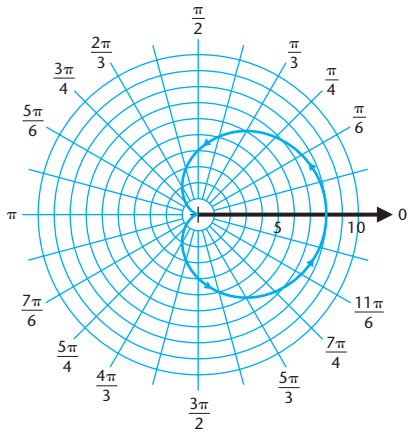
Sketch $r = 4 + 4\cos\theta$ using rapid sketching techniques with θ in radians.

SOLUTION

We set up a table that indicates how r varies as we let θ vary through each set of quadrant values:

θ Varies from	$\cos\theta$ Varies from	$4\cos\theta$ Varies from	$r = 4 + 4\cos\theta$ Varies from
0 to $\pi/2$	1 to 0	4 to 0	8 to 4
$\pi/2$ to π	0 to -1	0 to -4	4 to 0
π to $3\pi/2$	-1 to 0	-4 to 0	0 to 4
$3\pi/2$ to 2π	0 to 1	0 to 4	4 to 8

Notice that as θ increases from 0 to $\pi/2$, $\cos\theta$ decreases from 1 to 0, $4\cos\theta$ decreases from 4 to 0, and $r = 4 + 4\cos\theta$ decreases from 8 to 4, and so on. Sketching these values, we obtain the graph in Figure 7, called a **cardioid.**



$r = 4 + 4 \cos \theta$
Cardioid

› Figure 7

MATCHED PROBLEM 6

Sketch $r = 5 + 5 \sin \theta$ using rapid sketching techniques with θ in radians.

EXAMPLE 7 Rapid Polar Sketching

Sketch $r = 8 \cos 2\theta$ with θ in radians.

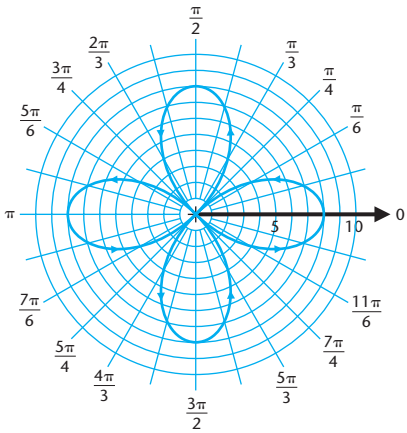
SOLUTION

Start by letting 2θ (instead of θ) range through each set of quadrant values. That is, start with values for 2θ in the second column of the table, fill in the table to the right, and then fill in the first column for θ.

Start with the second column.

θ Varies from	2θ Varies from	$\cos 2\theta$ Varies from	$r = 8 \cos 2\theta$ Varies from
0 to $\pi/4$	0 to $\pi/2$	1 to 0	8 to 0
$\pi/4$ to $\pi/2$	$\pi/2$ to π	0 to -1	0 to -8
$\pi/2$ to $3\pi/4$	π to $3\pi/2$	-1 to 0	-8 to 0
$3\pi/4$ to π	$3\pi/2$ to 2π	0 to 1	0 to 8
π to $5\pi/4$	2π to $5\pi/2$	1 to 0	8 to 0
$5\pi/4$ to $3\pi/2$	$5\pi/2$ to 3π	0 to -1	0 to -8
$3\pi/2$ to $7\pi/4$	3π to $7\pi/2$	-1 to 0	-8 to 0
$7\pi/4$ to 2π	$7\pi/2$ to 4π	0 to 1	0 to 8

As 2θ increases from 0 to $\pi/2$, θ increases from 0 to $\pi/4$, and r decreases from 8 to 0. As 2θ increases from $\pi/2$ to π, θ increases from $\pi/4$ to $\pi/2$, and r decreases from 0 to -8, and so on. Continue until the graph starts to repeat. Plotting the values, we obtain the graph in Figure 8, called a **four-leafed rose:**



$r = 8 \cos 2\theta$
Four-leafed rose

› Figure 8

MATCHED PROBLEM 7

Sketch $r = 6 \sin 2\theta$ with θ in radians.

We now turn to **graphing polar equations in a graphing calculator.** Example 8 illustrates the process.

EXAMPLE 8 Graphing in a Graphing Calculator

Graph each of the following polar equations in a graphing calculator (parts B and C are from Examples 6 and 7).

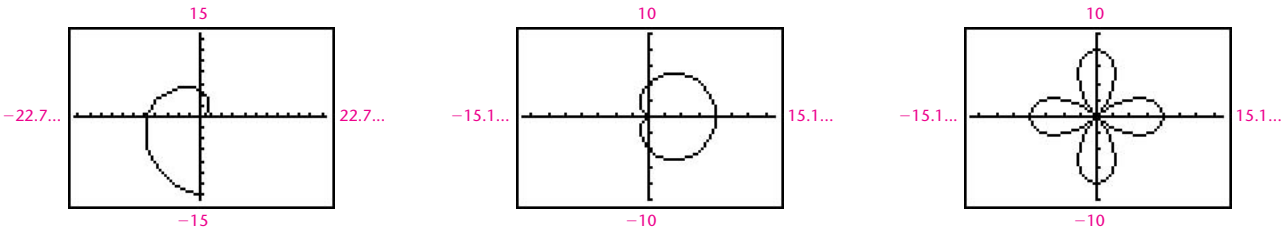
(A) $r = 3\theta, 0 \le \theta \le 3\pi/2$ (Archimedes' spiral)
(B) $r = 4 + 4 \cos \theta$ (cardioid)
(C) $r = 8 \cos 2\theta$ (four-leafed rose)

SOLUTIONS

Set the graphing calculator in polar mode and select polar coordinates and radian measure. Adjust window values to accommodate the whole graph. A squared graph is often desirable in showing the true shape of the curve, and is used here. Many

graphing calculators, including the one used here, do not show a polar grid. When using TRACE, many graphing calculators offer a choice between polar coordinates and rectangular coordinates for points on the polar curve. The graphs of the preceding equations are shown in Figure 9.

(A) $r = 3\theta, 0 \le \theta \le 3\pi/2$ (B) $r = 4 + 4\cos\theta$ (C) $r = 8\cos 2\theta$



› Figure 9

MATCHED PROBLEM 8

Graph each of the following polar equations in a graphing calculator.

(A) $r = 2\theta, 0 \le \theta \le 2\pi$
(B) $r = 5 + 5\sin\theta$
(C) $r = 6\sin 2\theta$

››› EXPLORE-DISCUSS 2

(A) Graph $r_1 = 10\sin\theta$ and $r_2 = 10\cos\theta$ in the same viewing window. Use TRACE on r_1 and estimate the polar coordinates where the two graphs intersect. Do the same thing for r_2. Which intersection point appears to have the same polar coordinates on each curve and consequently represents a simultaneous solution to both equations? Which intersection point appears to have different polar coordinates on each curve and consequently does not represent a simultaneous solution? Solve the system for r and θ.

(B) Explain how rectangular coordinate systems differ from polar coordinate systems relative to intersection points and simultaneous solutions of systems of equations in the respective systems.

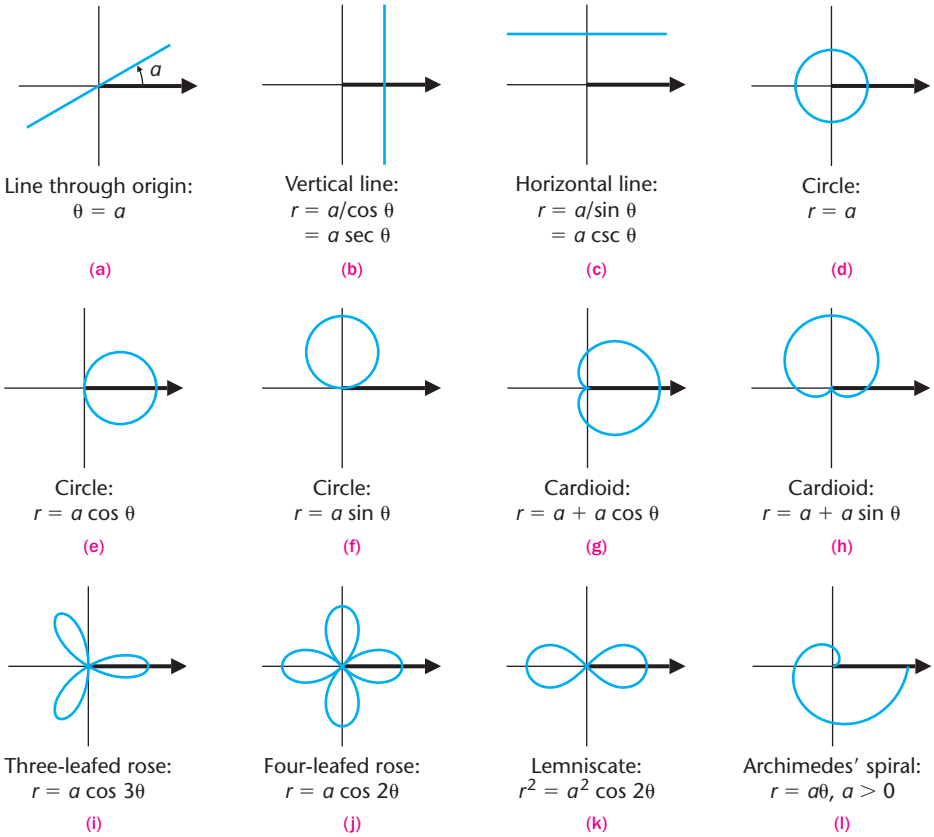
Some Standard Polar Curves

In a rectangular coordinate system, the simplest types of equations to graph are found by setting the rectangular variables x and y equal to constants:

$$x = a \qquad \text{and} \qquad y = b$$

The graphs are straight lines: The graph of $x = a$ is a vertical line, and the graph of $y = b$ is a horizontal line. A glance at Table 1 shows that horizontal and vertical lines do not have simple equations in polar coordinates.

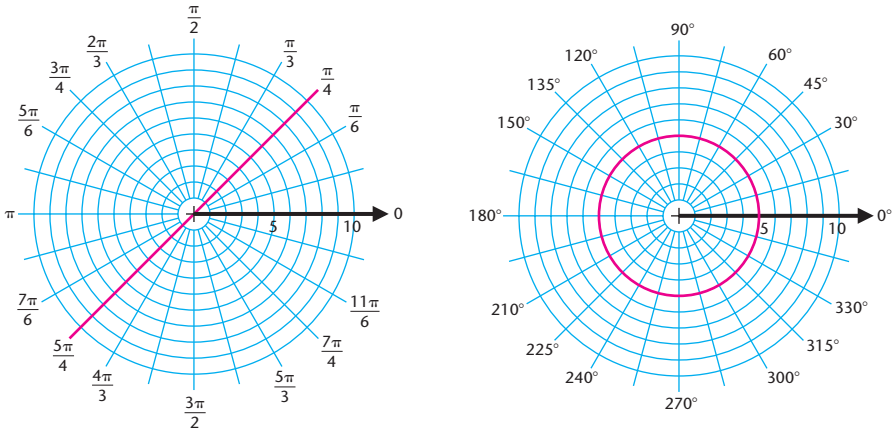
Table 1 Standard Polar Graphs



Two of the simplest types of polar equations to graph in a polar coordinate system are found by setting the polar variables r and θ equal to constants:

$$r = a \qquad \text{and} \qquad \theta = b$$

Figure 10 illustrates the graphs of $\theta = \pi/4$ and $r = 5$.



$\theta = \frac{\pi}{4}$ (short for $0r + \theta = \frac{\pi}{4}$)
Line through origin
(a)

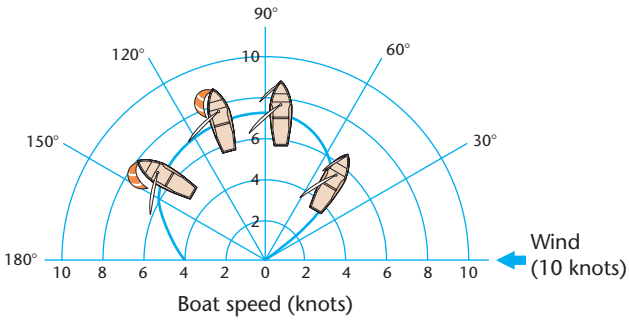
$r = 5$ (short for $0\theta + r = 5$)
Circle
(b)

› **Figure 10**

Table 1 illustrates a number of standard polar graphs and their equations. Polar graphing is often made easier if you have some idea of the final form.

› Application

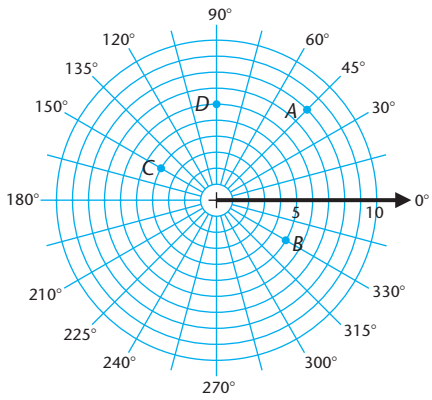
Serious sailboat racers make polar plots of boat speeds at various angles to the wind with various sail combinations at different wind speeds. With many polar plots for different sizes and types of sails at different wind speeds, they are able to accurately choose a sail for the optimum performance for different points of sail relative to any given wind strength. Figure 11 illustrates one such polar plot, where the maximum speed appears to be about 7.5 knots at 105° off the wind (with spinnaker sail set).



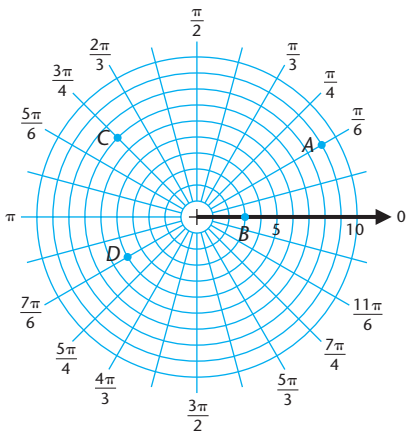
› **Figure 11** Polar diagram showing optimum sailing speed at different sailing angles to the wind.

ANSWERS TO MATCHED PROBLEMS

1. (A)



(B)

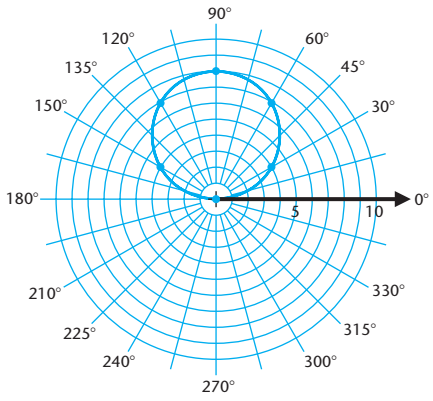


2. (A) (1.603, −8.528) (B) (7.588, 147.98°) **3.** $r = 6 \cos \theta$

4. $x^2 + y^2 + 2y = 0$

5. (A)

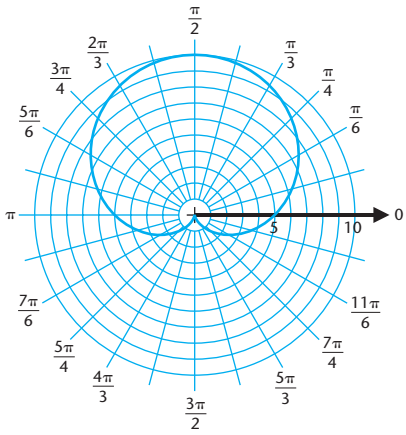
θ	r
0°	0.0
30°	4.0
60°	6.9
90°	8.0
120°	6.9
150°	4.0
180°	0.0
Graph Repeats	



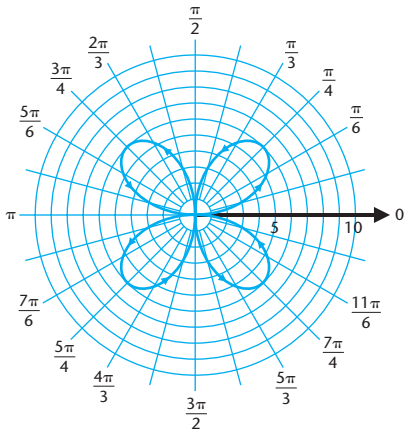
Circle:
$r = 8 \sin \theta$

(B) $x^2 + (y - 4)^2 = 4^2$, a circle with center at (0, 4) and radius 4

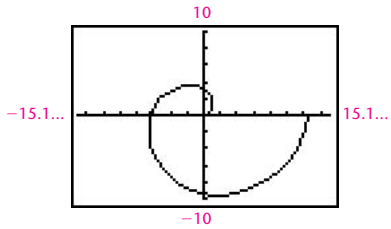
6. $r = 5 + 5 \sin \theta$, cardioid



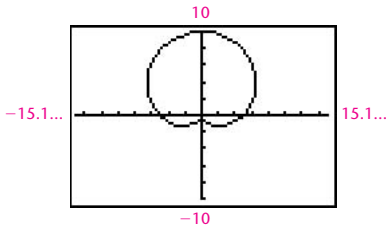
7. $r = 6 \sin 2\theta$, four-leafed rose



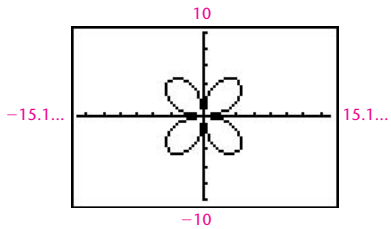
8. (A) $r = 2\theta$, $0 \le \theta \le 2\pi$



(B) $r = 5 + 5 \sin \theta$



(C) $r = 6 \sin 2\theta$



7-4 Exercises

1. In the context of polar coordinates, what is a pole?

2. What is a polar axis?

3. Explain why the point with rectangular coordinates (1, 0) has more than one set of polar coordinates.

4. If you are given the rectangular coordinates of a point, explain how you can find a set of polar coordinates of the same point.

5. If you are given a set of polar coordinates of a point, explain how you can find the rectangular coordinates of the same point.

6. Explain the difference between point-by-point plotting and rapid polar sketching.

Plot A, B, and C in Problems 7–14 in a polar coordinate system.

7. $A = (4, 0°)$, $B = (7, 180°)$, $C = (9, 45°)$

8. $A = (8, 0°)$, $B = (5, 90°)$, $C = (6, 30°)$

9. $A = (-4, 0°)$, $B = (-7, 180°)$, $C = (-9, 45°)$

10. $A = (-8, 0°)$, $B = (-5, 90°)$, $C = (-6, 30°)$

11. $A = (8, -\pi/3)$, $B = (4, -\pi/4)$, $C = (10, -\pi/6)$

12. $A = (6, -\pi/6)$, $B = (5, -\pi/2)$, $C = (8, -\pi/4)$

13. $A = (-6, -\pi/6)$, $B = (-5, -\pi/2)$, $C = (-8, -\pi/4)$

14. $A = (-6, -\pi/2)$, $B = (-5, -\pi/3)$, $C = (-8, -\pi/4)$

15. A point in a polar coordinate system has coordinates $(-5, 3\pi/4)$. Find all other polar coordinates for the point, $-2\pi \le \theta \le 2\pi$, and verbally describe how the coordinates are associated with the point.

16. A point in a polar coordinate system has coordinates $(6, -30°)$. Find all other polar coordinates for the point, $-360° \le \theta \le 360°$, and verbally describe how the coordinates are associated with the point.

Graph Problems 17 and 18 in a polar coordinate system using point-by-point plotting and the special values 0, $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$, $2\pi/3$, $3\pi/4$, $5\pi/6$, and π for θ.

17. $r = 10 \sin \theta$ **18.** $r = 10 \cos \theta$

Graph Problems 19–22 in a polar coordinate system.

19. $r = 8$ **20.** $r = 5$ **21.** $\theta = \pi/3$ **22.** $\theta = \pi/6$

In Problems 23–28, convert the polar coordinates to rectangular coordinates to three decimal places.

23. $(6, \pi/6)$ **24.** $(7, 2\pi/3)$

25. $(-2, 7\pi/8)$ **26.** $(3, -3\pi/7)$

27. $(-4.233, -2.084)$ **28.** $(-9.028, -0.663)$

In Problems 29–34, convert the rectangular coordinates to polar coordinates with θ in degree measure, $-180° < \theta \le 180°$, and $r \ge 0$.

29. (3.5, 7.1) **30.** (6.9, 4.7)

31. (22, −14) **32.** (16, −27)

33. (−7.33, −2.04) **34.** (−8.33, 4.29)

In Problems 35–44, use rapid graphing techniques to sketch the graph of each polar equation.

35. $r = 4 \sin \theta$ **36.** $r = 4 \cos \theta$

37. $r = 10 \sin 2\theta$ **38.** $r = 8 \cos 2\theta$

39. $r = 5 \cos 3\theta$ **40.** $r = 6 \sin 3\theta$

41. $r = 2 + 2 \sin \theta$ **42.** $r = 3 + 3 \cos \theta$

43. $r = 2 + 4 \sin \theta$ **44.** $r = 2 + 4 \cos \theta$

Problems 45–50 are exploratory problems requiring the use of a graphing calculator.

45. Graph each polar equation in its own viewing window: $r = 2 + 2 \sin \theta$, $r = 4 + 2 \sin \theta$, $r = 2 + 4 \sin \theta$.

46. Graph each polar equation in its own viewing window: $r = 2 + 2 \cos \theta$, $r = 4 + 2 \cos \theta$, $r = 2 + 4 \cos \theta$.

47. (A) Graph each polar equation in its own viewing window: $r = 4 \sin \theta$, $r = 4 \sin 3\theta$, $r = 4 \sin 5\theta$.
(B) What would you guess to be the number of leaves for $r = 4 \sin 7\theta$?
(C) What would you guess to be the number of leaves for $r = a \sin n\theta$, $a > 0$ and n odd?

48. (A) Graph each polar equation in its own viewing window: $r = 4 \cos \theta$, $r = 4 \cos 3\theta$, $r = 4 \cos 5\theta$.
(B) What would you guess to be the number of leaves for $r = 4 \cos 7\theta$?
(C) What would you guess to be the number of leaves for $r = a \cos n\theta$, $a > 0$ and n odd?

49. (A) Graph each polar equation in its own viewing window: $r = 4 \sin 2\theta$, $r = 4 \sin 4\theta$, $r = 4 \sin 6\theta$.
(B) What would you guess to be the number of leaves for $r = 4 \sin 8\theta$?
(C) What would you guess to be the number of leaves for $r = a \sin n\theta$, $a > 0$ and n even?

50. (A) Graph each polar equation in its own viewing window: $r = 4 \cos 2\theta$, $r = 4 \cos 4\theta$, $r = 4 \cos 6\theta$.
(B) What would you guess to be the number of leaves for $r = 4 \cos 8\theta$?
(C) What would you guess to be the number of leaves for $r = a \cos n\theta$, $a > 0$ and n even?

In Problems 51–56, change each rectangular equation to polar form.

51. $y^2 = 5y - x^2$ **52.** $6x - x^2 = y^2$

53. $y = x$ **54.** $x^2 + y^2 = 9$

55. $y^2 = 4x$ **56.** $2xy = 1$

In Problems 57–62, change each polar equation to rectangular form.

57. $r(3 \cos \theta - 4 \sin \theta) = -1$

58. $r(2 \cos \theta + \sin \theta) = 4$

59. $r = -2 \sin \theta$ **60.** $r = 8 \cos \theta$

61. $\theta = \pi/4$ **62.** $r = 4$

Problems 63 and 64 are exploratory problems requiring the use of a graphing calculator.

63. Graph $r = 1 + 2 \sin (n\theta)$ for various values of n, n a natural number. Describe how n is related to the number of large petals and the number of small petals on the graph and how the large and small petals are related to each other relative to n.

64. Graph $r = 1 + 2\cos(n\theta)$ for various values of n, n a natural number. Describe how n is related to the number of large petals and the number of small petals on the graph and how the large and small petals are related to each other relative to n.

In Problems 65–68 graph each system of equations on the same set of polar coordinate axes. Then solve the system simultaneously. [Note: Any solution (r_1, θ_1) to the system must satisfy each equation in the system and thus identifies a point of intersection of the two graphs. However, there may be other points of intersection of the two graphs that do not have any coordinates that satisfy both equations. This represents a major difference between the rectangular coordinate system and the polar coordinate system.]

65. $r = 4\cos\theta$
$r = -4\sin\theta$
$0 \le \theta \le \pi$

66. $r = 2\cos\theta$
$r = 2\sin\theta$
$0 \le \theta \le \pi$

67. $r = 6\cos\theta$
$r = 6\sin 2\theta$
$0° \le \theta \le 360°$

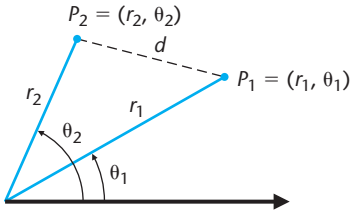
68. $r = 8\sin\theta$
$r = 8\cos 2\theta$
$0° \le \theta \le 360°$

APPLICATIONS

69. ANALYTIC GEOMETRY A distance formula for the distance between two points in a polar coordinate system follows directly from the law of cosines:

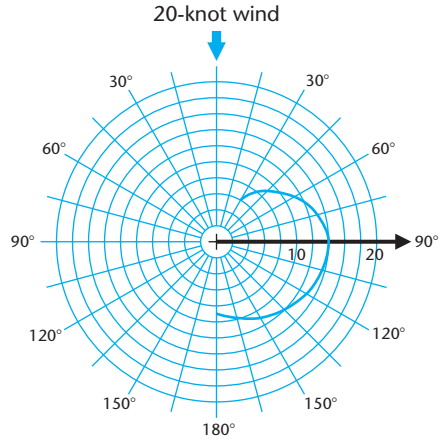
$$d^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)$$
$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$$

Find the distance (to three decimal places) between the two points $P_1 = (4, \pi/4)$ and $P_2 = (1, \pi/2)$.



70. ANALYTIC GEOMETRY Refer to Problem 69. Find the distance (to three decimal places) between the two points $P_1 = (2, 30°)$ and $P_2 = (3, 60°)$.

Problems 71–72 refer to the polar diagram in the figure. Polar diagrams of this type are used extensively by serious sailboat racers, and this polar diagram represents speeds in knots of a high-performance sailboat sailing at various angles to a wind blowing at 20 knots.



71. SAILBOAT RACING Referring to the figure, estimate to the nearest knot the speed of the sailboat sailing at the following angles to the wind: 30°, 75°, 135°, and 180°.

72. SAILBOAT RACING Referring to the figure, estimate to the nearest knot the speed of the sailboat sailing at the following angles to the wind: 45°, 90°, 120°, and 150°.

73. CONIC SECTIONS Using a graphing calculator, graph the equation

$$r = \frac{8}{1 - e\cos\theta}$$

for the following values of e (called the **eccentricity** of the conic) and identify each curve as a hyperbola, an ellipse, or a parabola.
(A) $e = 0.4$ (B) $e = 1$ (C) $e = 1.6$
(It is instructive to explore the graph for other positive values of e. See the Chapter 7 Group Activity.)

74. CONIC SECTIONS Using a graphing calculator, graph the equation

$$r = \frac{8}{1 - e\cos\theta}$$

for the following values of e and identify each curve as a hyperbola, an ellipse, or a parabola.
(A) $e = 0.6$ (B) $e = 1$ (C) $e = 2$

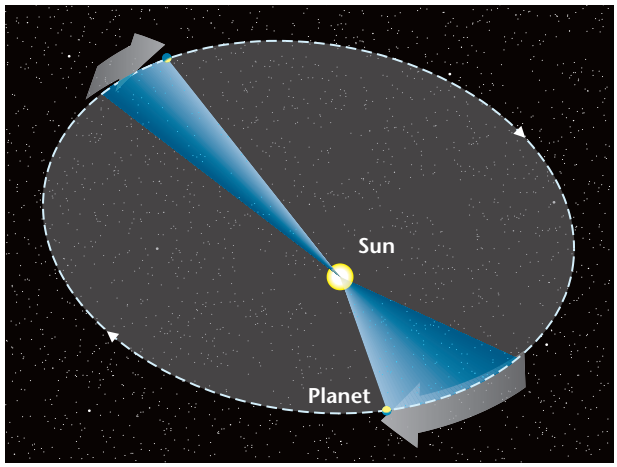
75. ASTRONOMY

(A) The planet Mercury travels around the sun in an elliptical orbit given approximately by

$$r = \frac{3.442 \times 10^7}{1 - 0.206 \cos \theta}$$

where r is measured in miles and the sun is at the pole. Graph the orbit. Use TRACE to find the distance from Mercury to the sun at **aphelion** (greatest distance from the sun) and at **perihelion** (shortest distance from the sun).

(B) Johannes Kepler (1571–1630) showed that a line joining a planet to the sun sweeps out equal areas in space in equal intervals in time (see the figure). Use this information to determine whether a planet travels faster or slower at aphelion than at perihelion. Explain your answer.



7-5 Complex Numbers and De Moivre's Theorem

- Rectangular Form
- Polar Form
- Multiplication and Division
- Powers—De Moivre's Theorem
- Roots
- Historical Note

In this section, we use the polar concepts studied in the last section to show how complex numbers can be written in polar form. Addition and subtraction of complex numbers are easy to carry out if the numbers are written in rectangular form. The operations of multiplication and division, however, become much simpler if complex numbers are written in polar form. A brief review of Section 2-4 on complex numbers should prove helpful.

Rectangular Form

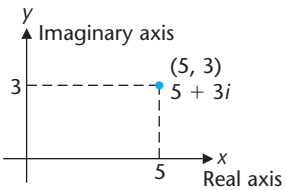
Recall from Section 2-4 that a complex number is any number that can be written in the form

$$x + yi$$

where x and y are real numbers and i is the imaginary unit. (We use $x + yi$ and $x + iy$ interchangeably; each has its advantages in keeping notation simple.) Therefore,

associated with each complex number $x + yi$ is a unique ordered pair of real numbers (x, y), and vice versa. For example,

$$5 + 3i \quad \text{corresponds to} \quad (5, 3)$$



› **Figure 1** Complex plane.

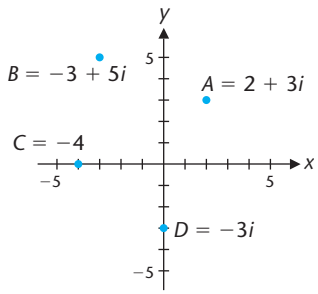
Associating these ordered pairs of real numbers with points in a rectangular coordinate system, we obtain a **complex plane** (Fig. 1). When complex numbers are associated with points in a rectangular coordinate system, we refer to the x axis as the **real axis** and the y axis as the **imaginary axis.** The complex number $x + yi$ is said to be in **rectangular form.**

EXAMPLE 1 Plotting in the Complex Plane

Plot the following complex numbers in a complex plane:

$$A = 2 + 3i \qquad B = -3 + 5i \qquad C = -4 \qquad D = -3i$$

SOLUTION



MATCHED PROBLEM 1

Plot the following complex numbers in a complex plane:

$$A = 4 + 2i \qquad B = 2 - 3i \qquad C = -5 \qquad D = 4i$$

››› EXPLORE-DISCUSS 1

On a *real number line* there is a one-to-one correspondence between the set of real numbers and the set of points on the line: each real number is associated with exactly one point on the line, and each point on the line is associated with exactly one real number. Does such a correspondence exist between the set of complex numbers and the set of points in a plane? Explain how a one-to-one correspondence can be established.

Polar Form

Each point (x, y) of the plane corresponds to a unique complex number z, namely, in *rectangular form,* $z = x + iy$. But the point (x, y) can also be specified by polar coordinates. Therefore, the complex number z can be given a polar form that depends on r and θ. The **polar form** of z is written $z = re^{i\theta}$. (When convenient, we write $re^{\theta i}$ in place of $re^{i\theta}$.)

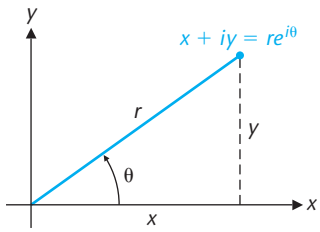
	Points	Complex numbers
Rectangular form	(x, y)	$x + iy$
Polar form	(r, θ)	$re^{i\theta}$

The point with rectangular coordinates (1, 1) has polar coordinates $(\sqrt{2}, \pi/4)$. (Why?) Therefore the complex number $z = 1 + i$ has the polar form $z = \sqrt{2}e^{i\pi/4}$. Many graphing calculators can convert a complex number in rectangular form to polar form and vice versa (see Fig. 2, where θ is in radians and numbers are displayed to two decimal places).

The polar–rectangular relationships of the last section imply the following connections between the rectangular and polar forms of a complex number.

Remark
In more advanced courses, the notation $e^{i\theta}$ is used to represent a value of a generalized version of the natural exponential function $f(x) = e^x$ with base $e \approx 2.718$. Our use of the notation $re^{i\theta}$ does not depend on that interpretation. Instead, $re^{i\theta}$ simply denotes the complex number that corresponds to the point with polar coordinates (r, θ). As is the case with polar coordinates, θ can be given in either radians or degrees. We will assume, however, that r and θ are chosen so that r is nonnegative.

POLAR–RECTANGULAR RELATIONSHIPS FOR COMPLEX NUMBERS



If $x + iy = re^{i\theta}$, then

$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$

$y = r\sin\theta \qquad \tan\theta = \dfrac{y}{x}, \qquad x \neq 0$

Therefore, $x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$ and $e^{i\theta} = \cos\theta + i\sin\theta$

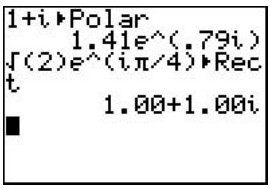


Figure 2 $1 + i = \sqrt{2}e^{i\pi/4}$.

If $z = re^{i\theta}$, then the number r is called the **modulus,** or **absolute value,** of z and is denoted by **mod** z or $|z|$. The angle θ (in radians or degrees) is called the **argument** of z and is denoted by **arg** z. Recall that (r, θ) and $(r, \theta + 2\pi)$ represent the same point in polar coordinates. Therefore, $z = re^{i\theta} = re^{i(\theta + 2\pi)}$. So the argument of a complex number is not unique. But we usually choose the argument θ so that $-\pi < \theta \leq \pi$ (or $-180° < \theta \leq 180°$).

EXAMPLE 2 From Rectangular to Polar Form

Write parts A–C in polar form, θ in radians, $-\pi < \theta \leq \pi$. Compute the modulus and arguments for parts A and B exactly; compute the modulus and argument for part C to two decimal places.

(A) $z_1 = 1 - i$ (B) $z_2 = -\sqrt{3} + i$ (C) $z = -5 - 2i$

SOLUTIONS

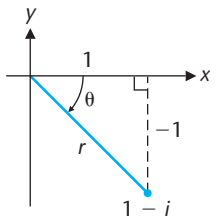
Locate in a complex plane first; then if x and y are associated with special angles, r and θ can often be determined by inspection.

(A) A sketch shows that z_1 is associated with a special 45° triangle (Fig. 3). By inspection, $r = \sqrt{2}$, $\theta = -\pi/4$ (not $7\pi/4$), and

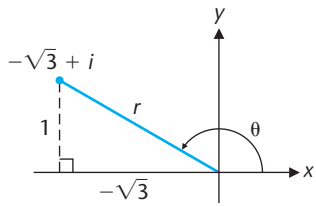
$$z_1 = \sqrt{2}e^{i(-\pi/4)}$$

(B) A sketch shows that z_2 is associated with a special 30°–60° triangle (Fig. 4). By inspection, $r = 2$, $\theta = 5\pi/6$, and

$$z_2 = 2e^{i(5\pi/6)}$$



› Figure 3



› Figure 4

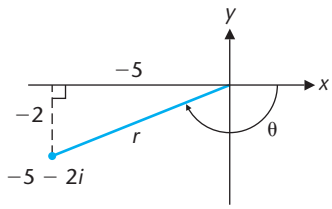
(C) A sketch shows that z_3 is not associated with a special triangle (Fig. 5). So, we proceed as follows:

$$r = \sqrt{(-5)^2 + (-2)^2} = 5.39 \quad \text{To two decimal places}$$
$$\theta = -\pi + \tan^{-1}\left(\tfrac{2}{5}\right) = -2.76 \quad \text{To two decimal places}$$

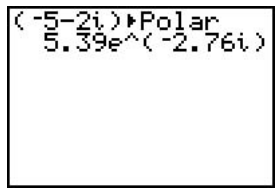
Therefore,

$$z_3 = 5.39e^{(-2.76)i} \quad \text{To two decimal places}$$

Figure 6 shows the same conversion done by a graphing calculator with a built-in conversion routine (with numbers displayed to two decimal places).



› Figure 5



› Figure 6 $(-5 - 2i) = 5.39e^{(-2.76)i}$.

MATCHED PROBLEM 2

Write parts A–C in polar form, θ in radians, $-\pi < \theta \leq \pi$. Compute the modulus and arguments for parts A and B exactly; compute the modulus and argument for part C to two decimal places.

(A) $-1 + i$ (B) $1 + i\sqrt{3}$ (C) $-3 - 7i$

EXAMPLE 3 **From Polar to Rectangular Form**

Write parts A–C in rectangular form. Compute the exact values for parts A and B; for part C, compute a and b for $a + bi$ to two decimal places.

(A) $z_1 = 2e^{(5\pi/6)i}$ (B) $z_2 = 3e^{(-60°)i}$ (C) $z_3 = 7.19e^{(-2.13)i}$

SOLUTIONS

(A) $x + iy = 2e^{(5\pi/6)i}$

$= 2[\cos(5\pi/6) + i\sin(5\pi/6)]$ $\cos\frac{5\pi}{6} = \frac{-\sqrt{3}}{2}, \sin\frac{5\pi}{6} = \frac{1}{2}$

$= 2\left(\frac{-\sqrt{3}}{2}\right) + i2\left(\frac{1}{2}\right)$ Simplify.

$= -\sqrt{3} + i$

(B) $x + iy = 3e^{(-60°)i}$

$= 3[\cos(-60°) + i\sin(-60°)]$ $\cos(-60°) = \frac{1}{2}, \sin(-60°) = \frac{-\sqrt{3}}{2}$

$= 3\left(\frac{1}{2}\right) + i3\left(\frac{-\sqrt{3}}{2}\right)$ Simplify.

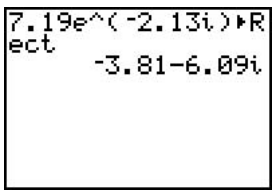
$= \frac{3}{2} - \frac{3\sqrt{3}}{2}i$

(C) $x + iy = 7.19e^{(-2.13)i}$

$= 7.19[\cos(-2.13) + i\sin(-2.13)]$

$= -3.81 - 6.09i$

Figure 7 shows the same conversion done by a graphing calculator with a built-in conversion routine.



› **Figure 7** $7.19e^{(-2.13)i} = -3.81 - 6.09i$.

››› EXPLORE-DISCUSS 2

If your calculator has a built-in polar-to-rectangular conversion routine, try it on $\sqrt{2}e^{45°i}$ and $\sqrt{2}e^{(\pi/4)i}$, then reverse the process to see if you get back where you started. (For complex numbers in exponential polar form, some calculators require θ to be in radian mode for calculations. Check your user's manual.)

MATCHED PROBLEM 3

Write parts A–C in rectangular form. Compute the exact values for parts A and B; for part C compute a and b for $a + bi$ to two decimal places.

(A) $z_1 = \sqrt{2}e^{(-\pi/2)i}$ (B) $z_2 = 3e^{120°i}$ (C) $z_3 = 6.49e^{(-2.08)i}$

››› EXPLORE-DISCUSS 3

Let $z_1 = \sqrt{3} + i$ and $z_2 = 1 + i\sqrt{3}$.

(A) Find z_1z_2 and z_1/z_2 using the rectangular forms of z_1 and z_2.

(B) Find z_1z_2 and z_1/z_2 using the polar forms of z_1 and z_2, θ in degrees. (Assume the product and quotient exponent laws hold for $e^{i\theta}$.)

(C) Convert the results from part B back to rectangular form and compare with the results in part A.

Multiplication and Division

There is a particular advantage in representing complex numbers in polar form: Multiplication and division become very easy. Theorem 1 provides the reason. (The polar form of a complex number obeys the product and quotient rules for exponents: $b^mb^n = b^{m+n}$ and $b^m/b^n = b^{m-n}$.)

THEOREM 1 Products and Quotients in Polar Form

If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then

1. $z_1z_2 = r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$
2. $\dfrac{z_1}{z_2} = \dfrac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \dfrac{r_1}{r_2}e^{i(\theta_1-\theta_2)}, \qquad r_2 \neq 0$

Theorem 1 says that to multiply two complex numbers, you multiply their moduli and add their arguments. Similarly, to divide two complex numbers, you divide their moduli and subtract their arguments.

We establish the multiplication property and leave the quotient property for Problem 74 in the exercises.

$$
\begin{aligned}
z_1z_2 &= r_1e^{i\theta_1}r_2e^{i\theta_2} && \text{Use polar–rectangular relationships.}\\
&= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) && \text{Multiply.}\\
&= r_1r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 && \text{Group real parts and imaginary parts.}\\
&\quad + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\\
&= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)] && \text{Use sum identities.}\\
&\quad + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\\
&= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] && \text{Use polar–rectangular relationships.}\\
&= r_1r_2e^{i(\theta_1+\theta_2)}
\end{aligned}
$$

EXAMPLE 4 Products and Quotients

If $z_1 = 8e^{45°i}$ and $z_2 = 2e^{30°i}$, find

(A) z_1z_2 (B) z_1/z_2

SOLUTIONS

(A) $z_1z_2 = 8e^{45°i}\cdot 2e^{30°i} = 8\cdot 2e^{i(45°+30°)}$* $= 16e^{75°i}$

(B) $\dfrac{z_1}{z_2} = \dfrac{8e^{45°i}}{2e^{30°i}} = \dfrac{8}{2}e^{i(45°-30°)} = 4e^{15°i}$

MATCHED PROBLEM 4

If $z_1 = 9e^{165°i}$ and $z_2 = 3e^{55°i}$, find

(A) z_1z_2 (B) z_1/z_2

Powers—De Moivre's Theorem

Abraham De Moivre (1667–1754), of French birth, spent most of his life in London tutoring, writing, and publishing mathematics. He belonged to many prestigious professional societies in England, Germany, and France. He was a close friend of Isaac Newton. The theorem that bears his name gives a formula for raising any complex number to the power n, where n is a natural number.

*Throughout the book, dashed boxes—called think boxes—are used to represent steps that may be performed mentally.

› THEOREM 2 De Moivre's Theorem

If $z = re^{i\theta}$ and n is a natural number, then $z^n = r^n e^{i(n\theta)}$.

De Moivre's theorem follows from the formula for the product of complex numbers in polar form. If $n = 2$ and $z = re^{i\theta}$, then

$$z^2 = re^{i\theta}re^{i\theta} = r^2e^{i(2\theta)}$$

In other words, to square a complex number, you square the modulus and double the argument. Similarly, to cube a complex number you cube the modulus and triple the argument. De Moivre's theorem says that to raise a complex number to the power n, you raise the modulus to the power n and multiply the argument by n.

EXAMPLE 5 The Natural Number Power of a Complex Number

Use De Moivre's theorem to find $(1 + i)^{10}$. Write the answer in exact rectangular form.

SOLUTION

First note that the polar form of $1 + i$ is $\sqrt{2}e^{45°i}$. Therefore,

$$\begin{aligned}
(1 + i)^{10} &= (\sqrt{2}e^{45°i})^{10} && \text{Use De Moivre's theorem.}\\
&= (\sqrt{2})^{10}e^{(10\cdot 45°)i} && \text{Simplify.}\\
&= 32e^{450°i} && \text{Change to rectangular form.}\\
&= 32(\cos 450° + i \sin 450°) && \cos 450° = 0,\ \sin 450° = 1\\
&= 32(0 + i) && \text{Simplify.}\\
&= 32i && \text{Rectangular form}
\end{aligned}$$

MATCHED PROBLEM 5

Use De Moivre's theorem to find $(1 + i\sqrt{3})^5$. Write the answer in exact polar form using degrees, and in exact rectangular form.

EXAMPLE 6 The Natural Number Power of a Complex Number

Use De Moivre's theorem to find $(-\sqrt{3} + i)^6$. Write the answer in exact rectangular form.

SOLUTION

First note that the polar form of $-\sqrt{3} + i$ is $2e^{150°i}$. Therefore,

$$
\begin{aligned}
(-\sqrt{3} + i)^6 &= (2e^{150°i})^6 && \text{Use De Moivre's theorem.}\\
&= 2^6 e^{(6\cdot 150°)i} && \text{Simplify.}\\
&= 64e^{900°i} && \text{Change to rectangular form.}\\
&= 64(\cos 900° + i \sin 900°) && \cos 900° = -1,\ \sin 900° = 0\\
&= 64(-1 + i0) && \text{Simplify.}\\
&= -64 && \text{Rectangular form}
\end{aligned}
$$

[*Note:* $-\sqrt{3} + i$ must be a sixth root of -64, because $(-\sqrt{3} + i)^6 = -64$.]

MATCHED PROBLEM 6

Use De Moivre's theorem to find $(1 - i\sqrt{3})^4$. Write the answer in exact polar form using degrees, and in exact rectangular form.

Roots

Let $n > 1$ be an integer. A complex number w is an ***n*th root of *z*** if $w^n = z$. For example, 2 and -2 are square roots (second roots) of 4 because $2^2 = 4$ and $(-2)^2 = 4$. Similarly, $3i$ and $-3i$ are square roots of -9 because $(3i)^2 = -9$ and $(-3i)^2 = -9$. The nth root theorem gives a formula for all of the nth roots of any nonzero complex number.

THEOREM 3 *n*th Root Theorem

Let $n > 1$ be an integer and let $z = re^{i\theta}$ be a nonzero complex number. Then z has n distinct nth roots given by

$$r^{1/n}e^{i(\theta/n + k360°/n)} \qquad k = 0, 1, \ldots, n - 1$$

The proof of Theorem 3 is left to Problems 75 and 76 in Exercises 7-5. The nth root theorem implies that every nonzero complex number z has two square roots, three cube roots, four fourth roots, and so on. Furthermore, all n of the nth roots of z have the same modulus, so they all lie on the same circle centered at the origin, and they are equally spaced around that circle.

EXAMPLE 7 Finding All Sixth Roots of a Complex Number

Find six distinct sixth roots of $-1 + i\sqrt{3}$, and plot them in a complex plane.

SOLUTION

First write $-1 + i\sqrt{3}$ in polar form:

$$-1 + i\sqrt{3} = 2e^{120°i}$$

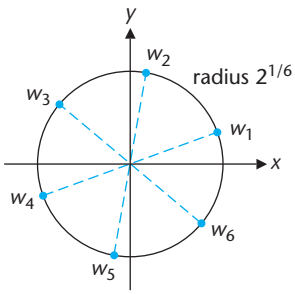
Using the nth-root theorem, all six roots are given by

$$2^{1/6}e^{(120°/6 + k360°/6)i} = 2^{1/6}e^{(20° + k60°)i} \qquad k = 0, 1, 2, 3, 4, 5$$

Therefore,

$$\begin{aligned} w_1 &= 2^{1/6}e^{(20° + 0\cdot 60)i} = 2^{1/6}e^{20°i} \\ w_2 &= 2^{1/6}e^{(20° + 1\cdot 60)i} = 2^{1/6}e^{80°i} \\ w_3 &= 2^{1/6}e^{(20° + 2\cdot 60)i} = 2^{1/6}e^{140°i} \\ w_4 &= 2^{1/6}e^{(20° + 3\cdot 60)i} = 2^{1/6}e^{200°i} \\ w_5 &= 2^{1/6}e^{(20° + 4\cdot 60)i} = 2^{1/6}e^{260°i} \\ w_6 &= 2^{1/6}e^{(20° + 5\cdot 60)i} = 2^{1/6}e^{320°i} \end{aligned}$$

All roots are easily graphed in the complex plane after the first root is located. The root points are equally spaced around a circle of radius $2^{1/6}$ at an angular increment of 60° from one root to the next (Fig. 8).



› Figure 8

MATCHED PROBLEM 7

Find five distinct fifth roots of $1 + i$. Leave the answers in exact polar form using degrees, and plot them in a complex plane.

EXAMPLE 8 Solving a Cubic Equation

Solve $x^3 + 1 = 0$. Write final answers in rectangular form, and plot them in a complex plane.

SOLUTION

$$\begin{aligned} x^3 + 1 &= 0 \\ x^3 &= -1 \end{aligned}$$

We see that x is a cube root of -1, and there are a total of three roots. To find the three roots, we first write -1 in polar form:

$$-1 = 1e^{180°i}$$

Using the nth-root theorem, all three cube roots of -1 are given by

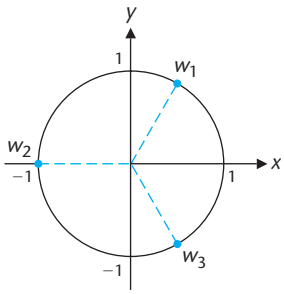
$$1^{1/3}e^{(180°/3+k360°/3)i} = 1e^{(60°+k120°)i} \qquad k = 0, 1, 2$$

Therefore,

$$w_1 = 1e^{60°i} = \cos 60° + i \sin 60° = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$
$$w_2 = 1e^{180°i} = \cos 180° + i \sin 180° = -1$$
$$w_3 = 1e^{300°i} = \cos 300° + i \sin 300° = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

(*Note:* This problem can also be solved using factoring and the quadratic formula—try it.)

The three roots are graphed in Figure 9.



› Figure 9

MATCHED PROBLEM 8

Solve $x^3 - 1 = 0$. Write final answers in exact rectangular form, and plot them in a complex plane.

› Historical Note

There is hardly an area in mathematics that does not have some imprint of the famous Swiss mathematician Leonhard Euler (1707–1783), who spent most of his productive life at the New St. Petersburg Academy in Russia and the Prussian Academy in Berlin. One of the most prolific writers in the history of the subject, he is credited with making the following familiar notations standard:

$f(x)$	function notation
e	natural logarithmic base
i	imaginary unit, $\sqrt{-1}$

For our immediate interest, he is also responsible for the extraordinary relationship

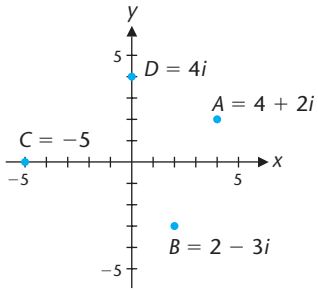
$$e^{i\theta} = \cos \theta + i \sin \theta$$

If we let $\theta = \pi$, we obtain an equation that relates five of the most important numbers in the history of mathematics:

$$e^{i\pi} + 1 = 0$$

ANSWERS TO MATCHED PROBLEMS

1.



2. (A) $\sqrt{2}e^{i(3\pi/4)}$
(B) $2e^{i(\pi/3)}$
(C) $7.62e^{(-1.98)i}$

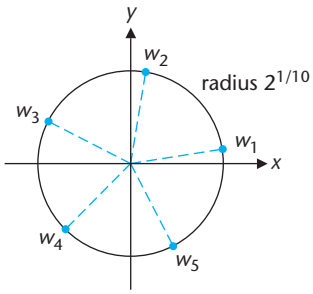
3. (A) $-i\sqrt{2}$ (B) $-\frac{3}{2}+\frac{3\sqrt{3}}{2}i$ (C) $-3.16-5.67i$

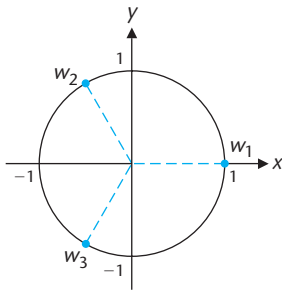
4. (A) $z_1z_2 = 27e^{220°i}$ (B) $z_1/z_2 = 3e^{110°i}$

5. $32e^{300°i} = 16 - i16\sqrt{3}$ **6.** $16e^{(-240°)i} = -8 + i8\sqrt{3}$

7. $w_1 = 2^{1/10}e^{9°i}$, $w_2 = 2^{1/10}e^{81°i}$,
$w_3 = 2^{1/10}e^{153°i}$, $w_4 = 2^{1/10}e^{225°i}$, $w_5 = 2^{1/10}e^{297°i}$

8. $1, -\frac{1}{2}+i\frac{\sqrt{3}}{2}, -\frac{1}{2}-i\frac{\sqrt{3}}{2}$





7-5 Exercises

1. What is the modulus of a complex number?

2. What is the argument of a complex number?

3. Explain how to locate the product of two complex numbers that lie on the unit circle.

4. Explain how to locate the quotient of two complex numbers that lie on the unit circle.

5. Explain how to locate the cube of a complex number that lies on the unit circle.

6. Explain how to locate the cube roots of a complex number that lies on the unit circle.

In Problems 7–14, plot each set of complex numbers in a complex plane.

7. $A = 3 + 4i, B = -2 - i, C = 2i$

8. $A = 4 + i, B = -3 + 2i, C = -3i$

9. $A = 3 - 3i, B = 4, C = -2 + 3i$

10. $A = -3, B = -2 - i, C = 4 + 4i$

11. $A = 2e^{(\pi/3)i}, B = \sqrt{2}e^{(\pi/4)i}, C = 4e^{(\pi/2)i}$

12. $A = 2e^{(\pi/6)i}, B = 4e^{\pi i}, C = \sqrt{2}e^{(3\pi/4)i}$

13. $A = 4e^{(-150°)i}, B = 3e^{20°i}, C = 5e^{(-90°)i}$

14. $A = 2e^{150°i}, B = 3e^{(-50°)i}, C = 4e^{75°i}$

In Problems 15–18, convert to the polar form $re^{i\theta}$. For Problems 15 and 16, choose θ in degrees, $-180° < \theta \le 180°$; for Problems 17 and 18 choose θ in radians, $-\pi < \theta \le \pi$. Compute the modulus and arguments for parts A and B exactly; compute the modulus and argument for part C to two decimal places.

15. (A) $\sqrt{3} + i$ (B) $-1 - i$ (C) $5 - 6i$

16. (A) $-1 + i\sqrt{3}$ (B) $-3i$ (C) $-7 - 4i$

17. (A) $-i\sqrt{3}$ (B) $-\sqrt{3} - i$ (C) $-8 + 5i$

18. (A) $\sqrt{3} - i$ (B) $-2 + 2i$ (C) $6 - 5i$

In Problems 19–22, change parts A–C to rectangular form. Compute the exact values for parts A and B; for part C compute a and b for $a + bi$ to two decimal places.

19. (A) $2e^{(\pi/3)i}$ (B) $\sqrt{2}e^{(-45°)i}$ (C) $3.08e^{2.44i}$

20. (A) $2e^{30°i}$ (B) $\sqrt{2}e^{(-3\pi/4)i}$ (C) $5.71e^{(-0.48)i}$

21. (A) $6e^{(\pi/6)i}$ (B) $\sqrt{7}e^{(-90°)i}$ (C) $4.09e^{(-122.88°)i}$

22. (A) $\sqrt{3}e^{(-\pi/2)i}$ (B) $\sqrt{2}e^{135°i}$ (C) $6.83e^{(-108.82°)i}$

In Problems 23–30, find z_1z_2 and z_1/z_2 in the polar form $re^{i\theta}$.

23. $z_1 = e^{195°i}, z_2 = e^{55°i}$ **24.** $z_1 = e^{75°i}, z_2 = e^{5°i}$

25. $z_1 = 7e^{82°i}, z_2 = 2e^{31°i}$ **26.** $z_1 = 6e^{132°i}, z_2 = 3e^{93°i}$

27. $z_1 = 5e^{52°i}, z_2 = 2e^{83°i}$ **28.** $z_1 = 3e^{67°i}, z_2 = 2e^{97°i}$

29. $z_1 = 3.05e^{1.76i}, z_2 = 11.94e^{2.59i}$

30. $z_1 = 7.11e^{0.79i}, z_2 = 2.66e^{1.07i}$

In Problems 31–38, use De Moivre's theorem to evaluate each. Leave answers in polar form.

31. $(e^{20°i})^5$ **32.** $(e^{50°i})^4$ **33.** $(2e^{30°i})^3$

34. $(5e^{15°i})^3$ **35.** $(\sqrt{2}e^{10°i})^6$ **36.** $(\sqrt{2}e^{15°i})^8$

37. $(1 + i\sqrt{3})^3$ **38.** $(\sqrt{3} + i)^8$

In Problems 39–44, find the value of each expression and write the final answer in exact rectangular form. (Verify the results in Problems 39–44 by evaluating each directly on a calculator.)

39. $(-\sqrt{3} - i)^4$ **40.** $(-1 + i)^4$

41. $(1 - i)^8$ **42.** $(-\sqrt{3} + i)^5$

43. $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3$ **44.** $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3$

For n and z as indicated in Problems 45–50, find all nth roots of z. Leave answers in the polar form $re^{i\theta}$ using degrees.

45. $z = 8e^{30°i}, n = 3$ **46.** $z = 8e^{45°i}, n = 3$

47. $z = 81e^{60i}, n = 4$ **48.** $z = 16e^{90°i}, n = 4$

49. $z = 1 - i, n = 5$ **50.** $z = -1 + i, n = 3$

For n and z as indicated in Problems 51–56, find all nth roots of z. Write answers in the polar form $re^{i\theta}$ using degrees, and plot in a complex plane.

51. $z = 8, n = 3$ **52.** $z = 1, n = 4$

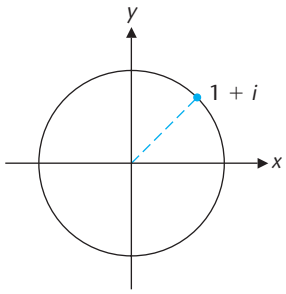
53. $z = -16, n = 4$ **54.** $z = -8, n = 3$

55. $z = i, n = 6$ **56.** $z = -i, n = 5$

57. (A) Show that $1 + i$ is a root of $x^4 + 4 = 0$. How many other roots does the equation have?

(B) The root $1 + i$ is located on a circle of radius $\sqrt{2}$ in the complex plane as indicated in the figure. Locate the other three roots of $x^4 + 4 = 0$ on the figure and explain geometrically how you found their location.

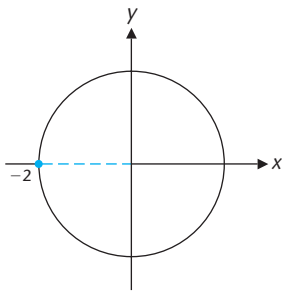
(C) Verify that each complex number found in part B is a root of $x^4 + 4 = 0$.



58. (A) Show that -2 is a root of $x^3 + 8 = 0$. How many other roots does the equation have?

(B) The root -2 is located on a circle of radius 2 in the complex plane as indicated in the figure. Locate the other two roots of $x^3 + 8 = 0$ on the figure and explain geometrically how you found their location.

(C) Verify that each complex number found in part B is a root of $x^3 + 8 = 0$.



In Problems 59–62, solve each equation for all roots. Write final answers in the polar form $re^{i\theta}$ using degrees, and in exact rectangular form.

59. $x^3 + 64 = 0$ **60.** $x^3 - 64 = 0$

61. $x^3 - 27 = 0$ **62.** $x^3 + 27 = 0$

In Problems 63–72, determine whether the statement is true or false. If true, explain why. If false, give a counterexample.

63. If two numbers lie on the real axis, then their product lies on the real axis.

64. If two numbers lie on the imaginary axis, then their quotient lies on the imaginary axis.

65. If z is a positive real number, then all of the fourth roots of z are real.

66. If z is a positive real number, then all of the square roots of z are real.

67. If w is a square root of 1, then w is a sixth root of 1.

68. If w is a sixth root of 1, then w is a square root of 1.

69. If w is both a cube root and a fourth root of a nonzero complex number z, then $|w| = 1$.

70. If w is both a cube root and a sixth root of a nonzero complex number z, then $|w| = 1$.

71. If w is both a cube root and a sixth root of a nonzero complex number z, then $w = 1$.

72. If w is both a cube root and a fourth root of a nonzero complex number z, then $w = 1$.

73. Suppose that z is a complex number that is not real. Explain why none of the nth roots of z lies on the x axis.

74. Prove

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

75. Show that

$$[r^{1/n} e^{(\theta/n + k360°/n)i}]^n = re^{i\theta}$$

for any natural number n and any integer k.

76. Show that

$$r^{1/n} e^{(\theta/n + k360°/n)i}$$

is the same number for $k = 0$ and $k = n$.

In Problems 77–80, write answers in the polar form $re^{i\theta}$ using degrees.

77. Find all complex zeros for $P(x) = x^5 - 32$.

78. Find all complex zeros for $P(x) = x^6 + 1$.

79. Solve $x^5 + 1 = 0$ in the set of complex numbers.

80. Solve $x^3 - i = 0$ in the set of complex numbers.

In Problems 81 and 82, write answers using exact rectangular forms.

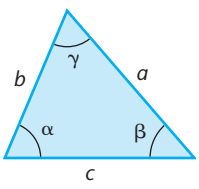
81. Write $P(x) = x^6 + 64$ as a product of linear factors.

82. Write $P(x) = x^6 - 1$ as a product of linear factors.

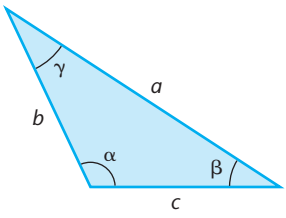
CHAPTER 7 Review

7-1 Law of Sines

An **oblique triangle** is a triangle without a right angle. An oblique triangle is **acute** if all angles are between 0° and 90° and **obtuse** if one angle is between 90° and 180°. The labeling convention shown in these figures is followed in this chapter.



Acute triangle



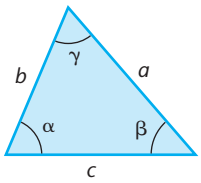
Obtuse triangle

The objective in Sections 7-1 and 7-2 is to solve an oblique triangle given any three of the six quantities indicated in either figure, if a solution exists. The law of sines, discussed in Section 7-1, and the law of cosines, discussed in Section 7-2, are used for this purpose. Accuracy in computation is governed by Table 1.

Table 1 Triangles and Significant Digits

Angle to nearest	Significant digits for side measure
1°	2
10′ or 0.1°	3
1′ or 0.01°	4
10″ or 0.001°	5

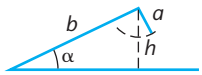
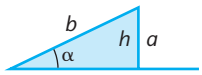
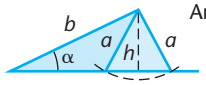
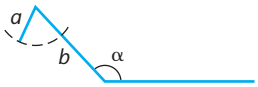
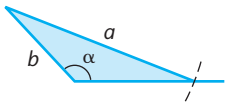
The **law of sines** is given as



$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

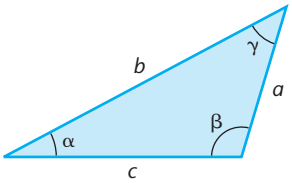
and is generally used to solve the ASA, AAS, and SSA cases for oblique triangles. The AAS case is easily reduced to the ASA case by solving for the third angle first. The SSA case has a number of variations, including the ambiguous case. These variations are summarized in Table 2. Note that the ambiguous case always results in two triangles, one obtuse and one acute.

Table 2 SSA Variations

α	$a[h = b \sin\alpha]$	Number of triangles	Figure
Acute	$0 < a < h$	0	
Acute	$a = h$	1	
Acute	$h < a < b$	2	 Ambiguous case
Acute	$a \geq b$	1	
Obtuse	$0 < a \leq b$	0	
Obtuse	$a > b$	1	

7-2 Law of Cosines

The **law of cosines** is given as

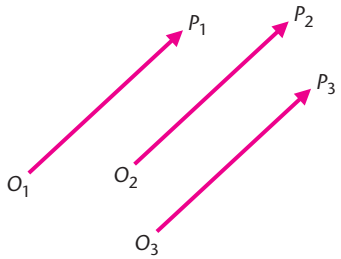


$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

and is generally used as the first step in solving the SAS and SSS cases for oblique triangles. After a side or angle is found using the law of cosines, it is usually easier to continue the solving process with the law of sines.

7-3 Vectors in the Plane

A **vector v** is a quantity that has both magnitude and direction. We picture a vector as an arrow from an initial point O to a terminal point P with this provision: arrows that have the same length (magnitude) and direction represent the same vector (see the figure).

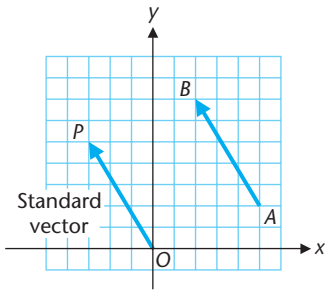


All three arrows represent the same vector **v**.

The vector **v** of the figure is also denoted by $\overrightarrow{O_1P_1}$ (or $\overrightarrow{O_2P_2}$ or $\overrightarrow{O_3P_3}$).

The **magnitude** of the vector $\mathbf{v} = \overrightarrow{OP}$, denoted by $|\mathbf{v}|$, $|\vec{v}|$, or $|\overrightarrow{OP}|$, is the length of the line segment OP. Two vectors have the **same direction** if they are parallel and point in the same direction. Two vectors have **opposite directions** if they are parallel and point in opposite directions. The **zero vector,** denoted by **0** or $\vec{0}$, has magnitude 0 and arbitrary direction. Two vectors are **equal** if they have the same magnitude and direction.

Any vector $\overrightarrow{AB}$ in a rectangular coordinate system can be translated so that its initial point is the origin O. The vector $\overrightarrow{OP}$ such that $\overrightarrow{OP} = \overrightarrow{AB}$ is said to be the **standard vector** for $\overrightarrow{AB}$, as shown in the figure.



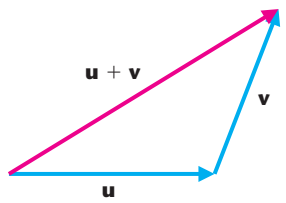
$\overrightarrow{OP}$ is the standard vector for $\overrightarrow{AB}$.

Referring to the figure, if $A = (x_a, y_a)$ and $B = (x_b, y_b)$, then the coordinates of the point P are given by $(x_P, y_p) = (x_b - x_a, y_b - y_a)$.

There is a one-to-one correspondence between vectors in a rectangular coordinate system and points in the system: Any vector $\overrightarrow{AB}$ is associated with the point $P = (x_p, y_p)$ such that $\overrightarrow{OP} = \overrightarrow{AB}$ and any point P is associated with the vector $\overrightarrow{OP}$. A vector can therefore be denoted by an ordered pair of real numbers. We use $\langle c, d \rangle$ to denote the vector with initial point $(0, 0)$ and terminal point (c, d). The real numbers c and d are called the **scalar components** of the vector $\langle c, d \rangle$. Two vectors $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$ are equal if their corresponding components are equal, that is, if $a = c$ and $b = d$. The zero vector is $\mathbf{0} = \langle 0, 0 \rangle$. The magnitude of the vector $\mathbf{u} = \langle a, b \rangle$ is given by

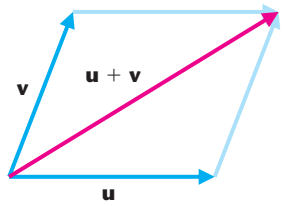
$$|\mathbf{u}| = \sqrt{a^2 + b^2}$$

The sum $\mathbf{u} + \mathbf{v}$ of two vectors **u** and **v** is defined by the **tail-to-tip rule:** Translate **v** so that its tail (initial point) is at the tip (terminal point) of **u.** Then, the vector from the tail of **u** to the tip of **v** is the sum, denoted $\mathbf{u} + \mathbf{v}$, of the vectors **u** and **v.**



Vector addition: tail-to-tip rule

If **u** and **v** are not parallel, the **parallelogram rule** gives an alternative description of $\mathbf{u} + \mathbf{v}$**:** The sum of two nonparallel vectors **u** and **v** is the diagonal formed using **u** and **v** as adjacent sides.



Vector addition: parallelogram rule

The vector $\mathbf{u} + \mathbf{v}$ is also called the **resultant** of the two vectors **u** and **v,** and **u** and **v** are called **vector components** of $\mathbf{u} + \mathbf{v}$**.**

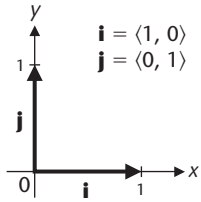
The **scalar product** $k\mathbf{u}$ of a scalar (real number) k and a vector **u** is the vector with magnitude $|k||\mathbf{u}|$ that has the same direction as **u** if k is positive and the opposite direction if k is negative.

Both the sum $\mathbf{u} + \mathbf{v}$ and the scalar product $k\mathbf{u}$ are easy to calculate if the scalar components of **u** and **v** are given. If $\mathbf{u} = \langle a, b \rangle$, $\mathbf{v} = \langle c, d \rangle$, and k is a scalar (real number), then

$$\mathbf{u} + \mathbf{v} = \langle a + c, b + d \rangle \quad \text{Vector addition}$$

$$k\mathbf{u} = \langle ka, kb \rangle \quad \text{Scalar multiplication}$$

Any vector that has magnitude 1 is called a **unit vector.** If **v** is an arbitrary nonzero vector, then $u = (1/|\mathbf{v}|)\mathbf{v}$**,** is a unit vector with the same direction as **v.** The unit vectors in the directions of the positive x axis and the positive y axis are denoted by **i** and **j,** respectively.



Every vector can be expressed in terms of the **i** and **j** unit vectors: $\langle a, b \rangle = a\mathbf{i} + b\mathbf{j}$**.**

The following algebraic properties of vector addition and scalar multiplication enable us to manipulate symbols representing vectors and scalars in much the same way we manipulate symbols that represent real numbers in algebra.

ALGEBRAIC PROPERTIES OF VECTORS

A. Addition Properties. For all vectors **u, v,** and **w:**

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	**Commutative Property**
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	**Associative Property**
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$	**Additive Identity**
4. $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$	**Additive Inverse**

B. Scalar Multiplication Properties. For all vectors **u** and **v** and all scalars m and n:

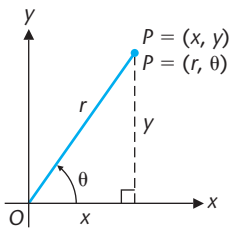
1. $m(n\mathbf{u}) = (mn)\mathbf{u}$	**Associative Property**
2. $m(\mathbf{u} + \mathbf{v}) = m\mathbf{u} + m\mathbf{v}$	**Distributive Property**
3. $(m + n)\mathbf{u} = m\mathbf{u} + n\mathbf{u}$	**Distributive Property**
4. $1\mathbf{u} = \mathbf{u}$	**Multiplicative Identity**

A vector that represents the direction and speed of an object in motion is called a **velocity vector.** The velocity of an airplane relative to the air is called the **apparent velocity,** and the velocity relative to the ground is called the **resultant,** or **actual, velocity.** The resultant velocity is the vector sum of the apparent velocity and wind velocity. Similar statements apply to objects in water subject to currents.

A vector that represents the direction and magnitude of an applied force is called a **force vector.** If an object is subjected to two forces, then the sum of these two forces, the **resultant force,** is a single force acting on the object in the same way as the two original forces taken together. An object at rest is said to be in **static equilibrium.** For an object to remain in static equilibrium, the sum of all the force vectors acting on the object must be the zero vector.

7-4 Polar Coordinates and Graphs

The figure illustrates a **polar coordinate system.** The fixed point O is called the **pole** or **origin,** and the horizontal arrow is called the **polar axis.** We have the following **relationships between rectangular coordinates (x, y) and polar coordinates (r, θ):**



$$r^2 = x^2 + y^2$$

$$\sin\theta = \frac{y}{r} \quad \text{or} \quad y = r\sin\theta$$

$$\cos\theta = \frac{x}{r} \quad \text{or} \quad x = r\cos\theta$$

$$\tan\theta = \frac{y}{x}$$

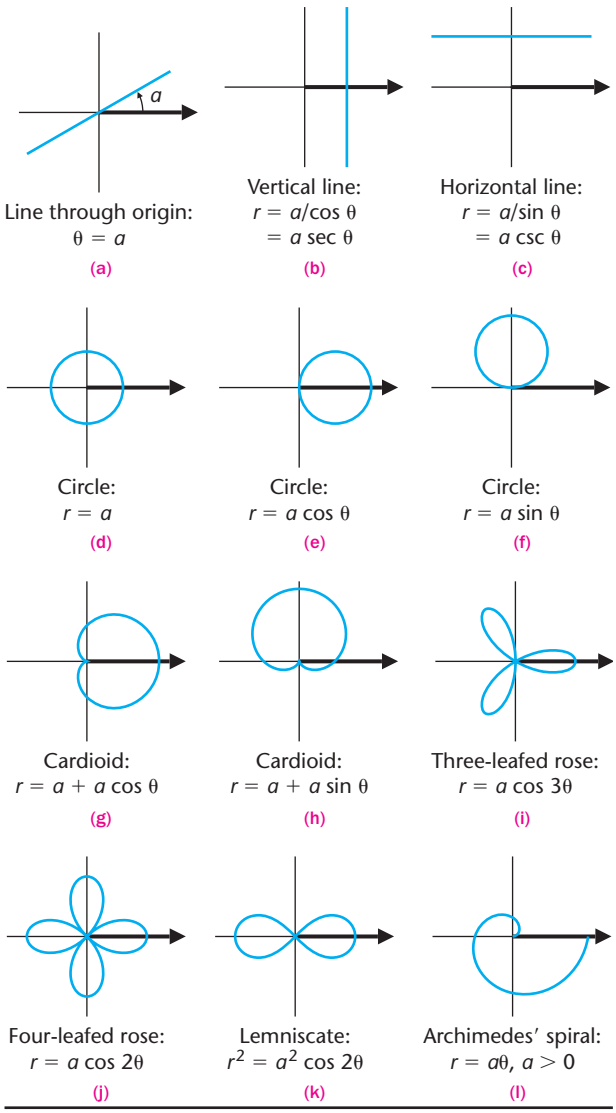
[*Note:* The signs of x and y determine the quadrant for θ. The angle θ is chosen so that $-\pi < \theta \le \pi$ or $-180° < \theta \le 180°$, unless directed otherwise.]

Polar graphs can be obtained by **point-by-point plotting** much in the same way graphs in rectangular coordinates are formed. Make a table of values that satisfy the polar equation, plot these points, then join them with a smooth curve.

Graphs can also be obtained by **rapid polar sketching.** If only a rough sketch of a polar equation involving $\sin\theta$ or $\cos\theta$ is desired, we can speed up the point-by-point graphing process by taking advantage of the uniform variation of $\sin\theta$ and $\cos\theta$ as θ moves through each set of quadrant values. Graphing calculators can produce polar graphs almost instantly.

The table shows some standard polar curves with their equations:

Standard Polar Graphs



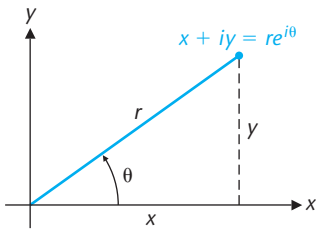
7-5 Complex Numbers and De Moivre's Theorem

Each point (x, y) of the plane corresponds to a unique complex number z. The **rectangular form** of z is written $z = x + iy$. The point (x, y) can also be specified by polar coordinates. Therefore the complex number z can be given a polar form that depends on r and θ. The **polar form** of z is written $z = re^{i\theta}$.

	Points	Complex numbers
Rectangular form	(x, y)	$x + iy$
Polar form	(r, θ)	$re^{i\theta}$

In advanced mathematics the notation $e^{i\theta}$ is used to represent a value of a generalized version of the natural exponential function $f(x) = e^x$ with base $e \approx 2.718$. Our use of the notation $re^{i\theta}$ does not depend on that interpretation. Instead, $re^{i\theta}$ simply denotes the complex number that corresponds to the point with polar coordinates (r, θ). As is the case with polar coordinates, θ can be given in either radians or degrees. We assume, however, that r and θ are chosen so that r is nonnegative.

The polar–rectangular relationships of Section 7-4 imply the following connections between the rectangular and polar forms of a complex number $z = x + iy = re^{i\theta}$:



$$x = r\cos\theta$$
$$r = \sqrt{x^2 + y^2}$$
$$y = r\sin\theta$$
$$\tan\theta = \frac{y}{x},\ x \neq 0$$

$x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$ and $e^{i\theta} = \cos\theta + i\sin\theta$

If $z = re^{i\theta}$, then the number r is called the **modulus,** or **absolute value,** of z and is denoted by **mod** z or $|z|$. The angle θ (in radians or degrees) is called the **argument** of z and is denoted by **arg** z**.** The argument of a complex number is not unique, but we usually choose the argument θ so that $-\pi < \theta \leq \pi$ (or $-180° < \theta \leq 180°$).

Products and **quotients** of complex numbers are easily calculated from their polar forms:

$$r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)} \qquad \textbf{Product}$$

$$\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)} \qquad \textbf{Quotient}$$

De Moivre's theorem gives a formula for raising any complex number to the power n where n is a natural number: If $z = re^{i\theta}$, then $z^n = r^ne^{i(n\theta)}$.

Let $n > 1$ be an integer. A complex number w is an ***n*th root of *z*** if $w^n = z$. The ***n*th root theorem** gives a formula for all of the nth roots of any nonzero complex number: If $z = re^{i\theta}$ is a nonzero complex number, then z has n distinct nth roots given by

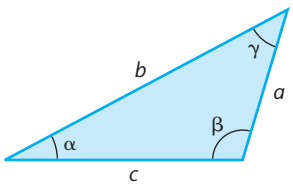
$$r^{1/n}e^{i(\theta/n + k360°/n)} \qquad k = 0, 1, \ldots, n - 1$$

Because all n of the nth roots of z have the same modulus, they all lie on the same circle centered at the origin, and they are equally spaced around that circle.

CHAPTER 7 Review Exercises

Work through all the problems in this chapter review and check answers in the back of the book. Answers to all review problems are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

Problems in this exercise use the following labeling of sides and angles:



In Problems 1–3, determine whether the information in each problem enables you to construct 0, 1, or 2 triangles. Do not solve the triangle.

1. $a = 11$ meters, $b = 3.7$ meters, $\alpha = 67°$

2. $c = 15$ centimeters, $\alpha = 97°$, $\beta = 84°$

3. $a = 18$ feet, $b = 22$ feet, $\alpha = 54°$

4. Referring to the figure at the beginning of the exercise, if $a = 52.6°$, $b = 57.1$ centimeters, and $c = 79.5$ centimeters, which of the two angles, β or γ, can you say for certain is acute and why?

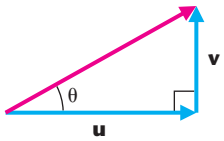
In Problems 5–7, solve each triangle, given the indicated information.

5. $\alpha = 67°$, $\beta = 38°$, and $c = 49$ meters

6. $\alpha = 15°$, $b = 9.1$ feet, and $c = 12$ feet

7. $\gamma = 121°$, $c = 11$ centimeters, and $b = 4.2$ centimeters

8. Given geometric vectors $\mathbf{u}$ and $\mathbf{v}$ as indicated in the figure, find $|\mathbf{u} + \mathbf{v}|$ and θ, given $|\mathbf{u}| = 160$ miles per hour and $|\mathbf{v}| = 55$ miles per hour.



9. Write the algebraic vector $\langle a, b\rangle$ corresponding to the geometric vector $\overrightarrow{AB}$ with endpoints $A = (2, 6)$ and $B = (5, -1)$.

10. Find the magnitude of the vector $\langle -3, -5\rangle$.

11. Sketch a graph of $\theta = \pi/6$ in a polar coordinate system.

12. Sketch a graph of $r = 6$ in a polar coordinate system.

13. Plot in a complex plane: $A = 3 + 5i$, $B = -1 - i$, $C = -3i$.

14. A point in a polar coordinate system has coordinates $(10, -30°)$. Find all other polar coordinates for the point, $-360° \le \theta \le 360°$, and verbally describe how the coordinates are associated with the point.

15. Plot in a complex plane: $A = 5e^{30°i}$, $B = 10e^{(-\pi/2)i}$, $C = 7e^{(3\pi/4)i}$.

16. (A) Change $1 - i\sqrt{3}$ to the polar form $re^{i\theta}$, $r \ge 0$, $-180° < \theta \le 180°$.

(B) Change $4e^{(-30°)i}$ to exact rectangular form.

17. (A) Find $[(-1/2) - (\sqrt{3}/2)i]^3$ using De Moivre's theorem. Write the final answer in exact rectangular form.

(B) Verify the results in part A with a calculator.

18. Find $(2e^{15°i})^4$ using De Moivre's theorem, and write the final answer in exact rectangular form.

19. Referring to the figure at the beginning of the exercise, if $a = 434$ meters, $b = 302$ meters, and $c = 197$ meters, then if the triangle has an obtuse angle which angle must it be and why?

In Problems 20–23, solve each triangle. If a problem does not have a solution, say so. If a triangle has two solutions, say so, and solve the obtuse case.

20. $\beta = 115.4°$, $a = 5.32$ centimeters, $c = 7.05$ centimeters

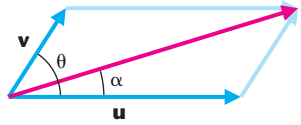
21. $\alpha = 63.2°$, $a = 179$ millimeters, $b = 205$ millimeters

22. $\alpha = 26.4°$, $a = 52.2$ kilometers, $b = 84.6$ kilometers

23. $a = 19.0$ inches, $b = 27.8$ inches, $c = 26.1$ inches

24. If four nonzero force vectors with different magnitudes and directions are acting on an object at rest, what must the sum of all four vectors be for the object to remain at rest?

25. Given geometric vectors $\mathbf{u}$ and $\mathbf{v}$ as indicated in the figure, find $|\mathbf{u} + \mathbf{v}|$ and α, given $|\mathbf{u}| = 75.2$ kilograms, $|\mathbf{v}| = 34.2$ kilograms, and $\theta = 57.2°$.



26. Express each vector in terms of $\mathbf{i}$ and $\mathbf{j}$ unit vectors:

(A) $\mathbf{u} = \langle -3, 9\rangle$ (B) $\mathbf{v} = \langle 0, -2\rangle$

For the indicated vectors in Problems 27 and 28, find

(A) $\mathbf{u} - \mathbf{v}$ (B) $3\mathbf{u} - \mathbf{v} + 2\mathbf{w}$

27. $\mathbf{u} = \langle -2, 3\rangle$, $\mathbf{v} = \langle 2, -4\rangle$, $\mathbf{w} = \langle -3, 0\rangle$

28. $\mathbf{u} = \mathbf{i} - 2\mathbf{j}$, $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j}$, $\mathbf{w} = -\mathbf{j}$

29. Find a unit vector $\mathbf{u}$ with the same direction as $\mathbf{v} = \langle -1, -3\rangle$.

In Problems 30–33, find the area of each triangle (to the same number of significant digits as the side with the least number of significant digits).

30. $a = 66$ feet, $b = 24$ feet, $\gamma = 90°$

31. $a = 34.5$ meters, $b = 29.8$ meters, $\gamma = 15°$

32. $a = 84$ yards, $b = 113$ yards, $c = 38$ yards

33. $\alpha = 47°$, $\beta = 19°$, $c = 56$ inches

In Problems 34–37, use rapid sketching techniques to sketch each graph in a polar coordinate system.

34. $r = 6 + 4\cos\theta$ **35.** $r = 8 + 8\sin\theta$

36. $r = 10\cos 2\theta$ **37.** $r = 8\sin 3\theta$

38. Graph $r = 6\cos\dfrac{\theta}{7}$ for $0 \le \theta \le 7\pi$.

39. Graph $r = 6\cos\dfrac{\theta}{9}$ for $0 \le \theta \le 9\pi$.

40. Graph $r = 8(\sin\theta)^{2n}$, for $n = 1$, 2, and 3. How many leaves do you expect the graph will have for arbitrary n?

41. Graph $r = 3/(1 - e\cos\theta)$ for the following values of e and identify each curve as an ellipse, a parabola, or a hyperbola:

(A) $e = 0.55$ (B) $e = 1$ (C) $e = 1.7$

42. Convert $x^2 + y^2 = 6x$ to polar form.

43. Convert $r = 5 \cos \theta$ to rectangular form.

44. Change the following complex numbers to the polar form $re^{i\theta}$, $r \geq 0$, $-180° < \theta \leq 180°$: $z_1 = -1 + i$, $z_2 = -1 - i\sqrt{3}$, $z_3 = 5$.

45. Change the following complex numbers to exact rectangular form: $z_1 = \sqrt{2}e^{(\pi/4)i}$, $z_2 = 3e^{210°i}$, $z_3 = 2e^{(-2\pi/3)i}$.

46. If $z_1 = 8e^{25°i}$ and $z_2 = 4e^{19°i}$, find

(A) z_1z_2 (B) z_1/z_2

Leave answers in the polar form $re^{i\theta}$.

47. (A) Write $(1 + i\sqrt{3})^4$ in exact rectangular form. Use De Moivre's theorem.

(B) Verify part A by evaluating $(1 + i\sqrt{3})^4$ directly on a calculator.

48. Find all cube roots of i. Write final answers in exact rectangular form, and locate the roots on a circle in the complex plane.

49. Find all cube roots of $-4\sqrt{3} + 4i$ exactly. Leave answers in the polar form $re^{i\theta}$ using degrees.

50. Show that $4e^{15°i}$ is a square root of $8\sqrt{3} + 8i$.

51. Change the rectangular coordinates $(5.17, -2.53)$ to polar coordinates to two decimal places, $r \geq 0$, $-180° < \theta \leq 180°$.

52. Change the polar coordinates $(5.81, -2.72)$ to rectangular coordinates to two decimal places.

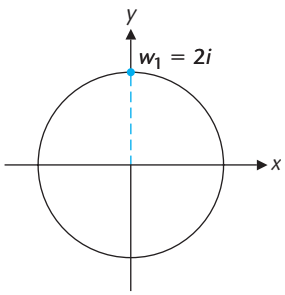
53. Change the complex number $-3.18 + 4.19i$ to the polar form $re^{i\theta}$ to two decimal places, $r \geq 0$, $-180° < \theta \leq 180°$.

54. Change the complex number $7.63e^{(-162.27°)i}$ to rectangular form $a + bi$, where a and b are computed to two decimal places.

55. (A) The cube root of a complex number is shown in the figure. Geometrically locate all other cube roots of the number on the figure, and explain how they were located.

(B) Determine geometrically the other cube roots of the number in exact rectangular form.

(C) Cube each cube root from parts A and B.



56. For an oblique triangle with $\alpha = 23.4°$, $b = 44.6$ millimeters, and a the side opposite angle α, determine a value k so that if $0 < a < k$, there is no solution; if $a = k$, there is one solution; and if $k < a < b$, there are two solutions.

57. Show that for any triangle

$$\frac{a^2 + b^2 + c^2}{2abc} = \frac{\cos \alpha}{a} + \frac{\cos \beta}{b} + \frac{\cos \gamma}{c}$$

58. Let $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$ be vectors and m a scalar; prove

(A) $(\mathbf{u} + \mathbf{v}) = (\mathbf{v} + \mathbf{u})$ (B) $m(\mathbf{u} + \mathbf{v}) = m\mathbf{u} + m\mathbf{v}$

59. Given the polar equation $r = 4 + 4 \cos(\theta/2)$,

(A) Sketch a graph of the equation using rapid graphing techniques.

(B) Verify the graph in part A on a graphing calculator.

60. (A) Graph $r = -8 \sin \theta$ and $r = 8 \cos \theta$, $0 \leq \theta \leq \pi$, in the same viewing window. Use TRACE to determine which intersection point has coordinates that satisfy both equations simultaneously.

(B) Solve the equations simultaneously to verify the results in part A.

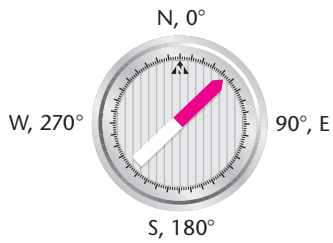
(C) Explain why the pole is not a simultaneous solution, even though the two curves intersect at the pole.

61. Find all solutions, real and imaginary, for $x^8 - 1 = 0$. Write roots in exact rectangular form.

62. Write $P(x) = x^3 - 8i$ as a product of linear factors.

APPLICATIONS

For Problems 63–65, use the navigational compass shown. Assume directions given in terms of north, east, south, and west are exact.



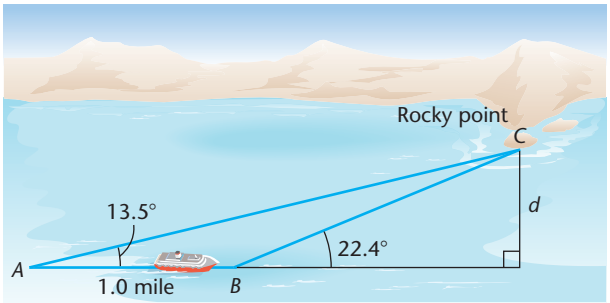
Navigational compass

63. NAVIGATION An airplane flies east at 256 miles per hour, and another airplane flies southeast at 304 miles per hour. After 2 hours, how far apart are the two planes?

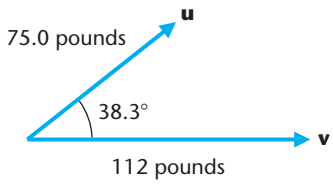
64. **NAVIGATION** An airplane flies with an airspeed of 450 miles per hour and a compass heading of 75°. If the wind is blowing at 65 miles per hour out of the north (from north to south), what is the plane's actual direction and speed relative to the ground? Compute direction to the nearest degree and speed to the nearest mile per hour.

65. **NAVIGATION** An airplane that can cruise at 500 miles per hour in still air is to fly due east. If the wind is blowing from the northeast at 50 miles per hour, what compass heading should the pilot choose? What will be the actual speed of the plane relative to the ground? Compute direction to the nearest degree and speed to the nearest mile per hour.

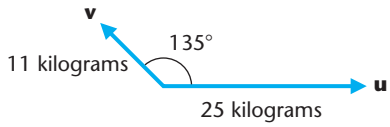
66. **COASTAL NAVIGATION** The owner of a pleasure boat cruising along a coast wants to pass a rocky point at a safe distance (see the figure). Sightings of the rocky point are made at A and at B, 1.0 mile apart. If the boat continues on the same course, how close will it come to the point? That is, find d in the figure to the nearest tenth of a mile.



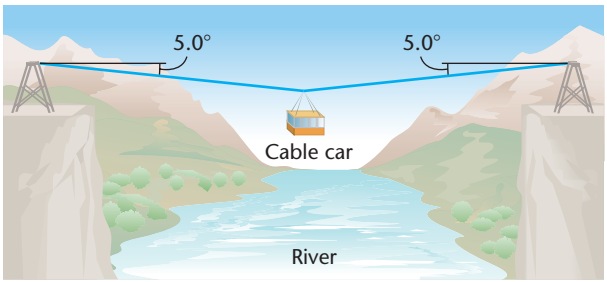
67. **FORCES** Two forces **u** and **v** are acting on an object as indicated in the figure. Find the direction and magnitude of the resultant force **u** + **v** relative to force **v.**



68. **STATIC EQUILIBRIUM** Two forces **u** and **v** are acting on an object as indicated in the figure. What third force **w** must be added to achieve static equilibrium? Give direction relative to **u.**



69. **ENGINEERING** A cable car weighing 1,000 pounds is used to cross a river (see the figure). What is the tension in each half of the cable when the car is located as indicated? Compute the answer to three significant digits.



70. **ASTRONOMY**

(A) The planet Mars travels around the sun in an elliptical orbit given approximately by

$$r = \frac{1.41 \times 10^8}{1 - 0.0934 \cos \theta} \qquad (1)$$

where r is measured in miles and the sun is at the pole. Graph the orbit. Use TRACE to find the distance (to three significant digits) from Mars to the sun at **aphelion** (greatest distance from the sun) and at **perihelion** (shortest distance from the sun).

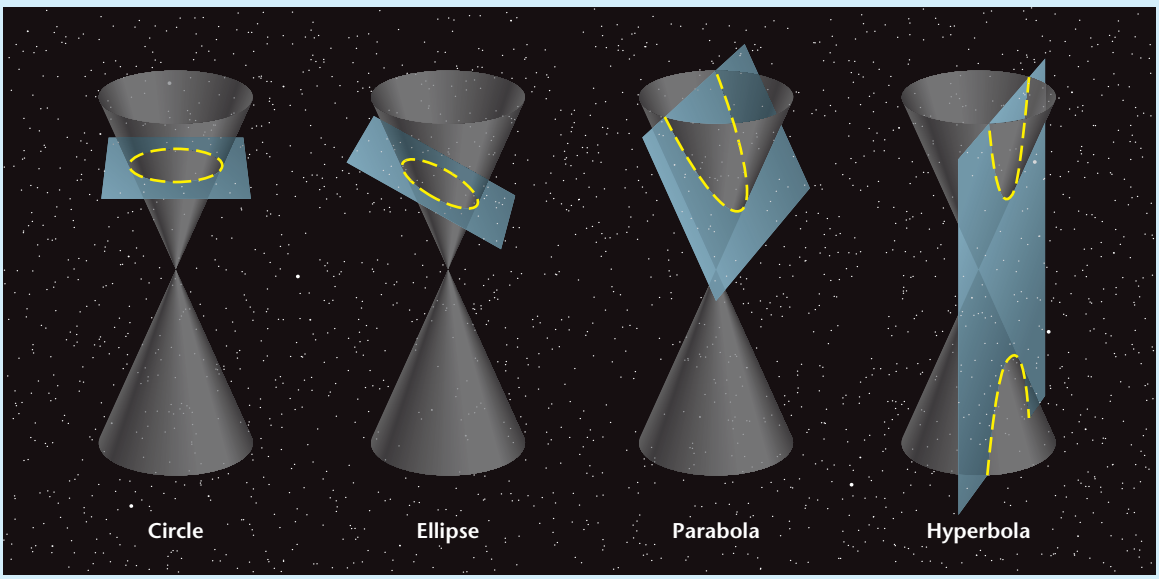
(B) Referring to equation (1), r is maximum when the denominator is minimum, and r is minimum when the denominator is maximum. Use this information to find the distance from Mars to the sun at aphelion and at perihelion.

CHAPTER 7

››› GROUP ACTIVITY Conic Sections and Planetary Orbits

I. Conic Sections in Polar Form

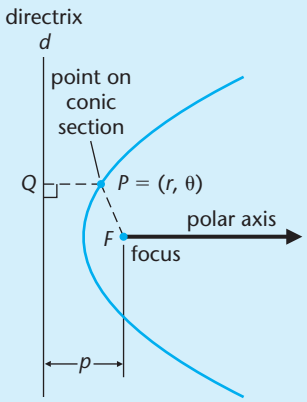
(A) Introduction to Conics. To understand orbits of planets, comets and other celestial bodies, you must know something of the nature and properties of conic sections. (Conic sections are treated in detail in Chapter 11. Here our treatment will be brief and limited to polar representations.) **Conic sections** get their name because the curves are formed by cutting a complete right circular cone of two nappes (the parts of the cone separated by a point) with a plane (Fig. 1). Any plane perpendicular to the axis of the cone cuts a section that is a **circle.** Tilt the plane slightly and the section becomes an **ellipse.** If the plane is parallel to one edge of the cone, it will cut only one nappe and the section will be a **parabola.** Tilt the plane further to the vertical, then it will cut both nappes of the cone and produce a **hyperbola** with two branches. Closed orbits of celestial bodies are ellipses or circles. Open (or escape) orbits of celestial bodies are parabolas or hyperbolas.



› **Figure 1** Conic sections.

(B) Conics and Eccentricity. Another way of defining conic sections is in terms of their eccentricity. Let F be a fixed point, called the **focus,** and let d be a fixed line, called the **directrix** (Fig. 2). For positive values of **eccentricity** e, a conic section can be defined as the set of points $\{P\}$ having the property that the ratio of the distance from P to the focus F to the distance from P to the directrix d is the constant e. As we will see, an ellipse, a parabola, or a hyperbola can be obtained by choosing e appropriately.

(C) Polar Representation of Conics. A unified treatment of conic sections can be obtained by use of the polar coordinate system. Polar equations of conics are used extensively in celestial mechanics to describe and analyze orbits of planets, comets, satellites, and other celestial bodies.



› **Figure 2** Conic sections.

Problem 1: Polar Equation of a Conic. Use the eccentricity definition of a conic section given in part B to show that the polar equation of a conic is given by

$$r = \frac{ep}{1 - e\cos\theta} \tag{1}$$

where p is the distance between the focus F and the directrix d, the pole of the polar axis is at F, and the polar axis is perpendicular to d and is pointing away from d (see Fig. 2).

Problem 2: Graphing Calculator Exploration, $0 < e < 1$. For $0 < e < 1$, use a graphing calculator to systematically explore the nature of the changes in the graph of equation (1) as you change the eccentricity e and the distance p. Summarize the results of holding e fixed and changing p, and the results of holding p fixed and changing e. For $0 < e < 1$, which conic section is produced?

Problem 3: Graphing Calculator Exploration, $e = 1$. For $e = 1$, use a graphing calculator to systematically explore the nature of the changes in the graph of equation (1) as you change the distance p. Summarize the results of holding e to 1 and changing p. For $e = 1$, which conic section is produced?

Problem 4: Graphing Calculator Exploration, $e > 1$. For, $e > 1$, use a graphing calculator to systematically explore the nature of the changes in the graph of equation (1) as you change the eccentricity e and the distance p. Summarize the results of holding e fixed and changing p, and the results of holding p fixed and changing e. For $e > 1$, which conic section is produced?

II. Planetary Orbits

We are now interested in finding polar equations for the orbits of specific planets where the sun is at the pole. Then these equations can be graphed in a graphing calculator and further questions about the orbits can be answered. The material in Table 1, found in the readily available *World Almanac* and rounded to three significant digits, gives us more than enough information to find the polar equation for any planet's orbit.

Table 1 The Planets

Planet	Eccentricity	Maximum distance from Sun (millions of miles)	Minimum distance from Sun (millions of miles)
Mercury	0.206	43.4	28.6
Venus	0.00677	67.7	66.8
Earth	0.0167	94.6	91.4
Mars	0.0934	155	129
Jupiter	0.0485	507	461
Saturn	0.0555	938	838
Uranus	0.0463	1,860	1,670
Neptune	0.00899	2,820	2,760

Problem 5: Polar Equations for the Orbits of Mercury, Earth, and Mars. In all cases, the polar axis intersects the planet's orbit at aphelion (the greatest distance from the sun).

(A) Show that Mercury's orbit is given approximately by

$$r = \frac{3.44 \times 10^7}{1 - 0.206 \cos \theta}$$

(B) Show that Earth's orbit is given approximately by

$$r = \frac{9.30 \times 10^7}{1 - 0.0167 \cos \theta}$$

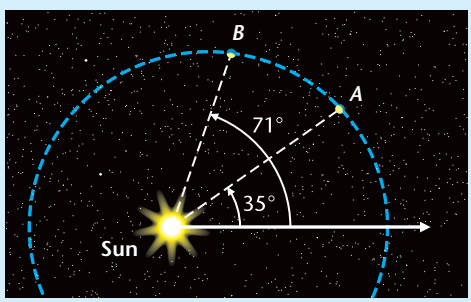
(C) Show that Mars' orbit is given approximately by

$$r = \frac{1.41 \times 10^8}{1 - 0.0934 \cos \theta}$$

Problem 6: Plotting the Orbits for Mercury, Earth, and Mars. Plot all three orbits (Mercury, Earth, and Mars) from the equations in parts A, B, and C in the same viewing window of a graphing calculator. Choose the window dimensions so that Mars' orbit fills up most of the window.

Problem 7: Finding Distances and Angles Related to Orbits. Figure 3 represents a schematic drawing showing Earth at two locations during its orbit. Find the straight-line distance between the position at *A* and the position at *B* to three significant digits. Find the measures of the angles *BAO* and *ABO* in degree measure to one decimal place. The Earth's orbit crosses the polar axis at aphelion (the greatest distance from the sun).

› **Figure 3** Earth's orbit.



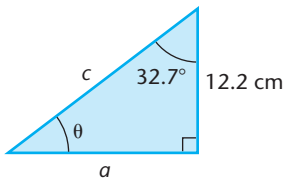
CHAPTERS 5–7

Cumulative Review

Work through all the problems in this cumulative review and check answers in the back of the book. Answers to all review problems, except verifications, are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

1. In a circle of radius 6 meters, find the length of an arc opposite an angle of 0.31 radians.

2. Solve the triangle.



3. In which quadrants is each positive?

(A) $\sin\theta$ (B) $\cos\theta$ (C) $\tan\theta$

4. If $(-3, 4)$ is on the terminal side of an angle θ, find

(A) $\cos\theta$ (B) $\csc\theta$ (C) $\tan\theta$

5. Find the reference angle associated with each angle θ:

(A) $-3\pi/4$ (B) $245°$ (C) $-30°$

6. Indicate the domain, range, and period of each.

(A) $y = \sin x$ (B) $y = \cos x$ (C) $y = \tan x$

7. Sketch a graph of $y = \cos x$, $-\pi/2 \le x \le 5\pi/2$.

8. Sketch a graph of $y = \tan x$, $-\pi/2 < x < 3\pi/2$.

9. Describe the meaning of a central angle in a circle with radian measure 2.

10. Describe the smallest shift of the graph of $y = \cos x$ to produce the graph of $y = \sin x$.

Verify each identity in Problems 11–14.

11. $\cot\theta \sec\theta = \csc\theta$

12. $\sec x - \cos x = \tan x \sin x$

13. $\sin(x - \pi/2) = -\cos x$

14. $\csc 2x = \frac{1}{2}\csc x \sec x$

15. Use a graphing calculator to test whether each of the following is an identity. If an equation appears to be an identity, verify it. If the equation does not appear to be an identity, find a value of x for which both sides are defined but are not equal.

(A) $\dfrac{\sin^2 x}{\cos x} + \cos x = \csc x$

(B) $\dfrac{\sin^2 x}{\cos x} + \cos x = \sec x$

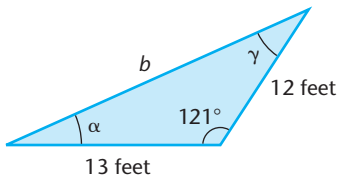
16. If in a triangle, $a = 32.5$ feet, $c = 77.2$ feet, and $\beta = 61.3°$, without solving the triangle or drawing any pictures, which of the two angles, α or γ, can you say for certain is acute and why?

Solve Problems 17 and 18 to four decimal places.

17. $\sin x = 0.3188$, $0 \le x \le 2\pi$

18. $\tan\theta = -4.076$, $-90° < \theta < 90°$

19. Solve the triangle.



20. Write the algebraic vector $\langle a, b\rangle$ corresponding to the geometric vector $\overrightarrow{AB}$ with endpoints $A = (-3, 2)$ and $B = (3, -1)$.

21. A point in a polar coordinate system has coordinates $(-5, 150°)$. Find all other polar coordinates for the point, $-360° \le \theta \le 360°$, and verbally describe how the coordinates are associated with the point.

22. Sketch a graph of $r = 6\cos\theta$ in a polar coordinate system.

23. Plot in a complex plane: $A = -3 + 4i$ and $B = 4e^{60°i}$.

24. Find $(2e^{10°i})^3$. Write the final answer in exact rectangular form.

25. Which of the following angles are coterminal with $150°$: $30°$, $-7\pi/6$, $870°$?

26. Change 1.31 radians to decimal degrees to two decimal places.

27. Which of the following have the same value as $\cos 8$?

(A) $\cos(8 \text{ rad})$ (B) $\cos 8°$ (C) $\cos(8 - 4\pi)$

Evaluate Problems 28–37 exactly without a calculator. If the function is not defined at the value, say so.

28. $\sin(-5\pi/6)$ **29.** $\tan(\pi/2)$ **30.** $\cot(7\pi/4)$

31. $\sec 330°$ **32.** $\cos^{-1}(-1)$ **33.** $\sin^{-1} 1.5$

34. $\arccos(-\frac{1}{2})$ **35.** $\sin(\sin^{-1} 0.55)$

36. $\cos[\sin^{-1}(-\frac{4}{5})]$ **37.** $\cos[\tan^{-1}(-2)]$

38. Evaluate to four significant digits using a calculator. If a function is not defined, say so.

(A) $\tan 84°12'55''$ (B) $\sec(-1.8409)$

(C) $\tan^{-1}(-84.32)$ (D) $\cos^{-1}(\tan 2.314)$

39. Sketch a graph of $y = 2 - 2\cos(\pi x/2)$, $-1 \le x \le 5$.

40. (A) Find the exact degree measure of $\theta = \cos^{-1}(-\sqrt{3}/2)$ without a calculator.

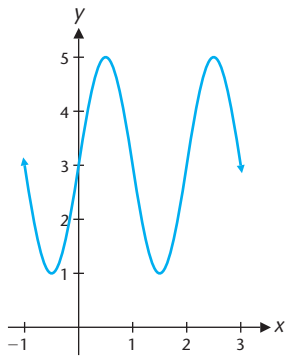
(B) Find the degree measure of $\theta = \sin^{-1}(-0.338)$ to three decimal places using a calculator.

41. Evaluate $\sin^{-1}(\sin 3)$ with a calculator set in radian mode, and explain why this does or does not illustrate a sine–inverse sine identity.

42. A circular point $P = (a, b)$ moves counterclockwise around the circumference of a unit circle starting at (1, 0) and stops after covering a distance of 11.205 units. Explain how you would find the coordinates of point P at its final position and how you would determine which quadrant P is in. Find the coordinates of P to three decimal places and the quadrant for the final position of P.

43. Explain the difference in solving the equation $\tan x = -24.5$ and evaluating $\tan^{-1}(-24.5)$.

44. Find an equation of the form $y = k + a\sin Bx$ that produces the graph shown.



45. Sketch a graph of $y = 3\sin(2x - \pi)$, $-\pi \le x \le 2\pi$. Indicate amplitude A, period P, and phase shift $P.S.$

46. Sketch a graph of $y = 2\tan(\pi x/2 - \pi/2)$, $0 < x < 4$. Indicate the period P and phase shift $P.S.$

47. Sketch a graph of $y = \sin x$ and $y = \csc x$ in the same coordinate system.

48. Describe the smallest left shift and/or reflection that transforms the graph of $y = \cot x$ into the graph of $y = \tan x$.

49. Graph $y = 1/(\cot^2 x + 1)$ in a graphing calculator that displays at least two full periods of the graph. Find an equation of the form $y = k + A\sin Bx$ or $y = k + A\cos Bx$ that has the same graph. Graph both equations in the same viewing window and use TRACE to verify that both graphs are the same.

50. Graph $y = (2 - 2\sin^2 x)/(\sin 2x)$ in a graphing calculator that displays at least two full periods of the graph. Find an equation of the form $y = A\tan Bx$ or $y = A\cot Bx$ that has the same graph. Graph both equations in the same viewing window and use TRACE to verify that both graphs are the same.

51. Given the equation $\sin 2x = 2\sin x$,

(A) Are $x = 0$ and $x = \pi$ solutions?

(B) Is the equation an identity or a conditional equation? Explain.

Verify each identity in Problems 52–57.

52. $\dfrac{\sin u}{1 + \cos u} + \cot u = \csc u$

53. $\sec x + \tan x = \dfrac{\cos x}{1 - \sin x}$

54. $\tan\dfrac{x}{2} = \csc x - \cot x$

55. $\csc^2\dfrac{x}{2} = 2\csc x(\csc x + \cot x)$

56. $\dfrac{2}{1 + \cos 2x} = \sec^2 x$

57. $\dfrac{\cos x + \cos y}{\sin x - \sin y} = \cot\dfrac{x - y}{2}$

[*Hint:* Use sum–product identities.]

58. Use a graphing calculator to test whether each of the following is an identity. If an equation appears to be an identity, verify it. If the equation does not appear to be an identity, find a value of x for which both sides are defined but are not equal.

(A) $\dfrac{\tan x}{2\tan x - \sin x} = \dfrac{1}{2 + \sin x}$

(B) $\dfrac{\tan x}{2\tan x - \sin x} = \dfrac{1}{2 - \cos x}$

59. Find $\cos(x - y)$ exactly without a calculator given $\sin x = (-2/\sqrt{5})$, $\cos y = (-2/\sqrt{5})$, x a quadrant IV angle, and y a quadrant III angle.

60. Compute the exact value of $\sin 2x$ and $\cos(x/2)$ without a calculator, given $\sin x = \frac{3}{5}$, $\pi/2 \le x \le \pi$.

In Problems 61 and 62, find the area of each triangle (to the same number of significant digits as the side with the least number of significant digits).

61. $a = 46$ feet, $b = 78$ feet, $\gamma = 56°$

62. $a = 4.92$ meters, $b = 7.33$ meters, $c = 5.76$ meters

Solve Problems 63 and 64 exactly without a calculator, θ in degrees and x real.

63. $2\sin^2\theta + \sin\theta = 1$, $0 \le \theta < 360°$

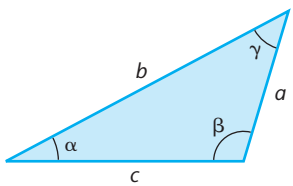
64. $\sin 2x = \sin x$, all real solutions

65. (A) Solve $\cot x = -2\cos x$ exactly, $0 \le x \le 2\pi$.

(B) Solve $\cot x = -2\cos x$ to three decimal places using a graphing calculator, $0 \le x \le 2\pi$.

66. Solve $2\cos x = x - \cos 2x$ to three decimal places for all real solutions using a graphing calculator.

In Problems 67–69, solve each triangle labeled as in the figure. If a problem does not have a solution, say so. If a triangle has two solutions, solve the obtuse case.



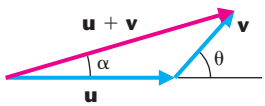
67. $a = 21.3$ meters, $b = 37.4$ meters, $c = 48.2$ meters

68. $\alpha = 125.4°$, $b = 25.4$ millimeters, $a = 20.3$ millimeters

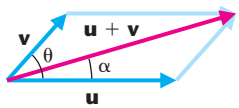
69. $\alpha = 52.9°$, $b = 37.1$ inches, $a = 34.4$ inches

70. Assume in a triangle that γ is acute, $a = 92.5$ centimeters, and $b = 43.4$ centimeters. Which of the angles, α or β, can you say for certain is acute and why?

71. Given vectors as indicated in the figures, find $|\mathbf{u} + \mathbf{v}|$ and α, given $|\mathbf{u}| = 25.3$ pounds, $|\mathbf{v}| = 13.4$ pounds, and $\theta = 48.3°$.



Tail-to-tip rule
(a)



Parallelogram rule
(b)

72. Find $2\mathbf{u} - \mathbf{v} + 3\mathbf{w}$ for,

(A) $\mathbf{u} = \langle -1, 2\rangle$, $\mathbf{v} = \langle 0, -2\rangle$, $\mathbf{w} = \langle 1, -1\rangle$

(B) $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$, $\mathbf{w} = 2\mathbf{j}$

73. Convert to polar form: $x^2 + y^2 = 8y$.

74. Convert $r = -4\cos\theta$ to rectangular form.

Use rapid sketching techniques to graph Problems 75 and 76 in a polar coordinate system.

75. $r = 4 + 4\cos\theta$ **76.** $r = 6\sin 3\theta$

77. Graph $r = 5(\cos 2\theta)^{2n}$, for $n = 1, 2$, and 3. How many leaves do you expect the graph will have for arbitrary n?

78. Graph $r = e^{(\cos\theta)} - 2\cos(4\theta)$ using a squared window and 0.05 for a step size for θ. The resulting curve is often referred to as *a butterfly curve.*

79. Change the rectangular coordinates $(-2.78, -3.19)$ to polar coordinates to two decimal places, $r \ge 0$, $-180° < \theta \le 180°$.

80. Change the polar coordinates $(6.22, -4.08)$ to rectangular coordinates to two decimal places.

81. Change $2e^{(-\pi/6)i}$ to exact rectangular form.

82. Change $z = -1 + i\sqrt{3}$ to the polar form $re^{i\theta}$, θ in degrees.

83. Compute $(1 - i\sqrt{3})^6$ using De Moivre's theorem and write the final answer in $a + bi$ form.

84. Find all cube roots of $-i$ exactly. Write final answers in the form $a + bi$, and locate the roots on a circle in the complex plane.

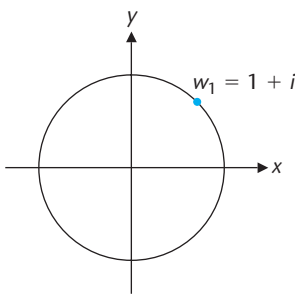
85. Change the complex number $-4.88 - 3.17i$ to the polar form $re^{i\theta}$ to two decimal places, $r \ge 0$, $-180° < \theta \le 180°$.

86. Change the complex number $6.97e^{163.87°i}$ to rectangular form $a + bi$, where a and b are computed to two decimal places.

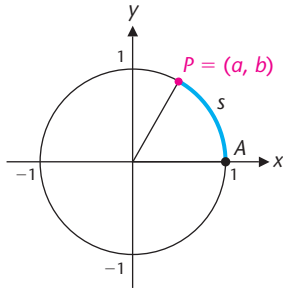
87. (A) The fourth root of a complex number is shown in the figure. Geometrically locate all other fourth roots of the number on the figure, and explain how they were located.

(B) Determine geometrically the other fourth roots of the number in exact rectangular form.

(C) Raise each fourth root from parts A and B to the fourth power.

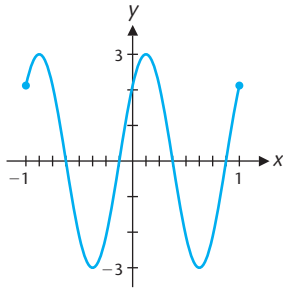


88. If, in the figure, the coordinates of A are (1, 0) and arc length s is 1.2 units, find the coordinates of P to three significant digits.



89. Sketch a graph of $y = 1 + \sec x$, $-3\pi/2 < x < 3\pi/2$.

90. The accompanying graph is a graph of an equation of the form $y = A\cos(Bx + C)$, $0 < -B/C < 1$. Find the equation by finding A, B, and C exactly. What are the period, amplitude, and phase shift?



91. Graph $1.6\sin 2x - 1.2\cos 2x$ in a graphing calculator. (Select the dimensions of a viewing window so that at least two periods are visible.) Find an equation of the form $y = A\sin(Bx + C)$ that has the same graph as the given equation. Find A and B exactly and C to three decimal places. Use the x intercept closest to the origin as the phase shift. To check your results graph both equations in the same viewing window and use TRACE while shifting back and forth between the two graphs.

92. Write $\csc(\cos^{-1} x)$ as an algebraic expression in x free of trigonometric or inverse trigonometric functions.

Solve Problems 93 and 94 without a calculator.

93. $\sin[2\cot^{-1}(\frac{3}{4})] = ?$

94. Given $\sec x = -5/3$, $\pi/2 \le x \le \pi$, find

(A) $\sin(x/2)$ (B) $\cos 2x$

95. (A) Solve $2\sin^2 x = 3\cos x$ exactly for all real solutions, $0 \le x \le 2\pi$.

(B) Solve $2\sin^2 x = 3\cos x$ to four decimal places using a graphing calculator, $0 \le x \le 2\pi$.

96. (A) Use rapid sketching techniques to sketch a graph of the polar equation $r^2 = 36\cos 2\theta$.

(B) Verify the graph in part A using a graphing calculator.

97. (A) Graph $r_1 = 2 + 2\cos\theta$ and $r_2 = 6\cos\theta$ in the same viewing window, $0 \le \theta \le 2\pi$.

(B) Use TRACE to determine how many times the graph of r_2 crosses the graph of r_1 as θ goes from 0 to 2π.

(C) Solve the two equations simultaneously to find the exact solutions for $0 \le \theta \le 2\pi$.

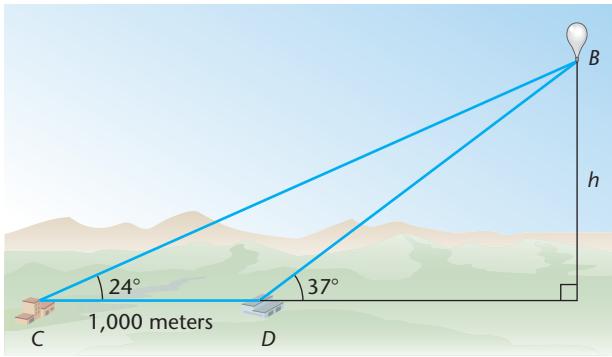
(D) Explain why the number of solutions found in part C does not agree with the number of times r_1 crosses r_2, $0 \le \theta \le 2\pi$.

98. Write $P(x) = x^3 + i$ as a product of linear factors.

APPLICATIONS

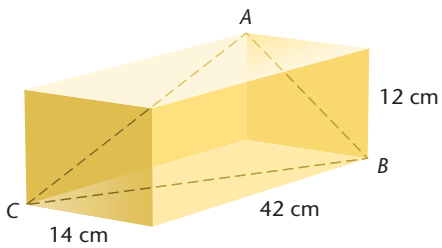
99. **ASTRONOMY** A line from the sun to the Earth sweeps out an angle of how many radians in 5 days?

100. **METEOROLOGY** A weather balloon is released and rises vertically. Two weather stations C and D in the same vertical plane as the balloon and 1,000 meters apart sight the balloon at the same time and record the information given in the figure. At the time of sighting, how high was the balloon to the nearest meter?



101. **GEOMETRY** Find the length to two decimal places of one side of a regular pentagon inscribed in a circle with radius 5 inches.

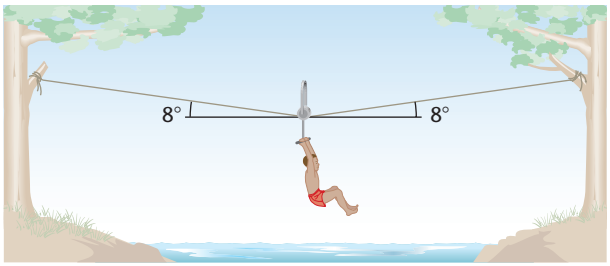
102. **GEOMETRY** Find $\angle ABC$ to the nearest degree in the rectangular solid shown in the figure.



103. **ELECTRICAL CIRCUIT** The current I in an alternating electrical circuit has an amplitude of 50 amperes and a period of $\frac{1}{110}$ second. If $I = 50$ amperes when $t = 0$, find an equation of the form $I = A \cos Bt$ that gives the current at time $t \geq 0$.

104. **NAVIGATION** An airplane flies with an airspeed of 260 miles per hour and a compass heading of 110°. If a 36 mile per hour wind is blowing out of the north, what is the plane's actual heading and ground speed? Compute direction to the nearest degree and ground speed to the nearest mile per hour.

105. **ENGINEERING** A 65-pound child glides across a small river on a homemade cable trolley (see the figure). What is the tension on each half of the support cable when the child is in the center? Compute answer to nearest pound.



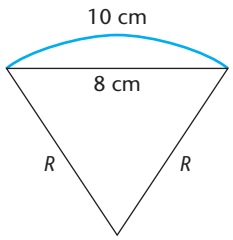
106. **GEOMETRY** A circular arc of 10 centimeters has a chord of 8 centimeters, as shown in the figure.

(A) Explain how the radius is given by the equation

$$\sin \frac{5}{R} = \frac{4}{R}$$

(B) What difficulties do you encounter in trying to solve the equation in part A exactly using algebraic and trigonometric methods?

(C) Show on a graphing calculator how to approximate the radius of the circle R, and find R to three decimal places.



107. **MODELING TEMPERATURE VARIATION** The 30-year average monthly temperature, in degrees Fahrenheit, for each month of the year for Washington, D.C., is given in Table 1 (from the *World Almanac*).

Table 1 Monthly Average Temperatures, Washington, D.C.

x (months)	y (temperature)
1	31
2	34
3	43
4	53
5	62
6	71
7	76
8	74
9	67
10	55
11	45
12	35

(A) Using 1 month as the basic unit of time, enter the data for a 2-year period in your graphing calculator and produce a scatter plot in the viewing window. Choose $25 \leq y \leq 80$ for the viewing window.

(B) It appears that a sine curve of the form

$$y = k + A \sin (Bx + C)$$

will closely model these data. The constants k, A, and B are easily determined from Table 1. To estimate C, visually estimate to one decimal place the smallest positive phase shift from the plot in part A. After determining A, B, k, and C, write the resulting equation. (Your value of C may differ slightly from the answer in the book.)

(C) Plot the results of parts A and B in the same viewing window. (An improved fit may result by adjusting your value of C a little.)

(D) If your graphing calculator has a sinusoidal regression feature, check your results from parts B and C by finding and plotting the regression equation.

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Modeling with Systems of Equations and Inequalities

MANY problems in business and the sciences can be modeled by a set of two or more equations (or inequalities), referred to as a system. In Section 2-2, we used two equations to construct supply and demand models for market equilibrium analysis. And in Section 2-7, we used two equations to construct revenue and cost models for break-even analysis. Each is an example of a system of two equations. A solution of a system must satisfy each of the equations (or inequalities). As the chapter contents indicate, we will consider three different types of systems, and we will introduce methods for finding all solutions of each type. Many applications of systems will be discussed, ending with the relatively new and very useful subject of linear programming.

CHAPTER

8

OUTLINE

8-1 Systems of Linear Equations in Two Variables

8-2 Systems of Linear Equations in Three Variables

8-3 Systems of Linear Inequalities

8-4 Linear Programming

Chapter 8 Review

Chapter 8 Group Activity: Heat Conduction

8-1 Systems of Linear Equations in Two Variables

- Systems of Equations
- Solving by Graphing
- Solving by Substitution
- Solving by Elimination by Addition
- Applications

In this section, we discuss both graphical and algebraic methods for solving systems of linear equations in two variables. Then we use systems of this type to construct and solve mathematical models for several applications.

Systems of Equations

To establish basic concepts, consider the following example. At a computer fair, student tickets cost \$2 and general admission tickets cost \$3. If a total of seven tickets are purchased for a total cost of \$18, how many of each type were purchased?

Let

$$x = \text{Number of student tickets}$$
$$y = \text{Number of general admission tickets}$$

Then

$$\begin{aligned} x + y &= 7 && \text{Total number of tickets purchased} \\ 2x + 3y &= 18 && \text{Total purchase cost} \end{aligned}$$

We now have a system of two linear equations in two variables. Thus, we can solve this problem by finding all pairs of numbers x and y that satisfy both equations.

In general, we are interested in solving linear systems of the type

$$\begin{aligned} ax + by &= h \\ cx + dy &= k \end{aligned} \qquad \text{System of two linear equations in two variables}$$

where x and y are variables, a, b, c, and d are real numbers called the **coefficients** of x and y, and h and k are real numbers called the **constant terms** in the equations. A pair of numbers $x = x_0$ and $y = y_0$ is a **solution** of this system if each equation is satisfied by the pair. The set of all such pairs of numbers is called the **solution set** for the system. To **solve** a system is to find its solution set.

Solving by Graphing

Recall that the graph of a linear equation is the line consisting of all ordered pairs that satisfy the equation. To solve the ticket problem by graphing, we graph both

equations in the same coordinate system. The coordinates of any points that the lines have in common must be solutions to the system, because they must satisfy both equations.

EXAMPLE 1 Solving a System by Graphing

Solve the ticket problem by graphing:

$$\begin{aligned} x + y &= 7 \\ 2x + 3y &= 18 \end{aligned}$$

SOLUTIONS

Hand-Drawn Solution

Find the x and y intercepts for each line.

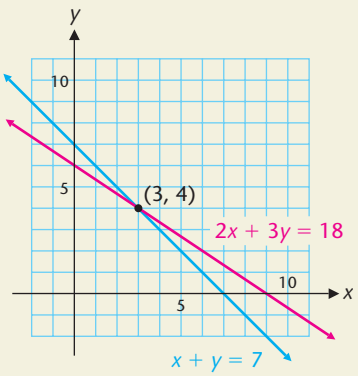
$x + y = 7$

x	y
0	7
7	0

$2x + 3y = 18$

x	y
0	6
9	0

Plot these points, graph the two lines, estimate the intersection point visually (Fig.1), and check the estimate.



› Figure 1

$x = 3$ Student tickets*

$y = 4$ General admission tickets

CHECK

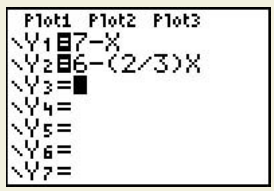
$$x + y = 7 \qquad 2x + 3y = 18$$
$$3 + 4 \stackrel{?}{=} 7 \qquad 2(3) + 3(4) \stackrel{?}{=} 18$$
$$7 \stackrel{\checkmark}{=} 7 \qquad 18 \stackrel{\checkmark}{=} 18$$

Graphing Calculator Solution

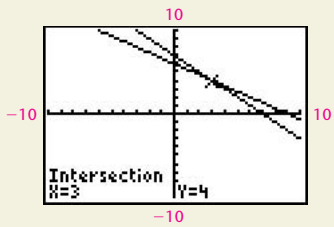
First, solve each equation for y:

$$x + y = 7 \qquad 2x + 3y = 18$$
$$y = 7 - x \qquad 3y = 18 - 2x$$
$$y = 6 - \tfrac{2}{3}x$$

Next, enter these functions in the equation editor of a graphing calculator (Fig. 2) and use the INTERSECT command to find the intersection point (Fig. 3).



› Figure 2



› Figure 3

From Figure 3, we see that the solution is

$x = 3$ Student tickets

$y = 4$ General admission tickets

*When the solution set for a linear system is a single point, we will follow the common practice of writing the solution as (3, 4) or as $x = 3$, $y = 4$, rather than the more formal expression $\{(3, 4)\}$.

MATCHED PROBLEM 1

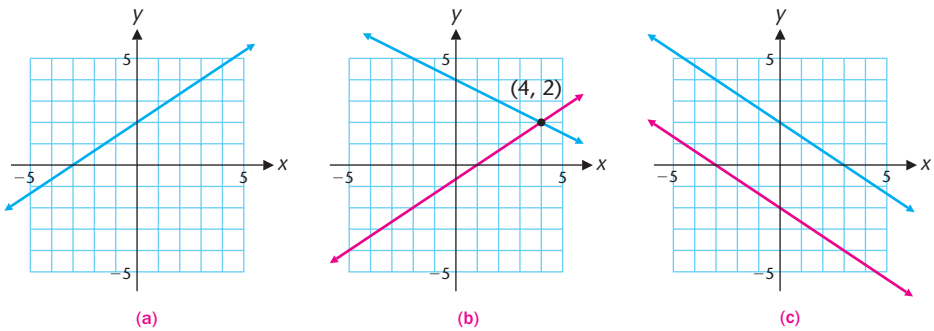
Solve two ways as in Example 1: $\begin{aligned} x - y &= 3 \\ x + 2y &= -3 \end{aligned}$

It is clear that Example 1 has exactly one solution, because the lines have exactly one point of intersection. In general, lines in a rectangular coordinate system are related to each other in one of three ways, as illustrated in Example 2.

EXAMPLE 2 Determining the Nature of Solutions

Match each of the following systems with one of the graphs in Figure 4 and discuss the nature of the solutions:

(A) $\begin{aligned} 2x - 3y &= 2 \\ x + 2y &= 8 \end{aligned}$ (B) $\begin{aligned} 4x + 6y &= 12 \\ 2x + 3y &= -6 \end{aligned}$ (C) $\begin{aligned} 2x - 3y &= -6 \\ -x + \tfrac{3}{2}y &= 3 \end{aligned}$



› Figure 4

SOLUTIONS

(A) Write each equation in slope–intercept form:

$$\begin{aligned} 2x - 3y &= 2 \\ -3y &= -2x + 2 \\ y &= \tfrac{2}{3}x - \tfrac{2}{3} \end{aligned} \qquad \begin{aligned} x + 2y &= 8 \\ 2y &= -x + 8 \\ y &= -\tfrac{1}{2}x + 4 \end{aligned}$$

The graphs of these two lines match graph (b). There is exactly one solution: $x = 4$, $y = 2$.

(B) $\begin{aligned} 4x + 6y &= 12 \\ 6y &= -4x + 12 \\ y &= -\tfrac{2}{3}x + 2 \end{aligned} \qquad \begin{aligned} 2x + 3y &= -6 \\ 3y &= -2x - 6 \\ y &= -\tfrac{2}{3}x - 2 \end{aligned}$

The graphs of these parallel lines match graph (c). There is no solution.

(C) $2x - 3y = -6$ $\quad -x + \frac{3}{2}y = 3$

$-3y = -2x - 6$ $\quad \frac{3}{2}y = x + 3$

$y = \frac{2}{3}x + 2$ $\quad y = \frac{2}{3}x + 2$

The graph of these identical lines match graph (a). There are an infinite number of solutions.

MATCHED PROBLEM 2

Solve each of the following systems by graphing:

(A) $2x + 3y = 12$
$x - 3y = -3$

(B) $x - 3y = -3$
$-2x + 6y = 12$

(C) $2x - 3y = 12$
$-x + \frac{3}{2}y = -6$

We now define some terms that can be used to describe the different types of solutions to systems of equations illustrated in Example 2.

› DEFINITION 1 Systems of Linear Equations: Basic Terms

A system of linear equations is **consistent** if it has one or more solutions and **inconsistent** if no solutions exist. Furthermore, a consistent system is said to be **independent** if it has exactly one solution (often referred to as the **unique solution**) and **dependent** if it has more than one solution.

Referring to the three systems in Example 2, the system in part A is consistent and independent, with the unique solution $x = 4$ and $y = 2$. The system in part B is inconsistent, with no solution. And the system in part C is consistent and dependent, with an infinite number of solutions: all the points on the two coinciding lines.

››› EXPLORE-DISCUSS 1

Can a consistent and dependent system have exactly two solutions? Exactly three solutions? Explain.

By geometrically interpreting a system of two linear equations in two variables, we gain useful information about what to expect in the way of solutions to the system. In general, any two lines in a rectangular coordinate plane must

intersect in exactly one point, be parallel, or coincide (have identical graphs). Thus, the systems in Example 2 illustrate the only three possible types of solutions for systems of two linear equations in two variables. These ideas are summarized in Theorem 1.

› THEOREM 1 Possible Solutions to a Linear System

The linear system

$$ax + by = h$$
$$cx + dy = k$$

must have

1. Exactly one solution — Consistent and independent

or

2. No solution — Inconsistent

or

3. Infinitely many solutions — Consistent and dependent

There are no other possibilities.

One drawback of finding a solution by graphing is the inaccuracy of hand-drawn graphs. A graphing calculator, however, can provide both a useful geometric interpretation and an accurate approximation of the solution to a system of linear equations in two variables.

› Solving by Substitution

There are a number of different algebraic techniques that can also be used to solve systems of linear equations in two variables. One of the simplest is the *substitution method.* To solve a system by **substitution,** we first choose one of the two equations in a system and solve for one variable in terms of the other. (We make a choice that avoids fractions, if possible.) Then we substitute the result in the other equation and solve the resulting linear equation in one variable. Finally, we substitute this result back into the expression obtained in the first step to find the second variable. We return to the ticket problem stated on p. 714 to illustrate this process.

EXAMPLE 3 Solving a System by Substitution

Use substitution to solve the ticket problem:

$$\begin{aligned} x + y &= 7 \\ 2x + 3y &= 18 \end{aligned}$$

SOLUTION

Solve either equation for one variable and substitute into the remaining equation. We choose to solve the first equation for y in terms of x:

$$\begin{aligned} x + y &= 7 && \text{Solve the first equation for } y \text{ in terms of } x. \\ y &= 7 - x && \text{Substitute into the second equation.} \\ 2x + 3y &= 18 \\ 2x + 3(7 - x) &= 18 && \text{Distribute 3.} \\ 2x + 21 - 3x &= 18 && \text{Collect } x \text{ terms on the left and constant terms on the right.} \\ -x &= -3 && \text{Multiply both sides by } -1. \\ x &= 3 \end{aligned}$$

Now, replace x with 3 in $y = 7 - x$ (this is called **back-substitution**):

$$\begin{aligned} y &= 7 - x \\ y &= 7 - 3 \\ y &= 4 \end{aligned}$$

Thus, the solution is three student tickets and four general admission tickets.

CHECK

$$\begin{aligned} x + y &= 7 & \qquad 2x + 3y &= 18 \\ 3 + 4 &\stackrel{?}{=} 7 & 2(3) + 3(4) &\stackrel{?}{=} 18 \\ 7 &\stackrel{\checkmark}{=} 7 & 18 &\stackrel{\checkmark}{=} 18 \end{aligned}$$

MATCHED PROBLEM 3

Solve by substitution and check:

$$\begin{aligned} x - y &= 3 \\ x + 2y &= -3 \end{aligned}$$

EXAMPLE 4 Solving a System by Substitution and by Graphing

Solve by substitution and by graphing: $2x - 3y = 7$
$3x - y = 7$

SOLUTION

Solution By Substitution

To avoid fractions, we choose to solve the second equation for y:

$3x - y = 7$	Solve for y in terms of x.
$-y = -3x + 7$	
$y = 3x - 7$	Substitute into first equation.
$2x - 3y = 7$	First equation.
$2x - 3(3x - 7) = 7$	Distribute -3.
$2x - 9x + 21 = 7$	Collect x terms on the left and constant terms on the right.
$-7x = -14$	Divide both sides by -7.
$x = 2$	Back substitute $x = 2$ in $y = 3x - 7$.
$y = 3x - 7$	
$y = 3(2) - 7$	
$y = -1$	

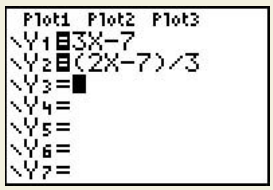
Thus, the solution is $x = 2$ and $y = -1$.

CHECK

$$2x - 3y = 7 \qquad 3x - y = 7$$
$$2(2) - 3(-1) \stackrel{?}{=} 7 \qquad 3(2) - (-1) \stackrel{?}{=} 7$$
$$7 \stackrel{\checkmark}{=} 7 \qquad 7 \stackrel{\checkmark}{=} 7$$

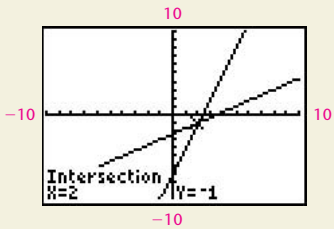
Solution by Graphing

Solve both equations for y and enter them in the equation editor of a graphing calculator (Fig. 5).



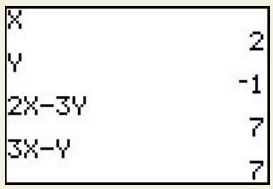
› Figure 5

Use the INTERSECT command on the calculator to find the intersection point (Fig. 6).



› Figure 6

A check of this solution is shown in Figure 7.



› Figure 7

MATCHED PROBLEM 4

Solve by substitution and by graphing: $3x - 4y = 18$
$2x + y = 1$

REMARK: In the solution to Example 4, you might wonder why we checked a solution produced by a graphing calculator. After all, we don't expect a graphing calculator to make an error. But the equations in the original system and the equations entered in Figure 5 are not identical. We might have made an error when solving the original equations for y. The check in Figure 7 eliminates this possibility.

››› EXPLORE-DISCUSS 2

Use substitution to solve each of the following systems. Discuss the nature of the solution sets you obtain.

$$\begin{aligned} x + 3y &= 4 \\ 2x + 6y &= 7 \end{aligned} \qquad \begin{aligned} x + 3y &= 4 \\ 2x + 6y &= 8 \end{aligned}$$

› Solving by Elimination by Addition

Now we turn to **elimination by addition.** This is probably the most important method of solution, since it is readily generalized to larger systems. The method involves the replacement of systems of equations with simpler *equivalent systems,* by performing appropriate operations, until we ontain a system with an obvious solution. **Equivalent systems** of equations are, as you would expect, systems that have exactly the same solution set. Theorem 2 lists operations that produce equivalent systems.

› THEOREM 2 Elementary Equation Operations Producing Equivalent Systems

A system of linear equations is transformed into an equivalent system if:

1. Two equations are interchanged.
2. An equation is multiplied by a nonzero constant.
3. A constant multiple of another equation is added to a given equation.

››› EXPLORE-DISCUSS 3

(A) Solve the following linear system by graphing.

$$\begin{aligned} x - y &= 1 \\ 3x + 4y &= 24 \end{aligned}$$

(B) Verify that multiplying the first equation by a nonzero constant k and adding the result to the second equation produces a new equation of the form:

$$(k + 3)x + (4 - k)y = k + 24$$

(C) Add the graph of the new equation for $k = -2$ to your graph in part A. Repeat for $k = -1$, $k = 1$, and $k = 2$. What do the graphs of all these lines have in common?

(D) Graph the original second equation ($3x + 4y = 4$) and the new equation with $k = -3$ in the same coordinate system. Find the solution of this system.

(E) Graph the original second equation ($3x + 4y = 4$) and the new equation with $k = 4$ in the same coordinate system. Find the solution of this system.

(F) Describe how you found the solution to the system in part D and to the system in part E. Discuss how the process of adding a nonzero constant times one equation to another equation can be used to solve systems. Why must the constant be nonzero?

Any one of the three operations in Theorem 2 can be used to produce an equivalent system, but operations 2 and 3 will be of most use to us now. Operation 1 becomes more important later in the book. The use of Theorem 2 is best illustrated by examples.

EXAMPLE 5 Solving a System Using Elimination by Addition

Solve using elimination by addition:

$$\begin{aligned} 3x - 2y &= 8 \\ 2x + 5y &= -1 \end{aligned}$$

SOLUTION

We use Theorem 2 to eliminate one of the variables and thus obtain a system with an obvious solution.

$$\begin{aligned} 3x - 2y &= 8 \\ 2x + 5y &= -1 \end{aligned}$$

If we multiply the top equation by 5, the bottom by 2, and then add, we can eliminate y.

$$\begin{array}{rr} 15x - 10y = & 40 \\ 4x + 10y = & -2 \\ \hline 19x = & 38 \end{array}$$ Divide both sides by 19.

$$x = 2$$

The equation $x = 2$ paired with either of the two original equations produces an equivalent system. Thus, we can back substitute $x = 2$ into either of the two original equations to solve for y. We choose the second equation.

$$\begin{aligned} 2(2) + 5y &= -1 \\ 5y &= -5 \\ y &= -1 \end{aligned}$$

SOLUTION

$$x = 2,\ y = -1, \text{ or } (2, -1).$$

CHECK

$$\begin{aligned} 3x - 2y &= 8 \\ 3(2) - 2(-1) &\stackrel{?}{=} 8 \\ 8 &\checkmark 8 \end{aligned} \qquad \begin{aligned} 2x + 5y &= -1 \\ 2(2) + 5(-1) &\stackrel{?}{=} -1 \\ -1 &\checkmark -1 \end{aligned}$$

MATCHED PROBLEM 5

Solve using elimination by addition: $\begin{aligned} 6x + 3y &= 3 \\ 5x + 4y &= 7 \end{aligned}$

Let's see what happens in the solution process when a system either has no solution or has infinitely many solutions. Consider the solutions to the following system:

$$\begin{aligned} 2x + 6y &= -3 \\ x + 3y &= -2 \end{aligned}$$

Solution by Substitution

Solve the second equation for x and substitute in the first equation.

$$\begin{aligned} x &= 2 - 3y \\ 2(2 - 3y) + 6y &= -3 \\ 4 - 6y + 6y &= -3 \\ 4 &= -3 \end{aligned}$$

Solution by Elimination

Multiply the second equation by -2 and add to the first equation.

$$\begin{array}{rr} 2x + 6y = & -3 \\ -2x - 6y = & -4 \\ \hline 0 = & -7 \end{array}$$

Both methods of solution lead to a contradiction. An assumption that the original system has solutions must be false. Thus, the system has no solution. The graphs of the equations are parallel lines and the system is inconsistent.

Now consider the system

$$\begin{aligned} x - \tfrac{1}{2}y &= 4 \\ -2x + y &= -8 \end{aligned}$$

Solution by Substitution

Solve the first equation for x and substitute in the second equation.

$$\begin{aligned} x &= \tfrac{1}{2}y + 4 \\ -2(\tfrac{1}{2}y + 4) + y &= -8 \\ -y - 8 + y &= -8 \\ -8 &= -8 \end{aligned}$$

Solution by Elimination

Multiply the first equation by 2 and add to the second equation.

$$\begin{aligned} 2x - y &= 8 \\ -2x + y &= -8 \\ \hline 0 &= 0 \end{aligned}$$

This time both solution methods lead to a statement that is always true. This implies that the two original equations are equivalent. That is, their graphs coincide. The system is dependent and has an infinite number of solutions. There are many different ways to represent this infinite solution set. For example,

$$S_1 = \{(x, y) \mid y = 2x - 8, x \text{ any real number}\}$$

and

$$S_2 = \{(x, y) \mid x = \tfrac{1}{2}y + 4, y \text{ any real number}\}$$

both represent the solutions to this system. For reasons that will become apparent later, it is customary to introduce a new variable, called a **parameter,** and express both variables in terms of this new variable. If we let $x = s$ and $y = 2s - 8$ in S_1, we can express the solution set as

$$\{(s, 2s - 8) \mid s \text{ any real number}\}$$

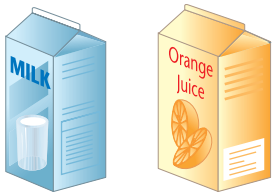
Some **particular solutions** to this system are

$s = -1$	$s = 2$	$s = 5$	$s = 9.4$
$(-1, -10)$	$(2, -4)$	$(5, 2)$	$(9.4, 10.8)$

› Applications

We now have three methods for solving systems of two equations in two variables: graphing, substitution, and elimination. You want to be proficient with all three methods. In the following application examples, we will choose one method to solve the system and use a second method to check our solution.

EXAMPLE 6 Diet



An individual wants to use milk and orange juice to increase the amount of calcium and vitamin A in her daily diet. An ounce of milk contains 38 milligrams of calcium and 56 micrograms* of vitamin A. An ounce of orange juice contains 5 milligrams of calcium and 60 micrograms of vitamin A. How many ounces of milk and orange juice should she drink each day to provide exactly 550 milligrams of calcium and 1,200 micrograms of vitamin A?

SOLUTIONS

First we define the relevant variables:

$$x = \text{Number of ounces of milk}$$
$$y = \text{Number of ounces of orange juice}$$

Next we summarize the given information in a table. It is convenient to organize the tables so that the quantities represented by the variables correspond to columns in the table (rather than to rows), as shown.

	Milk	Orange juice	Total needed
Calcium	38	5	550
Vitamin A	56	60	1,200

Now we use the information in the table to form equations involving x and y:

$$\begin{pmatrix}\text{Calcium in } x \text{ oz}\\ \text{of milk}\end{pmatrix} + \begin{pmatrix}\text{Calcium in } y \text{ oz}\\ \text{of orange juice}\end{pmatrix} = \begin{pmatrix}\text{Total calcium}\\ \text{needed (mg)}\end{pmatrix}$$
$$38x + 5y = 550$$

$$\begin{pmatrix}\text{Vitamin A in } x \text{ oz}\\ \text{of milk}\end{pmatrix} + \begin{pmatrix}\text{Vitamin A in } y \text{ oz}\\ \text{of orange juice}\end{pmatrix} = \begin{pmatrix}\text{Total vitamin A}\\ \text{needed (}\mu\text{g)}\end{pmatrix}$$
$$56x + 60y = 1{,}200$$

Algebraic Solution

We choose elimination to solve this system.

$$\begin{aligned} 38x + 5y &= 550 \\ 56x + 60y &= 1{,}200 \end{aligned}$$

If we multiply the first equation by -12 and then add, we can eliminate y.

$$\begin{aligned} -456x - 60y &= -6{,}600 \\ 56x + 60y &= 1{,}200 \\ \hline -400x &= -5{,}400 \end{aligned}$$

Divide both sides by -400.

$$x = 13.5$$

Graphical Solution

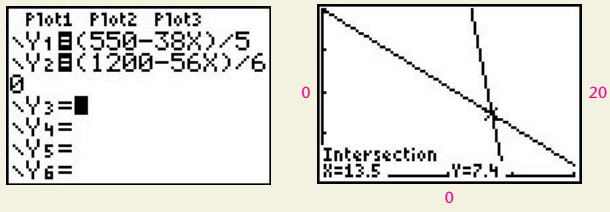
Solve each equation for y and enter in the equation editor of a graphing calculator [Fig. 8(a)]. Note that we did not simplify these equations before entering them. Calculators don't care if expressions are not simplified. Using the INTERSECT command [Fig. 8(b)] produces the same solution as in the algebraic solution.

*A microgram (μg) is one-millionth (10^{-6}) of a gram.

Now we can back substitute $x = 13.5$ in either equation (we choose the first equation) and solve for y.

$$\begin{aligned} 38(13.5) + 5y &= 550 \\ 5y &= 37 \\ y &= 7.4 \end{aligned}$$

Drinking 13.5 ounces of milk and 7.4 ounces of orange juice each day will provide the required amounts of calcium and vitamin A.



(a) (b)

› Figure 8

MATCHED PROBLEM 6

An individual wants to use cottage cheese and yogurt to increase the amount of protein and calcium in his daily diet. An ounce of cottage cheese contains 3 grams of protein and 12 milligrams of calcium. An ounce of yogurt contains 1 gram of protein and 44 milligrams of calcium. How many ounces of cottage cheese and yogurt should he eat each day to provide exactly 57 grams of protein and 840 milligrams of calcium?

EXAMPLE 7 Airspeed

An airplane makes the 2,400-mile trip from Washington, D.C., to San Francisco in 7.5 hours and makes the return trip in 6 hours. Assuming that the plane travels at a constant airspeed and that the wind blows at a constant rate from west to east, find the plane's airspeed and the wind rate.

SOLUTIONS

Let x represent the airspeed of the plane and let y represent the rate at which the wind is blowing (both in miles per hour). The ground speed of the plane is determined by combining these two rates; that is,

$$\begin{aligned} x - y &= \text{Ground speed flying east to west (headwind)} \\ x + y &= \text{Ground speed flying west to east (tailwind)} \end{aligned}$$

Applying the familiar formula $D = RT$ to each leg of the trip leads to the following system of equations:

$$\begin{aligned} 2{,}400 &= 7.5(x - y) && \text{From Washington to San Francisco} \\ 2{,}400 &= 6(x + y) && \text{From San Francisco to Washington} \end{aligned}$$

After simplification, we have

$$\begin{aligned} x - y &= 320 \\ x + y &= 400 \end{aligned}$$

Algebraic Solution

We choose elimination to solve this system. Add the two equations:

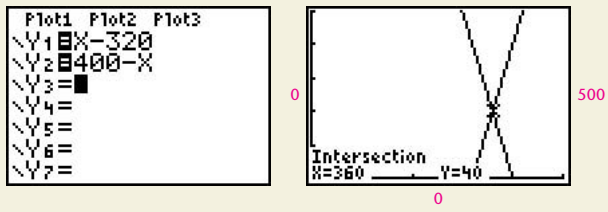
$$\begin{aligned} x - y &= 320 \\ x + y &= 400 \\ \hline 2x &= 720 \qquad \text{Divide both sides by 2.} \\ x &= 360 \text{ mph} \qquad \text{Airspeed} \end{aligned}$$

Back substitute $x = 360$ in the second equation and solve for y:

$$\begin{aligned} x + y &= 400 \\ 360 + y &= 400 \qquad \text{Subtract 360 from both sides.} \\ y &= 40 \text{ mph} \qquad \text{Wind rate} \end{aligned}$$

Graphical Solution

Solve each equation for y and enter in the equation editor of a graphing calculator [Fig. 9(a)]. Using the INTERSECT command [Fig. 9(b)] produces the same solution as in the algebraic solution.



(a) (b)

› Figure 9

MATCHED PROBLEM 7

A boat takes 8 hours to travel 80 miles upstream and 5 hours to return to its starting point. Find the speed of the boat in still water and the speed of the current.

EXAMPLE 8 Supply and Demand

The price–demand and price–supply equations for the sale of cherries each day in a particular city are

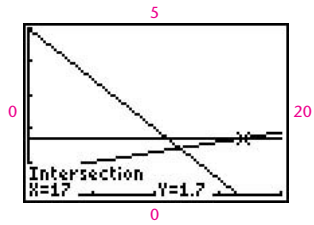
$$\begin{aligned} p &= -0.3q + 5 \qquad \text{Demand equation (consumer)} \\ p &= 0.06q + 0.68 \qquad \text{Supply equation (supplier)} \end{aligned}$$

where q represents the quantity in thousands of pounds and p represents the price per pound in dollars.

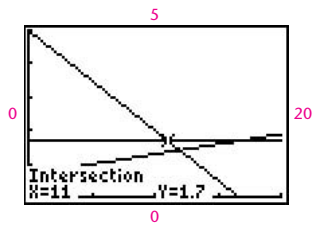
(A) Discuss the relationship between supply and demand when cherries are selling for \$1.70 per pound.

(B) Discuss the relationship between supply and demand when cherries are selling for \$1.10 per pound.

(C) Find the equilibrium quantity and the equilibrium price.



› Figure 10



› Figure 11

SOLUTIONS

(A) Enter $y_1 = -0.3x + 5$, $y_2 = 0.06x + 0.68$, and $y_3 = 1.70$. Use the intersect command to find the supply (Fig. 10) and the demand (Fig. 11) when the price is \$1.70.

From Figures 10 and 11, we find that at a price of \$1.70 per pound, suppliers are willing to supply 17,000 pounds of cherries, but consumers will purchase only 11,000 pounds. The supply exceeds the demand at this price, and the price will come down.

(B) Changing y_3 to $y_3 = 1.10$ and proceeding as before (details omitted), we find that at this price consumers will purchase 13,000 pounds of cherries, but suppliers will supply only 7,000 pounds. Thus, at \$1.10 per pound the demand exceeds the supply and the price will go up.

Algebraic Solution

(C) We choose to use substitution to solve this system.

$$p = -0.3q + 5$$
$$p = 0.06q + 0.68$$

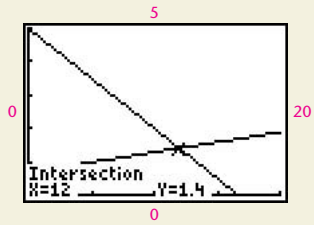
Since each equation has already been solved for p, we substitute $-0.3q + 5$ for p in the second equation and solve for q.

$$-0.3q + 5 = 0.06q + 0.68 \quad \text{Add } 0.3q \text{ and } -0.68 \text{ to both sides.}$$
$$5 - 0.68 = 0.06q + 0.3q \quad \text{Combine like terms.}$$
$$4.32 = 0.36q \quad \text{Divide both sides by } 0.36.$$
$$q = 12 \quad \text{Back substitute } q = 12 \text{ in } p = -0.3q + 5.$$
$$p = -0.3q + 5$$
$$p = -0.3(12) + 5$$
$$p = 1.4$$

The equilibrium quantity is 12,000 pounds and the equilibrium price is \$1.40 per pound. This agrees with our graphical solution.

Graphical Solution

(C) The equilibrium price is the price at which supply will equal demand (see Section 2-2) and the equilibrium quantity is the common value of supply and demand. Using the INTERSECT command (Fig. 12), we see that the equilibrium quantity is 12,000 pounds and the equilibrium price is \$1.40 per pound.



› Figure 12

MATCHED PROBLEM 8

The price–demand and price–supply equations for strawberries each day in a certain city are

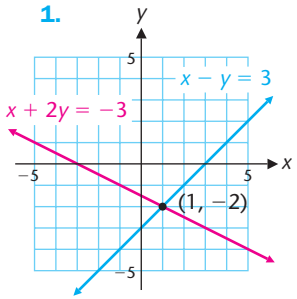
$$p = -0.2q + 4 \quad \text{Demand equation}$$
$$p = 0.04q + 1.84 \quad \text{Supply equation}$$

where q represents the quantity in thousands of pounds and p represents the price per pound in dollars.

Find the equilibrium quantity and the equilibrium price.

ANSWERS TO MATCHED PROBLEMS

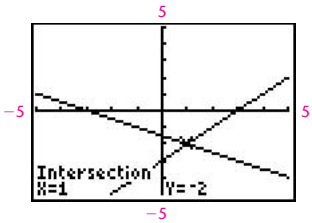
1.



$x = 1, y = -2$

Check:

$$x - y = 3$$
$$1 - (-2) \stackrel{?}{=} 3$$
$$3 \stackrel{\checkmark}{=} 3$$
$$x + 2y = -3$$
$$1 + 2(-2) \stackrel{?}{=} -3$$
$$-3 \stackrel{\checkmark}{=} -3$$



2. (A) (3, 2) or $x = 3$ and $y = 2$ (B) No solutions (C) Infinite number of solutions
3. $x = 1, y = -2$ **4.** $x = 2, y = -3$ **5.** $x = -1, y = 3$
6. 13.9 oz of cottage cheese, 15.3 oz of yogurt
7. Boat: 13 miles per hour; current: 3 miles per hour
8. Equilibrium quantity = 9,000 pounds; equilibrium price = \$2.20 per pound

8-1 Exercises

In Problems 1–6, "system" refers to a system of two linear equations in two variables.

1. Describe how the solution sets for a consistent system, for an inconsistent system, and for a dependent system differ.

2. Can a system have exactly two solutions? Explain.

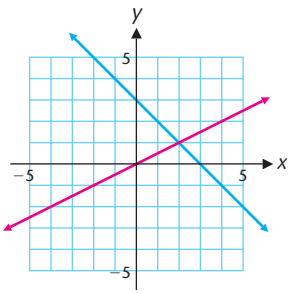
3. What is a parameter?

4. Explain in your own words how to solve a system using a graphing calculator.

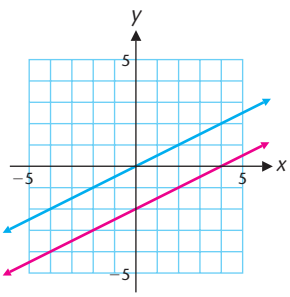
5. Explain in your own words how to solve a system using substitution.

6. Explain in your own words how to solve a system using elimination.

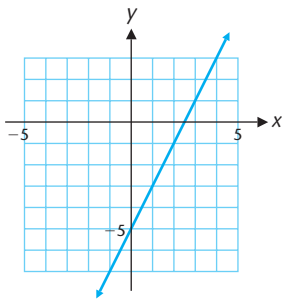
Match each system in Problems 7–10 with one of the following graphs, and use the graph to solve the system.



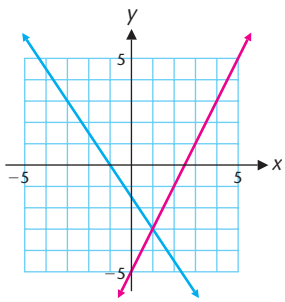
(a)



(b)



(c)



(d)

7. $2x - 4y = 8$
$x - 2y = 0$

8. $x + y = 3$
$x - 2y = 0$

9. $2x - y = 5$
$3x + 2y = -3$

10. $4x - 2y = 10$
$2x - y = 5$

Solve Problems 11–36.

11. $x + y = 7$
$x - y = 3$

12. $x - y = 2$
$x + y = 4$

13. $3x - 2y = 12$
$7x + 2y = 8$

14. $3x - y = 2$
$x + 2y = 10$

15. $3u + 5v = 15$
$6u + 10v = -30$

16. $m + 2n = 4$
$2m + 4n = -8$

17. $y = 2x + 3$
$y = 3x - 5$

18. $y = x + 4$
$y = 5x - 8$

19. $x - y = 4$
$x + 3y = 12$

20. $2x - y = 3$
$x + 2y = 14$

21. $3x - y = 7$
$2x + 3y = 1$

22. $2x + y = 6$
$x - y = -3$

23. $4x + 3y = 26$
$3x - 11y = -7$

24. $9x - 3y = 24$
$11x + 2y = 1$

25. $7m + 12n = -1$
$5m - 3n = 7$

26. $3p + 8q = 4$
$15p + 10q = -10$

27. $y = 0.08x$
$y = 100 + 0.04x$

28. $y = 0.07x$
$y = 80 + 0.05x$

29. $0.2u - 0.5v = 0.07$
$0.8u - 0.3v = 0.79$

30. $0.3s - 0.6t = 0.18$
$0.5s - 0.2t = 0.54$

31. $\frac{2}{5}x + \frac{3}{2}y = 2$
$\frac{7}{3}x - \frac{5}{4}y = -5$

32. $\frac{7}{2}x - \frac{5}{6}y = 10$
$\frac{2}{5}x + \frac{4}{3}y = 6$

33. $2x - 3y = -5$
$3x + 4y = 13$

34. $7x - 3y = 20$
$5x + 2y = 8$

35. $3.5x - 2.4y = 0.1$
$2.6x - 1.7y = -0.2$

36. $5.4x + 4.2y = -12.9$
$3.7x + 6.4y = -4.5$

In Problems 37 and 38, solve each system for p and q in terms of x and y. Explain how you could check your solution and then perform the check.

37. $x = 2 + p - 2q$
$y = 3 - p + 3q$

38. $x = -1 + 2p - q$
$y = 4 - p + q$

Problems 39 and 40 refer to the system

$$ax + by = h$$
$$cx + dy = k$$

where x and y are variables and a, b, c, d, h, and k are real constants.

39. Solve the system for x and y in terms of the constants a, b, c, d, h, and k. Clearly state any assumptions you must make about the constants during the solution process.

40. Discuss the nature of solutions to systems that do not satisfy the assumptions you made in Problem 39.

APPLICATIONS

41. **AIRSPEED** It takes a private airplane 8.75 hours to make the 2,100-mile flight from Atlanta to Los Angeles and 5 hours to make the return trip. Assuming that the wind blows at a constant rate from Los Angeles to Atlanta, find the airspeed of the plane and the wind rate.

42. **AIRSPEED** A plane carries enough fuel for 20 hours of flight at an airspeed of 150 miles per hour. How far can it fly into a 30 mph headwind and still have enough fuel to return to its starting point? (This distance is called the *point of no return.*)

43. **RATE–TIME** A crew of eight can row 20 kilometers per hour in still water. The crew rows upstream and then returns to its starting point in 15 minutes. If the river is flowing at 2 km/h, how far upstream did the crew row?

44. **RATE–TIME** It takes a boat 2 hours to travel 20 miles down a river and 3 hours to return upstream to its starting point. What is the rate of the current in the river?

45. **CHEMISTRY** A chemist has two solutions of hydrochloric acid in stock: a 50% solution and an 80% solution. How much of each should be used to obtain 100 milliliters of a 68% solution?

46. **BUSINESS** A jeweler has two bars of gold alloy in stock, one of 12 carats and the other of 18 carats (24-carat gold is pure gold, 12-carat is $\frac{12}{24}$ pure, 18-carat gold is $\frac{18}{24}$ pure, and so on). How many grams of each alloy must be mixed to obtain 10 grams of 14-carat gold?

47. **BREAK-EVEN ANALYSIS** It costs a small recording company $17,680 to prepare a compact disc. This is a one-time fixed cost that covers recording, package design, and so on. Variable costs, including such things as manufacturing, marketing, and royalties, are $4.60 per CD. If the CD is sold to music shops for $8 each, how many must be sold for the company to break even?

48. **BREAK-EVEN ANALYSIS** A videocassette manufacturer has determined that its weekly cost equation is $C = 3{,}000 + 10x$, where x is the number of cassettes produced and sold each week. If cassettes are sold for $15 each to distributors, how many must be sold each week for the manufacturer to break even?

49. FINANCE Suppose you have $12,000 to invest. If part is invested at 10% and the rest at 15%, how much should be invested at each rate to yield 12% on the total amount invested?

50. FINANCE An investor has $20,000 to invest. If part is invested at 8% and the rest at 12%, how much should be invested at each rate to yield 11% on the total amount invested?

51. PRODUCTION A supplier for the electronics industry manufactures keyboards and screens for graphing calculators at plants in Mexico and Taiwan. The hourly production rates at each plant are given in the table. How many hours should each plant be operated to fill an order for exactly 4,000 keyboards and exactly 4,000 screens?

Plant	Keyboards	Screens
Mexico	40	32
Taiwan	20	32

52. PRODUCTION A company produces Italian sausages and bratwursts at plants in Green Bay and Sheboygan. The hourly production rates at each plant are given in the table. How many hours should each plant be operated to exactly fill an order for 62,250 Italian sausages and 76,500 bratwursts?

Plant	Italian sausage	Bratwurst
Green Bay	800	800
Sheboygan	500	1,000

53. NUTRITION Animals in an experiment are to be kept on a strict diet. Each animal is to receive, among other things, 20 grams of protein and 6 grams of fat. The laboratory technician is able to purchase two food mixes of the following compositions: Mix A has 10% protein and 6% fat, mix B has 20% protein and 2% fat. How many grams of each mix should be used to obtain the right diet for a single animal?

54. NUTRITION A fruit grower can use two types of fertilizer in an orange grove, brand A and brand B. Each bag of brand A contains 8 pounds of nitrogen and 4 pounds of phosphoric acid. Each bag of brand B contains 7 pounds of nitrogen and 7 pounds of phosphoric acid. Tests indicate that the grove needs 720 pounds of nitrogen and 500 pounds of phosphoric acid. How many bags of each brand should be used to provide the required amounts of nitrogen and phosphoric acid?

55. SUPPLY AND DEMAND Suppose the supply and demand equations for printed T-shirts in a resort town for a particular week are

$$p = 0.007q + 3 \quad \text{Supply equation}$$
$$p = -0.018q + 15 \quad \text{Demand equation}$$

where p is the price in dollars and q is the quantity.
(A) Find the supply and the demand (to the nearest unit) if T-shirts are priced at $4 each. Discuss the stability of the T-shirt market at this price level.
(B) Find the supply and the demand (to the nearest unit) if T-shirts are priced at $8 each. Discuss the stability of the T-shirt market at this price level.
(C) Find the equilibrium price and quantity.
(D) Graph the two equations in the same coordinate system and identify the equilibrium point, supply curve, and demand curve.

56. SUPPLY AND DEMAND Suppose the supply and demand equations for printed baseball caps in a resort town for a particular week are

$$p = 0.006q + 2 \quad \text{Supply equation}$$
$$p = -0.014q + 13 \quad \text{Demand equation}$$

where p is the price in dollars and q is the quantity in hundreds.
(A) Find the supply and the demand (to the nearest unit) if baseball caps are priced at $4 each. Discuss the stability of the baseball cap market at this price level.
(B) Find the supply and the demand (to the nearest unit) if baseball caps are priced at $8 each. Discuss the stability of the baseball cap market at this price level.
(C) Find the equilibrium price and quantity.
(D) Graph the two equations in the same coordinate system and identify the equilibrium point, supply curve, and demand curve.

57. SUPPLY AND DEMAND At $0.60 per bushel, the daily supply for wheat is 450 bushels and the daily demand is 645 bushels. When the price is raised to $0.90 per bushel, the daily supply increases to 750 bushels and the daily demand decreases to 495 bushels. Assume that the supply and demand equations are linear.
(A) Find the supply equation.
(B) Find the demand equation.
(C) Find the equilibrium price and quantity.

58. SUPPLY AND DEMAND At $1.40 per bushel, the daily supply for soybeans is 1,075 bushels and the daily demand is 580 bushels. When the price falls to $1.20 per bushel, the daily supply decreases to 575 bushels and the daily demand increases to 980 bushels. Assume that the supply and demand equations are linear.
(A) Find the supply equation.
(B) Find the demand equation.
(C) Find the equilibrium price and quantity.

59. PHYSICS An object dropped off the top of a tall building falls vertically with constant acceleration. If s is the distance of the object above the ground (in feet) t seconds after its release, then s and t are related by an equation of the form

$$s = a + bt^2$$

where a and b are constants. Suppose the object is 180 feet above the ground 1 second after its release and 132 feet above the ground 2 seconds after its release.
(A) Find the constants a and b.
(B) How high is the building?
(C) How long does the object fall?

60. PHYSICS Repeat Problem 59 if the object is 240 feet above the ground after 1 second and 192 feet above the ground after 2 seconds.

61. EARTH SCIENCE An earthquake emits a primary wave and a secondary wave. Near the surface of the Earth the primary wave travels at about 5 miles per second and the secondary wave at about 3 miles per second. From the time lag between the two waves arriving at a given receiving station, it is possible to estimate the distance to the quake. (The *epicenter* can be located by obtaining distance bearings at three or more stations.) Suppose a station measured a time difference of 16 seconds between the arrival of the two waves. How long did each wave travel, and how far was the earthquake from the station?

62. EARTH SCIENCE A ship using sound-sensing devices above and below water recorded a surface explosion 6 seconds sooner by its underwater device than its above-water device. Sound travels in air at about 1,100 feet per second and in seawater at about 5,000 feet per second.
(A) How long did it take each sound wave to reach the ship?
(B) How far was the explosion from the ship?

8-2 Systems of Linear Equations in Three Variables

- Systems in Three Variables
- Using Elimination by Addition
- Recognizing Inconsistent and Dependent Systems
- Modeling with Systems in Three Variables

In this section, the elimination method discussed in the last section is extended to include systems with three variables.

Systems in Three Variables

Recall that any equation that can be written in the form

$$ax + by = c \qquad \text{For example, } 2x - 3y = 7$$

where a, b, and c are constants (not both a and b zero) is called a linear equation in two variables. Similarly, any equation that can be written in the form

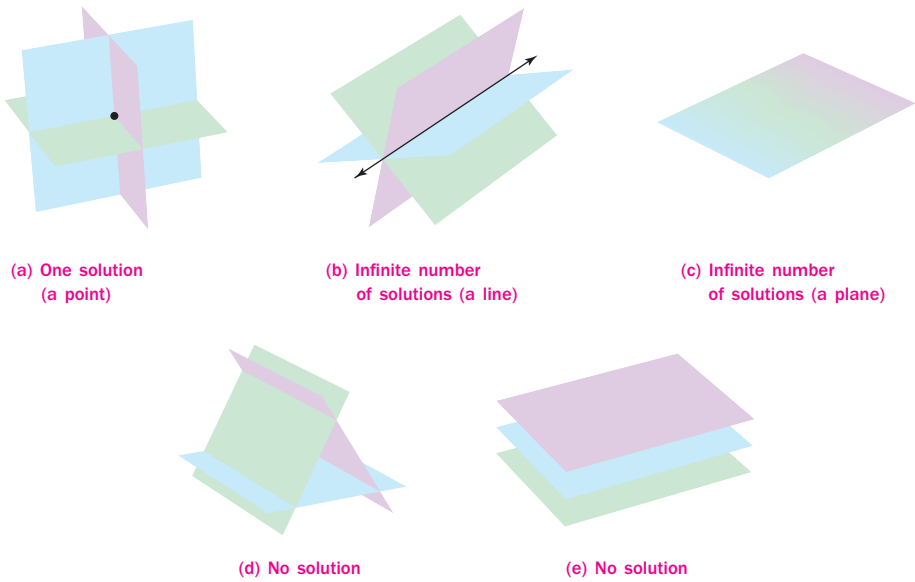
$$ax + by + cz = k \qquad \text{For example, } 3x - 4y - 7z = -9$$

where a, b, c, and k are constants (not all a, b, and c zero) is called a **linear equation in three variables.** (A similar definition holds for a linear equation in four or more variables.) Now that we know how to solve systems of linear equations in two variables, there is no reason to stop there. Systems of the form

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1 \\ a_2x + b_2y + c_2z &= k_2 \\ a_3x + b_3y + c_3z &= k_3 \end{aligned} \tag{1}$$

as well as higher-order systems are encountered frequently. In fact, systems of equations are so important in solving real-world problems that entire courses are devoted to this one topic. A triplet of numbers $x = x_0, y = y_0$, and $z = z_0$ [also written as an ordered triplet (x_0, y_0, z_0)] is a **solution** of system (1) if each equation is satisfied by this triplet. The set of all such ordered triplets of numbers is called the **solution set** of the system. If operations are performed on a system and the new system has the same solution set as the original, then the systems are said to be **equivalent.**

Linear equations in three variables represent planes in a three-dimensional space. Trying to visualize how three planes can intersect will give you insight as to what kind of solution sets are possible for system (1). Figure 1 shows several of the many ways in which three planes can intersect. It can be shown that system (1) will have exactly on solution [Fig. 1(a)], infinitely many solutions [Figs. 1(b) and 1(c)], or no solutions [Figs. 1(d) and (e)].



› **Figure 1** Three planes.

In this section, we use an extension of the method of elimination discussed in the last section to solve systems in the form of system (1). In Section 9-1 we will consider techniques for solving linear systems that are more compatible with computer solutions. In practice, most linear systems involving more than three variables are solved with the aid of a computer.

› Using Elimination by Addition

The general approach to solving a system with three variables is to use Theorem 2 in the last section to eliminate variables until an equivalent system with an obvious solution is obtained. To help you follow a solution, we will refer to the equations in a system as E_1, E_2, and so on. And we will show each equivalent system produced in the solution process.

EXAMPLE 1 Solution Using Elimination by Addition

$$\begin{aligned} x + 2y + 3z &= 2 && E_1\\ 3x - 5y - 4z &= 15 && E_2\\ -2x - 3y + 2z &= 2 && E_3 \end{aligned}$$

SOLUTION

Since the coefficient of x in E_1 is 1, our calculations will be simplified if we use E_1 to eliminate x from the other equations. First we eliminate x from E_2 by multiplying E_1 by -3 and adding the result to E_2, obtaining a new equation, which we will call E_4. Next, we replace E_2 in the original system with E_4 to obtain a system equivalent to the original system.

$$\begin{aligned} -3x - 6y - 9z &= -6 && -3E_1\\ 3x - 5y - 4z &= 15 && E_2\\ \hline -11y - 13z &= 9 && E_4 \end{aligned}$$

Equivalent System

$$\begin{aligned} x + 2y + 3z &= 2 && E_1\\ -11y - 13z &= 9 && E_4\\ -2x - 3y + 2z &= 2 && E_3 \end{aligned}$$

Now we use E_1 to eliminate x (the same variable eliminated from E_2) from E_3 by multiplying E_1 by 2 and adding the result to E_3, obtaining a new equation, which we will call E_5. Replacing E_3 with E_5 produces another equivalent system.

$$\begin{aligned} 2x + 4y + 6z &= 4 && 2E_1\\ -2x - 3y + 2z &= 2 && E_3\\ \hline y + 8z &= 6 && E_5 \end{aligned}$$

Equivalent System

$$\begin{aligned} x + 2y + 3z &= 2 && E_1\\ -11y - 13z &= 9 && E_4\\ y + 8z &= 6 && E_5 \end{aligned}$$

Next we use E_5 to eliminate y from E_4, obtaining E_6, and replace E_4 with E_6.

$$\begin{aligned} 11y + 88z &= 66 && 11E_5\\ -11y - 13z &= 9 && E_4\\ \hline 75z &= 75 && E_6 \end{aligned}$$

Equivalent System

$$\begin{aligned} x + 2y + 3z &= 2 && E_1\\ 75z &= 75 && E_6\\ y + 8z &= 6 && E_5 \end{aligned}$$

The last equivalent system is now in a form that we can solve. First solve E_6 for z.

$$\begin{aligned} 75z &= 75 && E_6\\ z &= 1 \end{aligned}$$

Next, back substitute $z = 1$ in E_5 and solve for y.

$$\begin{aligned} y + 8z &= 6 && E_5\\ y + 8(1) &= 6\\ y &= -2 \end{aligned}$$

Finally, back substitute $y = -2$ and $z = 1$ in E_1 and solve for x.

$$\begin{aligned} x + 2y + 3z &= 2 \qquad E_1 \\ x + 2(-2) + 3(1) &= 2 \\ x &= 3 \end{aligned}$$

The solution to the original system is $(3, -2, 1)$ or $x = 3, y = -2, z = 1$.

CHECK

To check the solution, we must check each equation in the original system:

$$\begin{aligned} x + 2y + 3z &= 2 \quad E_1 \\ 3 + 2(-2) + 3(1) &\stackrel{?}{=} 2 \\ 2 &\stackrel{\checkmark}{=} 2 \end{aligned} \qquad \begin{aligned} 3x - 5y - 4z &= 15 \quad E_2 \\ 3(3) - 5(-2) - 4(1) &\stackrel{?}{=} 15 \\ 15 &\stackrel{\checkmark}{=} 15 \end{aligned} \qquad \begin{aligned} -2x - 3y + 2z &= 2 \quad E_3 \\ -2(3) - 3(-2) + 2(1) &\stackrel{?}{=} 2 \\ 2 &\stackrel{\checkmark}{=} 2 \end{aligned}$$

MATCHED PROBLEM 1

Solve:

$$\begin{aligned} 2x + 3y - 5z &= -12 \\ 3x - 2y + 2z &= 1 \\ 4x - 5y - 4z &= -12 \end{aligned}$$

Recognizing Inconsistent and Dependent Systems

In the last section, we saw that obtaining an inconsistent equation (such as $0 = 1$) while solving a system of equations in two variables indicates that the system is inconsistent and has no solution. The same is true for systems in three variables. Once an inconsistent equation is obtained, the solution process can be stopped. The system is inconsistent and has no solution.

On the other hand, if an equation that is always true (such as $0 = 0$) is obtained, the system may be inconsistent or dependent. The solution process must proceed further to determine which is the case.

EXAMPLE 2 Using Elimination by Addition

Solve:

$$\begin{aligned} x + y + z &= 3 \qquad E_1 \\ x - y - 5z &= 1 \qquad E_2 \\ 2x + 3y + 5z &= 6 \qquad E_3 \end{aligned}$$

SOLUTION

Use E_1 to eliminate z from E_2 and replace E_2 with the result.

$$\begin{array}{rll} 5x + 5y + 5z = 15 & 5E_1 & \\ x - y - 5z = 1 & E_2 & \text{Equivalent System} \\ \hline 6x + 4y = 16 & E_4 & \end{array} \qquad \begin{array}{rll} x + y + z = 3 & E_1 \\ 6x + 4y = 16 & E_4 \\ 2x + 3y + 5z = 6 & E_3 \end{array}$$

Use E_1 to eliminate z from E_3 and replace E_3 with the result.

$$\begin{array}{rll} -5x - 5y - 5z = -15 & -5E_1 & \\ 2x + 3y + 5z = 6 & E_3 & \text{Equivalent System} \\ \hline -3x - 2y = -9 & E_5 & \end{array} \qquad \begin{array}{rll} x + y + z = 3 & E_1 \\ 6x + 4y = 16 & E_4 \\ -3x - 2y = -9 & E_5 \end{array}$$

Use E_5 to eliminate y from E_4 and replace E_4 with the result.

$$\begin{array}{rll} 6x + 4y = 16 & E_4 & \\ -6x - 4y = -18 & 2E_5 & \text{Equivalent System} \\ \hline 0 = -2 & E_6 & \end{array} \qquad \begin{array}{rll} x + y + z = 3 & E_1 \\ 0 = -2 & E_6 \\ -3x - 2y = -9 & E_5 \end{array}$$

Stop! We have obtained a contradiction. The original system is inconsistent and has no solution.

MATCHED PROBLEM 2

Solve:

$$\begin{aligned} 2x + 3y - 5z &= 3 \\ 3x - 2y + 2z &= 2 \\ x - 5y + 7z &= 1 \end{aligned}$$

EXAMPLE 3 **Using Elimination by Addition**

Solve:

$$\begin{array}{rll} x + y + z = 1 & E_1 \\ 2x + y - z = 3 & E_2 \\ 3x + y - 3z = 5 & E_3 \end{array}$$

SOLUTION

Use E_1 to eliminate y from E_2 and replace E_2 with the result.

$$\begin{array}{rrr} -x - y - z = -1 & -E_1 & \text{Equivalent System} \\ 2x + y - z = 3 & E_2 & x + y + z = 1 \quad E_1 \\ \hline x - 2z = 2 & E_4 & x - 2z = 2 \quad E_4 \\ & & 3x + y - 3z = 5 \quad E_3 \end{array}$$

Use E_1 to eliminate y from E_3 and replace E_3 with the result.

$$\begin{array}{rrr} -x - y - z = -1 & -E_1 & \text{Equivalent System} \\ 3x + y - 3z = 5 & E_3 & x + y + z = 1 \quad E_1 \\ \hline 2x - 4z = 4 & E_5 & x - 2z = 2 \quad E_4 \\ & & 2x - 4z = 4 \quad E_5 \end{array}$$

Use E_4 to eliminate z from E_5 and replace E_5 with the result.

$$\begin{array}{rrr} -2x + 4z = -4 & -2E_4 & \text{Equivalent System} \\ 2x - 4z = 4 & E_5 & x + y + z = 1 \quad E_1 \\ \hline 0 = 0 & E_6 & x - 2z = 2 \quad E_4 \end{array}$$

Since E_6 is true for all x, y, and z, it provides no information about the system's solution set and can be discarded. The solutions to the last equivalent system can be described by introducing a parameter (see the last section). If we let $z = s$, then, using E_4, we can write $x = 2s + 2$. Substituting for x and z in E_1 and solving for y, we have

$$\begin{array}{rr} x + y + z = 1 & E_1 \\ 2s + 2 + y + s = 1 & \\ y = -3s - 1 & \end{array}$$

The solution set is given by

$$\{(2s + 2, -3s - 1, s) \mid s \text{ any real number}\}$$

Remark on Checking
Notice that we did not check the solution to Example 2. It is impossible to check the solution to an inconsistent system. For each equation in the check of the dependent system in Example 3, the parameter s is eliminated and the final form is an equation that is true for any real number s. If this does not happen, the check fails and the solution is incorrect.

CHECK

$$\begin{array}{rr} x + y + z = 1 & E_1 \\ (2s + 2) + (-3s - 1) + s \stackrel{?}{=} 1 & \\ 2s - 3s + s + 2 - 1 \stackrel{?}{=} 1 & \\ 1 \stackrel{\checkmark}{=} 1 & \end{array}$$

$$\begin{array}{rr} 2x + y - z = 3 & E_2 \\ 2(2s + 2) + (-3s - 1) - s \stackrel{?}{=} 3 & \\ 4s + 4 - 3s - 1 - s \stackrel{?}{=} 3 & \\ 3 \stackrel{\checkmark}{=} 3 & \end{array}$$

$$3x + y - 3z = 5 \quad E_3$$
$$3(2s + 2) + (-3s - 1) - 3s \stackrel{?}{=} 5$$
$$6s + 6 - 3s - 1 - 3s \stackrel{?}{=} 5$$
$$5 \stackrel{\checkmark}{=} 5$$

MATCHED PROBLEM 3

Solve:

$$\begin{aligned} 3x + 2y + 4z &= 5 \\ 2x + y + 5z &= 2 \\ x + 6z &= -1 \end{aligned}$$

››› EXPLORE-DISCUSS 1

Refer to the solution to Example 3. The given representation of the solution set is not the only one. Which of the following is a representation of the solution set? Justify your answer.

(A) $\{(t, 2 - 1.5t, 0.5t - 1) | \ t \text{ any real number}\}$

(B) $\{(2u + 4, -2u - 3, u) | \ u \text{ any real number}\}$

Let $y = v$, where v is any real number, express x and z in terms of v, and find another representation of the solution set for Example 3.

› Modeling with Systems in Three Variables

EXAMPLE 4 **Production Scheduling**

A garment industry manufactures three shirt styles. Each style shirt requires the services of three departments as listed in the table. The cutting, sewing, and packaging departments have available a maximum of 1,160, 1,560, and 480 labor-hours per week, respectively. How many of each style shirt must be produced each week for the plant to operate at full capacity?

	Style A	Style B	Style C	Time available
Cutting department	0.2 hr	0.4 hr	0.3 hr	1,160 hr
Sewing department	0.3 hr	0.5 hr	0.4 hr	1,560 hr
Packaging department	0.1 hr	0.2 hr	0.1 hr	480 hr

SOLUTION

Let

$$x = \text{Number of style A shirts produced per week}$$
$$y = \text{Number of style B shirts produced per week}$$
$$z = \text{Number of style C shirts produced per week}$$

Then

$$\begin{aligned} 0.2x + 0.4y + 0.3z &= 1{,}160 && \text{Cutting department} \\ 0.3x + 0.5y + 0.4z &= 1{,}560 && \text{Sewing department} \\ 0.1x + 0.2y + 0.1z &= 480 && \text{Packaging department} \end{aligned}$$

We clear the system of decimals by multiplying each side of each equation by 10:

$$\begin{aligned} 2x + 4y + 3z &= 11{,}600 && E_1 \\ 3x + 5y + 4z &= 15{,}600 && E_2 \\ x + 2y + z &= 4{,}800 && E_3 \end{aligned}$$

Use E_3 to eliminate z from E_1 and replace E_1 with the result.

$$\begin{array}{rrl} 2x + 4y + 3z = & 11{,}600 & E_1 \\ -3x - 6y - 3z = & -14{,}400 & -3E_3 \\ \hline -x - 2y = & -2{,}800 & E_4 \end{array}$$

Equivalent System

$$\begin{array}{rrl} -x - 2y = & -2{,}800 & E_4 \\ 3x + 5y + 4z = & 15{,}600 & E_2 \\ x + 2y + z = & 4{,}800 & E_3 \end{array}$$

Use E_3 to eliminate z from E_2 and replace E_2 with the result.

$$\begin{array}{rrl} 3x + 5y + 4z = & 15{,}600 & E_2 \\ -4x - 8y - 4z = & -19{,}200 & -4E_3 \\ \hline -x - 3y = & -3{,}600 & E_4 \end{array}$$

Equivalent System

$$\begin{array}{rrl} -x - 2y = & -2{,}800 & E_4 \\ -x - 3y = & -3{,}600 & E_5 \\ x + 2y + z = & 4{,}800 & E_3 \end{array}$$

Use E_4 to eliminate x from E_5 and replace E_5 with the result.

$$\begin{array}{rrl} x + 2y = & 2{,}800 & -E_4 \\ -x - 3y = & -3{,}600 & E_5 \\ \hline -y = & -800 & E_6 \end{array}$$

Equivalent System

$$\begin{array}{rrl} -x - 2y = & -2{,}800 & E_4 \\ -y = & -800 & E_6 \\ x + 2y + z = & 4{,}800 & E_3 \end{array}$$

From E_6 we see that

$$\boldsymbol{y = 800}$$

Back substitute $y = 800$ in E_4 and solve for x.

$$\begin{aligned} -x - 2y &= -2{,}800 && E_4 \\ -x - 2(800) &= -2{,}800 \end{aligned}$$
$$\boldsymbol{x = 1{,}200}$$

Back substitute $x = 1{,}200$ and $y = 800$ in E_3 and solve for z.

$$\begin{aligned} x + 2y + z &= 4{,}800 \quad E_3 \\ 1{,}200 + 2(800) + z &= 4{,}800 \\ z &= 2{,}000 \end{aligned}$$

Each week, the company should produce 1,200 style A shirts, 800 style B shirts, and 2,000 style C shirts to operate at full capacity. You should check this solution.

MATCHED PROBLEM 4

Repeat Example 4 with the cutting, sewing, and packaging departments having available a maximum of 1,180, 1,560, and 510 labor-hours per week, respectively.

››› EXPLORE-DISCUSS 2

Refer to Example 4. Careful choice of the first variable to be eliminated may shorten the solution process. Solve Example 4 by first using E_3 to eliminate y from E_1 and E_2. Discuss relationships to look for that will reduce the calculations in a solution.

EXAMPLE 5 Data Analysis

In 2001 there were 110 million cell phone subscribers in the United States. This number grew to 128 million in 2002 and 141 million in 2003. Construct a model for these data by finding a quadratic function whose graph passes through the points (1, 110), (2, 128), and (3, 141). Use this model to estimate the number of subscribers in 2004 and 2005.

SOLUTIONS

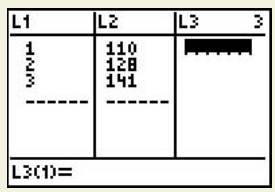
Algebraic Solution

Let $f(x) = ax^2 + bx + c$ be the quadratic model and evaluate f at each of the three data points to obtain three linear equations in a, b, and c.

$$\begin{aligned} x = 1: \quad a + b + c &= 110 \quad E_1 \\ x = 2: \quad 4a + 2b + c &= 128 \quad E_2 \\ x = 3: \quad 9a + 3b + c &= 141 \quad E_3 \end{aligned}$$

Graphical Solution

First we enter the three data points in the STAT, EDIT window (Fig. 2).



› Figure 2

Use E_1 to eliminate c from E_2 and replace E_2 with the result.

Equivalent System

$$\begin{array}{rrl} -a - b - c = -110 & -E_1 & a + b + c = 110 \quad E_1 \\ 4a + 2b + c = 128 & E_2 & 3a + b = 18 \quad E_4 \\ \hline 3a + b = 18 & E_4 & 9a + 3b + c = 141 \quad E_3 \end{array}$$

Use E_1 to eliminate c from E_3 and replace E_3 with the result.

Equivalent System

$$\begin{array}{rrl} -a - b - c = -110 & -E_1 & a + b + c = 110 \quad E_1 \\ 9a + 3b + c = 141 & E_3 & 3a + b = 18 \quad E_4 \\ \hline 8a + 2b = 31 & E_5 & 8a + 2b = 31 \quad E_5 \end{array}$$

Use E_4 to eliminate b from E_5 and replace E_5 with the result.

Equivalent System

$$\begin{array}{rrl} -6a - 2b = -36 & -2E_4 & a + b + c = 110 \quad E_1 \\ 8a + 2b = 31 & E_5 & 3a + b = 18 \quad E_4 \\ \hline 2a = -5 & E_6 & 2a = -5 \quad E_5 \end{array}$$

Solve E_6 for a.

$$2a = -5 \quad E_6$$
$$a = -2.5$$

Back substitute $a = -2.5$ in E_4 and solve for b.

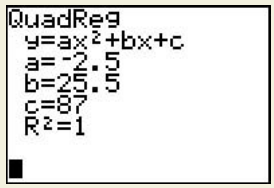
$$3a + b = 18 \quad E_4$$
$$3(-2.5) + b = 18$$
$$b = 25.5$$

Back substitute $a = -2.5$ and $b = 25.5$ in E_1 and solve for c.

$$a + b + c = 110 \quad E_1$$
$$-2.5 + 25.5 + c = 110$$
$$c = 87$$

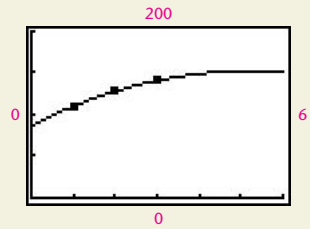
The quadratic model is $f(x) = -2.5x^2 + 25.5x + 87$. The estimate for subscribers in 2004 is $f(4) = 149$ million and in 2005 is $f(5) = 152$ million.

Next we use STAT, CALC, QUADREG to find the quadratic regression polynomial for this data (Fig. 3).



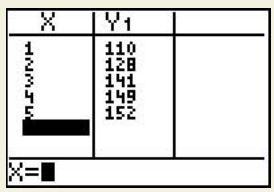
› Figure 3

From Figure 3, we see that the regression polynomial is $p(x) = -2.5x^2 + 25.5x + 87$. Now we graph $p(x)$ along with the data in Figure 2 (Fig. 4).



› Figure 4

Finally, we use the TABLE command to evaluate the polynomial $p(x)$ for $x = 1, 2, 3, 4$, and 5 (Fig. 5).



› Figure 5

The first three lines in Figure 5 verify that the graph of $p(x)$ does pass through the original three data points. The last two lines show that the estimate for subscribers in 2004 is 149 million and in 2005 is 152 million.

MATCHED PROBLEM 5

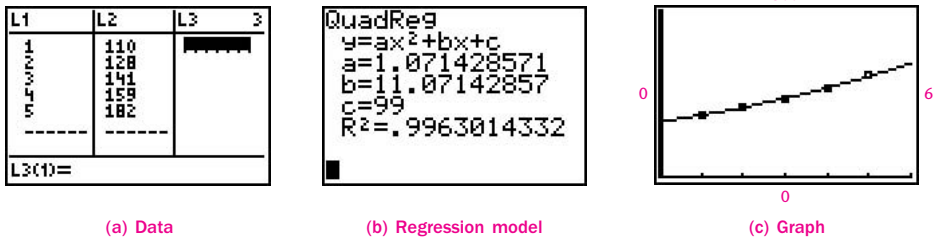
In 2001 there were 110 million Internet hosts in the United States. This number grew to 147 million in 2002 and 172 million in 2003. Construct a model for this data by finding a quadratic function whose graph passes through the points (1, 110), (2, 147), and (3, 172) and use this model to estimate the number of subscribers in 2004 and 2005.

EXAMPLE 6 Data Analysis

Refer to Example 5. Subsequent data indicated that there were 159 million subscribers in 2004 and 182 million in 2005. Add the points (4, 159) and (5, 182) to the data set in Example 5. Use a graphing calculator to find a quadratic regression model for all five data points. Graph the data and the model in the same viewing window.

SOLUTION

The data are shown in Figure 6(a), the regression model in Figure 6(b), and the graph in Figure 6(c).



(a) Data (b) Regression model (c) Graph

› Figure 6

MATCHED PROBLEM 6

Refer to Matched Problem 5. Subsequent data indicated that there were 233 Internet hosts in 2004 and 318 million in 2005. Add the points (4, 233) and (5, 318) to the data set in Matched Problem 5. Use a graphing calculator to find a quadratic regression model for all five data points. Graph the data and the model in the same viewing window.

››› EXPLORE-DISCUSS 3

(A) Use the quadratic polynomial in Example 5 to estimate the number of Internet subscribers in 2006 and 2007.

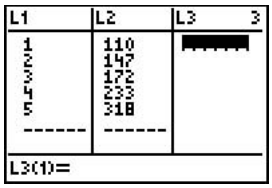
(B) Use the quadratic polynomial in Example 6 to estimate the number of Internet subscribers in 2006 and 2007.

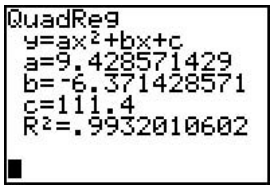
(C) Find estimates for the number of subscribers in 2006 and 2007 on the Internet or in a library. Compare these estimates with those found in parts A and B.

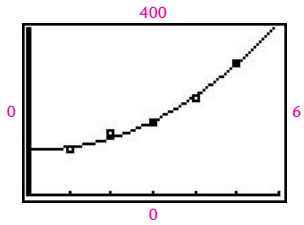
ANSWERS TO MATCHED PROBLEMS

1. $(-1, 0, 2)$ or $x = -1, y = 0, z = 2$ **2.** Inconsistent system with no solution **3.** $\{(-6s - 1, 7s + 4, s) \mid s \text{ is any real number}\}$ **4.** Each week, the company should produce 900 style A shirts, 1,300 style B shirts, and 1,600 style C shirts to operate at full capacity. **5.** The quadratic model is $f(x) = -6x^2 + 55x + 61$. The estimate for subscribers in 2004 is $f(4) = 185$ million and in 2005 is $f(5) = 186$ million.

6.







8-2 Exercises

1. Summarize the elimination procedure for solving a system of three linear equations in three variables in your own words.

2. What is an independent system? How does your summary in Problem 1 deal with independent systems?

3. What is a dependent system? How does your summary in Problem 1 deal with dependent systems?

4. What is an inconsistent system? How does your summary in Problem 1 deal with inconsistent systems?

Solve Problems 5–26 using elimination by addition.

5.
$$\begin{aligned} -2x \quad &= 2 \\ x - 3y \quad &= 2 \\ -x + 2y + 3z &= -7 \end{aligned}$$

6.
$$\begin{aligned} 2y + z &= -4 \\ x - 3y + 2z &= 9 \\ -y \quad &= 3 \end{aligned}$$

7.
$$\begin{aligned} 2y - z &= 2 \\ -4y + 2z &= 1 \\ x - 2y + 3z &= 0 \end{aligned}$$

8.
$$\begin{aligned} x + y - z &= 3 \\ x \quad - 2z &= 1 \\ y + z &= 2 \end{aligned}$$

9.
$$\begin{aligned} x - 3y \quad &= 2 \\ 2y + z &= -1 \\ x - y + z &= 1 \end{aligned}$$

10.
$$\begin{aligned} -4x + 3y \quad &= 1 \\ 8x - 6y \quad &4 \\ 2x - 4y + 3z &= 6 \end{aligned}$$

11.
$$\begin{aligned} 4y - z &= -13 \\ 3y + 2z &= 4 \\ 6x - 5y - 2z &= -18 \end{aligned}$$

12.
$$\begin{aligned} 2x + \quad z &= -5 \\ x - \quad 3z &= -6 \\ 4x + 2y - z &= -9 \end{aligned}$$

13.
$$\begin{aligned} 2x + y - z &= 5 \\ x - 2y - 2z &= 4 \\ 3x + 4y + 3z &= 3 \end{aligned}$$

14.
$$\begin{aligned} x - 3y + z &= 4 \\ -x + 4y - 4z &= 1 \\ 2x - y + 5z &= -3 \end{aligned}$$

15.
$$\begin{aligned} x - y + z &= 1 \\ 2x + y + z &= 6 \\ 7x - y + 5z &= 15 \end{aligned}$$

16.
$$\begin{aligned} 2x - y + 3z &= 7 \\ x + 2y - z &= -3 \\ 3x + y + 2z &= 2 \end{aligned}$$

17.
$$\begin{aligned} 2a + 4b + 3c &= -6 \\ a - 3b + 2c &= -15 \\ -a + 2b - c &= 9 \end{aligned}$$

18.
$$\begin{aligned} 3u - 2v + 3w &= 11 \\ 2u + 3v - 2w &= -5 \\ u + 4v - w &= -5 \end{aligned}$$

19.
$$\begin{aligned} 2x - 3y + 3z &= -5 \\ 3x + 2y - 5z &= 34 \\ 5x - 4y - 2z &= 23 \end{aligned}$$

20.
$$\begin{aligned} 3x - 2y - 4z &= -8 \\ 4x + 3y - 5z &= -5 \\ 6x - 5y + 2z &= -17 \end{aligned}$$

21.
$$\begin{aligned} x + 2y + z &= 2 \\ -2x + 3y - 2z &= -3 \\ x - 5y + z &= 2 \end{aligned}$$

22.
$$\begin{aligned} 2x + y + z &= 1 \\ 3x + 3y + 2z &= 1 \\ 7x - 4y + z &= 6 \end{aligned}$$

23.
$$\begin{aligned} -x + 2y - z &= -4 \\ 2x + 5y - 4z &= -16 \\ x + y - z &= -4 \end{aligned}$$

24.
$$\begin{aligned} x - 8y + 2z &= -1 \\ x - 3y + z &= 1 \\ 2x - 11y + 3z &= 2 \end{aligned}$$

25.
$$\begin{aligned} 2x + 3y + 2z &= 2 \\ 5x + 8y + 4z &= 3 \\ 4x + 7y + 2z &= 0 \end{aligned}$$

26.
$$\begin{aligned} 2x + 3y + 2z &= 5 \\ 4x + 5y + 2z &= 3 \\ 2x + 4y + 4z &= 12 \end{aligned}$$

APPLICATIONS

27. GEOMETRY Find a, b, and c so that the graph of the parabola with equation $y = ax^2 + bx + c$ passes through the points $(-2, 9)$, $(1, -9)$, and $(4, 9)$.

28. GEOMETRY Find a, b, and c so that the graph of the parabola with equation $y = ax^2 + bx + c$ passes through the points $(-1, -5)$, $(2, 7)$, and $(5, 1)$.

29. GEOMETRY Find a, b, and c so that the graph of the circle with equation $x^2 + y^2 + ax + by + c = 0$ passes through the points $(-1, 1)$, $(5, 1)$, and $(6, -6)$.

30. GEOMETRY Find a, b, and c so that the graph of the circle with equation $x^2 + y^2 + ax + by + c = 0$ passes through the points $(0, -2)$, $(-1, -3)$, and $(-7, 5)$.

31. MANUFACTURING A company manufactures three products, lawn mowers, snowblowers, and chain saws. The labor, material, and shipping costs for manufacturing one unit of each product are given in the table. The weekly allocations for labor, materials, and shipping are \$35,000, \$50,000, and \$20,000, respectively. How many of each type of product should be manufactured each week in order to exactly use the weekly allocations?

Product	Labor	Materials	Shipping
Lawn mower	\$20	\$35	\$15
Snowblower	\$30	\$50	\$25
Chain saw	\$45	\$40	\$10

32. MANUFACTURING A company manufactures three products, desk chairs, file cabinets, and printer stands. The labor, material, and shipping costs for manufacturing one unit of each product are given in the table. The weekly allocations for labor, materials, and shipping are \$21,100, \$31,500, and \$11,900, respectively. How many of each type of product should be manufactured each week in order to exactly use the weekly allocations?

Product	Desk chair	File cabinet	Printer stand
Labor	\$30	\$35	\$40
Materials	\$45	\$60	\$55
Shipping	\$25	\$20	\$15

33. SCHEDULING A company has plants located in Michigan, New York, and Ohio where it manufactures notebook computers, desktop computers, and servers. The number of units of each product that can be produced per day at each plant are given in the table. The company has orders for 2,150 notebook computers, 2,300 desktop computers, and 2,500 servers. How many days should the company operate each plant in order to exactly fill these orders?

Plant	Michigan	New York	Ohio
Notebook	10	70	60
Desktop	20	50	80
Server	40	30	90

34. SCHEDULING A company has plants located in Maine, Utah, and Oregon where it manufactures stoves, refrigerators, and dishwashers. The number of units of each product that can be produced per day at each plant are given in the table. The company has orders for 1,500 stoves, 2,350 refrigerators, and 2,400 dishwashers. How many days should the company operate each plant in order to exactly fill these orders? Set up a system of equations whose solution would answer this question and solve the system.

Plant	Stoves	Refrigerators	Dishwashers
Maine	30	70	60
Utah	20	50	50
Oregon	40	30	40

35. FINANCE Anne wants to invest \$100,000 to produce an annual income of \$4,400. A financial advisor recommends that she invest in treasury bonds that earn 4% annually, municipal bonds that earn 3.5% annually, and corporate bonds that earn 5% annually. As a risk control factor, the advisor recommends that the amount invested in corporate bonds should equal the total of the amounts invested in treasury bonds and municipal bonds. If Anne follows her advisor's recommendations, how much should she invest in each of the three types of bonds?

36. FINANCE Refer to Problem 35. After the rate for corporate bonds increases to 5.5% annually and the rates for treasury bonds and municipal bonds remain unchanged, Anne decides to reallocate her investment mix while still following her advisor's advice on risk control. How much should she invest in each of the three types of bonds to produce an annual income of \$4,700?

37. RESOURCE ALLOCATION Nina's Nut Shop produces a standard blend of peanuts, cashews, and walnuts. The wholesale prices for various nuts are given in the table. How many pounds of each nut should be used to produce 1,500 pounds of the standard blend at a cost of \$3,500 if the blend is 50% peanuts?

Wholesale Nut Prices

Item	Price per pound
Almonds	$5.00
Cashews	$4.00
Peanuts	$1.40
Pecans	$5.40
Walnuts	$2.90

38. **RESOURCE ALLOCATION** Refer to Problem 37. Nina's also produces a deluxe blend of almonds, cashews, and pecans. How many pounds of each nut should be used to produce 940 pounds of the deluxe blend at a cost of $4,300 if the blend is 50% cashews?

39. **POPULATION GROWTH** The U.S. population was approximately 75 million in 1900, 150 million in 1950, and 275 million in 2000. Construct a model for this data by finding a quadratic function whose graph passes through the points (0, 75), (50, 150), and (100, 275). Use this model to estimate the population in 2050.

40. **POPULATION GROWTH** The population of California was approximately 24 million in 1980, 30 million in 1990, and 34 million in 2000. Construct a model for this data by finding a quadratic function whose graph passes through the points (0, 24), (10, 30), and (20, 34). Use this model to estimate the population in 2010.

41. **LIFE EXPECTANCY** The life expectancy for females born during 1980–1985 was approximately 77.6 years. This grew to 78 years during 1985–1990 and to 78.6 years during 1990–1995. Construct a model for this data by finding a quadratic function whose graph passes through the points (0, 77.6), (5, 78), and (10, 78.6). Use this model to estimate the life expectancy for females born between 1995 and 2000 and for those born between 2000 and 2005.

42. **LIFE EXPECTANCY** The life expectancy for males born during 1980–1985 was approximately 70.7 years. This grew to 71.1 years during 1985–1990 and to 71.8 years during 1990–1995. Construct a model for this data by finding a quadratic function whose graph passes through the points (0, 70.7), (5, 71.1), and (10, 71.8). Use this model to estimate the life expectancy for males born between 1995 and 2000 and for those born between 2000 and 2005.

43. **LIFE EXPECTANCY** Refer to Problem 41. Subsequent data indicated that life expectancy grew to 79.1 years for females born during 1995–2000 and to 79.7 years for females born during 2000–2005. Add the points (15, 79.1) and (20, 79.7) to the data set in Problem 41. Use a graphing calculator to find a quadratic regression model for all five data points. Graph the data and the model in the same viewing window.

44. **LIFE EXPECTANCY** Refer to Problem 42. Subsequent data indicated that life expectancy grew to 73.2 years for males born during 1995–2000 and to 74.3 years for males born during 2000–2005. Add the points (15, 73.2) and (20, 74.3) to the data set in Problem 42. Use a graphing calculator to find a quadratic regression model for all five data points. Graph the data and the model in the same viewing window.

8-3 Systems of Linear Inequalities

- Graphing Linear Inequalities in Two Variables
- Solving Systems of Linear Inequalities
- Mathematical Modeling with Systems of Linear Inequalities

Many applications of mathematics involve systems of inequalities rather than systems of equations. A graph is often the most convenient way to represent the solutions of a system of inequalities in two variables. In this section we discuss techniques for graphing both a single linear inequality in two variables and a system of linear inequalities in two variables.

Graphing Linear Inequalities in Two Variables

We know how to graph first-degree equations such as

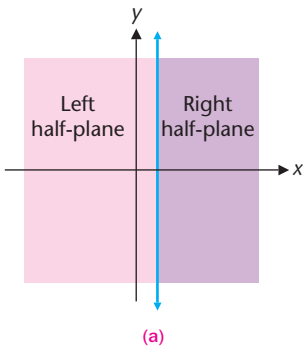
$$y = 2x - 3 \qquad \text{and} \qquad 2x - 3y = 5$$

but how do we graph first-degree inequalities such as

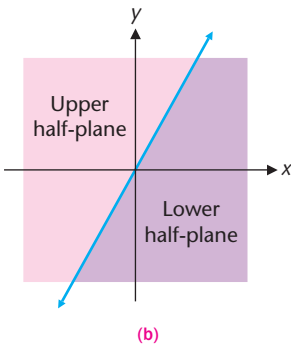
$$y \le 2x - 3 \qquad \text{and} \qquad 2x - 3y > 5$$

Actually, graphing these inequalities is almost as easy as graphing the equations. But before we begin, we must discuss some important subsets of a plane in a rectangular coordinate system.

A line divides a plane into two halves called **half-planes.** A vertical line divides a plane into **left** and **right half-planes** [Fig. 1(a)]; a nonvertical line divides a plane into **upper** and **lower half-planes** [Fig. 1(b)].



(a)



(b)

Figure 1 Half-planes.

EXPLORE-DISCUSS 1

Consider the following linear equation and related linear inequalities:

$$(1)\ 2x - 3y = 12 \qquad (2)\ 2x - 3y < 12 \qquad (3)\ 2x - 3y > 12$$

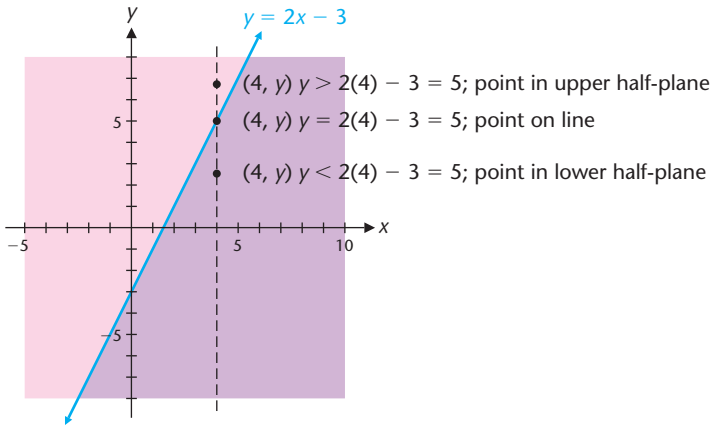
(A) Graph the line with equation (1).

(B) Find the point on this line with x coordinate 3 and draw a vertical line through this point. Discuss the relationship between the y coordinates of the points on this line and statements (1), (2), and (3).

(C) Repeat part B for $x = -3$. For $x = 9$.

(D) Based on your observations in parts B and C, write a verbal description of all the points in the plane that satisfy equation (1), those that satisfy inequality (2), and those that satisfy inequality (3).

Now let's investigate the half-planes determined by the linear equation $y = 2x - 3$. We start by graphing $y = 2x - 3$ (Fig. 2). For any given value of x, there is exactly one value for y such that (x, y) lies on the line. For the same x, if the point (x, y) is below the line, then $y < 2x - 3$. Thus, the lower half-plane corresponds to the solution of the inequality $y < 2x - 3$. Similarly, the upper half-plane corresponds to the solution of the inequality $y > 2x - 3$, as shown in Figure 2.

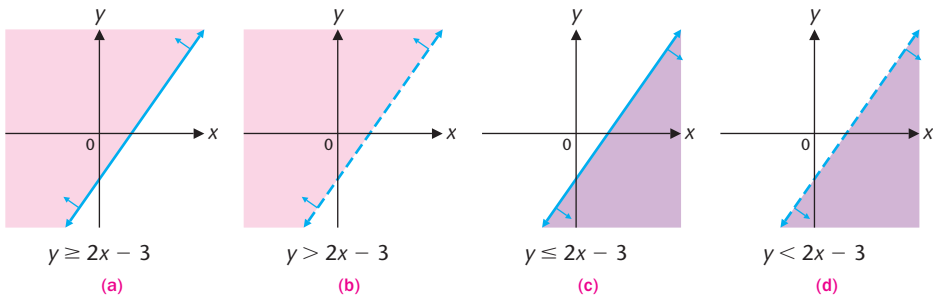


› Figure 2

The four inequalities formed from the equation $y = 2x - 3$ by replacing the $=$ sign by $\geq$, $>$, $\leq$, and $<$, respectively, are

$$y \geq 2x - 3 \qquad y > 2x - 3 \qquad y \leq 2x - 3 \qquad y < 2x - 3$$

The graph of each is a half-plane. The line $y = 2x - 3$, called the **boundary line** for the half-plane, is included for $\geq$ and $\leq$ and excluded for $>$ and $<$. In Figure 3, the half-planes are indicated with small arrows on the graph of $y = 2x - 3$ and then graphed as shaded regions. Included boundary lines are shown as solid lines, and excluded boundary lines are shown as dashed lines.

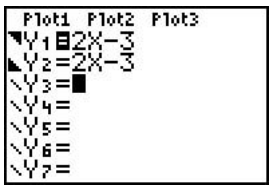


› Figure 3

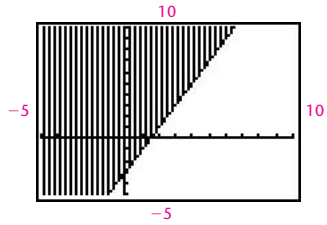
Most graphing calculators give the user the option to shade the region above or below an equation by changing the icon to the left of the equation* [Fig. 4(a)]. Graphing y_1 with the shade-above option selected produces the graph in Figure 4(b) and graphing the same function in y_2 with the shade-below option produces the graph in Figure 4(c). Note that although it is possible to graph a dashed line on a graphing calculator, it is not possible to distinguish between a dashed line and a solid line when using the shading options. We will indicate in words if the boundary line is not part of a solution graphed on a graphing calculator.

*On most graphing calculators, you can select a shading option by moving the cursor to the icon at the left side of the Y=screen and pressing ENTER repeatedly to toggle through the various choices.

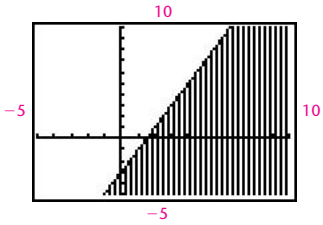
› Figure 4



Shade above y_1
Shade below y_2
(a)



$y \geq 2x - 3$
(b)



$y \leq 2x - 3$
(c)

› THEOREM 1 Graphs of Linear Inequalities in Two Variables

The graph of a linear inequality

$$Ax + By < C \qquad \text{or} \qquad Ax + By > C$$

with $B \neq 0$, is either the upper half-plane or the lower half-plane (but not both) determined by the line $Ax + By = C$.

If $B = 0$, then the graph of

$$Ax < C \qquad \text{or} \qquad Ax > C$$

is either the left half-plane or the right half-plane (but not both) determined by the line $Ax = C$.

As a consequence of Theorem 1, we state simple and quick procedures for graphing a linear inequality by hand and on a graphing calculator.

› ALGEBRAIC PROCEDURE FOR GRAPHING LINEAR INEQUALITIES IN TWO VARIABLES

Step 1. Graph $Ax + By = C$ as a dashed line if equality is not included in the original statement or as a solid line if equality is included.

Step 2. Choose a test point anywhere in the plane not on the line and substitute the coordinates into the inequality. The origin (0, 0) often requires the least computation.

Step 3. The graph of the original inequality includes the half-plane containing the test point if the inequality is satisfied by that point, or the half-plane not containing that point if the inequality is not satisfied by that point.

› GRAPHING CALCULATOR PROCEDURE FOR GRAPHING LINEAR INEQUALITIES IN TWO VARIABLES

Step 1. Solve the inequality for y.

Step 2. Enter the equation of the boundary line and select a shading option as follows:

$$\left.\begin{array}{c} y > mx + b \\ \text{or} \\ y \geq mx + b \end{array}\right\} \text{Select shade above.}$$

$$\left.\begin{array}{c} y < mx + b \\ \text{or} \\ y \leq mx + b \end{array}\right\} \text{Select shade below.}$$

Step 3. Graph the solution.

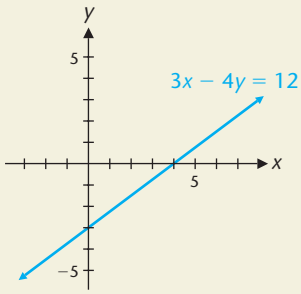
EXAMPLE 1 Graphing a Linear Inequality

Graph: $3x - 4y \le 12$

SOLUTIONS

Algebraic Solution

Step 1. Graph $3x - 4y = 12$ as a solid line, because equality is included in the original statement (Fig. 5).



› Figure 5

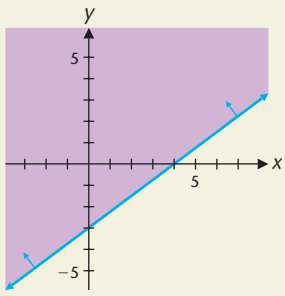
Step 2. Pick a convenient test point above or below the line. The origin (0, 0) requires the least computation. Substituting (0, 0) into the inequality

$$3x - 4y \le 12$$

$$3(0) - 4(0) = 0 \le 12 \quad *$$

produces a true statement; therefore, (0, 0) is in the solution set.

Step 3. The line $3x - 4y = 12$ and the half-plane containing the origin form the graph of $3x - 4y \le 12$ (Fig. 6).



› Figure 6

Graphing Calculator Solution

Step 1. Solve the inequality for y.

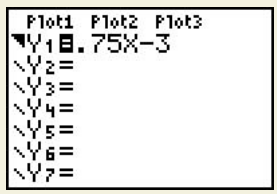
$$3x - 4y \le 12$$

$$-4y \le -3x + 12$$

$$\frac{-4y}{-4} \ge \frac{-3x}{-4} + \frac{12}{-4}$$

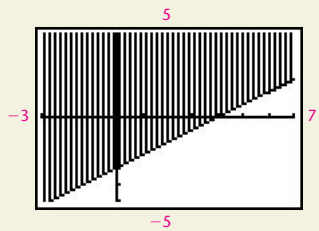
$$y \ge 0.75x - 3$$

Step 2. Enter $y_1 = 0.75x - 3$ and select the shade-above option (Fig. 7).



› Figure 7

Step 3. Graph the solution (Fig. 8).



› Figure 8

The boundary line is part of the solution.

*The dashed "think boxes" are used to enclose steps that may be performed mentally.

MATCHED PROBLEM 1

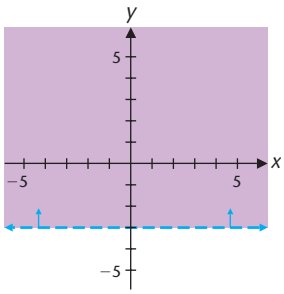
Graph by hand and on a graphing calculator: $2x + 3y < 6$

EXAMPLE 2 Graphing a Linear Inequality

Graph: (A) $y > -3$ (B) $2x \le 5$

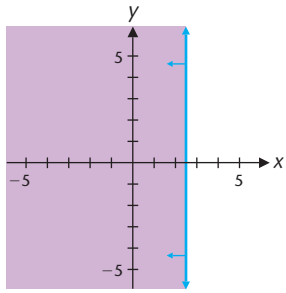
SOLUTIONS

(A) The graph of $y > -3$ is shown in Figure 9.



› Figure 9

(B) The graph of $2x \le 5$, or equivalently $x \le \frac{5}{2}$, is shown in Figure 10.



› Figure 10

MATCHED PROBLEM 2

Graph: (A) $y \le 2$ (B) $3x > -8$

› Solving Systems of Linear Inequalities

We now consider systems of linear inequalities such as

$$\begin{aligned} x + y &\ge 6 \\ 2x - y &\ge 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 2x + y &\le 22 \\ x + y &\le 13 \\ 2x + 5y &\le 50 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

We wish to solve such systems graphically—that is, to find the graph of all ordered pairs of real numbers (x, y) that simultaneously satisfy all the inequalities in the system. The graph is called the **solution region** for the system. To find the solution region, we graph each inequality in the system and then take the intersection of all the graphs.

To simplify the discussion that follows, we will consider only systems of linear inequalities where equality is included in each statement in the system.

EXAMPLE 3 Solving a System of Linear Inequalities

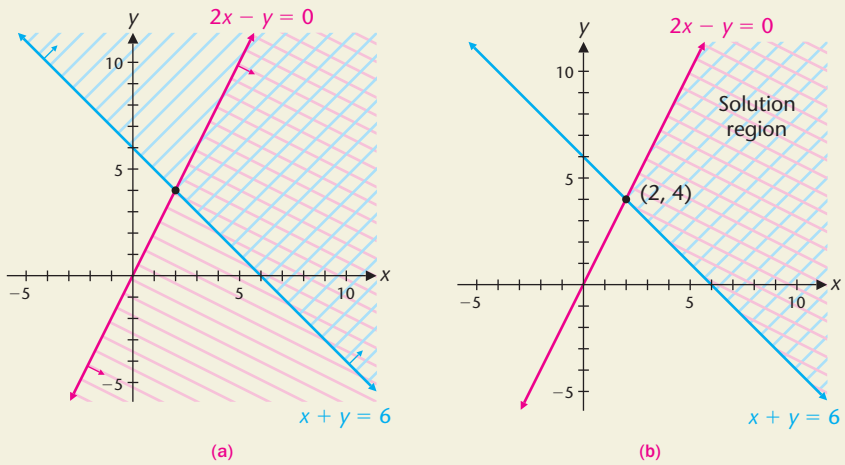
Graph the solution region for the following system of linear inequalities:

$$x + y \geq 6$$
$$2x - y \geq 0$$

SOLUTIONS

Algebraic Solution

First, graph the line $x + y = 6$ and shade the region that satisfies the inequality $x + y \geq 6$. This region is shaded in blue in Figure 11(a). Next, graph the line $2x - y = 0$ and shade the region that satisfies the inequality $2x - y \geq 0$. This region is shaded in red in Figure 11(a). The solution region for the system of inequalities is the intersection of these two regions. This is the region shaded in both red and blue in Figure 11(a), which is redrawn in Figure 11(b) with only the solution region shaded for clarity. The coordinates of any point in the shaded region of Figure 11(b) specify a solution to the system. For example, the points (2, 4), (6, 3), and (7.43, 8.56) are three of infinitely many solutions, as can be easily checked. The intersection point (2, 4) can be obtained by solving the equations $x + y = 6$ and $2x - y = 0$ simultaneously.



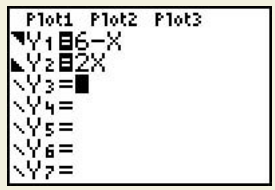
› Figure 11

Graphing Calculator Solution

Solve each inequality for y:

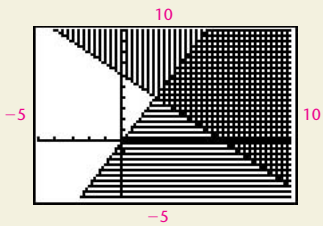
$$x + y \geq 6$$
$$y \geq 6 - x$$
$$2x - y \geq 0$$
$$-y \geq -2x$$
$$y \leq 2x$$

Enter $y_1 = 6 - x$ and select the shade-above option (Fig. 12). Enter $y_2 = 2x$ and select the shade-below option (Fig. 12).



› Figure 12

Graph the inequalities (Fig. 13). The region shaded with both horizontal and vertical lines is the solution region.



› Figure 13

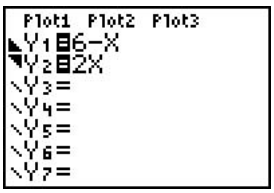
MATCHED PROBLEM 3

Graph the solution region for the following system of linear inequalities two ways, as in Example 3:

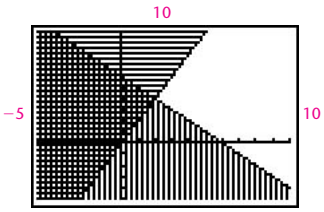
$$\begin{aligned} 3x + y &\le 21 \\ x - 2y &\le 0 \end{aligned}$$

››› EXPLORE-DISCUSS 2

Refer to Example 3. Graph each boundary line and shade the regions obtained by reversing each inequality. That is, shade the region of the plane that corresponds to the inequality $x + y < 6$ and then shade the region that corresponds to the inequality $2x - y < 0$. What portion of the plane is left unshaded? Compare this method with the one used in the solution to Example 3.



(a)



(b)

› Figure 14

The method of solving inequalities investigated in Explore-Discuss 2 works very well on a graphing calculator that enables the user to shade above and below a graph. Referring to Example 3, the unshaded region in Figure 14(b) corresponds to the solution region in Figure 13.

The points of intersection of the lines that form the boundary of a solution region play a fundamental role in the solution of linear programming problems, which are discussed in the next section.

› DEFINITION 1 Corner Point

A **corner point** of a solution region is a point in the solution region that is the intersection of two boundary lines.

The point (2, 4) is the only corner point of the solution region in Example 3; see Figure 11(b).

EXAMPLE 4 Solving a System of Linear Inequalities

Graph the solution region for the following system of linear inequalities, and find the corner points.

$$\begin{aligned} 2x + y &\le 22 \\ x + y &\le 13 \\ 2x + 5y &\le 50 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

SOLUTIONS

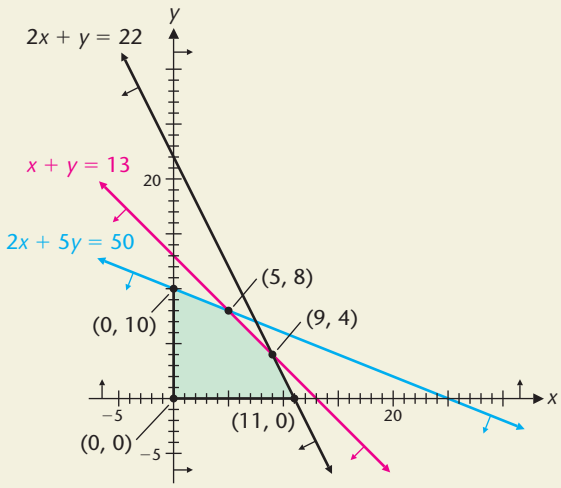
Algebraic Solution

The inequalities $x \geq 0$ and $y \geq 0$, called **nonnegative restrictions,** occur frequently in applications involving systems of inequalities because x and y often represent quantities that can't be negative—number of units produced, number of hours worked, and the like. The solution region lies in the first quadrant, and we can restrict our attention to that portion of the plane. First we graph the lines.

$$\begin{aligned} 2x + y &= 22 \\ x + y &= 13 \\ 2x + 5y &= 50 \end{aligned}$$

Find the x and y intercepts of each line; then sketch the line through these points, as shown in Figure 15.

Next, choosing (0, 0) as a test point, we see that the graph of each of the first three inequalities in the system consists of its corresponding line and the half-plane lying below it. Thus, the solution region of the system consists of the points in the first quadrant that simultaneously lie below all three of these lines (Fig. 15).



› Figure 15

The corner points (0, 0), (0, 10), and (11, 0) can be determined from the graph. The other two corner points are determined as follows:

Solve the system

$$\begin{aligned} 2x + 5y &= 50 \\ x + y &= 13 \end{aligned}$$

to obtain (5, 8).

Solve the system

$$\begin{aligned} 2x + y &= 22 \\ x + y &= 13 \end{aligned}$$

to obtain (9, 4).

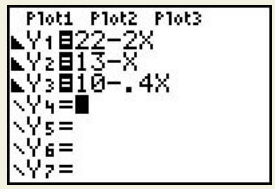
Note that the lines $2x + 5y = 50$ and $2x + y = 22$ also intersect, but the intersection point is not part of the solution region, and hence, is not a corner point.

Graphing Calculator Solution

Solve each inequality for y.

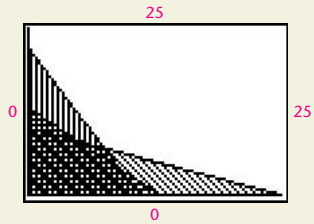
$$\begin{aligned} 2x + y &\leq 22 \\ y &\leq 22 - 2x \\ x + y &\leq 13 \\ y &\leq 13 - x \\ 2x + 5y &\leq 50 \\ 5y &\leq 50 - 2x \\ y &\leq 10 - 0.4x \end{aligned}$$

Enter the equation of each boundary line and select the desired shading option (Fig.16).



› Figure 16

The nonnegative restrictions, $x \geq 0$ and $y \geq 0$, indicate that the solution region lies in the first quadrant. To restrict the graph on a graphing calculator to the first quadrant, simply choose Xmin = 0 and Ymin = 0. The solution region is the five-sided polygon in the lower left corner of the screen that is shaded horizontally, vertically, and diagonally (Fig. 17).



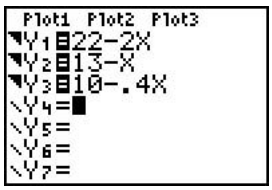
› Figure 17

The corner points are (0, 0), (0, 10), (11, 0), (9, 4), and (5, 8). The last two were found by using the INTERSECT command (details omitted).

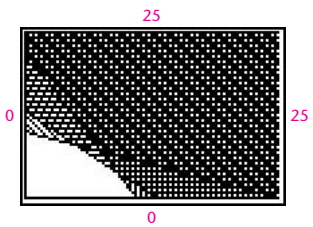
MATCHED PROBLEM 4

Graph the solution region of the following system of linear inequalities two ways, as in Example 4.

$$\begin{aligned} 5x + y &\geq 20 \\ x + y &\geq 12 \\ x + 3y &\geq 18 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$



(a)



(b)

› Figure 18

If you find it difficult to recognize the solution region in Figure 17, you might want to consider the technique of shading the complement discussed in Explore-Discuss 2 and illustrated in Figure 18.

If we compare the solution regions of Examples 3 and 4, we see that there is a fundamental difference between these two regions. We can draw a circle around the solution region in Example 4. However, it is impossible to include all the points in the solution region in Example 3 in any circle, no matter how large we draw it. This leads to the following definition.

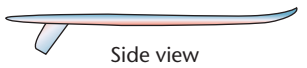
› **DEFINITION 2 Bounded and Unbounded Solution Regions**

A solution region of a system of linear inequalities is **bounded** if it can be enclosed within a circle. If it cannot be enclosed within a circle, then it is **unbounded.**

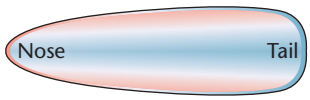
Thus, the solution region for Example 4 is bounded and the solution region for Example 3 is unbounded. This definition will be important in the next section.

› Mathematical Modeling with Systems of Linear Inequalities

EXAMPLE 5 Production Scheduling



Side view



Top view

A manufacturer of surfboards makes a standard model and a competition model. Each standard board requires 6 labor-hours for fabricating and 1 labor-hour for finishing. Each competition board requires 8 labor-hours for fabricating and 3 labor-hours for finishing. The maximum labor-hours available per week in the fabricating and finishing departments are 120 and 30, respectively. What combinations of boards can be produced each week so as not to exceed the number of labor-hours available in each department per week?

SOLUTION

To clarify relationships, we summarize the information in the following table:

	Standard model (labor-hours per board)	Competition model (labor-hours per board)	Maximum labor-hours available per week
Fabricating	6	8	120
Finishing	1	3	30

Let

$$x = \text{Number of standard boards produced per week}$$
$$y = \text{Number of competition boards produced per week}$$

These variables are restricted as follows:

Fabricating department restriction:

$$\begin{pmatrix}\text{Weekly fabricating}\\ \text{time for } x\\ \text{standard boards}\end{pmatrix} + \begin{pmatrix}\text{Weekly fabricating}\\ \text{time for } y\\ \text{competition boards}\end{pmatrix} \le \begin{pmatrix}\text{Maximum labor-hours}\\ \text{available per week}\end{pmatrix}$$
$$6x + 8y \le 120$$

Finishing department restriction:

$$\begin{pmatrix}\text{Weekly finishing}\\ \text{time for } x\\ \text{standard boards}\end{pmatrix} + \begin{pmatrix}\text{Weekly finishing}\\ \text{time for } y\\ \text{competition boards}\end{pmatrix} \le \begin{pmatrix}\text{Maximum labor-hours}\\ \text{available per week}\end{pmatrix}$$
$$1x + 3y \le 30$$

Because it is not possible to manufacture a negative number of boards, x and y also must satisfy the nonnegative restrictions

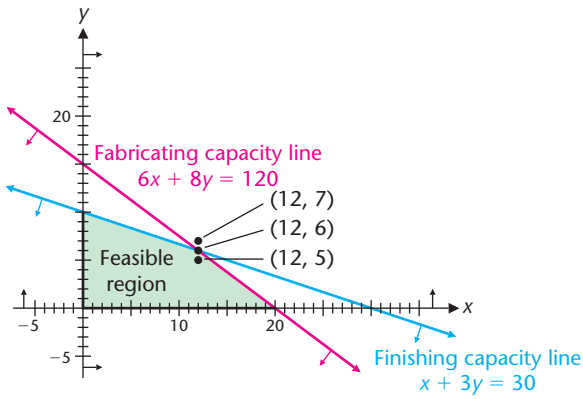
$$x \ge 0$$
$$y \ge 0$$

Thus, x and y must satisfy the following system of linear inequalities:

$$\begin{aligned} 6x + 8y &\le 120 && \text{Fabricating department restriction}\\ x + 3y &\le 30 && \text{Finishing department restriction}\\ x &\ge 0 && \text{Nonnegative restriction}\\ y &\ge 0 && \text{Nonnegative restriction}\end{aligned}$$

Graphing this system of linear inequalities, we obtain the set of **feasible solutions,** or the **feasible region,** as shown in Figure 19. For problems of this type and for the linear programming problems we consider in the next section, solution regions are often referred to as feasible regions. Any point within the shaded area, including the boundary

lines, represents a possible production schedule. Any point outside the shaded area represents an impossible schedule. For example, it would be possible to produce 12 standard boards and 5 competition boards per week, but it would not be possible to produce 12 standard boards and 7 competition boards per week (see Fig. 19).



› Figure 19

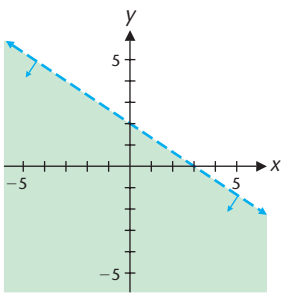
MATCHED PROBLEM 5

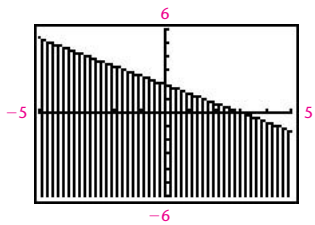
Repeat Example 5 using 5 hours for fabricating a standard board and a maximum of 27 labor-hours for the finishing department.

REMARK: Refer to Example 5. How do we interpret a production schedule of 10.5 standard boards and 4.3 competition boards? It is not possible to manufacture a fraction of a board. But it is possible to *average* 10.5 standard and 4.3 competition boards per week. In general, we will assume that all points in the feasible region represent acceptable solutions, although noninteger solutions might require special interpretation.

ANSWERS TO MATCHED PROBLEMS

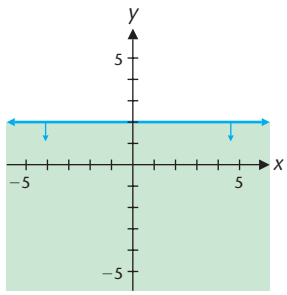
1.



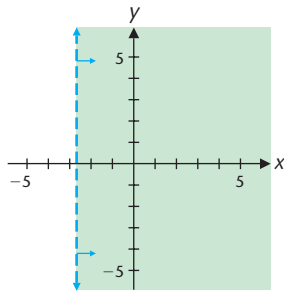


The boundary line is not part of the solution.

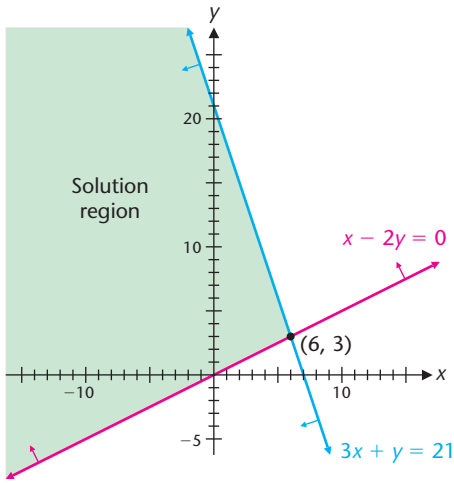
2. (A)

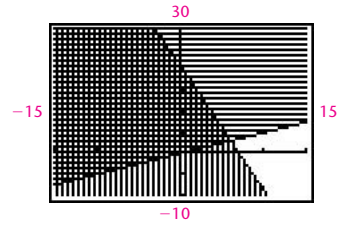


(B)



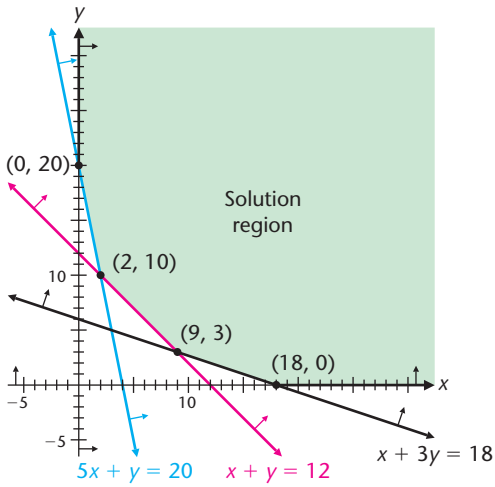
3.

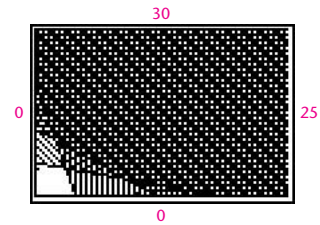




The boundary of the solution region is part of the solution region.

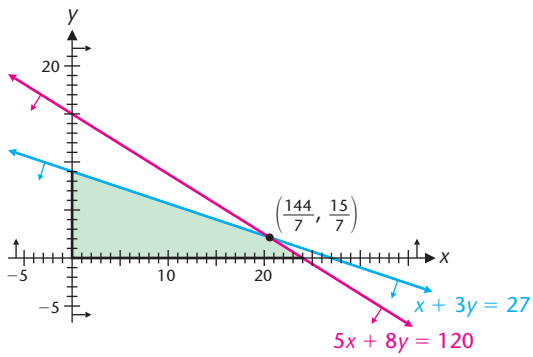
4.





The boundary of the solution region is part of the solution region.

5.



8-3 Exercises

1. If $m \neq 0$, describe geometrically the graphs of

(A) $y = mx + b$ (B) $y < mx + b$ (C) $y > mx + b$

2. If $m \neq 0$, describe geometrically the graphs of

(A) $y \leq mx + b$ (B) $y \geq mx + b$

3. What are nonnegativity restrictions? How do they affect the graph of a system of linear inequalities?

4. In an applied problem, if x is the number of tables produced and y is the number of chairs produced, how would you interpret the solution $x = 12.6$ and $y = 23.2$?

In Problems 5–14, graph the solution region for each inequality and write a verbal description of the solution region.

5. $2x - 3y < 6$

6. $3x + 4y < 12$

7. $3x + 2y \geq 18$

8. $3y - 2x \geq 24$

9. $y \leq \frac{2}{3}x + 5$

10. $y \geq \frac{1}{3}x - 2$

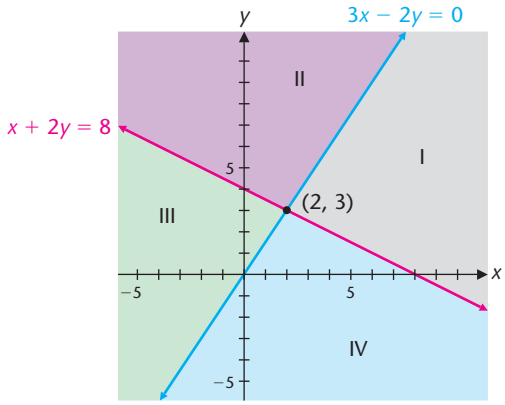
11. $y < 8$

12. $x > -5$

13. $-3 \leq y < 2$

14. $-1 < x \leq 3$

In Problems 15–18, match the solution region of each system of linear inequalities with one of the four regions shown in the figure.



15. $x + 2y \leq 8$
$3x - 2y \geq 0$

16. $x + 2y \geq 8$
$3x - 2y \leq 0$

17. $x + 2y \geq 8$
$3x - 2y \geq 0$

18. $x + 2y \leq 8$
$3x - 2y \leq 0$

In Problems 19–24, graph the solution region for each system of linear inequalities.

19. $x \geq 5$
$y \leq 6$

20. $x \leq 4$
$y \geq 2$

21. $3x + y \geq 6$
$x \leq 4$

22. $3x + 4y \leq 12$
$y \geq -3$

23. $x - 2y \leq 12$
$2x + y \geq 4$

24. $2x + 5y \leq 20$
$x - 5y \leq -5$

Problems 25–28 require a graphing calculator that gives the user the option of shading above or below a graph.

(A) Graph the boundary lines in a standard viewing window and shade the region that contains the points that satisfy each inequality.

(B) Repeat part A, but this time shade the region that contains the points that do not satisfy each inequality (see Explore-Discuss 2)

Explain how you can recognize the solution region in each graph.

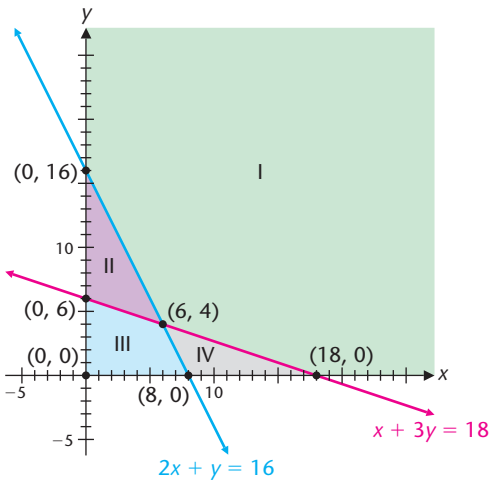
25. $x + y \leq 5$
$2x - y \leq 1$

26. $x - 2y \leq 1$
$x + 3y \geq 12$

27. $2x + y \geq 4$
$3x - y \leq 7$

28. $3x + y \geq -2$
$x - 2y \geq -6$

In Problems 29–32, match the solution region of each system of linear inequalities with one of the four regions shown in the figure below. Identify the corner points of each solution region.



29. $x + 3y \leq 18$
$2x + y \geq 16$
$x \geq 0$
$y \geq 0$

30. $x + 3y \leq 18$
$2x + y \leq 16$
$x \geq 0$
$y \geq 0$

31. $x + 3y \geq 18$
$2x + y \geq 16$
$x \geq 0$
$y \geq 0$

32. $x + 3y \geq 18$
$2x + y \leq 16$
$x \geq 0$
$y \geq 0$

In Problems 33–52, graph the solution region for each system, and indicate whether each solution region is bounded or unbounded. Find the coordinates of each corner point.

33. $2x + 3y \leq 6$
$x \geq 0$
$y \geq 0$

34. $4x + 3y \leq 12$
$x \geq 0$
$y \geq 0$

35. $4x + 5y \geq 20$
$x \geq 0$
$y \geq 0$

36. $5x + 6y \geq 30$
$x \geq 0$
$y \geq 0$

37. $2x + y \leq 8$
$x + 3y \leq 12$
$x \geq 0$
$y \geq 0$

38. $x + 2y \leq 10$
$3x + y \leq 15$
$x \geq 0$
$y \geq 0$

39. $4x + 3y \geq 24$
$2x + 3y \geq 18$
$x \geq 0$
$y \geq 0$

40. $x + 2y \geq 8$
$2x + y \geq 10$
$x \geq 0$
$y \geq 0$

41. $2x + y \leq 12$
$x + y \leq 7$
$x + 2y \leq 10$
$x \geq 0$
$y \geq 0$

42. $3x + y \leq 21$
$x + y \leq 9$
$x + 3y \leq 21$
$x \geq 0$
$y \geq 0$

43. $x + 2y \geq 16$
$x + y \geq 12$
$2x + y \geq 14$
$x \geq 0$
$y \geq 0$

44. $3x + y \geq 30$
$x + y \geq 16$
$x + 3y \geq 24$
$x \geq 0$
$y \geq 0$

45. $x + y \leq 11$
$5x + y \geq 15$
$x + 2y \geq 12$

46. $4x + y \leq 32$
$x + 3y \leq 30$
$5x + 4y \geq 51$

47. $3x + 2y \geq 24$
$3x + y \leq 15$
$x \geq 4$

48. $3x + 4y \leq 48$
$x + 2y \geq 24$
$y \leq 9$

49. $x + y \leq 10$
$3x + 5y \geq 15$
$3x - 2y \leq 15$
$-5x + 2y \leq 6$

50. $3x - y \geq 1$
$-x + 5y \geq 9$
$x + y \leq 9$
$y \leq 5$

51. $16x + 13y \leq 119$
$12x + 16y \geq 101$
$-4x + 3y \leq 11$

52. $8x + 4y \leq 41$
$-15x + 5y \leq 19$
$2x + 6y \geq 37$

APPLICATIONS

53. MANUFACTURING—RESOURCE ALLOCATION A manufacturing company makes two types of water skis: a trick ski and a slalom ski. The trick ski requires 6 labor-hours for fabricating and 1 labor-hour for finishing. The slalom ski requires 4 labor-hours for fabricating and 1 labor-hour for finishing. The maximum labor-hours available per day for fabricating and finishing are 108 and 24, respectively. If x is the number of trick skis and y is the number of slalom skis produced per day, write a system of inequalities that indicates appropriate restrictions on x and y. Graph the set of feasible solutions for the number of each type of ski that can be produced.

54. MANUFACTURING—RESOURCE ALLOCATION A furniture manufacturing company manufactures dining room tables and chairs. A table requires 8 labor-hours for assembling and 2 labor-hours for finishing. A chair requires 2 labor-hours for assembling and 1 labor-hour for finishing. The maximum labor-hours available per day for assembly and finishing are 400 and 120, respectively. If x is the number of tables and y is the number of chairs produced per day, write a system of inequalities that indicates appropriate restrictions on x and y. Graph the set of feasible solutions for the number of tables and chairs that can be produced.

55. MANUFACTURING—RESOURCE ALLOCATION Refer to Problem 53. The company makes a profit of \$50 on each trick ski and a profit of \$60 on each slalom ski.
(A) If the company makes 10 trick and 10 slalom skis per day, the daily profit will be \$1,100. Are there other feasible production schedules that will result in a daily profit of \$1,100? How are these schedules related to the graph of the line $50x + 60y = 1{,}100$?
(B) Find a feasible production schedule that will produce a daily profit greater than \$1,100 and repeat part A for this schedule.
(C) Discuss methods for using lines like those in parts A and B to find the largest possible daily profit.

56. MANUFACTURING—RESOURCE ALLOCATION Refer to Problem 54. The company makes a profit of \$50 on each table and a profit of \$15 on each chair.
(A) If the company makes 20 tables and 20 chairs per day, the daily profit will be \$1,300. Are there other feasible production schedules that will result in a daily profit of \$1,300? How are these schedules related to the graph of the line $50x + 15y = 1{,}300$?
(B) Find a feasible production schedule that will produce a daily profit greater than \$1,300 and repeat part A for this schedule.
(C) Discuss methods for using lines like those in parts A and B to find the largest possible daily profit.

57. NUTRITION—PLANTS A farmer can buy two types of fertilizer, mix A and mix B. Each cubic yard of mix A contains 20 pounds of phosphoric acid, 30 pounds of nitrogen, and 5 pounds of potash. Each cubic yard of mix B contains 10 pounds of phosphoric acid, 30 pounds of nitrogen, and 10 pounds of potash. The minimum requirements are 460 pounds of phosphoric acid, 960 pounds of nitrogen, and 220 pounds of potash. If x is the number of cubic yards of mix A used and y is the number of cubic yards of mix B used, write a system of inequalities that indicates appropriate restrictions on x and y. Graph the set of feasible solutions for the amount of mix A and mix B that can be used.

58. NUTRITION A dietitian in a hospital is to arrange a special diet using two foods. Each ounce of food M contains 30 units of calcium, 10 units of iron, and 10 units of vitamin A. Each ounce

of food N contains 10 units of calcium, 10 units of iron, and 30 units of vitamin A. The minimum requirements in the diet are 360 units of calcium, 160 units of iron, and 240 units of vitamin A. If x is the number of ounces of food M used and y is the number of ounces of food N used, write a system of linear inequalities that reflects the conditions indicated. Graph the set of feasible solutions for the amount of each kind of food that can be used.

59. SOCIOLOGY A city council voted to conduct a study on inner-city community problems. A nearby university was contacted to provide sociologists and research assistants. Each sociologist will spend 10 hours per week collecting data in the field and 30 hours per week analyzing data in the research center. Each research assistant will spend 30 hours per week in the field and 10 hours per week in the research center. The minimum weekly labor-hour requirements are 280 hours in the field and 360 hours in the research center. If x is the number of sociologists hired for the study and y is the number of research assistants hired for the study, write a system of linear inequalities that indicates appropriate restrictions on x and y. Graph the set of feasible solutions.

60. PSYCHOLOGY In an experiment on conditioning, a psychologist uses two types of Skinner (conditioning) boxes with mice and rats. Each mouse spends 10 minutes per day in box A and 20 minutes per day in box B. Each rat spends 20 minutes per day in box A and 10 minutes per day in box B. The total maximum time available per day is 800 minutes for box A and 640 minutes for box B. We are interested in the various numbers of mice and rats that can be used in the experiment under the conditions stated. If x is the number of mice used and y is the number of rats used, write a system of linear inequalities that indicates appropriate restrictions on x and y. Graph the set of feasible solutions.

8-4 Linear Programming

- A Linear Programming Problem
- Linear Programming—A General Description
- Mathematical Modeling and Linear Programming

Several problems in the last section are related to the general type of problems called *linear programming problems.* Linear programming is a mathematical process that has been developed to help management in decision making, and it has become one of the most widely used and best known tools of management science and industrial engineering. We will use an intuitive graphical approach based on the techniques discussed in the last section to illustrate this process for problems involving two variables.

The American mathematician George B. Dantzig (1914–2005) formulated the first linear programming problem in 1947 and introduced a solution technique, called the *simplex method,* that does not rely on graphing and is readily adaptable to computer solutions. Today, it is quite common to use a computer to solve applied linear programming problems involving thousands of variables and thousands of inequalities.

A Linear Programming Problem

We begin our discussion with an example that will lead to a general procedure for solving linear programming problems in two variables.

EXAMPLE 1 Production Scheduling

A manufacturer of fiberglass camper tops for pickup trucks makes a compact model and a regular model. Each compact top requires 5 hours from the fabricating department and 2 hours from the finishing department. Each regular top requires 4 hours from the fabricating department and 3 hours from the finishing department. The maximum

labor-hours available per week in the fabricating department and the finishing department are 200 and 108, respectively. If the company makes a profit of $40 on each compact top and $50 on each regular top, how many tops of each type should be manufactured each week to maximize the total weekly profit, assuming all tops can be sold? What is the maximum profit?

SOLUTION

This is an example of a linear programming problem. To see relationships more clearly, we summarize the manufacturing requirements, objectives, and restrictions in the table:

	Compact model (labor-hours per top)	Regular model (labor-hours per top)	Maximum labor-hours available per week
Fabricating	5	4	200
Finishing	2	3	108
Profit per top	$40	$50	

We now proceed to formulate a *mathematical model* for the problem and then to solve it using graphical methods.

OBJECTIVE FUNCTION

The *objective* of management is to *decide* how many of each camper-top model should be produced each week to *maximize* profit. Let

$$\left.\begin{aligned} x &= \text{Number of compact tops produced per week} \\ y &= \text{Number of regular tops produced per week} \end{aligned}\right\} \text{ Decision variables}$$

The following function gives the total profit P for x compact tops and y regular tops manufactured each week:

$$P = 40x + 50y \qquad \text{Objective function}$$

Mathematically, management needs to decide on values for the **decision variables** (x and y) that achieve its objective, that is, maximizing the **objective function** (profit) $P = 40x + 50y$. It appears that the profit can be made as large as we like by manufacturing more and more tops—or can it?

CONSTRAINTS

Any manufacturing company, no matter how large or small, has manufacturing limits imposed by available resources, plant capacity, demand, and so forth. These limits are referred to as **problem constraints.**

Fabricating department constraint:

$$\begin{pmatrix}\text{Weekly fabricating}\\ \text{time for } x\\ \text{compact tops}\end{pmatrix} + \begin{pmatrix}\text{Weekly fabricating}\\ \text{time for } y\\ \text{regular tops}\end{pmatrix} \le \begin{pmatrix}\text{Maximum labor-hours}\\ \text{available per week}\end{pmatrix}$$

$$5x \quad + \quad 4y \quad \le \quad 200$$

Finishing department constraint:

$$\begin{pmatrix}\text{Weekly finishing}\\ \text{time for } x\\ \text{compact tops}\end{pmatrix} + \begin{pmatrix}\text{Weekly finishing}\\ \text{time for } y\\ \text{regular tops}\end{pmatrix} \le \begin{pmatrix}\text{Maximum labor-hours}\\ \text{available per week}\end{pmatrix}$$

$$2x + 3y \le 108$$

Nonnegative constraints: It is not possible to manufacture a negative number of tops; thus, we have the **nonnegative constraints**

$$x \ge 0$$
$$y \ge 0$$

which we usually write in the form

$$x, y \ge 0$$

MATHEMATICAL MODEL

We now have a **mathematical model** for the problem under consideration:

$$\begin{aligned} &\text{Maximize} && P = 40x + 50y && \text{Objective function}\\ &\text{Subject to} && 5x + 4y \le 200 \\ &&& 2x + 3y \le 108 && \text{Problem constraints}\\ &&& x, y \ge 0 && \text{Nonnegative constraints}\end{aligned}$$

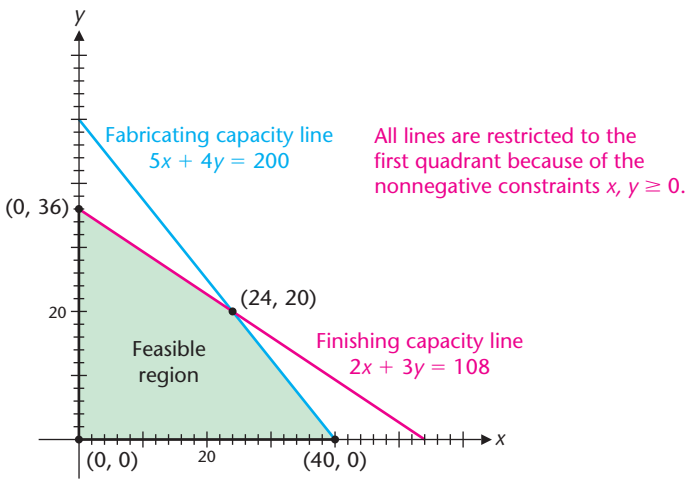
GRAPHICAL SOLUTION

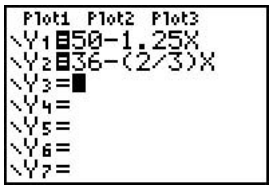
The feasible region is the solution of the system of linear inequality constraints (see the last section). We can graph the feasible region by hand (Figure 1) or on a graphing calculator (Figure 2).

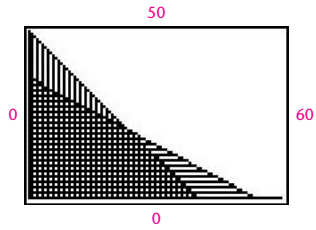
By choosing a production schedule (x, y) from the feasible region, a profit can be determined using the objective function $P = 40x + 50y$. For example, if $x = 24$ and $y = 10$, then the profit for the week is

$$P = 40(24) + 50(10) = \$1{,}460$$

› Figure 1







› Figure 2

Corner point (x, y)	Objective function $P = 40x + 50y$	
(0, 0)	0	
(0, 36)	1,800	
(24, 20)	1,960	Maximum value of P
(40, 0)	1,600	

Or if $x = 15$ and $y = 20$, then the profit for the week is

$$P = 40(15) + 50(20) = \$1{,}600$$

The question is, out of all possible production schedules (x, y) from the feasible region, which schedule(s) produce(s) the maximum profit? Such a schedule, if it exists, is called an **optimal solution** to the problem because it produces the maximum value of the objective function and is in the feasible region. It is not practical to use point-by-point checking to find the optimal solution. Even if we consider only points with integer coordinates, there are over 800 such points in the feasible region for this problem. Instead, we use the theory that has been developed to solve linear programming problems. Using advanced techniques, it can be shown that:

If the feasible region is bounded, then one or more of the corner points of the feasible region is an optimal solution to the problem.

The maximum value of the objective function is unique; however, there can be more than one feasible production schedule that will produce this unique value. We will have more to say about this later in this section.

Because the feasible region for this problem is bounded, at least one of the corner points, (0, 0), (0, 36), (24, 20), or (40, 0), is an optimal solution. To find which one, we evaluate $P = 40x + 50y$ at each corner point and choose the corner point that produces the largest value of P. It is convenient to organize these calculations in a table, as shown in the margin.

Examining the values in the table, we see that the maximum value of P at a corner point is $P = 1{,}960$ at $x = 24$ and $y = 20$. Because the maximum value of P over the entire feasible region must always occur at a corner point, we conclude that the maximum profit is \$1,960 when 24 compact tops and 20 regular tops are produced each week.

MATCHED PROBLEM 1

We now convert the surfboard problem discussed in the last section into a linear programming problem. A manufacturer of surfboards makes a standard model and a competition model. Each standard board requires 6 labor-hours for fabricating and 1 labor-hour for finishing. Each competition board requires 8 labor-hours for fabricating and 3 labor-hours for finishing. The maximum labor-hours available per week in the fabricating and finishing departments are 120 and 30, respectively. If the company makes a profit of \$40 on each standard board and \$75 on each competition board, how many boards of each type should be manufactured each week to maximize the total weekly profit?

(A) Identify the decision variables.

(B) Write the objective function P.

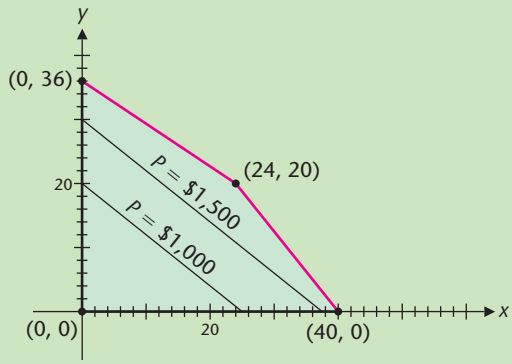
(C) Write the problem constraints and the nonnegative constraints.

(D) Graph the feasible region, identify the corner points, and evaluate P at each corner point.

(E) How many boards of each type should be manufactured each week to maximize the profit? What is the maximum profit?

››› EXPLORE-DISCUSS 1

Refer to Example 1. If we assign the profit P in $P = 40x + 50y$ a particular value and plot the resulting equation in the coordinate system shown in Figure 1, we obtain a **constant-profit line (isoprofit line).** Every point in the feasible region on this line represents a production schedule that will produce the same profit. Figure 3 shows the constant-profit lines for $P = \$1{,}000$ and $P = \$1{,}500$.



› Figure 3

(A) How are all the constant-profit lines related?

(B) Place a straightedge along the constant-profit line for $P = \$1{,}000$ and slide it as far as possible in the direction of increasing profit without changing its slope and without leaving the feasible region. Explain how this process can be used to identify the optimal solution to a linear programming problem.

(C) If P is changed to $P = 25x + 75y$, graph the constant-profit lines for $P = \$1{,}000$ and $P = \$1{,}500$, and use a straightedge to identify the optimal solution. Check your answer by evaluating P at each corner point.

(D) Repeat part C for $P = 75x + 25y$.

› Linear Programming—A General Description

The linear programming problems considered in Example 1 and Matched Problem 1 were *maximization problems,* where we wanted to maximize profits. The same technique can be used to solve *minimization problems,* where, for example, we may want to minimize costs. Before considering additional examples, we state a few general definitions.

A **linear programming problem** is one that is concerned with finding the **optimal value** (maximum or minimum value) of a linear *objective function* of the form

$$z = ax + by$$

where the **decision variables** x and y are subject to **problem constraints** in the form of linear inequalities and to **nonnegative constraints** $x, y \geq 0$. The set of points

satisfying both the problem constraints and the nonnegative constraints is called the **feasible region** for the problem. Any point in the feasible region that produces the optimal value of the objective function over the feasible region is called an **optimal solution.**

Theorem 1 is fundamental to the solving of linear programming problems.

THEOREM 1 Fundamental Theorem of Linear Programming

Let S be the feasible region for a linear programming problem, and let $z = ax + by$ be the objective function. If S is bounded, then z has both a maximum and a minimum value on S and each of these occurs at a corner point of S. If S is unbounded, then a maximum or minimum value of z on S may not exist. However, if either does exist, then it must occur at a corner point of S.

We will not consider any problems with unbounded feasible regions in this brief introduction. If a feasible region is bounded, then Theorem 1 provides the basis for the following simple procedure for solving the associated linear programming problem:

SOLUTION OF LINEAR PROGRAMMING PROBLEMS

Step 1. Form a mathematical model for the problem:

(A) Introduce decision variables and write a linear objective function.

(B) Write problem constraints in the form of linear inequalities.

(C) Write nonnegative constraints.

Step 2. Graph the feasible region and find the corner points.

Step 3. Evaluate the objective function at each corner point to determine the optimal solution.

Before considering additional applications, we use this procedure to solve a linear programming problem where the model has already been determined.

EXAMPLE 2 Solving a Linear Programming Problem

Minimize and maximize $z = 5x + 15y$

Subject to $x + 3y \le 60$

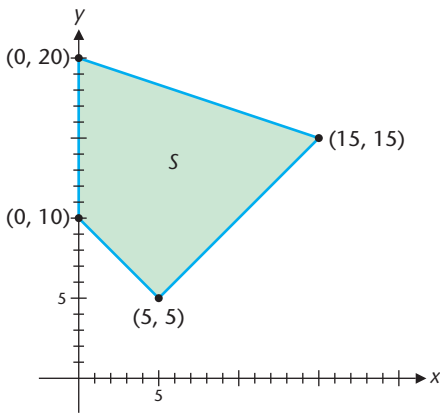
$x + y \ge 10$

$x - y \le 0$

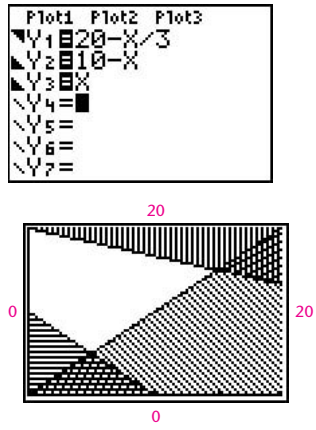
$x, y \ge 0$

SOLUTION

This problem is a combination of two linear programming problems—a minimization problem and a maximization problem. Because the feasible region is the same for both problems, we can solve these problems together. To begin, we graph the system of linear inequalities formed by the constraints given in the example (details omitted), we obtain the feasible region S shown in Figure 4. Figure 5 shows S graphed on a graphing calculator using the complement method discussed in Explore-Discuss 2 in the last section.



› Figure 4



› Figure 5

Next, we evaluate the objective function at each corner point, with the results given in the table:

Corner point (x, y)	Objective function $z = 5x + 15y$	
(0, 10)	150	
(0, 20)	300	Maximum value
(15, 15)	300	Maximum value
(5, 5)	100	Minimum value

Multiple optimal solutions

Examining the values in the table, we see that the minimum value of z on the feasible region S is 100 at (5, 5). Thus, (5, 5) is the optimal solution to the minimization problem. The maximum value of z on the feasible region S is 300, which occurs at (0, 20) and at (15, 15). Thus, the maximization problem has *multiple optimal solutions*. In general,

If two corner points are both optimal solutions of the same type (both produce the same maximum value or both produce the same minimum value) to a linear programming problem, then any point on the line segment joining the two corner points is also an optimal solution of that type.

It can be shown that this is the only way that an optimal value occurs at more than one point.

MATCHED PROBLEM 2

Minimize and maximize $z = 10x + 5y$
Subject to $2x + y \geq 40$
$3x + y \leq 150$
$2x - y \geq 0$
$x, y \geq 0$

Mathematical Modeling and Linear Programming

Now we consider another application where we must first find the mathematical model and then find its solution.

EXAMPLE 3 Agriculture

	Pounds per cubic yard	
	Mix *A*	Mix *B*
Nitrogen	10	5
Potash	8	24
Phosphoric acid	9	6

A farmer can use two types of fertilizer, mix *A* and mix *B*. The amounts (in pounds) of nitrogen, phosphoric acid, and potash in a cubic yard of each mix are given in the table. Tests performed on the soil in a large field indicate that the field needs at least 840 pounds of potash and at least 350 pounds of nitrogen. The tests also indicate that no more than 630 pounds of phosphoric acid should be added to the field. A cubic yard of mix *A* costs \$7, and a cubic yard of mix *B* costs \$9. How many cubic yards of each mix should the farmer add to the field to supply the necessary nutrients at minimal cost?

SOLUTION

Let

x = Number of cubic yards of mix *A* added to the field
y = Number of cubic yards of mix *B* added to the field
} Decision variables

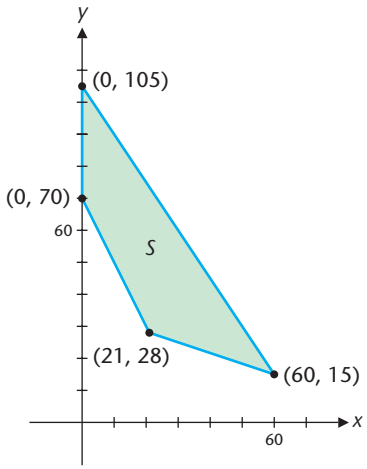
We form the linear objective function

$$C = 7x + 9y$$

which gives the cost of adding x cubic yards of mix *A* and y cubic yards of mix *B* to the field. Using the data in the table and proceeding as in Example 1, we formulate the mathematical model for the problem:

Minimize	$C = 7x + 9y$	Objective function
Subject to	$10x + 5y \geq 350$	Nitrogen constraint
	$8x + 24y \geq 840$	Potash constraint
	$9x + 6y \leq 630$	Phosphoric acid constraint
	$x, y \geq 0$	Nonnegative constraints

The feasible region S and its corner points are shown in Figure 6 (graphing details omitted).



› Figure 6

Corner point (x, y)	Objective function $C = 7x + 9y$	
(0, 105)	945	
(0, 70)	630	
(21, 28)	399	Minimum value of C
(60, 15)	555	

Next, we evaluate the objective function at each corner point, as shown in the table. The optimal value is $C = 399$ at the corner point (21, 28). Thus, the farmer should add 21 cubic yards of mix A and 28 cubic yards of mix B at a cost of \$399. This will result in adding the following nutrients to the field:

Nitrogen:	$10(21) + 5(28) = 350$ pounds
Potash:	$8(21) + 24(28) = 840$ pounds
Phosphoric acid:	$9(21) + 6(28) = 357$ pounds

All the nutritional requirements are satisfied.

MATCHED PROBLEM 3

Repeat Example 3 if the tests indicate that the field needs at least 400 pounds of nitrogen with all other conditions remaining the same.

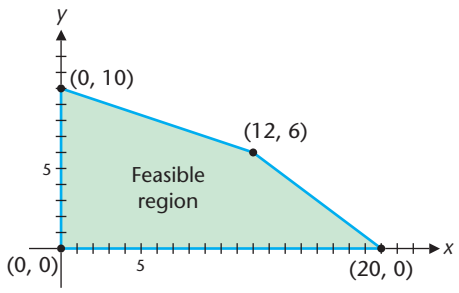
ANSWERS TO MATCHED PROBLEMS

1. (A) x = Number of standard boards manufactured each week
y = Number of competition boards manufactured each week

(B) $P = 40x + 75y$

(C) $6x + 8y \le 120$ Fabricating constraint
$x + 3y \le 30$ Finishing constraint
$x, y \ge 0$ Nonnegative constraints

(D)



Corner point (x, y)	Objective function $P = 40x + 75y$
(0, 0)	0
(0, 10)	750
(12, 6)	930
(20, 0)	800

(E) 12 standard boards and 6 competition boards for a maximum profit of \$930

2. Max $z = 600$ at (30, 60); min $z = 200$ at (10, 20) and (20, 0) (multiple optimal solutions)

3. 27 cubic yards of mix A, 26 cubic yards of mix B; min $C = \$423$

8-4 Exercises

In Problems 1–6, explain how each term is used in the description of a linear programming program.

1. Objective function

2. Decision variables

3. Problem constraints

4. Nonnegativity constraints

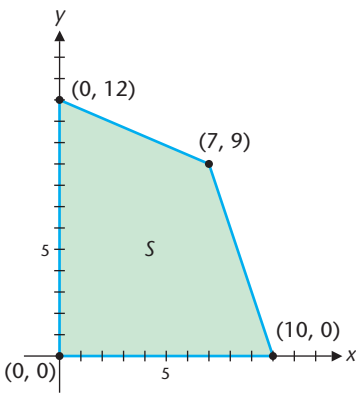
5. Feasible region

6. Optimal value and optimal solution

7. Can a linear programming problem have more than one optimal value? Explain.

8. Can a linear programming problem have more than one optimal value? Explain.

In Problems 9–12, find the maximum value of each objective function over the feasible region S shown in the figure.



9. $z = x + y$ **10.** $z = 4x + y$ **11.** $z = 3x + 7y$

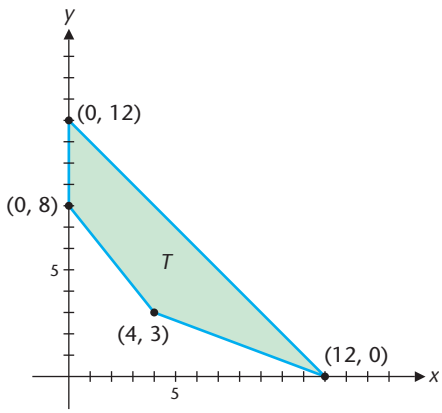
12. $z = 9x + 3y$

Problems 13–16 refer to the feasible region S shown and the constant-profit lines discussed in Explore-Discuss 1. For each objective function, draw the line that passes through the feasible point (5, 5) and use the straightedge method from Explore-Discuss 1 to find the maximum value. Check your answer by evaluating the objective function at each corner point.

13. $z = x + 2y$ **14.** $z = 3x + y$ **15.** $z = 7x + 2y$

16. $z = 2x + 8y$

In Problems 17–20, find the minimum value of each objective function over the feasible region T shown in the figure.



17. $z = 7x + 4y$ **18.** $z = 7x + 9y$ **19.** $z = 3x + 8y$

20. $z = 5x + 4y$

Problems 21–24 refer to the feasible region T shown. For each objective function, draw the constant-value line that passes through the feasible point (5, 5) and use the straightedge method from Explore-Discuss 1 to find the minimum value. Check your answer by evaluating the objective function at each corner point.

21. $z = x + 2y$ **22.** $z = 2x + y$ **23.** $z = 5x + 4y$

24. $z = 2x + 8y$

In Problems 25–38, solve the linear programming problems.

25. Maximize $z = 3x + 2y$
Subject to $x + 2y \le 10$
$3x + y \le 15$
$x, y \ge 0$

26. Maximize $z = 4x + 5y$
Subject to $2x + y \le 12$
$x + 3y \le 21$
$x, y \ge 0$

27. Minimize $z = 3x + 4y$
Subject to $2x + y \ge 8$
$x + 2y \le 10$
$x, y \ge 0$

28. Minimize $z = 2x + y$
Subject to $4x + 3y \ge 24$
$4x + y \le 16$
$x, y \ge 0$

29. Maximize $z = 3x + 4y$
Subject to $x + 2y \le 24$
$x + y \le 14$
$2x + y \le 24$
$x, y \ge 0$

30. Maximize $z = 5x + 3y$
Subject to $3x + y \le 24$
$x + y \le 10$
$x + 3y \le 24$
$x, y \ge 0$

31. Minimize $z = 5x + 6y$
Subject to $x + 4y \ge 20$
$4x + y \ge 20$
$x + y \le 20$
$x, y \ge 0$

32. Minimize $z = x + 2y$
Subject to $2x + 3y \ge 30$
$3x + 2y \ge 30$
$x + y \le 15$
$x, y \ge 0$

33. Minimize and maximize $z = 25x + 50y$
Subject to $x + 2y \le 120$
$x + y \ge 60$
$x - 2y \ge 0$
$x, y \ge 0$

34. Minimize and maximize $z = 15x + 30y$
Subject to $x + 2y \ge 100$
$2x - y \le 0$
$2x + y \le 200$
$x, y \ge 0$

35. Minimize and maximize $z = 25x + 15y$
Subject to $4x + 5y \ge 100$
$3x + 4y \le 240$
$x \le 60$
$y \le 45$
$x, y \ge 0$

36. Minimize and maximize $z = 25x + 30y$
Subject to $2x + 3y \ge 120$
$3x + 2y \le 360$
$x \le 80$
$y \le 120$
$x, y \ge 0$

37. Maximize $P = 525x_1 + 478x_2$
Subject to $275x_1 + 322x_2 \le 3{,}381$
$350x_1 + 340x_2 \le 3{,}762$
$425x_1 + 306x_2 \le 4{,}114$
$x_1, x_2 \ge 0$

38. Maximize $P = 300x_1 + 460x_2$
Subject to $245x_1 + 452x_2 \le 4{,}181$
$290x_1 + 379x_2 \le 3{,}888$
$390x_1 + 299x_2 \le 4{,}407$
$x_1, x_2 \ge 0$

39. The corner points for the feasible region determined by the problem constraints

$$2x + y \le 10$$
$$x + 3y \le 15$$
$$x, y \ge 0$$

are $O = (0, 0)$, $A = (5, 0)$, $B = (3, 4)$, and $C = (0, 5)$. If $z = ax + by$ and $a, b > 0$, determine conditions on a and b that ensure that the maximum value of z occurs

(A) Only at A (B) Only at B
(C) Only at C (D) At both A and B
(E) At both B and C

40. The corner points for the feasible region determined by the problem constraints

$$x + y \ge 4$$
$$x + 2y \ge 6$$
$$2x + 3y \le 12$$
$$x, y \ge 0$$

are $A = (6, 0)$, $B = (2, 2)$, and $C = (0, 4)$. If $z = ax + by$ and $a, b > 0$, determine conditions on a and b that ensure that the minimum value of z occurs

(A) Only at A (B) Only at B
(C) Only at C (D) At both A and B
(E) At both B and C

APPLICATIONS

41. RESOURCE ALLOCATION A manufacturing company makes two types of water skis, a trick ski and a slalom ski. The relevant manufacturing data are given in the table.

(A) If the profit on a trick ski is \$40 and the profit on a slalom ski is \$30, how many of each type of ski should be manufactured each day to realize a maximum profit? What is the maximum profit?

(B) Discuss the effect on the production schedule and the maximum profit if the profit on a slalom ski decreases to \$25 and all other data remains the same.

(C) Discuss the effect on the production schedule and the maximum profit if the profit on a slalom ski increases to \$45 and all other data remain the same.

	Trick ski (labor-hours per ski)	Slalom ski (labor-hours per ski)	Maximum labor-hours available per day
Fabricating department	6	4	108
Finishing department	1	1	24

42. PSYCHOLOGY In an experiment on conditioning, a psychologist uses two types of Skinner boxes with mice and rats. The amount of time (in minutes) each mouse and each rat spends in each box per day is given in the table. What is the maximum total number of mice and rats that can be used in this experiment? How many mice and how many rats produce this maximum?

	Mice (minutes)	Rats (minutes)	Max. time available per day (minutes)
Skinner box A	10	20	800
Skinner box B	20	10	640

43. PURCHASING A trucking firm wants to purchase a maximum of 15 new trucks that will provide at least 36 tons of additional shipping capacity. A model A truck holds 2 tons and costs \$15,000. A model B truck holds 3 tons and costs \$24,000. How many trucks of each model should the company purchase to provide the additional shipping capacity at minimal cost? What is the minimal cost?

44. TRANSPORTATION The officers of a high school senior class are planning to rent buses and vans for a class trip. Each bus can transport 40 students, requires 3 chaperones, and costs \$1,200 to rent. Each van can transport 8 students, requires 1 chaperone, and costs \$100 to rent. The officers want to be able to accommodate at least 400 students with no more than 36 chaperones. How many vehicles of each type should they rent to minimize the transportation costs? What are the minimal transportation costs?

45. RESOURCE ALLOCATION A furniture company manufactures dining room tables and chairs. Each table requires 8 hours from the assembly department and 2 hours from the finishing department and contributes a profit of \$90. Each chair requires 2 hours from the assembly department and 1 hour from the finishing department and contributes a profit of \$25. The maximum labor-hours available each day in the assembly and finishing departments are 400 and 120, respectively.

(A) How many tables and how many chairs should be manufactured each day to maximize the daily profit? What is the maximum daily profit?

(B) Discuss the effect on the production schedule and the maximum profit if the marketing department of the company decides that the number of chairs produced should be at least four times the number of tables produced.

46. RESOURCE ALLOCATION An electronics firm manufactures two types of personal computers, a desktop model and a portable model. The production of a desktop computer requires a capital expenditure of \$400 and 40 hours of labor. The production of a portable computer requires a capital expenditure of \$250 and 30 hours of labor. The firm has \$20,000 capital and 2,160 labor-hours available for production of desktop and portable computers.

(A) What is the maximum number of computers the company is capable of producing?

(B) If each desktop computer contributes a profit of \$320 and each portable contributes a profit of \$220, how much profit will the company make by producing the maximum number of computers determined in part A? Is this the maximum profit? If not, what is the maximum profit?

47. POLLUTION CONTROL Because of new federal regulations on pollution, a chemical plant introduced a new process to supplement or replace an older process used in the production of a particular chemical. The older process emitted 20 grams of sulfur dioxide and 40 grams of particulate matter into the atmosphere for each gallon of chemical produced. The new process emits 5 grams of sulfur dioxide and 20 grams of particulate matter for each gallon produced. The company makes a profit of 60¢ per gallon and 20¢ per gallon on the old and new processes, respectively.

(A) If the regulations allow the plant to emit no more than 16,000 grams of sulfur dioxide and 30,000 grams of particulate matter daily, how many gallons of the chemical should be

produced by each process to maximize daily profit? What is the maximum daily profit?
(B) Discuss the effect on the production schedule and the maximum profit if the regulations restrict emissions of sulfur dioxide to 11,500 grams daily and all other data remain unchanged.
(C) Discuss the effect on the production schedule and the maximum profit if the regulations restrict emissions of sulfur dioxide to 7,200 grams daily and all other data remain unchanged.

48. SOCIOLOGY A city council voted to conduct a study on inner-city community problems. A nearby university was contacted to provide a maximum of 40 sociologists and research assistants. Allocation of time and cost per week are given in the table.
(A) How many sociologists and research assistants should be hired to meet the weekly labor-hour requirements and minimize the weekly cost? What is the weekly cost?
(B) Discuss the effect on the solution in part A if the council decides that they should not hire more sociologists than research assistants and all other data remain unchanged.

	Sociologist (labor-hours)	Research assistant (labor-hours)	Minimum labor-hours needed per week
Fieldwork	10	30	280
Research center	30	10	360
Cost per week	$500	$300	

49. PLANT NUTRITION A fruit grower can use two types of fertilizer in her orange grove, brand *A* and brand *B*. The amounts (in pounds) of nitrogen, phosphoric acid, potash, and chloride in a bag of each mix are given in the table. Tests indicate that the grove needs at least 480 pounds of phosphoric acid, at least 540 pounds of potash, and at most 620 pounds of chloride. If the grower always uses a combination of bags of brand *A* and brand *B* that will satisfy the constraints of phosphoric acid, potash, and chloride, discuss the effect that this will have on the amount of nitrogen added to the field.

	Pounds per bag	
	Brand *A*	Brand *B*
Nitrogen	6	7
Phosphoric acid	2	4
Potash	6	3
Chloride	3	4

50. DIET A dietitian in a hospital is to arrange a special diet composed of two foods, *M* and *N*. Each ounce of food *M* contains 16 units of calcium, 5 units of iron, 6 units of cholesterol, and 8 units of vitamin A. Each ounce of food *N* contains 4 units of calcium, 25 units of iron, 4 units of cholesterol, and 4 units of vitamin A. The diet requires at least 320 units of calcium, at least 575 units of iron, and at most 300 units of cholesterol. If the dietitian always selects a combination of foods *M* and *N* that will satisfy the constraints for calcium, iron, and cholesterol, discuss the effects that this will have on the amount of vitamin A in the diet.

CHAPTER 8 Review

8-1 Systems of Linear Equations in Two Variables

A system of two linear equations with two variables is a system of the form

$$\begin{aligned} ax + by &= h \\ cx + dy &= k \end{aligned} \tag{1}$$

where x and y are variables, a, b, c, and d are real numbers called the **coefficients** of x and y, and h and k are real numbers called the **constant terms** in the equations. The ordered pair of numbers (x_0, y_0) is a **solution** to system (1) if each equation is satisfied by the pair. The set of all such ordered pairs of numbers is called the **solution set** for the system. To **solve** a system is to find its solution set.

In general, a system of linear equations has exactly one solution, no solution, or infinitely many solutions. A system of linear equations is **consistent** if it has one or more solutions and **inconsistent** if no solutions exist. A consistent system is said to be **independent** if it has exactly one solution and **dependent** if it has more than one solution.

To solve system (1) by *graphing* (by hand), graph both equations in the same coordinate system, make an estimate of the coordinates of the solution, and then check to see if your estimate is correct.

To solve system (1) by *graphing* (with a calculator), solve each equation for y, graph these equations in the same viewing window, and use the intersect command.

To solve a system by **substitution,** solve either equation for either variable, substitute in the other equation, solve the resulting linear equation in one variable, and then substitute this value into the expression obtainbed in the first step to find the other variable.

Two systems of equations are **equivalent** if both have the same solution set. To solve a system of equations by **elimination by addition,** use Theorem 2 to find a simpler equivalent system whose solution is obvious.

As stated in Theorem 2, a system of linear equations is transformed into an equivalent system if:

1. Two equations are interchanged.

2. An equation is multiplied by a nonzero constant.

3. A constant multiple of another equation is added a to a given equation.

The solution set S of a dependent system is often expressed in terms of a **parameter.** Any element is S is called a **particular solution.**

8-2 Systems of Linear Equations in Three Variables

Any equation that can be writeen in the form

$$ax + by + cz = k$$

where a, b, c, and k are constants (not all a, b, and c zero) is called a **linear equation in three variables.** The method of elimination by addition discussed in the last section is extended to systems of linear equations in three variables.

8-3 Systems of Linear Inequalities

A graph is often the most convenient way to represent the solution of a linear inequality in two variables or of a system of linear inequalities in two variables.

A vertical line divides a plane into **left** and **right half-planes.** A nonvertical line divides a plane into **upper** and **lower half-planes.** If A, B, and C are real numbers with A and B not both zero, then the **graph of the linear inequality**

$$Ax + By < C \quad \text{or} \quad Ax + By > C$$

with $B \neq 0$, is either the upper half-plane or the lower half-plane (but not both) determined by the line $Ax + By = C$. If $B = 0$, then the graph of

$$Ax < C \quad \text{or} \quad Ax > C$$

is either the left half-plane or the right half-plane (but not both) determined by the line $Ax = C$. There are two *step-by-step procedures for graphing a linear inequality in two variables:*

Algebraic Procedure

Step 1. Graph $Ax + By = C$ as a broken line if equality is not included in the original statement or as a solid line if equality is included.

Step 2. Choose a test point anywhere in the plane not on the line and substitute the coordinates into the inequality. The origin (0, 0) often requires the least computation.

Step 3. The graph of the original inequality includes the half-plane containing the test point if the inequality is satisfied by that point, or the half-plane not containing that point if the inequality is not satisfied by that point.

Graphing Calculator Procedure

Step 1. Solve the inequality for y.

Step 2. Enter the equation of the boundary line and select a shading option as follows:

$$\left.\begin{array}{c} y > mx + b \\ \text{or} \\ y \geq mx + b \end{array}\right\} \text{ Select shade above.}$$

$$\left.\begin{array}{c} y < mx + b \\ \text{or} \\ y \leq mx + b \end{array}\right\} \text{ Select shade below.}$$

Step 3. Graph the solution.

We now turn to systems of linear inequalities in two variables. The **solution to a system of linear inequalities in two variables** is the set of all ordered pairs of real numbers that simultaneously satisfy all the inequalities in the system. The graph is called the **solution region.** In many applications the solution region is also referred to as the **feasible region.** To **find the solution region,** we graph each inequality in the system and then take the intersection of all the graphs. A **corner point** of a solution region is a point in the solution region that is the intersection of two **boundary lines.** A solution region is **bounded** if it can be enclosed within a circle. If it cannot be enclosed within a circle, then it is **unbounded.**

8-4 Linear Programming

Linear programming is a mathematical process that has been developed to help management in decision making, and it has become one of the most widely used and best-known tools of management science and industrial engineering.

A **linear programming problem** is one that is concerned with finding the **optimal value** (maximum or minimum value) of a linear **objective function** of the form $z = ax + by$, where the **decision variables** x and y are subject to **problem constraints** in the form of linear inequalities and **nonnegative constraints**

$x, y \geq 0$. The set of points satisfying both the problem constraints and the nonnegative constraints is called the *feasible region* for the problem. Any point in the feasible region that produces the optimal value of the objective function over the feasible region is called an **optimal solution.** The **fundamental theorem of linear programming** is basic to the solving of linear programming problems: Let S be the feasible region for a linear programming problem, and let $z = ax + by$ be the objective function. If S is bounded, then z has both a maximum and a minimum value on S and each of these occurs at a corner point of S. If S is unbounded, then a maximum or minimum value of z on S may not exist. However, if either does exist, then it must occur at a corner point of S.

Problems with unbounded feasible regions are not considered in this brief introduction. The fundamental theorem leads to a simple *step-by-step solution to linear programming problems with a bounded feasible region*:

Step 1. Form a mathematical model for the problem:
- (A) Introduce decision variables and write a linear objective function.
- (B) Write problem constraints in the form of linear inequalities.
- (C) Write nonnegative constraints.

Step 2. Graph the feasible region and find the corner points.

Step 3. Evaluate the objective function at each corner point to determine the optimal solution.

If two corner points are both optimal solutions of the same type (both produce the same maximum value or both produce the same minimum value) to a linear programming problem, then any point on the line segment joining the two corner points is also an optimal solution of that type.

CHAPTER 8

Work through all the problems in this chapter review and check answers in the back of the book. Answers to all review problems are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

A *Solve Problems 1 and 2 by substitution.*

1. $y = 4x - 9$
$y = -x + 6$
$x = 3, y = 3$ *(8-1)*

2. $3x + 2y = 5$
$4x - y = 14$
$x = 3, y = -2$ *(8-1)*

Solve Problems 3–6 by graphing.

3. $3x - 2y = 8$
$x + 3y = -1$
$x = 2, y = -1$ *(8-1)*

4. $2x + y = 4$
$-2x + 7y = 9$
$x = 1.1875, y = 1.625$ *(8-1)*

5. $3x - 4y \geq 24$

6. $2x + y \leq 2$
$x + 2y \geq -2$

Solve Problems 7–11 using elimination by addition.

7. $2x + y = 7$
$3x - 2y = 0$ (2, 3) *(8-1)*

8. $3x - 6y = 5$
$-2x + 4y = 1$
No solution (inconsistent) *(8-1)*

9. $4x - 3y = -8$
$-2x + \frac{3}{2}y = 4$

10. $x - 3y + z = 4$
$-x + 4y - 4z = 1$
$2x - y + 5z = -3$
(3, −1, −2) *(8-2)*

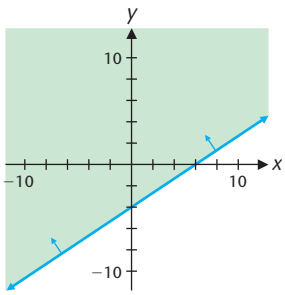
11. $2x + y - z = 5$
$x - 2y - 2z = 4$
$3x + 4y + 3z = 3$ (2, 0, −1) *(8-2)*

In Exercises 12 and 13, state the linear inequality whose graph is given in the figure. Write the boundary line equation in the form $Ax + By = C$, with A, B, and C integers, before stating the inequality.

Review Exercises

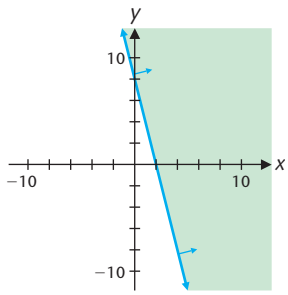
**Additional answers can be found in the Instructor Answer Appendix.*

12.



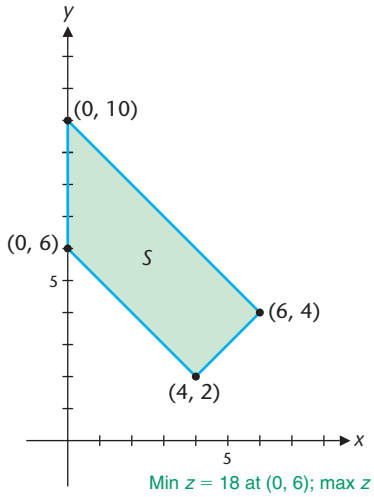
$2x - 3y = 12; 2x - 3y \leq 12$ *(8-3)*

13.



$4x + y = 8; 4x + y \geq 8$ *(8-3)*

14. Find the maximum and minimum values of $z = 5x + 3y$ over the feasible region S shown in the figure.



Min $z = 18$ at (0, 6); max $z = 42$ at (6, 4) *(8-4)*

B *Solve Problems 15–18 using elimination by addition.*

15.
$$\begin{aligned} x + 2y + 3z &= 1 \\ 2x + 3y + 4z &= 3 \\ x + 2y + z &= 3 \end{aligned}$$
$x = 2, y = 1, z = -1$ *(8-2)*

16.
$$\begin{aligned} x + 2y - z &= 2 \\ 2x + 3y + z &= -3 \\ 3x + 5y \quad z &= -1 \end{aligned}$$

17.
$$\begin{aligned} x - 2y &= 1 \\ 2x - y &= 0 \\ x - 3y &= -2 \end{aligned}$$
No solution *(8-2)*

18.
$$\begin{aligned} x + 2y - z &= 2 \\ 3x - y + 2z &= -3 \\ 4x + y + z &= 7 \end{aligned}$$
No solution *(8-2)*

Solve the systems in Problems 19–21 graphically, and indicate whether each solution region is bounded or unbounded. Find the coordinates of each corner point.

19.
$$\begin{aligned} 2x + y &\le 8 \\ 2x + 3y &\le 12 \\ x, y &\ge 0 \end{aligned}$$

20.
$$\begin{aligned} 2x + y &\ge 8 \\ x + 3y &\ge 12 \\ x, y &\ge 0 \end{aligned}$$

21.
$$\begin{aligned} x + y &\le 20 \\ x + 4y &\ge 20 \\ x - y &\ge 0 \end{aligned}$$

Solve the linear programming problems in Problems 22–26.

22. Maximize $z = 7x + 9y$
Subject to
$$\begin{aligned} x + 2y &\le 8 \\ 2x + y &\le 10 \\ x, y &\ge 0 \end{aligned}$$
Max $z = 45$ at (4, 2) *(8-4)*

23. Minimize $z = 5x + 10y$
Subject to
$$\begin{aligned} x + 3y &\le 20 \\ 3x + y &\ge 15 \\ x + 2y &\ge 15 \\ x, y &\ge 0 \end{aligned}$$
Min $z = 75$ at (3, 5) and (15, 0) (multiple optimal solutions) (15)

24. Minimize and maximize $z = 5x + 8y$
Subject to
$$\begin{aligned} x + 2y &\le 20 \\ 3x + y &\le 15 \\ x + y &\ge 7 \\ x, y &\ge 0 \end{aligned}$$
Min $z = 44$ at (4, 3); max $z = 82$ at (2, 9) *(8-4)*

C **25.** Solve using elimination by addition:
$$\begin{aligned} x_1 + x_2 + x_3 &= 7{,}000 \\ 0.04x_1 + 0.05x_2 + 0.06x_3 &= 360 \\ 0.04x_1 + 0.05x_2 - 0.06x_3 &= 120 \end{aligned}$$
$x_1 = 1{,}000, x_2 = 4{,}000, x_3 = 2{,}000$ *(8-3)*

26. Maximize $z = 30x + 20y$
Subject to
$$\begin{aligned} 1.2x + 0.6y &\le 960 \\ 0.04x + 0.03y &\le 36 \\ 0.2x + 0.3y &\le 270 \\ x, y &\ge 0 \end{aligned}$$
Max $z = 26{,}000$ at (600, 400) *(8-4)*

APPLICATIONS

27. **BUSINESS** A container holds 120 packages. Some of the packages weigh $\frac{1}{2}$ pound each, and the rest weigh $\frac{1}{3}$ pound each. If the total contents of the container weigh 48 pounds, how many are there of each type of package?

$\frac{1}{2}$-lb packages: 48, $\frac{1}{3}$-lb packages: 72 *(8-1)*

28. **GEOMETRY** Find the dimensions of a rectangle with an area of 48 square meters and a perimeter of 28 meters.

6 m by 8 m *(8-1)*

29. **DIET** A laboratory assistant wishes to obtain a food mix that contains, among other things, 27 grams of protein, 5.4 grams of fat, and 19 grams of moisture. He has available mixes A, B, and C with the compositions listed in the table. How many grams of each mix should be used to get the desired diet mix?

Mix	Protein (%)	Fat (%)	Moisture (%)
A	30	3	10
B	20	5	20
C	10	4	10

40 g mix A, 60 g mix B, 30 g mix C *(8-2)*

30. **RESOURCE ALLOCATION** A company uses two machines to solder circuit boards, a selective soldering machine and a wave soldering machine. A circuit board for a calculator needs 4 minutes in the selective soldering machine and 2 minutes on the wave soldering machine, while a circuit board for a toaster requires 3 minutes in the selective soldering machine and 1 minute on the wave soldering machine. (Source: Universal Electronics)

(A) How many boards for calculators and toasters can be produced if the selective soldering machine is available for 5 hours? Express your answer as a linear inequality with appropriate nonnegative restrictions and draw its graph.

(B) How many boards for calculators and toasters can be produced if the wave soldering machine is available for 2 hours? Express your answer as a linear inequality with appropriate nonnegative restrictions and draw its graph.

31. **RESOURCE ALLOCATION** North Star Sail Loft manufactures regular and competition sails. Each regular sail takes 1 labor-hour to cut and 3 labor-hours to sew. Each competition sail takes 2 labor-hours to cut and 4 labor-hours to sew. There are 140 labor-hours available in the cutting department and 360 labor-hours available in the sewing department.

(A) If the loft makes a profit of \$60 on each regular sail and \$100 on each competition sail, how many sails of each type should the company manufacture to maximize its profit? What is the maximum profit?

(B) An increase in the demand for competition sails causes the profit on a competition sail to rise to \$125. Discuss the effect of this change on the number of sails manufactured and on the maximum profit.

(C) A decrease in the demand for competition sails causes the profit on a competition sail to drop to \$75. Discuss the effect of this change on the number of sails manufactured and on the maximum profit.

32. **NUTRITION—ANIMALS** A special diet for laboratory animals is to contain at least 800 units of vitamins, at least 800 units

of minerals, and at most 1,300 calories. There are two feed mixes available, mix A and mix B. A gram of mix A contains 5 units of vitamins, 2 units of minerals, and 4 calories. A gram of mix B contains 2 units of vitamins, 4 units of minerals, and 4 calories.

(A) If mix A costs \$0.07 per gram and mix B costs \$0.04 per gram, how many grams of each mix should be used to satisfy the requirements of the diet at minimal cost? What is the minimal cost?

(B) If the price of mix B decreases to \$0.02 per gram, discuss the effect of this change on the solution in part A.

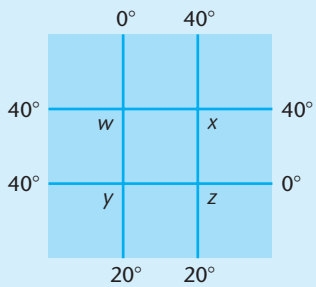
(C) If the price of mix B increases to \$0.15 per gram, discuss the effect of this change on the solution in part A.

CHAPTER 8

››› GROUP ACTIVITY Heat Conduction

A metal grid consists of four thin metal bars. The end of each bar of the grid is kept at a constant temperature, as shown in Figure 1. We assume that the temperature at each intersection point in the grid is the average of the temperatures at the four adjacent points in the grid (adjacent points are either other intersection points or ends of bars). Thus, the temperature w at the intersection point in the upper left-hand corner of the grid must satisfy

$$w = \frac{1}{4}(\underset{\text{Left}}{40} + \underset{\text{Above}}{0} + \underset{\text{Right}}{x} + \underset{\text{Below}}{y})$$



› Figure 1

which can be rewritten as

$$w - 0.25x - 0.25y = 10 \qquad E_1$$

(A) Find equations E_2, E_3, and E_4 for the temperature at each of the other three intersection points.

(B) The equations E_1, E_2, E_3, and E_4 form a system of four linear equations in the four variables w, x, y, and z. To solve this system, use E_1 to eliminate w in the other three equations. This produces a system of three equations in x, y, and z. Use elimination by addition to solve this system. Back-substitute x, y, and z in E_1 to find w.

(C) Add your solutions to Figure 1 and check that the temperature at each point is the average of the temperatures at the adjacent points.

Matrices and Determinants

MATRICES are both a very ancient and a very current mathematical concept. References to matrices and systems of equations can be found in Chinese manuscripts dating back to around 200 B.C. More recently, the advent of computers has led to an increased use of matrices in a wide variety of applications. In this chapter, you will learn about matrix operations, the properties of these operations, and how to use matrices to solve real-world applications.

CHAPTER

9

OUTLINE

9-1 Systems of Linear Equations: Gauss–Jordan Elimination

9-2 Matrix Operations

9-3 Inverse of a Square Matrix

9-4 Matrix Equations and Systems of Linear Equations

9-5 Determinants

9-6 Properties of Determinants

9-7 Determinants and Cramer's Rule

Chapter 9 Review

Chapter 9 Group Activity: Using Matrices to Find Cost, Revenue, and Profit

Cumulative Review Exercise for Chapters 8 and 9

9-1 Systems of Linear Equations: Gauss–Jordan Elimination

- Performing Row Operations on Matrices
- Recognizing Reduced Matrices
- Solving Systems by Gauss–Jordan Elimination
- Mathematical Modeling

As we learned in Section 8-1, linear systems with two variables can be solved by graphing, substitution, or elimination by addition. The elimination method was extended to solving linear systems with three variables in Section 8-2. (It's a good idea to review Section 8-2 right now.) In this section, a new mathematical object, a *matrix,* and the elimination method are combined. The result is a method called *Gauss–Jordan elimination* that can be used to solve linear systems with any number of equations and any number of variables. This new method is easy to implement on a computer. In fact, most graphing calculators, including the TI-84 and TI-86 have a built-in command that performs all the calculations related to Gauss–Jordan elimination.

Performing Row Operations on Matrices

In solving systems of equations using elimination by addition, the coefficients of the variables and the constant terms played a central role. The process can be made more efficient for generalization and computer work by the introduction of a mathematical form called a *matrix*. A **matrix** is a rectangular array of numbers written within brackets. Two examples are

$$A = \begin{bmatrix} 1 & -3 & 7 \\ 5 & 0 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} -5 & 4 & 11 \\ 0 & 1 & 6 \\ -2 & 12 & 8 \\ -3 & 0 & -1 \end{bmatrix} \tag{1}$$

Each number in a matrix is called an **element** of the matrix. Matrix A has six elements arranged in two rows and three columns. Matrix B has 12 elements arranged in four rows and three columns. If a matrix has m rows and n columns, it is called an $\boldsymbol{m \times n}$ **matrix** (read "m by n matrix"). The expression $m \times n$ is called the **size** of the matrix, and the numbers m and n are called the **dimensions** of the matrix. It is important to note that the number of rows is always given first. Referring to equation (1), A is a 2×3 matrix and B is a 4×3 matrix. A matrix with n rows and n columns is called a **square matrix of order** $\boldsymbol{n}$**.** A matrix with only one column is called a **column**

matrix, and a matrix with only one row is called a **row matrix.** These definitions are illustrated by the following:

$$\underset{3\times 3}{\begin{bmatrix} 0.5 & 0.2 & 1.0 \\ 0.0 & 0.3 & 0.5 \\ 0.7 & 0.0 & 0.2 \end{bmatrix}} \qquad \underset{4\times 1}{\begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}} \qquad \underset{1\times 4}{\begin{bmatrix} 2 & \frac{1}{2} & 0 & -\frac{2}{3} \end{bmatrix}}$$

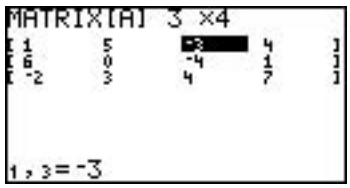
Square matrix of order 3 | Column matrix | Row matrix

The **position** of an element in a matrix is the row and column containing the element. This is usually denoted using **double subscript notation** a_{ij}, where i is the row and j is the column containing the element a_{ij}, as illustrated next:

$$A = \begin{bmatrix} 1 & 5 & -3 & 4 \\ 6 & 0 & -4 & 1 \\ -2 & 3 & 4 & 7 \end{bmatrix} \qquad \begin{aligned} a_{11} &= 1, a_{12} = 5, a_{13} = -3, a_{14} = 4 \\ a_{21} &= 6, a_{22} = 0, a_{23} = -4, a_{24} = 1 \\ a_{31} &= -2, a_{32} = 3, a_{33} = 4, a_{34} = 7 \end{aligned}$$

Note that a_{12} is read "a sub one two," not "a sub twelve." The elements $a_{11} = 1$, $a_{22} = 0$, and $a_{33} = 4$ make up the *principal diagonal* of A. In general, the **principal diagonal** of a matrix A consists of the elements a_{ii}, $i = 1, 2, \ldots n$.

Most graphing calculators are capable of storing and manipulating matrices. Figure 1 shows matrix A displayed in the matrix editing screen of a TI-84 graphing calculator. The size of the matrix is given at the top of the screen, and the position of the currently selected element is given at the bottom. Notice that a comma is used in the notation for the position. This is common practice on graphing calculators but not in mathematical literature.*



› **Figure 1** Matrix notation on a TI-83 graphing calculator.

The coefficients and constant terms in a system of linear equations can be used to form several matrices of interest to our work. Related to the system

$$\begin{aligned} x + 5y - 3z &= 4 \\ 6x \qquad - 4z &= 1 \\ -2x + 3y + 4z &= 7 \end{aligned} \tag{2}$$

*The onscreen display of A was too large to fit on the screen of a TI-84, so we pasted together two screen dumps to form Figure 1. When this happens on your graphing calculator, you will have to scroll left and right and/or up and down to see the entire matrix.

are the following matrices:

Coefficient matrix Constant matrix Augmented coefficient matrix

$$\begin{bmatrix} 1 & 5 & -3 \\ 6 & 0 & -4 \\ -2 & 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix} \quad \left[\begin{array}{ccc|c} 1 & 5 & -3 & 4 \\ 6 & 0 & -4 & 1 \\ -2 & 3 & 4 & 7 \end{array}\right]$$

Notice that 0 is used for the coefficient of the missing y variable in the second equation in system (2). The *augmented coefficient matrix* will be used in this section. The other matrices will be used in later sections. The **augmented coefficient matrix** contains the essential parts of the system—both the coefficients and the constants. The vertical bar is included only as a visual aid to help us separate the coefficients from the constant terms. (Matrices entered and displayed on a graphing calculator will not display this line.) For ease of generalization to the larger systems in the following sections, we now change the notation for the variables in system (2) to a subscript form (we would soon run out of letters, but we will not run out of subscripts). That is, in place of x, y, and z, we will use x_1, x_2, and x_3 respectively, and system (2) will be written as

$$\begin{aligned} x_1 + 5x_2 - 3x_3 &= 4 \\ 6x_1 \qquad - 4x_3 &= 1 \\ -2x_1 + 3x_2 + 4x_3 &= 7 \end{aligned}$$

In Sections 8-1 and 8-2, we used E_i to denote the equations in a linear system. Now we use R_i to denote the rows and C_i to denote the columns, respectively, in a matrix, as illustrated below for system (2).

$$\begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 5 & -3 & 4 \\ 6 & 0 & -4 & 1 \\ -2 & 3 & 4 & 7 \end{array}\right] \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \tag{3}$$

Matrix (3) contains the essential parts of system (2). Our objective is to learn how to manipulate augmented matrices in order to solve linear systems. This new solution process is a direct outgrowth of the elimination method discussed in Sections 8-1 and 8-2.

EXAMPLE 1 Forming an Augmented Coefficient Matrix

Write the augmented coefficient matrix corresponding to each of the following systems.

(A) $\begin{aligned} 2x_1 - 4x_2 &= 5 \\ -3x_1 + x_2 &= -6 \end{aligned}$ (B) $\begin{aligned} -3x_1 \qquad + 2x_3 &= -4 \\ 7x_1 - 5x_2 + 3x_3 &= 0 \end{aligned}$ (C) $\begin{aligned} 2x_1 - x_2 \qquad &= 4 \\ 3x_1 \qquad - 5x_3 &= 6 \\ -2x_2 + x_3 &= -3 \end{aligned}$

SOLUTIONS

(A) $\left[\begin{array}{cc|c} 2 & -4 & 5 \\ -3 & 1 & -6 \end{array}\right]$ (B) $\left[\begin{array}{ccc|c} -3 & 0 & 2 & -4 \\ 7 & -5 & 3 & 0 \end{array}\right]$ (C) $\left[\begin{array}{ccc|c} 2 & -1 & 0 & 4 \\ 3 & 0 & -5 & 6 \\ 0 & -2 & 1 & -3 \end{array}\right]$

MATCHED PROBLEM 1

Write the augmented coefficient matrix corresponding to each of the following systems.

(A) $\begin{aligned} -x_1 + 2x_2 &= -3 \\ 3x_1 - 5x_2 &= 8 \end{aligned}$ (B) $\begin{aligned} -2x_2 + 2x_3 &= -4 \\ 7x_1 - 5x_2 + 3x_3 &= 0 \end{aligned}$ (C) $\begin{aligned} 2x_1 - x_2 + x_3 &= 4 \\ 3x_1 + 4x_2 &= 6 \\ x_1 + 5x_3 &= -3 \end{aligned}$

Recall that two linear systems are said to be equivalent if they have the same solution set. In Theorem 2, Section 8-1, we used the operations listed next to transform linear systems into equivalent systems:

(A) Two equations are interchanged.
(B) An equation is multiplied by a nonzero constant.
(C) A constant multiple of one equation is added to another equation.

Paralleling this approach, we now say that two augmented matrices are **row-equivalent,** denoted by the symbol $\sim$ between the two matrices, if they are augmented matrices of equivalent systems of equations. How do we transform augmented matrices into row-equivalent matrices? We use Theorem 1, which is a direct consequence of operations (A), (B), and (C).

THEOREM 1 Elementary Row Operations Producing Row-Equivalent Matrices

An augmented matrix is transformed into a row-equivalent matrix if any of the following **row operations** is performed:

1. Two rows are interchanged ($R_i \leftrightarrow R_j$).
2. A row is multiplied by a nonzero constant ($kR_i \rightarrow R_i$).
3. A constant multiple of one row is added to another row ($kR_j + R_i \rightarrow R_i$).

[*Note:* The arrow means "replaces."]

EXAMPLE 2 Row Operations

Perform each of the indicated row operations on the following augmented coefficient matrix.

$$\left[\begin{array}{cc|c} 1 & -4 & 3 \\ 2 & 4 & -8 \end{array}\right]$$

(A) $R_1 \leftrightarrow R_2$ (B) $\frac{1}{2}R_2 \rightarrow R_2$ (C) $(-2)R_1 + R_2 \rightarrow R_2$

SOLUTIONS

(A) $\left[\begin{array}{cc|c} 2 & 4 & -8 \\ 1 & -4 & 3 \end{array}\right]$ (B) $\left[\begin{array}{cc|c} 1 & -4 & 3 \\ 1 & 2 & -4 \end{array}\right]$ (C) $\left[\begin{array}{cc|c} 1 & -4 & 3 \\ 0 & 12 & -14 \end{array}\right]$

MATCHED PROBLEM 2

Perform each of the indicated row operations on the following augmented coefficient matrix.

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 3 & -6 & -3 \end{array}\right]$$

(A) $R_1 \leftrightarrow R_2$ (B) $\frac{1}{3}R_2 \rightarrow R_2$ (C) $(-3)R_1 + R_2 \rightarrow R_2$

Recognizing Reduced Matrices

The goal of the elimination process is to transform a system of equations into an equivalent system whose solution is evident. Now our goal is to use a sequence of matrix row operations to transform an augmented coefficient matrix into a simpler equivalent matrix that corresponds to a system with an obvious solution. Example 3 illustrates the process of interpreting the solution of a system given its augmented coefficient matrix.

EXAMPLE 3 Interpreting an Augmented Coefficient Matrix

Write the system corresponding to each of the following augmented coefficient matrices and find its solution.

(A) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 0 \end{array}\right]$ (B) $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -4 \\ 0 & 1 & -3 & 6 \\ 0 & 0 & 0 & 1 \end{array}\right]$ (C) $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -4 \\ 0 & 1 & -3 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$

SOLUTIONS

(A) The corresponding system is

$$\begin{aligned} x_1 &= -4 \\ x_2 &= 6 \\ x_3 &= 0 \end{aligned}$$

and $(-4, 6, 0)$ is the solution.

(B) The corresponding system is

$$\begin{aligned} x_1 \quad &+ \quad 2x_3 = -4 \\ x_2 - \quad &3x_3 = \quad 6 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot \; &x_3 = \quad 1 \end{aligned}$$

The third equation, $0 = 1$, is a contradiction, so the system has no solutions.

(C) The first two rows of this augmented coefficient matrix correspond to the system

$$\begin{aligned} x_1 \quad &+ 2x_3 = -4 \\ x_2 - \; &3x_3 = \quad 6 \end{aligned}$$

The third row corresponds to the equation 0 = 0, which is always true and can be discarded.

This is a dependent system with an infinite number of solutions. Introducing a parameter s, we can write

$$\begin{aligned} x_1 \quad &+ 2s = -4 \\ x_2 - \; &3s = \quad 6 \\ x_3 = \; &s \end{aligned} \quad \text{or} \quad \begin{aligned} x_1 &= -2s - 4 \\ x_2 &= \quad 3s + 6 \\ x_3 &= \quad s \end{aligned}$$

So the solution set is

$$\{(-2s - 4, 3s + 6, s) \mid s \text{ any real number}\}$$

MATCHED PROBLEM 3

Write the system corresponding to each of the following augmented coefficient matrices and find its solution.

$$\text{(A)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \text{(B)} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 5 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{(C)} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 5 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

››› EXPLORE-DISCUSS 1

If an augmented coefficient matrix contains a row where every element on the left of the vertical line is 0 and the single element on the right is a nonzero number, what can you say about the solution of the corresponding system?

Since there is no limit on the number of variables or the number of equations in a linear system, it is not feasible to explicitly list all the simpler forms of augmented coefficient matrices that correspond to systems with an obvious solution. Instead, we

state a general definition of a simplified form called a *reduced matrix* that can be applied to all matrices and systems, regardless of size.

› DEFINITION 1 Reduced Matrix

A matrix is in **reduced form*** if:

1. Each row consisting entirely of 0's is below any row having at least one nonzero element.
2. The leftmost nonzero element in each row is 1.
3. The column containing the leftmost 1 of a given row has 0's above and below the 1.
4. The leftmost 1 in any row is to the right of the leftmost 1 in the preceding row.

For example, each of the following matrices is in reduced form. You should verify that each matrix satisfies all four conditions in Definition 1.

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{cccc|c} 1 & 4 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

EXAMPLE 4 Reduced Forms

The matrices shown next are not in reduced form. Indicate which condition in the definition is violated for each matrix. State the row operation(s) required to transform the matrix to reduced form, and find the reduced form.

$$\text{(A)} \left[\begin{array}{cc|c} 0 & 1 & -2 \\ 1 & 0 & 3 \end{array}\right] \qquad \text{(B)} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\text{(C)} \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 1 & -2 \end{array}\right] \qquad \text{(D)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

SOLUTIONS

(A) Condition 4 is violated: The leftmost 1 in row 2 is not to the right of the leftmost 1 in row 1. Perform the row operation $R_1 \leftrightarrow R_2$ to obtain the reduced form:

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

*The reduced form we have defined here is often referred to as the *reduced row-echelon form* to distinguish it from other reduced forms. Most graphing calculators use the abbreviation RREF to refer to this reduced form.

(B) Condition 3 is violated: The column containing the leftmost 1 in row 2 does not have a zero above the 1. Perform the row operation $2R_2 + R_1 \rightarrow R_1$ to obtain the reduced form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

(C) Condition 1 is violated: The second row contains all zeros, and it is not below any row having at least one nonzero element. Perform the row operation $R_2 \leftrightarrow R_3$ to obtain the reduced form:

$$\left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

(D) Condition 2 is violated: The leftmost nonzero element in row 2 is not a 1. Perform the row operation $\frac{1}{2}R_2 \rightarrow R_2$ to obtain the reduced form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & -5 \end{array}\right]$$

MATCHED PROBLEM 4

These matrices are not in reduced form. Indicate which condition in the definition is violated for each matrix. State the row operation(s) required to transform the matrix to reduced form and find the reduced form.

(A) $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 3 & -6 \end{array}\right]$ (B) $\left[\begin{array}{ccc|c} 1 & 5 & 4 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(C) $\left[\begin{array}{ccc|c} 0 & 1 & 0 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$ (D) $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{array}\right]$

› Solving Systems by Gauss–Jordan Elimination

We are now ready to outline the **Gauss–Jordan elimination** method for solving systems of linear equations. The method systematically transforms an augmented matrix into a reduced form. The system corresponding to a reduced augmented coefficient matrix is called a **reduced system.** As we will see, reduced systems are easy to solve.

The Gauss–Jordan elimination method is named after the German mathematician Carl Friedrich Gauss (1777–1885) and the German geodesist Wilhelm Jordan (1842–1899). Gauss, one of the greatest mathematicians of all time, used a method of solving systems of equations that was later generalized by Jordan to solve problems in large-scale surveying.

EXAMPLE 5 Solving a System Using Gauss–Jordan Elimination

Solve by Gauss–Jordan elimination:

$$\begin{aligned} 2x_1 - 2x_2 + x_3 &= 3 \\ 3x_1 + x_2 - x_3 &= 7 \\ x_1 - 3x_2 + 2x_3 &= 0 \end{aligned}$$

SOLUTION

Write the augmented matrix and follow the steps indicated at the right to produce a reduced form.

Need a 1 here.

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & 3 \\ 3 & 1 & -1 & 7 \\ 1 & -3 & 2 & 0 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

Step 1: Choose the leftmost nonzero column and get a 1 at the top.

Need 0's here.

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 3 & 1 & -1 & 7 \\ 2 & -2 & 1 & 3 \end{array}\right] \quad \begin{array}{l} (-3)R_1 + R_2 \to R_2 \\ (-2)R_1 + R_3 \to R_3 \end{array}$$

Step 2: Use multiples of the row containing the 1 from step 1 to get zeros in all remaining places in the column containing this 1.

Need a 1 here.

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 10 & -7 & 7 \\ 0 & 4 & -3 & 3 \end{array}\right] \quad 0.1R_2 \to R_2$$

Step 3: Repeat step 1 with the *submatrix* formed by (mentally) deleting the top row.

Need 0's here.

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 4 & -3 & 3 \end{array}\right] \quad \begin{array}{l} 3R_2 + R_1 \to R_1 \\ (-4)R_2 + R_3 \to R_3 \end{array}$$

Step 4: Repeat step 2 with the *entire matrix.*

Need a 1 here.

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -0.1 & 2.1 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 0 & -0.2 & 0.2 \end{array}\right] \quad (-5)R_3 \to R_3$$

Step 3: Repeat step 1 with the *submatrix* formed by (mentally) deleting the top two rows.

Need 0's here.

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -0.1 & 2.1 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad \begin{array}{l} 0.1R_3 + R_1 \to R_1 \\ 0.7R_3 + R_2 \to R_2 \end{array}$$

Step 4: Repeat step 2 with the *entire matrix.*

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

The matrix is now in reduced form, and we can proceed to solve the corresponding reduced system.

$$\begin{aligned} x_1 &= 2 \\ x_2 &= 0 \\ x_3 &= -1 \end{aligned}$$

The solution to this system is $x_1 = 2, x_2 = 0, x_3 = -1$. You should check this solution in the original system.

GAUSS–JORDAN ELIMINATION

Step 1. Choose the leftmost nonzero column and use appropriate row operations to get a 1 at the top.

Step 2. Use multiples of the row containing the 1 from step 1 to get zeros in all remaining places in the column containing this 1.

Step 3. Repeat step 1 with the **submatrix** formed by (mentally) deleting the row used in step 2 and all rows above this row.

Step 4. Repeat step 2 with the **entire matrix,** including the mentally deleted rows. Continue this process until the entire matrix is in reduced form.

[*Note:* If at any point in this process we obtain a row with all zeros to the left of the vertical line and a nonzero number to the right, we can stop, since we will have a contradiction: $0 = n, n \neq 0$. We can then conclude that the system has no solution.]

MATCHED PROBLEM 5

Solve by Gauss–Jordan elimination:

$$\begin{aligned} 3x_1 + x_2 - 2x_3 &= 2 \\ x_1 - 2x_2 + x_3 &= 3 \\ 2x_1 - x_2 - 3x_3 &= 3 \end{aligned}$$

REMARKS

1. Even though each matrix has a unique reduced form, the sequence of steps (algorithm) presented here for transforming a matrix into a reduced form is not unique. That is, other sequences of steps (using row operations) can produce a reduced matrix. (For example, it is possible to use row operations in such a way that computations involving fractions are minimized.) But we emphasize again that we are not interested in the most efficient hand methods for transforming small matrices into reduced forms. Our main interest is in giving you a little experience with a method that is suitable for solving large-scale systems on a computer or graphing utility.
2. Figure 2 illustrates the solution of Example 5 on a TI-84 graphing calculator using the built-in RREF (reduced row-echelon form) command for finding reduced forms. Notice that in row 2 and column 4 of the reduced form the graphing calculator has displayed the very small number −3.5E − 13 instead of the exact value 0. This is a common occurrence on a graphing calculator and causes no problems. Just replace any very small numbers displayed in scientific notation with 0.

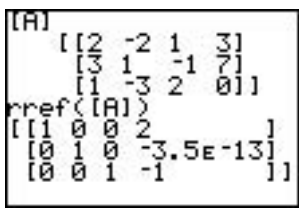


Figure 2 Using RREF on a TI-84 graphing calculator.

EXAMPLE 6 Solving a System Using Gauss–Jordan Elimination

Solve by Gauss–Jordan elimination:

$$\begin{aligned} 2x_1 - 4x_2 + x_3 &= -4 \\ 4x_1 - 8x_2 + 7x_3 &= 2 \\ -2x_1 + 4x_2 - 3x_3 &= 5 \end{aligned}$$

SOLUTIONS

Algebraic Solution

$$\left[\begin{array}{ccc|c} 2 & -4 & 1 & -4 \\ 4 & -8 & 7 & 2 \\ -2 & 4 & -3 & 5 \end{array}\right] \quad 0.5R_1 \to R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 0.5 & -2 \\ 4 & -8 & 7 & 2 \\ -2 & 4 & -3 & 5 \end{array}\right] \quad \begin{array}{l} (-4)R_1 + R_2 \to R_2 \\ 2R_1 + R_3 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 0.5 & -2 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & -2 & 1 \end{array}\right] \quad 0.2R_2 \to R_2$$

Note that column 3 is the leading nonzero column in this submatrix.

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 0.5 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{array}\right] \quad \begin{array}{l} (-0.5)R_2 + R_1 \to R_1 \\ 2R_2 + R_3 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

We stop the Gauss–Jordan elimination, although the matrix is not in reduced form, because the last row produces a contradiction: $0x_1 + 0x_2 + 0x_3 = 5$

The system is inconsistent and has no solution.

Graphing Calculator Solution

Enter the augmented coefficient matrix and use the RREF command to find the reduced form (Fig. 3).

```
A
      [[2   -4 1   -4]
       [4   -8 7   2 ]
       [-2 4   -3 5 ]]
rref A
          [[1 -2 0 0]
           [0 0  1 0]
           [0 0  0 1]]
```

› Figure 3

The last row of the reduced form in Figure 3 indicates an inconsistent system with no solution. Notice that the graphing calculator program does not stop when a contradiction first occurs, as we did in the algebraic solution, but continues on to find the reduced form.

MATCHED PROBLEM 6

Solve by Gauss–Jordan elimination:

$$\begin{aligned} 2x_1 - 4x_2 - x_3 &= -8 \\ 4x_1 - 8x_2 + 3x_3 &= 4 \\ -2x_1 + 4x_2 + x_3 &= 11 \end{aligned}$$

EXAMPLE 7 Solving a System Using Gauss–Jordan Elimination

Solve by Gauss–Jordan elimination:

$$\begin{aligned} 3x_1 + 6x_2 - 9x_3 &= 15 \\ 2x_1 + 4x_2 - 6x_3 &= 10 \\ -2x_1 - 3x_2 + 4x_3 &= -6 \end{aligned}$$

SOLUTIONS

First we find the reduced form of the augmented coefficient matrix.

Algebraic Method

$$\left[\begin{array}{ccc|c} 3 & 6 & -9 & 15 \\ 2 & 4 & -6 & 10 \\ -2 & -3 & 4 & -6 \end{array}\right] \quad \tfrac{1}{3}R_1 \to R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 2 & 4 & -6 & 10 \\ -2 & -3 & 4 & -6 \end{array}\right] \quad \begin{array}{l}(-2)R_1 + R_2 \to R_2 \\ 2R_1 + R_3 \to R_3\end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 4 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

Note that we must interchange rows 2 and 3 to obtain a nonzero entry at the top of the second column of this submatrix.

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad (-2)R_2 + R_1 \to R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix is now in reduced form.

Graphing Calculator Method

Enter the augmented coefficient matrix and use RREF (Fig. 4).

```
A
      [[3  6  -9 15]
       [2  4  -6 10]
       [-2 -3 4  -6]]
rref A
      [[1 0 1  -3]
       [0 1 -2 4 ]
       [0 0 0  0 ]]
```

› Figure 4

Using the reduced form from either of the preceding methods, we write the corresponding system and solve.

$$\begin{aligned} x_1 + x_3 &= -3 \\ x_2 - 2x_3 &= 4 \end{aligned}$$

We discard the equation corresponding to the third (all 0) row in the reduced form, because it is satisfied by all values of x_1, x_2, and x_3.

Note that the leading variable in each equation appears in one and only one equation. We solve for the leading variables x_1 and x_2 in terms of the remaining variable x_3:

$$\begin{aligned} x_1 &= -x_3 - 3 \\ x_2 &= 2x_3 + 4 \end{aligned}$$

This dependent system has an infinite number of solutions. We will use a parameter to represent all the solutions. If we let $x_3 = t$, then for any real number t,

$$\begin{aligned} x_1 &= -t - 3 \\ x_2 &= 2t + 4 \\ x_3 &= t \end{aligned}$$

You should check that $(-t - 3, 2t + 4, t)$ is a solution of the original system for any real number t. Some particular solutions are

$t = 0$	$t = -2$	$t = 3.5$
$(-3, 4, 0)$	$(-1, 0, -2)$	$(-6.5, 11, 3.5)$

MATCHED PROBLEM 7

Solve by Gauss–Jordan elimination:

$$\begin{aligned} 2x_1 - 2x_2 - 4x_3 &= -2 \\ 3x_1 - 3x_2 - 6x_3 &= -3 \\ -2x_1 + 3x_2 + x_3 &= 7 \end{aligned}$$

In general,

If the number of leading 1s in a reduced augmented coefficient matrix is less than the number of variables in the system and there are no contradictions, then the system is dependent and has infinitely many solutions.

There are many different ways to use the reduced augmented coefficient matrix to describe the infinite number of solutions of a dependent system. We will always proceed as follows: Solve each equation in a reduced system for its leftmost variable and then introduce a different parameter for each remaining variable. As the solution to Example 7 illustrates, this method produces a concise and useful representation of the solutions to a dependent system. Example 8 illustrates a dependent system where two parameters are required to describe the solution.

››› EXPLORE-DISCUSS 2

Explain why the definition of reduced form ensures that each leading variable in a reduced system appears in one and only one equation and no equation contains more than one leading variable. Discuss methods for determining if a consistent system is independent or dependent by examining the reduced form.

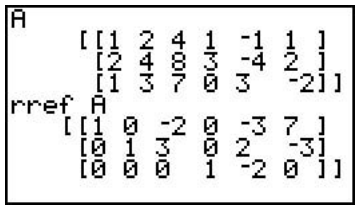
EXAMPLE 8 Solving a System Using Gauss–Jordan Elimination

Solve by using RREF on a graphing calculator:

$$\begin{aligned} x_1 + 2x_2 + 4x_3 + x_4 - x_5 &= 1 \\ 2x_1 + 4x_2 + 8x_3 + 3x_4 - 4x_5 &= 2 \\ x_1 + 3x_2 + 7x_3 + 3x_5 &= -2 \end{aligned}$$

SOLUTION

The augmented coefficient matrix and its reduced form are shown in Figure 5. Write the corresponding reduced system and solve.



› Figure 5

$$\begin{aligned} x_1 - 2x_3 - 3x_5 &= 7 \\ x_2 + 3x_3 + 2x_5 &= -3 \\ x_4 - 2x_5 &= 0 \end{aligned}$$

Solve for the leading variables x_1, x_2, and x_4 in terms of the remaining variables x_3 and x_5.

$$\begin{aligned} x_1 &= 2x_3 + 3x_5 + 7 \\ x_2 &= -3x_3 - 2x_5 - 3 \\ x_4 &= 2x_5 \end{aligned}$$

If we let $x_3 = s$ and $x_5 = t$, then for any real numbers s and t,

$$\begin{aligned} x_1 &= 2s + 3t + 7 \\ x_2 &= -3s - 2t - 3 \\ x_3 &= s \\ x_4 &= 2t \\ x_5 &= t \end{aligned}$$

is a solution. The check is left for you to perform. ◉

MATCHED PROBLEM 8

Solve by using RREF on a graphing utility:

$$\begin{aligned} x_1 - x_2 + 2x_3 - 2x_5 &= 3 \\ -2x_1 + 2x_2 - 4x_3 - x_4 + x_5 &= -5 \\ 3x_1 - 3x_2 + 7x_3 + x_4 - 4x_5 &= 6 \end{aligned}$$

◉

Mathematical Modeling

Dependent systems of linear equations provide an excellent opportunity to discuss mathematical modeling in a little more detail. The process of using mathematics to solve real-world problems can be broken down into three steps (Fig. 6):

Step 1. *Construct* a mathematical model whose solution will provide information about the real-world problem.

Step 2. *Solve* the mathematical model.

Step 3. *Interpret* the solution to the mathematical model in terms of the original real-world problem.

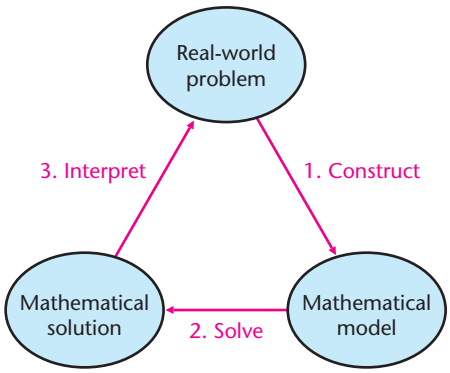


Figure 6

In more complex problems, this cycle may have to be repeated several times to obtain the required information about the real-world problem.

EXAMPLE 9 Purchasing

A chemical manufacturer wants to purchase a fleet of 24 railroad tank cars with a combined carrying capacity of 250,000 gallons. Tank cars with three different carrying capacities are available: 6,000 gallons, 8,000 gallons, and 18,000 gallons. How many of each type of tank car should be purchased?

SOLUTION

Let

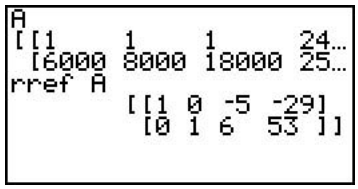
$$x_1 = \text{Number of 6,000-gallon tank cars}$$
$$x_2 = \text{Number of 8,000-gallon tank cars}$$
$$x_3 = \text{Number of 18,000-gallon tank cars}$$

Then

$$\begin{aligned} x_1 + x_2 + x_3 &= 24 && \text{Total number of tank cars} \\ 6{,}000x_1 + 8{,}000x_2 + 18{,}000x_3 &= 250{,}000 && \text{Total carrying capacity} \end{aligned}$$

The augmented coefficient matrix and its reduced form are shown in Figure 7. Write the corresponding reduced system and solve.

› Figure 7



$$\begin{aligned} x_1 - 5x_3 &= -29 \quad \text{or} \quad x_1 = 5x_3 - 29 \\ x_2 + 6x_3 &= 53 \quad \text{or} \quad x_2 = -6x_3 + 53 \end{aligned}$$

Let $x_3 = t$. Then for t any real number,

$$\begin{aligned} x_1 &= 5t - 29 \\ x_2 &= -6t + 53 \\ x_3 &= t \end{aligned}$$

is a solution—or is it? Because the variables in this system represent the number of tank cars purchased, the values of x_1, x_2, and x_3 must be nonnegative integers. Thus, the third equation requires that t must be a nonnegative integer. The first equation requires that $5t - 29 \geq 0$, so t must be at least 6. The middle equation requires that $-6t + 53 \geq 0$, so t can be no larger than 8. Thus, 6, 7, and 8 are the only possible values for t. There are only three possible combinations that meet the company's specifications of 24 tank cars with a total carrying capacity of 250,000 gallons, as shown in Table 1.

Table 1

t	6,000-gallon tank cars x_1	8,000-gallon tank cars x_2	18,000-gallon tank cars x_3
6	1	17	6
7	6	11	7
8	11	5	8

The final choice would probably be influenced by other factors. For example, the company might want to minimize the cost of the 24 tank cars.

MATCHED PROBLEM 9

A commuter airline wants to purchase a fleet of 30 airplanes with a combined carrying capacity of 960 passengers. The three available types of planes carry 18, 24, and 42 passengers, respectively. How many of each type of plane should be purchased?

ANSWERS TO MATCHED PROBLEMS

1. (A) $\left[\begin{array}{cc|c} -1 & 2 & -3 \\ 3 & -5 & 8 \end{array}\right]$ (B) $\left[\begin{array}{ccc|c} 0 & -2 & 2 & -4 \\ 7 & -5 & 3 & 0 \end{array}\right]$ (C) $\left[\begin{array}{ccc|c} 2 & -1 & 1 & 4 \\ 3 & 4 & 0 & 6 \\ 1 & 0 & 5 & -3 \end{array}\right]$

2. (A) $\left[\begin{array}{cc|c} 3 & -6 & -3 \\ 1 & -2 & 3 \end{array}\right]$ (B) $\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 1 & -2 & -1 \end{array}\right]$ (C) $\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 0 & -12 \end{array}\right]$

3. (A) $x_1 = 5, x_2 = -7, x_3 = 0$ or $(5, -7, 0)$
(B) $x_1 = 3s + 5, x_2 = -4s - 7, x_3 = s$, s any real number; or
$\{(3s + 5, -4s - 7, s) \mid s \text{ any real number}\}$ (C) No solution

4. (A) Condition 2 is violated: The 3 in row 2 and column 2 should be a 1. Perform the operation $\frac{1}{3}R_2 \to R_2$ to obtain:

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -2 \end{array}\right]$$

(B) Condition 3 is violated: The 5 in row 1 and column 2 should be a 0. Perform the operation $(-5)R_2 + R_1 \to R_1$ to obtain:

$$\left[\begin{array}{ccc|c} 1 & 0 & -6 & 8 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(C) Condition 4 is violated: The leftmost 1 in the second row is not to the right of the leftmost 1 in the first row. Perform the operation $R_1 \leftrightarrow R_2$ to obtain:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

(D) Condition 1 is violated: The all-zero second row should be at the bottom. Perform the operation $R_2 \leftrightarrow R_3$ to obtain:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

5. $x_1 = 1, x_2 = -1, x_3 = 0$ or $(1, -1, 0)$

6. Inconsistent; no solution

7. $x_1 = 5t + 4, x_2 = 3t + 5, x_3 = t$, t any real number

8. $x_1 = s + 7, x_2 = s, x_3 = t - 2, x_4 = -3t - 1, x_5 = t$, s and t any real numbers

9.

t	18-passenger planes x_1	24-passenger planes x_2	42-passenger planes x_3
14	2	14	14
15	5	10	15
16	8	6	16
17	11	2	17

9-1 Exercises

1. What is the size of a matrix?

2. What is a row matrix? What is its size?

3. What is a column matrix? What is its size?

4. What is a square matrix?

5. What does a_{ij} mean?

6. What is the principal diagonal of a matrix?

7. What is an augmented coefficient matrix?

8. What operations can you perform on an augmented coefficient matrix to produce a row-equivalent matrix?

9. What is a reduced matrix and how is it used to solve a system of linear equations?

10. Describe the Gauss–Jordan Elimination process in your own words.

In Problems 11–18, indicate whether each matrix is in reduced form.

11. $\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 2 & 6 \end{array}\right]$

12. $\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -3 \end{array}\right]$

13. $\left[\begin{array}{ccc|c} 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

14. $\left[\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

15. $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -5 \\ 1 & 0 & 0 & 4 \end{array}\right]$

16. $\left[\begin{array}{ccc|c} 1 & -2 & 4 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

17. $\left[\begin{array}{cccc|c} 0 & 1 & 6 & 0 & -8 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$

18. $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

In Problems 19–26, write the linear system corresponding to each reduced augmented matrix and solve.

19. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$

20. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$

21. $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

22. $\left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

23. $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$

24. $\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right]$

25. $\left[\begin{array}{cccc|c} 1 & -2 & 0 & -3 & -5 \\ 0 & 0 & 1 & 3 & 2 \end{array}\right]$

26. $\left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 4 \\ 0 & 1 & -1 & 2 & -1 \end{array}\right]$

Use row operations to change each matrix in Problems 27–32 to reduced form.

27. $\left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 1 & 3 \end{array}\right]$

28. $\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 2 & -4 \end{array}\right]$

29. $\left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 3 & -6 \end{array}\right]$

30. $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -2 & 2 \end{array}\right]$

31. $\left[\begin{array}{ccc|c} 1 & 2 & -2 & -1 \\ 0 & 3 & -6 & 1 \\ 0 & -1 & 2 & -\frac{1}{3} \end{array}\right]$

32. $\left[\begin{array}{ccc|c} 0 & -2 & 8 & 1 \\ 2 & -2 & 6 & -4 \\ 0 & -1 & 4 & \frac{1}{2} \end{array}\right]$

Solve Problems 33–52 using Gauss–Jordan elimination.

33.
$$\begin{aligned} 2x_1 + 4x_2 - 10x_3 &= -2 \\ 3x_1 + 9x_2 - 21x_3 &= 0 \\ x_1 + 5x_2 - 12x_3 &= 1 \end{aligned}$$

34.
$$\begin{aligned} 3x_1 + 5x_2 - x_3 &= -7 \\ x_1 + x_2 + x_3 &= -1 \\ 2x_1 + 11x_3 &= 7 \end{aligned}$$

35.
$$\begin{aligned} 3x_1 + 8x_2 - x_3 &= -18 \\ 2x_1 + x_2 + 5x_3 &= 8 \\ 2x_1 + 4x_2 + 2x_3 &= -4 \end{aligned}$$

36.
$$\begin{aligned} 2x_1 + 7x_2 + 15x_3 &= -12 \\ 4x_1 + 7x_2 + 13x_3 &= -10 \\ 3x_1 + 6x_2 + 12x_3 &= -9 \end{aligned}$$

37.
$$\begin{aligned} 2x_1 - x_2 - 3x_3 &= 8 \\ x_1 - 2x_2 &= 7 \end{aligned}$$

38.
$$\begin{aligned} 2x_1 + 4x_2 - 6x_3 &= 10 \\ 3x_1 + 3x_2 - 3x_3 &= 6 \end{aligned}$$

39.
$$\begin{aligned} 2x_1 - x_2 &= 0 \\ 3x_1 + 2x_2 &= 7 \\ x_1 - x_2 &= -1 \end{aligned}$$

40.
$$\begin{aligned} 2x_1 - x_2 &= 0 \\ 3x_1 + 2x_2 &= 7 \\ x_1 - x_2 &= -2 \end{aligned}$$

41.
$$\begin{aligned} 3x_1 - 4x_2 - x_3 &= 1 \\ 2x_1 - 3x_2 + x_3 &= 1 \\ x_1 - 2x_2 + 3x_3 &= 2 \end{aligned}$$

42.
$$\begin{aligned} 3x_1 + 7x_2 - x_3 &= 11 \\ x_1 + 2x_2 - x_3 &= 3 \\ 2x_1 + 4x_2 - 2x_3 &= 10 \end{aligned}$$

43.
$$\begin{aligned} -2x_1 + x_2 + 3x_3 &= -7 \\ x_1 - 4x_2 + 2x_3 &= 0 \\ x_1 - 3x_2 + x_3 &= 1 \end{aligned}$$

44.
$$\begin{aligned} 2x_1 + 5x_2 + 4x_3 &= -7 \\ -4x_1 - 5x_2 + 2x_3 &= 9 \\ -2x_1 - x_2 + 4x_3 &= 3 \end{aligned}$$

45. $\begin{aligned} 2x_1 - 2x_2 - 4x_3 &= -2 \\ -3x_1 + 3x_2 + 6x_3 &= 3 \end{aligned}$

46. $\begin{aligned} 2x_1 + 8x_2 - 6x_3 &= 4 \\ -3x_1 - 12x_2 + 9x_3 &= -6 \end{aligned}$

47. $\begin{aligned} 4x_1 - x_2 + 2x_3 &= 3 \\ -4x_1 + x_2 - 3x_3 &= -10 \\ 8x_1 - 2x_2 + 9x_3 &= -1 \end{aligned}$

48. $\begin{aligned} 4x_1 - 2x_2 + 2x_3 &= 5 \\ -6x_1 + 3x_2 - 3x_3 &= -2 \\ 10x_1 - 5x_2 - 9x_3 &= 4 \end{aligned}$

49. $\begin{aligned} 2x_1 - 5x_2 - 3x_3 &= 7 \\ -4x_1 + 10x_2 + 2x_3 &= 6 \\ 6x_1 - 15x_2 - x_3 &= -19 \end{aligned}$

50. $\begin{aligned} -4x_1 + 8x_2 + 10x_3 &= -6 \\ 6x_1 - 12x_2 - 15x_3 &= 9 \\ -8x_1 + 14x_2 + 19x_3 &= -8 \end{aligned}$

51. $\begin{aligned} 5x_1 - 3x_2 + 2x_3 &= 13 \\ 2x_1 - x_2 - 3x_3 &= 1 \\ 4x_1 - 2x_2 + 4x_3 &= 12 \end{aligned}$

52. $\begin{aligned} 4x_1 - 2x_2 + 3x_3 &= 3 \\ 3x_1 - x_2 - 2x_3 &= -10 \\ 2x_1 + 4x_2 - x_3 &= -1 \end{aligned}$

53. Consider a consistent system of three linear equations in three variables. Discuss the nature of the solution set for the system if the reduced form of the augmented coefficient matrix has

(A) One leading 1 (B) Two leading 1s
(C) Three leading 1s (D) Four leading 1s

54. Consider a system of three linear equations in three variables. Give an example of two reduced forms that are not row equivalent if the system is

(A) Consistent and dependent
(B) Inconsistent

In Problems 55–58, discuss the relationship between the number of solutions of the system and the constant k.

55. $\begin{aligned} x_1 - x_2 &= 4 \\ 3x_1 + kx_2 &= 7 \end{aligned}$

56. $\begin{aligned} x_1 + 2x_2 &= 4 \\ -2x_1 + kx_2 &= -8 \end{aligned}$

57. $\begin{aligned} x_1 + kx_2 &= 3 \\ 2x_1 + 6x_2 &= 6 \end{aligned}$

58. $\begin{aligned} x_1 + kx_2 &= 3 \\ 2x_1 + 4x_2 &= 8 \end{aligned}$

In Problems 59–64, solve using Gauss–Jordan elimination. Use RREF on a graphing calculator to find the reduced forms.

59. $\begin{aligned} x_1 + 2x_2 - 4x_3 - x_4 &= 7 \\ 2x_1 + 5x_2 - 9x_3 - 4x_4 &= 16 \\ x_1 + 5x_2 - 7x_3 - 7x_4 &= 13 \end{aligned}$

60. $\begin{aligned} 2x_1 + 4x_2 + 5x_3 + 4x_4 &= 8 \\ x_1 + 2x_2 + 2x_3 + x_4 &= 3 \end{aligned}$

61. $\begin{aligned} x_1 - x_2 + 3x_3 - 2x_4 &= 1 \\ -2x_1 + 4x_2 - 3x_3 + x_4 &= 0.5 \\ 3x_1 - x_2 + 10x_3 - 4x_4 &= 2.9 \\ 4x_1 - 3x_2 + 8x_3 - 2x_4 &= 0.6 \end{aligned}$

62. $\begin{aligned} x_1 + x_2 + 4x_3 + x_4 &= 1.3 \\ -x_1 + x_2 - x_3 &= 1.1 \\ 2x_1 + x_3 + 3x_4 &= -4.4 \\ 2x_1 + 5x_2 + 11x_3 + 3x_4 &= 5.6 \end{aligned}$

63. $\begin{aligned} x_1 - 2x_2 + x_3 + x_4 + 2x_5 &= 2 \\ -2x_1 + 4x_2 + 2x_3 + 2x_4 - 2x_5 &= 0 \\ 3x_1 - 6x_2 + x_3 + x_4 + 5x_5 &= 4 \\ -x_1 + 2x_2 + 3x_3 + x_4 + x_5 &= 3 \end{aligned}$

64. $\begin{aligned} x_1 - 3x_2 + x_3 + x_4 + 2x_5 &= 2 \\ -x_1 + 5x_2 + 2x_3 + 2x_4 - 2x_5 &= 0 \\ 2x_1 - 6x_2 + 2x_3 + 2x_4 + 4x_5 &= 4 \\ -x_1 + 3x_2 - x_3 - x_5 &= -3 \end{aligned}$

APPLICATIONS

Solve Problems 65–78 using Gauss–Jordan elimination.

65. **PUZZLE** A friend of yours came out of the post office after spending \$14.00 on 15¢, 20¢, and 35¢ stamps. If she bought 45 stamps in all, how many of each type did she buy?

66. **PUZZLE** A parking meter accepts only nickels, dimes, and quarters. If the meter contains 32 coins with a total value of \$6.80, how many of each type are there?

67. **CHEMISTRY** A chemist can purchase a 10% saline solution in 500 cubic centimeter containers, a 20% saline solution in 500 cubic centimeter containers, and a 50% saline solution in 1,000 cubic centimeter containers. He needs 12,000 cubic centimeters of 30% saline solution. How many containers of each type of solution should he purchase in order to form this solution?

68. **CHEMISTRY** Repeat Problem 67 if the 50% saline solution is available only in 1,500 cubic centimeter containers.

69. **GEOMETRY** Find a, b, and c so that the graph of the parabola with equation $y = a + bx + cx^2$ passes through the points $(-2, 3)$, $(-1, 2)$, and $(1, 6)$.

70. **GEOMETRY** Find a, b, and c so that the graph of the parabola with equation $y = a + bx + cx^2$ passes through the points $(1, 3)$, $(2, 2)$, and $(3, 5)$.

71. **GEOMETRY** Find a, b, and c so that the graph of the circle with equation $x^2 + y^2 + ax + by + c = 0$ passes through the points $(6, 2)$, $(4, 6)$, and $(-3, -1)$.

72. **GEOMETRY** Find a, b, and c so that the graph of the circle with equation $x^2 + y^2 + ax + by + c = 0$ passes through the points $(-4, 1)$, $(-1, 2)$, and $(3, -6)$.

73. **PRODUCTION SCHEDULING** A small manufacturing plant makes three types of inflatable boats: one-person, two-person, and four-person models. Each boat requires the services of three departments, as listed in the table. The cutting, assembly, and packaging departments have available a maximum of 380, 330, and 120 labor-hours per week, respectively.

	One-person boat	Two-person boat	Four-person boat
Cutting department	0.5 h	1.0 h	1.5 h
Assembly department	0.6 h	0.9 h	1.2 h
Packaging department	0.2 h	0.3 h	0.5 h

(A) How many boats of each type must be produced each week for the plant to operate at full capacity?
(B) How is the production schedule in part A affected if the packaging department is no longer used?
(C) How is the production schedule in part A affected if the four-person boat is no longer produced?

74. PRODUCTION SCHEDULING Repeat Problem 73 assuming the cutting, assembly, and packaging departments have available a maximum of 350, 330, and 115 labor-hours per week, respectively.

75. NUTRITION A dietitian in a hospital is to arrange a special diet using three basic foods. The diet is to include exactly 340 units of calcium, 180 units of iron, and 220 units of vitamin A. The number of units per ounce of each nutrient for each of the foods is indicated in the table.

	Units per ounce		
	Food *A*	Food *B*	Food *C*
Calcium	30	10	20
Iron	10	10	20
Vitamin A	10	30	20

(A) How many ounces of each food must be used to meet the diet requirements?
(B) How is the diet in part A affected if food *C* is not used?
(C) How is the diet in part A affected if the vitamin A requirement is dropped?

76. NUTRITION Repeat Problem 75 if the diet is to include exactly 400 units of calcium, 160 units of iron, and 240 units of vitamin A.

77. AGRICULTURE A farmer can buy four types of fertilizer. Each barrel of mix *A* contains 30 pounds of phosphoric acid, 50 pounds of nitrogen, and 30 pounds of potash; each barrel of mix *B* contains 30 pounds of phosphoric acid, 75 pounds of nitrogen, and 20 pounds of potash; each barrel of mix *C* contains 30 pounds of phosphoric acid, 25 pounds of nitrogen, and 20 pounds of potash; and each barrel of mix *D* contains 60 pounds of phosphoric acid, 25 pounds of nitrogen, and 50 pounds of potash. Soil tests indicate that a particular field needs 900 pounds of phosphoric acid, 750 pounds of nitrogen, and 700 pounds of potash. How many barrels of each type of fertilizer should the farmer mix together to supply the necessary nutrients for the field?

78. ANIMAL NUTRITION In a laboratory experiment, rats are to be fed five packets of food containing a total of 80 units of vitamin E. There are four different brands of food packets that can be used. A packet of brand *A* contains 5 units of vitamin E, a packet of brand *B* contains 10 units of vitamin E, a packet of brand *C* contains 15 units of vitamin E, and a packet of brand *D* contains 20 units of vitamin E. How many packets of each brand should be mixed and fed to the rats?

9-2 Matrix Operations

- Adding and Subtracting Matrices
- Multiplying a Matrix by a Number
- Finding the Product of Two Matrices

In the last section, we introduced basic matrix terminology and solved systems of equations by performing row operations on augmented coefficient matrices. Matrices have many other useful applications and possess an interesting mathematical structure in their own right. As we will see, matrix addition and multiplication are similar to real number addition and multiplication in many respects, but there are some important differences.

› Adding and Subtracting Matrices

Before we can discuss arithmetic operations for matrices, we have to define equality for matrices. Two matrices are **equal** if they have the same size and their corresponding elements are equal. For example,

$$\overset{2\times 3}{\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}} = \overset{2\times 3}{\begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}} \quad \text{if and only if} \quad \begin{matrix} a=u & b=v & c=w \\ d=x & e=y & f=z \end{matrix}$$

The **sum of two matrices** of the same size is a matrix with elements that are the sums of the corresponding elements of the two given matrices.

Addition is not defined for matrices of different sizes.

EXAMPLE 1 Adding Matrices

Add:

(A) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix}$

(B) $\begin{bmatrix} 2 & -3 & 0 \\ 1 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 \\ -3 & 2 & 5 \end{bmatrix}$

(C) $\begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -3 & 5 \\ -1 & 4 \end{bmatrix}$

SOLUTIONS

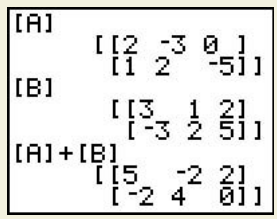
(A) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} (a+w) & (b+x) \\ (c+y) & (d+z) \end{bmatrix}$

(B) Algebraic Solution

$$\begin{bmatrix} 2 & -3 & 0 \\ 1 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 \\ -3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} (2+3) & (-3+1) & (0+2) \\ (1-3) & (2+2) & (-5+5) \end{bmatrix}^{*}$$

$$= \begin{bmatrix} 5 & -2 & 2 \\ -2 & 4 & 0 \end{bmatrix}$$

(B) Graphing Calculator Solution



› Figure 1

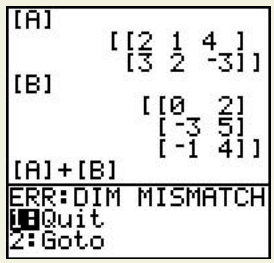
*The dashed "think boxes" are used to enclose steps that may be performed mentally.

(C) Algebraic Solution

$$\begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -3 & 5 \\ -1 & 4 \end{bmatrix}$$

Because the first matrix is 2×3 and the second is 3×2, this sum is not defined.

(C) Graphing Calculator Solution



› Figure 2

MATCHED PROBLEM 1

Add: $\begin{bmatrix} 3 & 2 \\ -1 & -1 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 1 & -1 \\ 2 & -2 \end{bmatrix}$

Because we add two matrices by adding their corresponding elements, it follows from the properties of real numbers that matrices of the same size are commutative and associative relative to addition. That is, if A, B, and C are matrices of the same size, then

$$A + B = B + A \quad \text{Commutative}$$
$$(A + B) + C = A + (B + C) \quad \text{Associative}$$

A matrix with elements that are all 0s is called a **zero matrix.** For example, the following are zero matrices of different sizes:

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[*Note:* "0" can be used to denote the zero matrix of any size.]

The **negative of a matrix *M*,** denoted by $-M$, is a matrix with elements that are the negatives of the elements in M. Thus, if

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$-M = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$

Note that $M + (-M) = 0$ (a zero matrix).

If A and B are matrices of the same size, then we define **subtraction** as follows:

$$A - B = A + (-B)$$

Thus, to subtract matrix B from matrix A, we simply subtract corresponding elements.

EXAMPLE 2 Subtracting Matrices

Subtract: $\begin{bmatrix} 3 & -2 \\ 5 & 0 \end{bmatrix} - \begin{bmatrix} -2 & 2 \\ 3 & 4 \end{bmatrix}$

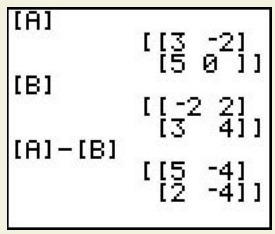
SOLUTIONS

Algebraic Solution

$$\begin{bmatrix} 3 & -2 \\ 5 & 0 \end{bmatrix} - \begin{bmatrix} -2 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3-(-2) & -2-2 \\ 5-3 & 0-4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -4 \\ 2 & -4 \end{bmatrix}$$

Graphing Calculator Solution



› Figure 3

MATCHED PROBLEM 2

Subtract: $[2 \quad -3 \quad 5] - [3 \quad -2 \quad 1]$

EXAMPLE 3 Solving a Matrix Equation

Find a, b, c, and d so that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -2 & 4 \end{bmatrix}$$

SOLUTION

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -2 & 4 \end{bmatrix}$$ Subtract the matrices on the left side.

$$\begin{bmatrix} a-2 & b-(-1) \\ c-(-5) & d-6 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -2 & 4 \end{bmatrix}$$ Simplify.

$$\begin{bmatrix} a-2 & b+1 \\ c+5 & d-6 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -2 & 4 \end{bmatrix}$$ Use the definition of equality to change this matrix equation into four real number equations.

$$a-2=4 \quad b+1=3 \quad c+5=-2 \quad d-6=4$$

$$a=6 \quad b=2 \quad c=-7 \quad d=10$$

MATCHED PROBLEM 3

Find a, b, c, and d so that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} -4 & 2 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 8 & 2 \end{bmatrix}$$

››› EXPLORE-DISCUSS 1

Solve Example 3 using a graphing calculator. Compare the algebraic solution given in the text and your graphing calculator solution. Which do you prefer?

› Multiplying a Matrix by a Number

The **product of a number k and a matrix M,** denoted by kM, is a matrix formed by multiplying each element of M by k.

EXAMPLE 4 Multiplying a Matrix by a Number

Multiply: $-2\begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 3 \\ 0 & -1 & -2 \end{bmatrix}$

SOLUTIONS

Algebraic Solution

$$-2\begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 3 \\ 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} -2(3) & -2(-1) & -2(0) \\ -2(-2) & -2(1) & -2(3) \\ -2(0) & -2(-1) & -2(-2) \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 2 & 0 \\ 4 & -2 & -6 \\ 0 & 2 & 4 \end{bmatrix}$$

Graphing Calculator Solution

```
[A]
    [[3   -1  0 ]
     [-2  1   3 ]
     [0   -1  -2]]
-2[A]
    [[-6  2   0 ]
     [4   -2  -6]
     [0   2   4 ]]
```

› Figure 4

MATCHED PROBLEM 4

Find: $10\begin{bmatrix} 1.3 \\ 0.2 \\ 3.5 \end{bmatrix}$

››› EXPLORE-DISCUSS 2

Multiplication of two numbers can be interpreted as repeated addition if one of the numbers is a positive integer. That is,

$$2a = a + a \qquad 3a = a + a + a \qquad 4a = a + a + a + a$$

and so on. Discuss this interpretation for the product of an integer k and a matrix M. Use specific examples to illustrate your remarks.

We now consider an application that uses various matrix operations.

EXAMPLE 5 Sales and Commissions

Ms. Fong and Mr. Petris are salespeople for a new car agency that sells only two models. August was the last month for this year's models, and next year's models were introduced in September. Gross dollar sales for each month are given in the following matrices:

$$\begin{array}{c} \\ \text{Fong} \\ \text{Petris} \end{array} \begin{array}{c} \begin{array}{cc} \text{AUGUST SALES} & \\ \text{Compact} & \text{Luxury} \end{array} \\ \begin{bmatrix} \$36{,}000 & \$72{,}000 \\ \$72{,}000 & \$0 \end{bmatrix} \end{array} = A \qquad \begin{array}{c} \begin{array}{cc} \text{SEPTEMBER SALES} & \\ \text{Compact} & \text{Luxury} \end{array} \\ \begin{bmatrix} \$144{,}000 & \$288{,}000 \\ \$180{,}000 & \$216{,}000 \end{bmatrix} \end{array} = B$$

For example, Ms. Fong had \$36,000 in compact sales in August and Mr. Petris had \$216,000 in luxury car sales in September.

(A) What are the combined dollar sales in August and September for each salesperson and each model?

(B) What was the increase in dollar sales from August to September?

(C) If both salespeople receive a 3% commission on gross dollar sales, compute the commission for each salesperson for each model sold in September.

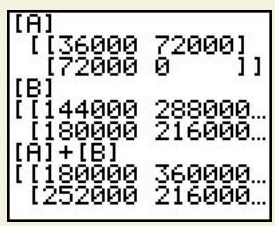
SOLUTIONS

(A) Algebraic Solution

$$A + B = \begin{bmatrix} 36{,}000 & 72{,}000 \\ 72{,}000 & 0 \end{bmatrix} + \begin{bmatrix} 144{,}000 & 288{,}000 \\ 180{,}000 & 216{,}000 \end{bmatrix}$$

$$= \begin{array}{c} \\ \text{Fong} \\ \text{Petris} \end{array} \begin{array}{c} \begin{array}{cc} \text{Compact} & \text{Luxury} \end{array} \\ \begin{bmatrix} \$180{,}000 & \$360{,}000 \\ \$252{,}000 & \$216{,}000 \end{bmatrix} \end{array}$$

(A) Graphing Calculator Solution



› Figure 5

(B) Algebraic Solution

$$B - A = \begin{bmatrix} 144{,}000 & 288{,}000 \\ 180{,}000 & 216{,}000 \end{bmatrix} - \begin{bmatrix} 36{,}000 & 72{,}000 \\ 72{,}000 & 0 \end{bmatrix}$$

Compact Luxury

$$= \begin{bmatrix} \$108{,}000 & \$216{,}000 \\ \$108{,}000 & \$216{,}000 \end{bmatrix}$$ Fong, Petris

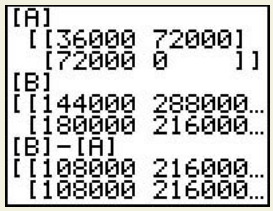
(C) Algebraic Solution

$$0.03B = \begin{bmatrix} (0.03)(144{,}000) & (0.03)(288{,}000) \\ (0.03)(180{,}000) & (0.03)(216{,}000) \end{bmatrix}$$

Compact Luxury

$$= \begin{bmatrix} \$4{,}320 & \$8{,}640 \\ \$5{,}400 & \$6{,}480 \end{bmatrix}$$ Fong, Petris

(B) Graphing Calculator Solution



› Figure 6

(C) Graphing Calculator Solution

› Figure 7

MATCHED PROBLEM 5

Repeat Example 4 with

$$A = \begin{bmatrix} \$72{,}000 & \$72{,}000 \\ \$36{,}000 & \$72{,}000 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \$180{,}000 & \$216{,}000 \\ \$144{,}000 & \$216{,}000 \end{bmatrix}$$

Example 5 involved an agency with only two salespeople and two models. A more realistic problem might involve 20 salespeople and 15 models. Problems of this size are often solved with the aid of a spreadsheet on a personal computer. Figure 8 illustrates a computer spreadsheet solution for Example 5.

	A	B	C	D	E	F	G
1		Compact	Luxury	Compact	Luxury	Compact	Luxury
2		August Sales		September Sales		September Commissions	
3	Fong	$36,000	$72,000	$144,000	$288,000	$4,320	$8,640
4	Petris	$72,000	$0	$180,000	$216,000	$5,400	$6,480
5		Combined Sales		Sales Increases			
6	Fong	$180,000	$360,000	$108,000	$216,000		
7	Petris	$252,000	$216,000	$108,000	$216,000		

› Figure 8

› Finding the Product of Two Matrices

Now we are going to introduce a matrix multiplication that may at first seem rather strange. In spite of its apparent strangeness, this operation is well-founded in the general theory of matrices and, as we will see, is extremely useful in many practical problems.

Historically, matrix multiplication was introduced by the English mathematician Arthur Cayley (1821–1895) in studies of linear equations and linear transformations. In the section after next, you will see how matrix multiplication is central to the process of expressing systems of equations as matrix equations and to the process of solving matrix equations. Matrix equations and their solutions provide us with an alternate method of solving linear systems with the same number of variables as equations.

We start by defining the product of two special matrices, a row matrix and a column matrix.

› DEFINITION 1 Product of a Row Matrix and a Column Matrix

The **product** of a $1 \times n$ row matrix and an $n \times 1$ column matrix is a 1×1 matrix given by

$$\overset{1 \times n}{[a_1 \ a_2 \cdots a_n]} \overset{n \times 1}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}} = [a_1b_1 + a_2b_2 + \cdots + a_nb_n]$$

Note that the number of elements in the row matrix and in the column matrix must be the same for the product to be defined.

EXAMPLE 6 Product of a Row Matrix and a Column Matrix

Multiply: $[2 \quad -3 \quad 0]\begin{bmatrix} -5 \\ 2 \\ -2 \end{bmatrix}$

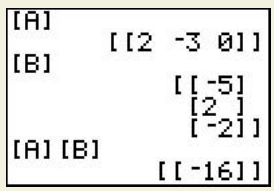
SOLUTION

Algebraic Solution

$$[2 \quad -3 \quad 0]\begin{bmatrix} -5 \\ 2 \\ -2 \end{bmatrix} = [(2)(-5) + (-3)(2) + (0)(-2)]$$

$$= [-10 + (-6) + 0] = [-16]$$

Graphing Calculator Solution



› Figure 9

MATCHED PROBLEM 6

$$[-1 \quad 0 \quad 3 \quad 2]\begin{bmatrix} 2 \\ 3 \\ 4 \\ -1 \end{bmatrix} = ?$$

Refer to Example 6. The distinction between the real number -16 and the 1×1 matrix $[-16]$ is a technical one, and it is common to see 1×1 matrices written as real numbers without brackets. In the work that follows, we will frequently refer to 1×1 matrices as real numbers and omit the brackets whenever it is convenient to do so.

EXAMPLE 7 Production Scheduling

A factory produces a slalom water ski that requires 4 labor-hours in the fabricating department and 1 labor-hour in the finishing department. Fabricating personnel receive \$10 per hour, and finishing personnel receive \$8 per hour.
Total labor cost per ski is given by the product

SOLUTIONS

$$[4 \quad 1]\begin{bmatrix} 10 \\ 8 \end{bmatrix} = [(4)(10) + (1)(8)] = [40 + 8] = [48] \text{ or } \$48 \text{ per ski}$$

MATCHED PROBLEM 7

If the factory in Example 7 also produces a trick water ski that requires 6 labor-hours in the fabricating department and 1.5 labor-hours in the finishing department. Find the total labor cost per ski by multiplying an appropriate row matrix and column matrix.

We now use the product of a $1 \times n$ row matrix and an $n \times 1$ column matrix to extend the definition of matrix product to more general matrices.

DEFINITION 2 Matrix Product

If A is an $m \times p$ matrix and B is a $p \times n$ matrix, then the **matrix product** of A and B, denoted AB or $A \cdot B$, is an $m \times n$ matrix whose element in the ith row and jth column is the real number obtained from the product of the ith row of A and the jth column of B. If the number of columns in A does not equal the number of rows in B, then the matrix product AB is **not defined.**

It is important to check sizes before starting the multiplication process. If A is an $a \times b$ matrix and B is a $c \times d$ matrix, then if $b = c$, the product AB will exist and will be an $a \times d$ matrix (Fig. 10). If $b \neq c$, then the product AB does not exist.

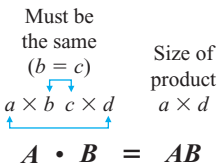


Figure 10

The definition is not as complicated as it might first seem. An example should help clarify the process. For

$$A = \begin{bmatrix} 2 & 3 & -1 \\ -2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$$

A is 2×3, B is 3×2, and so AB is 2×2. To find the first row of AB, we take the product of the first row of A with every column of B and write each result as a real number, not a 1×1 matrix. The second row of AB is computed in the same manner. The four products of row and column matrices used to produce the four elements in AB are shown in the dashed box. These products are usually calculated mentally, or with the aid of a calculator, and need not be written out. The shaded portions highlight the steps involved in computing the element in the first row and second column of AB.

$$\overset{2\times 3}{\begin{bmatrix} 2 & 3 & -1 \\ -2 & 1 & 2 \end{bmatrix}} \overset{3\times 2}{\begin{bmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}} = \begin{bmatrix} [2 \quad 3 \quad -1]\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} & [2 \quad 3 \quad -1]\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \\ [-2 \quad 1 \quad 2]\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} & [-2 \quad 1 \quad 2]\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (2)(1) + (3)(2) + (-1)(-1) & (2)(3) + (3)(0) + (-1)(2) \\ (-2)(1) + (1)(2) + (2)(-1) & (-2)(3) + (1)(0) + (2)(2) \end{bmatrix}$$

$$= \overset{2\times 2}{\begin{bmatrix} 9 & 4 \\ -2 & -2 \end{bmatrix}}$$

EXAMPLE 8 Matrix Product

Given

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

Find each product that is defined:

(A) AB (B) BA (C) CD (D) DC

SOLUTIONS

(A) Algebraic Solution

$$AB = \overset{3\times 2}{\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}} \overset{2\times 4}{\begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix}}$$

$$= \begin{bmatrix} 4 & -1 & 2 & 2 \\ 1 & -1 & 0 & 1 \\ 3 & 3 & 4 & -1 \end{bmatrix}$$

(A) Graphing Calculator Solution

```
[A]
          [[2   1]
           [1   0]
           [-1  2]]
[B]
     [[1  -1  0  1]
      [2  1   2  0]]
[A][B]
     [[4  -1  2  2 ]
      [1  -1  0  1 ]
      [3  3   4  -1]]
```

› Figure 11

(B) Algebraic Solution

$$BA = \overset{2\times 4}{\begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix}} \overset{3\times 2}{\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}}$$

Because B has four columns and A has three rows, the product BA is not defined.

(C) Algebraic Solution

$$CD = \begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 40 \\ -10 & -20 \end{bmatrix}$$

(D) Algebraic Solution

$$DC = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

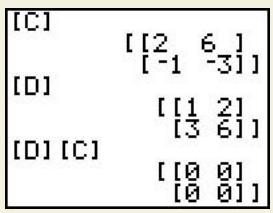
(B) Graphing Calculator Solution

› Figure 12

(C) Graphing Calculator Solution

› Figure 13

(D) Graphing Calculator Solution



› Figure 14

MATCHED PROBLEM 8

Find each product, if it is defined:

(A) $\begin{bmatrix} -1 & 0 & 3 & -2 \\ 1 & 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}$

(B) $\begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 3 & -2 \\ 1 & 2 & 2 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$

(D) $\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$

(E) $\begin{bmatrix} 3 & -2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$

(F) $\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 3 & -2 & 1 \end{bmatrix}$

In the arithmetic of real numbers it does not matter in which order we multiply; for example, $5 \times 7 = 7 \times 5$. In matrix multiplication, however, it does make a difference. That is, AB does not always equal BA, even if both multiplications are defined and both products are the same size (see Example 8, parts C and D).

Thus,

Matrix multiplication is not commutative.

Also, AB may be zero with neither A nor B equal to zero (see Example 8, part D). Thus,

The zero product property does not hold for matrix multiplication.

(The zero product property, namely, $ab = 0$ if and only if $a = 0$ or $b = 0$, does hold for all real numbers a and b.)

Just as we used the familiar algebraic notation AB to represent the product of matrices A and B, we use the notation A^2 for AA, the product of A with itself, A^3 for AAA, and so on.

››› EXPLORE-DISCUSS 3

In addition to the commutative and zero product properties, there are other significant differences between real number multiplication and matrix multiplication.

(A) In real number multiplication, the only real number whose square is 0 is the real number 0 ($0^2 = 0$). Find at least one 2×2 matrix A with all elements nonzero such that $A^2 = 0$, where 0 is the 2×2 zero matrix.

(B) In real number multiplication, the only nonzero real number that is equal to its square is the real number 1 ($1^2 = 1$). Find at least one 2×2 matrix A with all elements nonzero such that $A^2 = A$.

We will continue our discussion of properties of matrix multiplication later in this chapter. Now we consider an application of matrix multiplication.

EXAMPLE 9 Labor Costs

Let us combine the time requirements for slalom and trick water skis discussed in Example 7 and Matched Problem 7 into one matrix:

$$\begin{array}{l} \\ \text{Trick ski} \\ \text{Slalom ski} \end{array} \begin{array}{c} \begin{array}{cc} \text{Labor-hours per ski} & \\ \text{Assembly department} & \text{Finishing department} \end{array} \\ \begin{bmatrix} 6 \text{ h} & 1.5 \text{ h} \\ 4 \text{ h} & 1 \text{ h} \end{bmatrix} \end{array} = L$$

Now suppose that the company has two manufacturing plants, X and Y, in different parts of the country and that the hourly wages for each department are given in the following matrix:

$$\begin{matrix} & \begin{matrix}\text{Plant } X & \text{Plant } Y\end{matrix} \\ \begin{matrix}\text{Assembly department} \\ \text{Finishing department}\end{matrix} & \begin{bmatrix}\$10 & \$12 \\ \$\ 8 & \$10\end{bmatrix} \end{matrix} = H \quad \text{Hourly wages}$$

Because H and L are both 2×2 matrices, we can take the product of H and L in either order and the result will be a 2×2 matrix:

$$HL = \begin{bmatrix}10 & 12 \\ 8 & 10\end{bmatrix}\begin{bmatrix}6 & 1.5 \\ 4 & 1\end{bmatrix} = \begin{bmatrix}108 & 27 \\ 88 & 22\end{bmatrix}$$

$$LH = \begin{bmatrix}6 & 1.5 \\ 4 & 1\end{bmatrix}\begin{bmatrix}10 & 12 \\ 8 & 10\end{bmatrix} = \begin{bmatrix}72 & 87 \\ 48 & 58\end{bmatrix}$$

How can we interpret the elements in these products? Let's begin with the product HL. The element 108 in the first row and first column of HL is the product of the first row matrix of H and the first column matrix of L:

$$\begin{matrix}\text{Plant } X & \text{Plant } Y \\ [10 & 12]\end{matrix}\begin{bmatrix}6 \\ 4\end{bmatrix} \begin{matrix}\text{Trick} \\ \text{Slalom}\end{matrix} = 10(6) + 12(4) = 60 + 48 = 108$$

Notice that \$60 is the labor cost for assembling a trick ski at plant X and \$48 is the labor cost for assembling a slalom ski at plant Y. Although both numbers represent labor costs, it makes no sense to add them together. They do not pertain to the same type of ski or to the same plant. Thus, although the product HL happens to be defined mathematically, it has no useful interpretation in this problem.

Now let's consider the product LH. The element 72 in the first row and first column of LH is given by the following product:

$$\begin{matrix}\text{Assembly} & \text{Finishing}\end{matrix}$$
$$[6 \quad 1.5]\begin{bmatrix}10 \\ 8\end{bmatrix} \begin{matrix}\text{Assembly} \\ \text{Finishing}\end{matrix} = 6(10) + 1.5(8)$$
$$= 60 + 12 = 72$$

where \$60 is the labor cost for assembling a trick ski at plant X and \$12 is the labor cost for finishing a trick ski at plant X. Thus, the sum is the total labor cost for producing a trick ski at plant X. The other elements in LH also represent total labor costs, as indicated by the row and column labels shown below:

$$LH = \begin{matrix} & \begin{matrix}\text{Plant } X & \text{Plant } Y\end{matrix} \\ \begin{matrix}\text{Trick ski} \\ \text{Slalom ski}\end{matrix} & \begin{bmatrix}\$72 & \$87 \\ \$48 & \$58\end{bmatrix} \end{matrix} \quad \text{Labor costs per ski}$$

MATCHED PROBLEM 9

Refer to Example 9. The company wants to know how many hours to schedule in each department to produce 1,000 trick skis and 2,000 slalom skis. These production requirements can be represented by either of the following matrices:

$$P = \begin{matrix}\text{Trick skis} & \text{Slalom skis}\end{matrix}\begin{bmatrix}1{,}000 & 2{,}000\end{bmatrix} \qquad Q = \begin{matrix}\\ \text{Trick skis} \\ \text{Slalom skis}\end{matrix}\begin{bmatrix}1{,}000 \\ 2{,}000\end{bmatrix}$$

Using the labor-hour matrix L from Example 9, find PL or LQ, whichever has a meaningful interpretation for this problem, and label the rows and columns accordingly.

Figure 15 shows a solution to Example 9 on a spreadsheet.

	A	B	C	D	E	F
1		Labor-hours per ski			Hourly wages	
2		Fabricating	Finishing		Plant X	Plant Y
3	Trick ski	6	1.5	Fabricating	$10	$12
4	Slalom ski	4	1	Finishing	$8	$10
5		Labor costs per ski				
6		Plant X	Plant Y			
7	Trick ski	$72	$87			
8	Slalom ski	$48	$58			

› **Figure 15** Matrix multiplication in a spreadsheet.

››› CAUTION ›››

Example 9 and Matched Problem 9 illustrate an important point about matrix multiplication. Even if you are using a graphing calculator to perform the calculations in a matrix product, it is still necessary for you to know the definition of matrix multiplication so that you can interpret the results correctly.

ANSWERS TO MATCHED PROBLEMS

1. $\begin{bmatrix}1 & 5\\ 0 & -2\\ 2 & 1\end{bmatrix}$ **2.** $[-1 \quad -1 \quad 4]$ **3.** $a = -6, b = 7, c = 9, d = -1$

4. $\begin{bmatrix}13\\ 2\\ 35\end{bmatrix}$ **5.** (A) $\begin{bmatrix}\$252{,}000 & \$288{,}000\\ \$180{,}000 & \$288{,}000\end{bmatrix}$ (B) $\begin{bmatrix}\$108{,}000 & \$144{,}000\\ \$108{,}000 & \$144{,}000\end{bmatrix}$

(C) $\begin{bmatrix}\$5{,}400 & \$6{,}480\\ \$4{,}320 & \$6{,}480\end{bmatrix}$ **6.** $[8]$ **7.** $[6 \quad 1.5]\begin{bmatrix}10\\ 8\end{bmatrix} = [72]$ or \$72

8. (A) Not defined (B) $\begin{bmatrix} 2 & 2 & -1 & 2 \\ 1 & 6 & 12 & -4 \\ -1 & 0 & 3 & -2 \end{bmatrix}$ (C) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (D) $\begin{bmatrix} -6 & -12 \\ 3 & 6 \end{bmatrix}$

(E) [11] (F) $\begin{bmatrix} 12 & -8 & 4 \\ 6 & -4 & 2 \\ 9 & -6 & 3 \end{bmatrix}$

9. $PL = \begin{matrix} \\ \text{Labor-hours} \end{matrix} \begin{matrix} \begin{matrix} \text{Assembly} & \text{Finishing} \end{matrix} \\ \begin{bmatrix} 14{,}000 & 3{,}500 \end{bmatrix} \end{matrix}$

9-2 Exercises

1. What conditions must matrices A and B satisfy so that $A + B$ exists?

2. What conditions must matrices A and B satisfy so that AB exists?

3. What conditions must matrices A and B satisfy so that BA exists?

4. What conditions must matrices A and B satisfy so that both AB and BA exist?

5. What is the negative of a matrix?

6. How do you subtract two matrices?

7. How do you multiply a matrix by a number?

8. If A is a $1 \times n$ matrix and B is an $n \times 1$ matrix, how do you find the product AB? What is the size of AB?

9. If A is a $1 \times n$ matrix and B is an $n \times 1$ matrix, how do you find the product BA? What is the size of BA?

10. Describe the operation of matrix multiplication in your own words.

Perform the indicated operations in Problems 11–28, if possible.

11. $\begin{bmatrix} 5 & -2 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 7 \\ 1 & -6 \end{bmatrix}$

12. $\begin{bmatrix} 0 & 8 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 9 & -4 \\ 7 & 5 \end{bmatrix}$

13. $\begin{bmatrix} 4 & 0 \\ -2 & 3 \\ 8 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 0 & 5 \\ 4 & -6 \end{bmatrix}$

14. $\begin{bmatrix} 6 & -2 & 3 \\ 4 & -8 & -7 \end{bmatrix} + \begin{bmatrix} 3 & 9 & -1 \\ 6 & -2 & 4 \end{bmatrix}$

15. $\begin{bmatrix} 4 & 0 \\ -2 & 3 \\ 8 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 4 \\ 2 & 5 & -6 \end{bmatrix}$

16. $\begin{bmatrix} 6 & -2 & 3 \\ 4 & -8 & -7 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 9 & -2 \\ -1 & 4 \end{bmatrix}$

17. $\begin{bmatrix} 5 & -1 & 0 \\ 4 & 6 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 4 & -6 \\ 3 & 5 & -5 \end{bmatrix}$

18. $\begin{bmatrix} 6 & 2 \\ -4 & 1 \\ 3 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 5 \\ -7 & 2 \\ -1 & 0 \end{bmatrix}$ **19.** $4\begin{bmatrix} 3 & -4 & 7 \\ -2 & 9 & 5 \end{bmatrix}$

20. $5\begin{bmatrix} -7 & 3 & 0 & 9 \\ 4 & -5 & 6 & 2 \end{bmatrix}$ **21.** $\begin{bmatrix} 5 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \end{bmatrix}$

22. $\begin{bmatrix} -2 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ -8 \end{bmatrix}$ **23.** $\begin{bmatrix} -6 & 3 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

24. $\begin{bmatrix} 3 & 7 \\ -1 & -9 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ **25.** $\begin{bmatrix} 5 & 1 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix}$

26. $\begin{bmatrix} -2 & 7 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 0 & 5 \end{bmatrix}$ **27.** $\begin{bmatrix} 8 & -3 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$

28. $\begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 9 & -2 \\ -4 & -1 \end{bmatrix}$

Find the products in Problems 29–36.

29. $\begin{bmatrix} 4 & -2 \end{bmatrix}\begin{bmatrix} -5 \\ -3 \end{bmatrix}$ **30.** $\begin{bmatrix} 2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix}$

31. $\begin{bmatrix} -5 \\ -3 \end{bmatrix}\begin{bmatrix} 4 & -2 \end{bmatrix}$ **32.** $\begin{bmatrix} 3 \\ -4 \end{bmatrix}\begin{bmatrix} 2 & -1 \end{bmatrix}$

33. $\begin{bmatrix} 3 & -2 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ **34.** $\begin{bmatrix} 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

35. $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}\begin{bmatrix} 3 & -2 & -4 \end{bmatrix}$ **36.** $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \end{bmatrix}$

Problems 37–54 refer to the following matrices.

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \end{bmatrix} \quad B = \begin{bmatrix} -3 & 1 \\ 2 & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 0 & 2 \\ 4 & -3 & 1 \\ -2 & 3 & 5 \end{bmatrix} \quad D = \begin{bmatrix} 3 & -2 \\ 0 & -1 \\ 1 & 2 \end{bmatrix}$$

37. CA **38.** AC **39.** BA

40. AB **41.** C^2 **42.** B^2

43. $C + DA$ **44.** $B + AD$ **45.** $0.2CD$

46. $0.1DB$ **47.** $2DB + 5CD$ **48.** $3BA + 4AC$

49. $(-1)AC + 3DB$ **50.** $(-2)BA + 6CD$

51. CDA **52.** ACD

53. DBA **54.** BAD

In Problems 55 and 56, use a graphing calculator to calculate B, B^2, B^3, . . . and AB, AB^2, AB^3, ... Describe any patterns you observe in each sequence of matrices.

55. $A = [0.3 \quad 0.7]$ and $B = \begin{bmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix}$

56. $A = [0.4 \quad 0.6]$ and $B = \begin{bmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{bmatrix}$

57. Find a, b, c, and d so that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

58. Find w, x, y, and z so that

$$\begin{bmatrix} 4 & -2 \\ -3 & 0 \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 0 & 5 \end{bmatrix}$$

59. Find x and y so that

$$\begin{bmatrix} 3x & 5 \\ -1 & 4x \end{bmatrix} + \begin{bmatrix} 2y & -3 \\ -6 & -y \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ -7 & 2 \end{bmatrix}$$

60. Find x and y so that

$$\begin{bmatrix} 4 & 2x \\ -4x & -3 \end{bmatrix} + \begin{bmatrix} -5 & -3y \\ 5y & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

In Problems 61 and 62, let a, b, and c be any nonzero real numbers, let

$$A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

61. If $A^2 = 0$, how are a, b, and c related? Use this relationship to provide several examples of 2×2 matrices with no zero entries whose square is the zero matrix.

62. If $A^2 = I$, how are a, b, and c related? Use this relationship to provide several examples of 2×2 matrices with no zero entries whose square is the matrix I.

Problems 63 and 64 refer to the matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

63. If $AB = 0$, how are a, b, c, and d related? Use this relationship to provide several examples of 2×2 matrices A with no zero entries that satisfy $AB = 0$.

64. If $BA = 0$, how are a, b, c, and d related? Use this relationship to provide several examples of 2×2 matrices A with no zero entries that satisfy $BA = 0$.

65. Find x and y so that

$$\begin{bmatrix} 1 & 3 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} y & 7 \\ y & -6 \end{bmatrix}$$

66. Find x and y so that

$$\begin{bmatrix} x & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} y & y \\ 2 & 1 \end{bmatrix}$$

67. Find a, b, c, and d so that

$$\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 6 & -5 \\ 7 & -7 \end{bmatrix}$$

68. Find a, b, c, and d so that

$$\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

69. A square matrix is a **diagonal matrix** if all elements not on the principal diagonal are zero. Thus, a 2×2 diagonal matrix has the form

$$A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

where a and d are any real numbers. Discuss the validity of each of the following statements. If the statement is always true, explain why. If not, give examples.

(A) If A and B are 2×2 diagonal matrices, then $A + B$ is a 2×2 diagonal matrix.

(B) If A and B are 2×2 diagonal matrices, then $A + B = B + A$.

(C) If A and B are 2×2 diagonal matrices, then AB is a 2×2 diagonal matrix.

(D) If A and B are 2×2 diagonal matrices, then $AB = BA$.

70. A square matrix is an **upper triangular matrix** if all elements below the principal diagonal are zero. Thus, a 2×2 upper triangular matrix has the form

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

where a, b, and d are any real numbers. Discuss the validity of each of the following statements. If the statement is always true, explain why. If not, give examples.

(A) If A and B are 2×2 upper triangular matrices, then $A + B$ is a 2×2 upper triangular matrix.

(B) If A and B are 2×2 upper triangular matrices, then $A + B = B + A$.
(C) If A and B are 2×2 upper triangular matrices, then AB is a 2×2 upper triangular matrix.
(D) If A and B are 2×2 upper triangular matrices, then $AB = BA$.

APPLICATIONS

71. **COST ANALYSIS** A company with two different plants manufactures guitars and banjos. Its production costs for each instrument are given in the following matrices:

$$\begin{array}{l} \\ \text{Materials} \\ \text{Labor} \end{array} \begin{array}{c} \begin{array}{cc} \text{Plant } X & \\ \text{Guitar} & \text{Banjo} \end{array} \\ \begin{bmatrix} \$30 & \$25 \\ \$60 & \$80 \end{bmatrix} \end{array} = A \qquad \begin{array}{c} \begin{array}{cc} \text{Plant } Y & \\ \text{Guitar} & \text{Banjo} \end{array} \\ \begin{bmatrix} \$36 & \$27 \\ \$54 & \$74 \end{bmatrix} \end{array} = B$$

Find $\frac{1}{2}(A + B)$, the average cost of production for the two plants.

72. **COST ANALYSIS** If both labor and materials at plant X in Problem 71 are increased 20%, find $\frac{1}{2}(1.2A + B)$, the new average cost of production for the two plants.

73. **MARKUP** An import car dealer sells three models of a car. Current dealer invoice price (cost) and the retail price for the basic models and the indicated options are given in the following two matrices (where "Air" means air conditioning):

Dealer invoice price

$$\begin{array}{l} \\ \text{Model } A \\ \text{Model } B \\ \text{Model } C \end{array} \begin{array}{c} \begin{array}{cccc} \text{Basic Car} & \text{Air} & \text{AM/FM radio} & \text{Cruise control} \end{array} \\ \begin{bmatrix} \$10{,}400 & \$682 & \$215 & \$182 \\ \$12{,}500 & \$721 & \$295 & \$182 \\ \$16{,}400 & \$827 & \$443 & \$192 \end{bmatrix} \end{array} = M$$

Retail price

$$\begin{array}{l} \\ \text{Model } A \\ \text{Model } B \\ \text{Model } C \end{array} \begin{array}{c} \begin{array}{cccc} \text{Basic Car} & \text{Air} & \text{AM/FM radio} & \text{Cruise control} \end{array} \\ \begin{bmatrix} \$13{,}900 & \$783 & \$263 & \$215 \\ \$15{,}000 & \$838 & \$395 & \$236 \\ \$18{,}300 & \$967 & \$573 & \$248 \end{bmatrix} \end{array} = N$$

We define the markup matrix to be $N - M$ (**markup** is the difference between the retail price and the dealer invoice price). Suppose the value of the dollar has had a sharp decline and the dealer invoice price is to have an across-the-board 15% increase next year. To stay competitive with domestic cars, the dealer increases the retail prices only 10%. Calculate a markup matrix for next year's models and the indicated options. (Compute results to the nearest dollar.)

74. **MARKUP** Referring to Problem 73, what is the markup matrix resulting from a 20% increase in dealer invoice prices and an increase in retail prices of 15%? (Compute results to the nearest dollar.)

75. **LABOR COSTS** A company with manufacturing plants located in different parts of the country has labor-hour and wage requirements for the manufacturing of three types of inflatable boats as given in the following two matrices:

Labor-hours per boat

$$M = \begin{array}{c} \begin{array}{ccc} \text{Cutting department} & \text{Assembly department} & \text{Packaging department} \end{array} \\ \begin{bmatrix} 0.6\text{ h} & 0.6\text{ h} & 0.2\text{ h} \\ 1.0\text{ h} & 0.9\text{ h} & 0.3\text{ h} \\ 1.5\text{ h} & 1.2\text{ h} & 0.4\text{ h} \end{bmatrix} \end{array} \begin{array}{l} \text{One-person boat} \\ \text{Two-person boat} \\ \text{Four-person boat} \end{array}$$

Hourly wages

$$N = \begin{array}{c} \begin{array}{cc} \text{Plant I} & \text{Plant II} \end{array} \\ \begin{bmatrix} \$8 & \$9 \\ \$10 & \$12 \\ \$5 & \$6 \end{bmatrix} \end{array} \begin{array}{l} \text{Cutting department} \\ \text{Assembly department} \\ \text{Packaging department} \end{array}$$

(A) Find the labor costs for a one-person boat manufactured at plant I.
(B) Find the labor costs for a four-person boat manufactured at plant II.
(C) Discuss possible interpretations of the elements in the matrix products MN and NM.
(D) If either of the products MN or NM has a meaningful interpretation, find the product and label its rows and columns.

76. **INVENTORY VALUE** A personal computer retail company sells five different computer models through three stores located in a large metropolitan area. The inventory of each model on hand in each store is summarized in matrix M. Wholesale (W) and retail (R) values of each model computer are summarized in matrix N.

Model

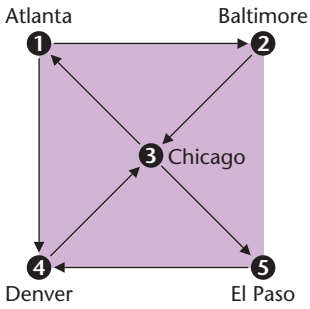
$$M = \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 4 & 2 & 3 & 7 & 1 \\ 2 & 3 & 5 & 0 & 6 \\ 10 & 4 & 3 & 4 & 3 \end{bmatrix} \end{array} \begin{array}{l} \text{Store 1} \\ \text{Store 2} \\ \text{Store 3} \end{array}$$

$$N = \begin{array}{c} \begin{array}{cc} W & R \end{array} \\ \begin{bmatrix} \$700 & \$840 \\ \$1{,}400 & \$1{,}800 \\ \$1{,}800 & \$2{,}400 \\ \$2{,}700 & \$3{,}300 \\ \$3{,}500 & \$4{,}900 \end{bmatrix} \end{array} \begin{array}{l} A \\ B \\ C \\ D \\ E \end{array}$$

(A) What is the retail value of the inventory at store 2?
(B) What is the wholesale value of the inventory at store 3?
(C) Discuss possible interpretations of the elements in the matrix products MN and NM.
(D) If either of the products MN or NM has a meaningful interpretation, find the product and label its rows and columns.
(E) Discuss methods of matrix multiplication that can be used to find the total inventory of each model on hand at all three stores. State the matrices that can be used, and perform the necessary operations.

(F) Discuss methods of matrix multiplication that can be used to find the total inventory of all five models at each store. State the matrices that can be used, and perform the necessary operations.

77. AIRFREIGHT A nationwide airfreight service has connecting flights between five cities, as illustrated in the figure. To represent this schedule in matrix form, we construct a 5×5 **incidence matrix** A, where the rows represent the origins of each flight and the columns represent the destinations. We place a 1 in the ith row and jth column of this matrix if there is a connecting flight from the ith city to the jth city and a 0 otherwise. We also place 0s on the principal diagonal, because a connecting flight with the same origin and destination does not make sense.



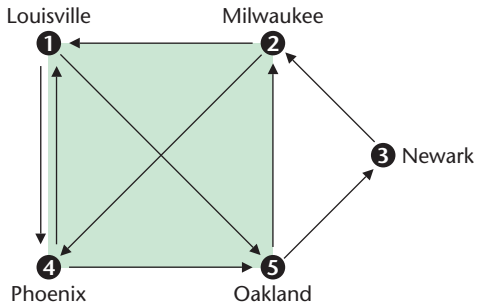
Destination

Origin	1	2	3	4	5
1	0	1	0	1	0
2	0	0	1	0	0
3	1	0	0	0	1
4	0	0	1	0	0
5	0	0	0	1	0

$= A$

Now that the schedule has been represented in the mathematical form of a matrix, we can perform operations on this matrix to obtain information about the schedule.
(A) Find A^2. What does the 1 in row 2 and column 1 of A^2 indicate about the schedule? What does the 2 in row 1 and column 3 indicate about the schedule? In general, how would you interpret each element off the principal diagonal of A^2? [*Hint:* Examine the diagram for possible connections between the ith city and the jth city.]
(B) Find A^3. What does the 1 in row 4 and column 2 of A^3 indicate about the schedule? What does the 2 in row 1 and column 5 indicate about the schedule? In general, how would you interpret each element off the principal diagonal of A^3?
(C) Compute A, $A + A^2$, $A + A^2 + A^3$, . . ., until you obtain a matrix with no zero elements (except possibly on the principal diagonal), and interpret.

78. AIRFREIGHT Find the incidence matrix A for the flight schedule illustrated in the figure. Compute A, $A + A^2$, $A + A^2 + A^3$, . . ., until you obtain a matrix with no zero elements (except possibly on the principal diagonal), and interpret.



79. POLITICS In a local election, a group hired a public relations firm to promote its candidate in three ways: telephone, house calls, and letters. The cost per contact is given in matrix M:

Cost per contact

$$M = \begin{bmatrix} \$0.80 \\ \$1.50 \\ \$0.40 \end{bmatrix} \begin{matrix} \text{Telephone} \\ \text{House call} \\ \text{Letter} \end{matrix}$$

The number of contacts of each type made in two adjacent cities is given in matrix N:

	Telephone	House call	Letter	
$N =$	1,000	500	5,000	Berkeley
	2,000	800	8,000	Oakland

(A) Find the total amount spent in Berkeley.
(B) Find the total amount spent in Oakland.
(C) Discuss possible interpretations of the elements in the matrix products MN and NM.
(D) If either of the products MN or NM has a meaningful interpretation, find the product and label its rows and columns.
(E) Discuss methods of matrix multiplication that can be used to find the total number of telephone calls, house calls, and letters. State the matrices that can be used, and perform the necessary operations.
(F) Discuss methods of matrix multiplication that can be used to find the total number of contacts in Berkeley and in Oakland. State the matrices that can be used, and perform the necessary operations.

80. NUTRITION A nutritionist for a cereal company blends two cereals in different mixes. The amounts of protein, carbohydrate, and fat (in grams per ounce) in each cereal are given by matrix M. The amounts of each cereal used in the three mixes are given by matrix N.

	Cereal A	Cereal B	
$M =$	4 g/oz	2 g/oz	Protein
	20 g/oz	16 g/oz	Carbohydrate
	3 g/oz	1 g/oz	Fat

	Mix X	Mix Y	Mix Z	
$N =$	15 oz	10 oz	5 oz	Cereal A
	5 oz	10 oz	15 oz	Cereal B

(A) Find the amount of protein in mix X.
(B) Find the amount of fat in mix Z.
(C) Discuss possible interpretations of the elements in the matrix products MN and NM.
(D) If either of the products MN or NM has a meaningful interpretation, find the product and label its rows and columns.

81. DOMINANCE RELATION To rank players for an upcoming tennis tournament, a club decides to have each player play one set with every other player. The results are given in the table.

Player	Defeated
1. Aaron	Charles, Dan, Elvis
2. Bart	Aaron, Dan, Elvis
3. Charles	Bart, Dan
4. Dan	Frank
5. Elvis	Charles, Dan, Frank
6. Frank	Aaron, Bart, Charles

(A) Express the outcomes as an incidence matrix A by placing a 1 in the ith row and jth column of A if player i defeated player j and a 0 otherwise (see Problem 77).
(B) Compute the matrix $B = A + A^2$
(C) Discuss matrix multiplication methods that can be used to find the sum of each of the rows in B. State the matrices that can be used and perform the necessary operations.
(D) Rank the players from strongest to weakest. Explain the reasoning behind your ranking.

82. DOMINANCE RELATION Each member of a chess team plays one match with every other player. The results are given in the table.

Player	Defeated
1. Anne	Diane
2. Bridget	Anne, Carol, Diane
3. Carol	Anne
4. Diane	Carol, Erlene
5. Erlene	Anne, Bridget, Carol

(A) Express the outcomes as an incidence matrix A by placing a 1 in the ith row and jth column of A if player i defeated player j and a 0 otherwise (see Problem 77).
(B) Compute the matrix $B = A + A^2$.
(C) Discuss matrix multiplication methods that can be used to find the sum of each of the rows in B. State the matrices that can be used and perform the necessary operations.
(D) Rank the players from strongest to weakest. Explain the reasoning behind your ranking.

9-3 Inverse of a Square Matrix

- Identity Matrix for Multiplication
- Inverse of a Square Matrix
- Application: Cryptography

In this section we introduce the identity matrix and the inverse of a square matrix. These matrix forms, along with matrix multiplication, are then used to solve some systems of equations written in matrix form in the next section.

Identity Matrix for Multiplication

We know that for any real number a

$$(1)a = a(1) = a$$

The number 1 is called the *identity* for real number multiplication. Does the set of all matrices of a given dimension have an identity element for multiplication? That is, if M is an arbitrary $m \times n$ matrix, does M have an identity element I such that $IM = MI = M$? The answer in general is no. However, the set of all **square matrices of order n** (matrices with n rows and n columns) does have an identity.

› **DEFINITION 1** Identity Matrix

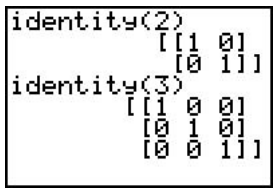
The **identity matrix for multiplication** for the set of all square matrices of order n is the square matrix of order n, denoted by I, with 1s along the principal diagonal (from upper left corner to lower right corner) and 0s elsewhere.

For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are the identity matrices for all square matrices of order 2 and 3, respectively.

Most graphing utilities have a built-in command for generating the identity matrix of a given order (Fig. 1).



› **Figure 1** Identity matrices.

EXAMPLE 1 Identity Matrix Multiplication

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

(B) $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$

(D) $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$

MATCHED PROBLEM 1

Multiply:

(A) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ 4 & 6 \end{bmatrix}$ and $\begin{bmatrix} 3 & -5 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -7 \\ 2 & 4 \\ 6 & -8 \end{bmatrix}$ and $\begin{bmatrix} 5 & -7 \\ 2 & 4 \\ 6 & -8 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

In general, we can show that if M is a square matrix of order n and I is the identity matrix of order n, then

$$IM = MI = M$$

If M is an $m \times n$ matrix that is not square ($m \neq n$), then it is still possible to multiply M on the left and on the right by an identity matrix, but not with the same-size identity matrix (see Example 1, parts C and D). To avoid the complications involved with associating two different identity matrices with each nonsquare matrix, we restrict our attention in this section to square matrices.

››› EXPLORE-DISCUSS 1

The only real number solutions to the equation $x^2 = 1$ are $x = 1$ and $x = -1$.

(A) Show that $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ satisfies $A^2 = I$, where I is the 2×2 identity matrix.

(B) Show that $B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ satisfies $B^2 = I$.

(C) Find a 2×2 matrix with all elements nonzero whose square is the 2×2 identity matrix.

› Inverse of a Square Matrix

In the set of real numbers, we know that for each real number a, except 0, there exists a real number a^{-1} such that

$$a^{-1}a = 1$$

The number a^{-1} is called the *inverse* of the number a relative to multiplication, or the *multiplicative inverse* of a. For example, 2^{-1} is the multiplicative inverse of 2, because $2^{-1}(2) = 1$. We use this idea to define the *inverse of a square matrix.*

› DEFINITION 2 Inverse of a Square Matrix

If A is a square matrix of order n and if there exists a matrix A^{-1} (read "A inverse") such that

$$A^{-1}A = AA^{-1} = I$$

then A^{-1} is called the **multiplicative inverse of A** or, more simply, the **inverse of A.** If no such matrix exists, then A is said to be a **singular matrix.**

››› EXPLORE-DISCUSS 2

Let $A = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 0.25 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$ $C = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$

(A) How are the entries in A and B related?

(B) Find AB. Is B the inverse of A?

(C) Find AC. Is C the inverse of A?

The multiplicative inverse of a nonzero real number a also can be written as $1/a$. This notation is not used for matrix inverses.

Let's use Definition 2 to find A^{-1}, if it exists, for

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

We are looking for

$$A^{-1} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

such that

$$AA^{-1} = A^{-1}A = I$$

Thus, we write

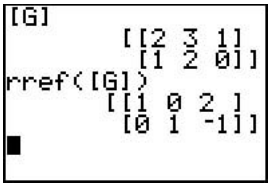
$$\overset{A}{\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}} \overset{A^{-1}}{\begin{bmatrix} a & c \\ b & d \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

and try to find a, b, c, and d so that the product of A and A^{-1} is the identity matrix I. Multiplying A and A^{-1} on the left side, we obtain

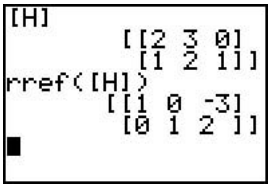
$$\begin{bmatrix} (2a + 3b) & (2c + 3d) \\ (a + 2b) & (c + 2d) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is true only if

System G	System H
$2a + 3b = 1$	$2c + 3d = 0$
$a + 2b = 0$	$c + 2d = 1$



(a) System *G*



(b) System *H*

› Figure 2

Solving these two systems (Fig. 2), we find that $a = 2, b = -1, c = -3$, and $d = 2$. Thus,

$$A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

as is easily checked:

$$\overset{A}{\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}} \overset{A^{-1}}{\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \overset{A^{-1}}{\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}} \overset{A}{\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}}$$

Unlike nonzero real numbers, inverses do not always exist for nonzero square matrices. For example, if

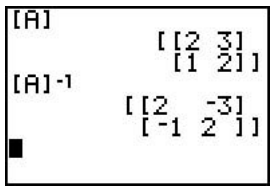
$$B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

then, proceeding as before, we are led to the systems

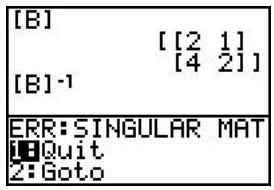
$$\begin{aligned} 2a + b &= 1 \\ 4a + 2b &= 0 \end{aligned} \qquad \begin{aligned} 2c + d &= 0 \\ 4c + 2d &= 1 \end{aligned}$$

These systems are both inconsistent and have no solution (verify this by solving both systems, as in Figure 2). Hence, B^{-1} does not exist, and B is a singular matrix.

Most graphing calculators can find matrix inverses and can identify singular matrices. Figure 3 shows the calculation of A^{-1} for the matrix A discussed earlier. Figure 4 shows the error message that results when the inverse operation is applied to the singular matrix B discussed earlier.



› Figure 3



› Figure 4

› Figure 5

Note that the inverse operation is performed by pressing the x^{-1} key. Entering [A] ∧ (−1) results in an error message (Fig. 5).

Being able to find inverses, when they exist, leads to direct and simple solutions to many practical problems. In the next section, for example, we will show how inverses can be used to solve systems of linear equations.

The algebraic method outlined for finding the inverse, if it exists, gets very involved for matrices of order larger than 2. Now that we know what we are looking for, we can use augmented matrices, as in Section 9-1, to make the process more efficient. Details are illustrated in Example 2.

EXAMPLE 2 **Finding an Inverse**

Find the inverse, if it exists, of

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}$$

SOLUTION

We start as before and write

$$\overset{A}{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}} \overset{A^{-1}}{\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

This is true only if

$$\begin{array}{rrr} a - b + c = 1 & d - e + f = 0 & g - h + i = 0 \\ 2b - c = 0 & 2e - f = 1 & 2h - i = 0 \\ 2a + 3b = 0 & 2d + 3e = 0 & 2g + 3h = 1 \end{array}$$

Now we write augmented matrices for each of the three systems:

$$\overset{\text{First}}{\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & -1 & 0 \\ 2 & 3 & 0 & 0 \end{array}\right]} \quad \overset{\text{Second}}{\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \\ 2 & 3 & 0 & 0 \end{array}\right]} \quad \overset{\text{Third}}{\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right]}$$

Because each matrix to the left of the vertical bar is the same, exactly the same row operations can be used on each augmented matrix to transform it into a reduced form. We can speed up the process substantially by combining all three augmented matrices into the single augmented matrix form

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right] = [A \mid I] \tag{1}$$

We now try to perform row operations on matrix (1) until we obtain a row-equivalent matrix that looks like matrix (2):

$$\overset{I}{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & a & d & g \\ 0 & 1 & 0 & b & e & h \\ 0 & 0 & 1 & c & f & i \end{array}\right]} \overset{B}{} = [I \mid B] \tag{2}$$

If this can be done, then the new matrix to the right of the vertical bar is A^{-1}! Now let's try to transform matrix (1) into a form like that of matrix (2). We follow the same sequence of steps as in the solution of linear systems by Gauss–Jordan elimination (see Section 9-1):

$$\begin{array}{l} \qquad\qquad\quad A \qquad\qquad\quad I \\ \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \quad (-2)R_1 + R_3 \to R_3 \\ \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & 5 & -2 & -2 & 0 & 1 \end{array}\right] \quad \tfrac{1}{2}R_2 \to R_2 \\ \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 5 & -2 & -2 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 + R_1 \to R_1 \\ \\ (-5)R_2 + R_3 \to R_3 \end{array} \\ \sim \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -2 & -\frac{5}{2} & 1 \end{array}\right] \quad 2R_3 \to R_3 \\ \sim \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -4 & -5 & 2 \end{array}\right] \quad \begin{array}{l} (-\frac{1}{2})\, R_3 + R_1 \to R_1 \\ \tfrac{1}{2}R_3 + R_2 \to R_2 \end{array} \\ \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 3 & -1 \\ 0 & 1 & 0 & -2 & -2 & 1 \\ 0 & 0 & 1 & -4 & -5 & 2 \end{array}\right] = [I \,|\, B] \end{array}$$

Converting back to systems of equations equivalent to our three original systems (we won't have to do this step in practice), we have

$$\begin{array}{lll} a = 3 & d = 3 & g = -1 \\ b = -2 & e = -2 & h = 1 \\ c = -4 & f = -5 & i = 2 \end{array}$$

And these are just the elements of A^{-1} that we are looking for! Hence,

$$A^{-1} = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

Note that this is the matrix to the right of the vertical line in the last augmented matrix.

CHECK

Because the definition of matrix inverse requires that

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I \tag{3}$$

it appears that we must compute both $A^{-1}A$ and AA^{-1} to check our work. However, it can be shown that if one of the equations in (3) is satisfied, then the other is also satisfied. Thus, for checking purposes it is sufficient to compute either $A^{-1}A$ or AA^{-1}—we don't need to do both.

$$A^{-1}A = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

MATCHED PROBLEM 2

Let $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

(A) Form the augmented matrix $[A \mid I]$.

(B) Use row operations to transform $[A \mid I]$ into $[I \mid B]$.

(C) Verify by multiplication that $B = A^{-1}$.

The procedure used in Example 2 can be used to find the inverse of any square matrix, if the inverse exists, and will also indicate when the inverse does not exist. These ideas are summarized in Theorem 1.

› THEOREM 1 Inverse of a Square Matrix *A*

If $[A \mid I]$ is transformed by row operations into $[I \mid B]$, then the resulting matrix B is A^{-1}. If, however, we obtain all 0s in one or more rows to the left of the vertical line, then A^{-1} does not exist.

››› EXPLORE-DISCUSS 3

(A) Suppose that the square matrix A has a row of all zeros. Explain why A has no inverse.

(B) Suppose that the square matrix A has a column of all zeros. Explain why A has no inverse.

EXAMPLE 3 Finding a Matrix Inverse

Find A^{-1}, given $A = \begin{bmatrix} 4 & -1 \\ -6 & 2 \end{bmatrix}$

SOLUTIONS

Algebraic Solution

$$\left[\begin{array}{cc|cc} 4 & -1 & 1 & 0 \\ -6 & 2 & 0 & 1 \end{array}\right] \quad \tfrac{1}{4}R_1 \to R_1$$

$$\sim \left[\begin{array}{cc|cc} 1 & -\frac{1}{4} & \frac{1}{4} & 0 \\ -6 & 2 & 0 & 1 \end{array}\right] \quad 6R_1 + R_2 \to R_2$$

$$\sim \left[\begin{array}{cc|cc} 1 & -\frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & 1 \end{array}\right] \quad 2R_2 \to R_2$$

$$\sim \left[\begin{array}{cc|cc} 1 & -\frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 1 & 3 & 2 \end{array}\right] \quad \tfrac{1}{4}R_2 + R_1 \to R_1$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & \frac{1}{2} \\ 0 & 1 & 3 & 2 \end{array}\right]$$

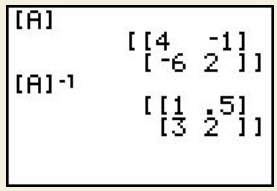
Thus,

$$A^{-1} = \begin{bmatrix} 1 & \frac{1}{2} \\ 3 & 2 \end{bmatrix}$$

Check by showing $A^{-1}A = I$.

Graphing Calculator Solution

Enter A and use the inverse key (Fig. 6).



› Figure 6

From Figure 6, we see that

$$A^{-1} = \begin{bmatrix} 1 & 0.5 \\ 3 & 2 \end{bmatrix}$$

MATCHED PROBLEM 3

Find A^{-1}, given $A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

EXAMPLE 4 Finding an Inverse

Find B^{-1}, if it exists, given $B = \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$

SOLUTIONS

Algebraic Solution

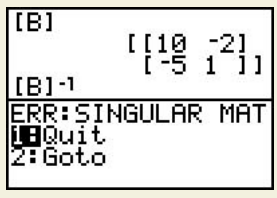
$$\left[\begin{array}{cc|cc} 10 & -2 & 1 & 0 \\ -5 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -\frac{1}{5} & \frac{1}{10} & 0 \\ -5 & 1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & -\frac{1}{5} & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{2} & 1 \end{array}\right]$$

We have all 0s in the second row to the left of the vertical line. Therefore, B^{-1} does not exist.

Graphing Calculator Solution

Enter B and use the inverse key (Fig. 7).



› Figure 7

From Figure 7, we see that B is a singular matrix and B^{-1} does not exist.

MATCHED PROBLEM 4

Find B^{-1}, if it exists, given $B = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

Application: Cryptography

Matrix inverses can be used to provide a simple and effective procedure for encoding and decoding messages. To begin, we assign the numbers 1 to 26 to the letters in the alphabet, as shown below. We also assign the number 27 to a blank to provide for space between words. (A more sophisticated code could include both uppercase and lowercase letters and punctuation symbols.)

A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	2	3	4	5	6	7	8	9	10	11	12	13	14
O	P	Q	R	S	T	U	V	W	X	Y	Z	Blank	
15	16	17	18	19	20	21	22	23	24	25	26	27	

Thus, the message I LOVE MATH corresponds to the sequence

9 27 12 15 22 5 27 13 1 20 8

Any matrix whose elements are positive integers and whose inverse exists can be used as an **encoding matrix.** For example, to use the 2×2 matrix

$$A = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}$$

to encode the above message, first we divide the numbers in the sequence into groups of 2 and use these groups as the columns of a matrix B with two rows:

$$B = \begin{bmatrix} 9 & 12 & 22 & 27 & 1 & 8 \\ 27 & 15 & 5 & 13 & 20 & 27 \end{bmatrix}$$

Proceed down the columns, not across the rows.

(Notice that we added an extra blank at the end of the message to make the columns come out even.) Then we multiply this matrix on the left by A:

$$AB = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 9 & 12 & 22 & 27 & 1 & 8 \\ 27 & 15 & 5 & 13 & 20 & 27 \end{bmatrix}$$

$$= \begin{bmatrix} 117 & 93 & 103 & 147 & 64 & 113 \\ 153 & 120 & 130 & 187 & 85 & 148 \end{bmatrix}$$

The coded message is

117 153 93 120 103 130 147 187 64 85 113 148

This message can be decoded simply by putting it back into matrix form and multiplying on the left by the **decoding matrix** A^{-1}. Because A^{-1} is easily determined if A is

known, the encoding matrix A is the only key needed to decode messages encoded in this manner. Although simple in concept, codes of this type can be very difficult to crack.

EXAMPLE 5 Cryptography

The message

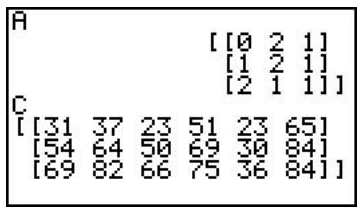
31 54 69 37 64 82 23 50 66 51 69 75 23 30 36 65 84 84

was encoded with the matrix A shown below. Decode this message.

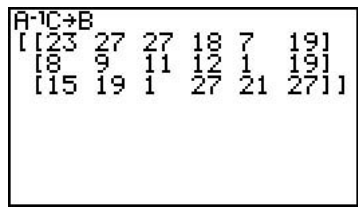
$$A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

SOLUTION

We begin by entering the 3×3 encoding matrix A (Fig. 8). Then we enter the coded message in the columns of a matrix C with three rows (Fig. 8). If B is the matrix containing the uncoded message, then B and C are related by $C = AB$. To find B, we multiply both sides of the equation $C = AB$ by A^{-1} (Fig. 9).



› Figure 8



› Figure 9

Writing the numbers in the columns of this matrix in sequence and using the correspondence between numbers and letters noted earlier produces the decoded message:

23	8	15	27	9	19	27	11	1	18	12	27	7	1	21	19	19	27
W	H	O		I	S		K	A	R	L		G	A	U	S	S	

The answer to this question can be found somewhere in this chapter.

MATCHED PROBLEM 5

The message

46 84 85 55 101 100 59 95 132 25 42 53 52 91 90 43 71 83 19 37 25

was encoded with the matrix A shown below. Decode this message.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

ANSWERS TO MATCHED PROBLEMS

1. (A) $\begin{bmatrix} 3 & -5 \\ 4 & 6 \end{bmatrix}$ (B) $\begin{bmatrix} 5 & -7 \\ 2 & 4 \\ 6 & -8 \end{bmatrix}$

2. (A) $\left[\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ (B) $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right]$

(C) $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. $\begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix}$ **4.** Does not exist **5.** Who is Wilhelm Jordan

9-3 Exercises

1. What is an identity matrix?

2. What is the (multiplicative) inverse of a real number? Does every real number have an inverse?

3. What is the (multiplicative) inverse of a matrix? Does every matrix have an inverse?

4. What is a singular matrix?

5. Describe the process for finding the inverse of a matrix by hand.

6. Describe the process for finding the inverse of a matrix on a graphing calculator.

Perform the indicated operations in Problems 7–14.

7. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$ **8.** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 0 & 2 \end{bmatrix}$

9. $\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **10.** $\begin{bmatrix} 4 & -3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 & 3 \\ 2 & 4 & -2 \\ 5 & 1 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & 0 & 2 \\ 1 & 1 & 5 \\ 2 & -1 & 7 \end{bmatrix}$

13. $\begin{bmatrix} -2 & 1 & 3 \\ 2 & 4 & -2 \\ 5 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

14. $\begin{bmatrix} -3 & 0 & 2 \\ 1 & 1 & 5 \\ 2 & -1 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

In Problems 15–24, examine the product of the two matrices to determine if each is the inverse of the other.

15. $\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}; \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ **16.** $\begin{bmatrix} -2 & -1 \\ -4 & 2 \end{bmatrix}; \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$

17. $\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}; \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ **18.** $\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}; \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$

19. $\begin{bmatrix} -5 & 2 \\ -8 & 3 \end{bmatrix}; \begin{bmatrix} 3 & -2 \\ 8 & -5 \end{bmatrix}$ **20.** $\begin{bmatrix} 7 & 4 \\ -5 & -3 \end{bmatrix}; \begin{bmatrix} 3 & 4 \\ -5 & -7 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}; \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 0 & 1 \\ -3 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

23. $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}; \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 0 & -1 \\ -3 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

In Problems 25–44, given A, find A^{-1}, if it exists. Check each inverse by showing $A^{-1}A = I$.

25. $\begin{bmatrix} 1 & 9 \\ 0 & 1 \end{bmatrix}$ **26.** $\begin{bmatrix} 0 & -1 \\ -1 & 3 \end{bmatrix}$ **27.** $\begin{bmatrix} -1 & -2 \\ 2 & 5 \end{bmatrix}$

28. $\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$ **29.** $\begin{bmatrix} -5 & 7 \\ 2 & -3 \end{bmatrix}$ **30.** $\begin{bmatrix} 11 & 4 \\ 3 & 1 \end{bmatrix}$

31. $\begin{bmatrix} 3 & 9 \\ 2 & 6 \end{bmatrix}$ **32.** $\begin{bmatrix} 2 & -4 \\ -3 & 6 \end{bmatrix}$ **33.** $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

34. $\begin{bmatrix} -5 & 4 \\ 4 & -3 \end{bmatrix}$ **35.** $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ **36.** $\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

37. $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 5 & 9 \\ 1 & 1 & -2 \end{bmatrix}$ **38.** $\begin{bmatrix} 1 & -1 & 1 \\ -2 & 3 & 2 \\ 3 & -3 & 2 \end{bmatrix}$

39. $\begin{bmatrix} 2 & 2 & -1 \\ 0 & 4 & -1 \\ -1 & -2 & 1 \end{bmatrix}$ **40.** $\begin{bmatrix} 4 & 2 & -1 \\ 1 & 1 & -1 \\ -3 & -1 & 1 \end{bmatrix}$

41. $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$ **42.** $\begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

43. $\begin{bmatrix} 1 & 5 & 10 \\ 0 & 1 & 4 \\ 1 & 6 & 15 \end{bmatrix}$ **44.** $\begin{bmatrix} 1 & -5 & -10 \\ 0 & 1 & 6 \\ 1 & -4 & -3 \end{bmatrix}$

45. Discuss the existence of A^{-1} for 2×2 diagonal matrices of the form

$$A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

46. Discuss the existence of A^{-1} for 2×2 upper triangular matrices of the form

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

47. Find A^{-1} and A^2 for each of the following matrices.

(A) $A = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}$

(B) $A = \begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix}$

48. Based on your observations in Problem 47, if $A = A^{-1}$ for a square matrix A, what is A^2? Give a mathematical argument to support your conclusion.

49. Find $(A^{-1})^{-1}$ for each of the following matrices.

(A) $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$

(B) $A = \begin{bmatrix} 5 & 5 \\ -1 & 3 \end{bmatrix}$

50. Based on your observations in Problem 49, if A^{-1} exists for a square matrix A, what is $(A^{-1})^{-1}$? Give a mathematical argument to support your conclusion.

51. Find $(AB)^{-1}$, $A^{-1}B^{-1}$, and $B^{-1}A^{-1}$ for each of the following pairs of matrices.

(A) $A = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$

(B) $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 2 \\ 2 & 1 \end{bmatrix}$

52. Based on your observations in Problem 51, which of the following is a true statement? Give a mathematical argument to support your conclusion.
(A) $(AB)^{-1} = A^{-1}B^{-1}$
(B) $(AB)^{-1} = B^{-1}A^{-1}$

APPLICATIONS

Problems 53–56 refer to the encoding matrix $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$

53. CRYPTOGRAPHY Encode the message CAT IN THE HAT with the matrix A given here.

54. CRYPTOGRAPHY Encode the message FOX IN SOCKS with the matrix A given here.

55. CRYPTOGRAPHY The following message was encoded with the matrix A given here. Decode this message.

111 43 40 15 177 68 50 19 116 45 86
29 62 22 121 43 68 27

56. CRYPTOGRAPHY The following message was encoded with the matrix A given here. Decode this message.

99 38 154 58 115 43 121 43 20 7 149
56 86 29 196 73 99 38

Problems 57–60 refer to the encoding matrix

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 1 & 1 & 1 & 2 & 1 \end{bmatrix}$$

57. CRYPTOGRAPHY Encode the message DWIGHT DAVID EISENHOWER with the matrix B given here.

58. CRYPTOGRAPHY Encode the message JOHN FITZGERALD KENNEDY with the matrix B given here.

59. CRYPTOGRAPHY The following message was encoded with the matrix B given here. Decode this message.

41 84 82 44 74 25 56 67 20 54 43 54
89 39 102 44 67 86 44 90 68 135 136
81 149

60. CRYPTOGRAPHY The following message was encoded with the matrix B given here. Decode this message.

22 15 57 5 47 54 58 89 45 84 46 80
87 53 96 51 68 116 39 113 68 135 136
81 149

9-4 Matrix Equations and Systems of Linear Equations

- Solving Matrix Equations
- Using Matrix Inverses to Solve Systems of Equations
- Application

The identity matrix and inverse matrix discussed in the last section can be put to immediate use in the solving of certain simple matrix equations. Being able to solve a matrix equation gives us another important method of solving a system of equations having the same number of variables as equations. If the system either has fewer variables than equations or more variables than equations, then we must return to the Gauss–Jordan method of elimination.

Solving Matrix Equations

EXPLORE-DISCUSS 1

Let a, b, and c be real numbers, with $a \neq 0$. Solve each equation for x.

(A) $ax = b$ **(B)** $ax + b = c$

Solving simple matrix equations follows very much the same procedures used in solving real number equations. We have, however, less freedom with matrix equations, because matrix multiplication is not commutative. In solving matrix equations, we will be guided by the properties of matrices summarized in Theorem 1.

THEOREM 1 Basic Properties of Matrices

Assuming all products and sums are defined for the indicated matrices A, B, C, I, and 0, then

Addition Properties	
Associative:	$(A + B) + C = A + (B + C)$
Commutative:	$A + B = B + A$
Additive Identity:	$A + 0 = 0 + A = A$
Additive Inverse:	$A + (-A) = (-A) + A = 0$
Multiplication Properties	
Associative Property:	$A(BC) = (AB)C$
Multiplicative Identity:	$AI = IA = A$
Multiplicative Inverse:	If A is a square matrix and A^{-1} exists, then $AA^{-1} = A^{-1}A = I$.
Combined Properties	
Left Distributive:	$A(B + C) = AB + AC$
Right Distributive:	$(B + C)A = BA + CA$
Equality	
Addition:	If $A = B$, then $A + C = B + C$.
Left Multiplication:	If $A = B$, then $CA = CB$.
Right Multiplication:	If $A = B$, then $AC = BC$.

The process of solving certain types of simple matrix equations is best illustrated by an example.

EXAMPLE 1 Solving a Matrix Equation

Given an $n \times n$ matrix A and $n \times 1$ column matrices B and X, solve $AX = B$ for X. Assume all necessary inverses exist.

SOLUTION

We are interested in finding a column matrix X that satisfies the matrix equation $AX = B$. To solve this equation, we multiply both sides, on the left, by A^{-1}, assuming it exists, to isolate X on the left side.

$$
\begin{aligned}
AX &= B && \text{Use the left multiplication property.}\\
A^{-1}(AX) &= A^{-1}B && \text{Associative property}\\
(A^{-1}A)X &= A^{-1}B && A^{-1}A = I\\
IX &= A^{-1}B && IX = X\\
X &= A^{-1}B
\end{aligned}
$$

›››› CAUTION ››››

1. Do not mix the left multiplication property and the right multiplication property. If $AX = B$, then

$$A^{-1}(AX) \neq BA^{-1}$$

2. Matrix division is not defined. If a, b, and x are real numbers, then the solution of $ax = b$ can be written either as $x = a^{-1}b$ or as $x = \frac{b}{a}$. But if A, B, and X are matrices, the solution of $AX = B$ must be written as $X = A^{-1}B$. The expression $\frac{B}{A}$ is not defined for matrices.

MATCHED PROBLEM 1

Given an $n \times n$ matrix A and $n \times 1$ column matrices B, C, and X, solve $AX + C = B$ for X. Assume all necessary inverses exist.

Using Matrix Inverses to Solve Systems of Equations

We now show how independent systems of linear equations with the same number of variables as equations can be solved by first converting the system into a matrix equation of the form $AX = B$ and using $X = A^{-1}B$ as obtained in Example 1.

EXAMPLE 2 Using Inverses to Solve Systems of Equations

Use matrix inverse methods to solve the system

$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ 2x_2 - x_3 &= 1 \\ 2x_1 + 3x_2 &= 1 \end{aligned} \tag{1}$$

SOLUTIONS

The inverse of the coefficient matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}$$

provides an efficient method for solving this system. To see how, we convert system (1) into a matrix equation:

$$\underset{A}{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}} \underset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \tag{2}$$

Check that matrix equation (2) is equivalent to system (1) by finding the product of the left side and then equating corresponding elements on the left with those on the right. Now you see another important reason for defining matrix multiplication as we did.

We are interested in finding a column matrix X that satisfies the matrix equation $AX = B$. In Example 1 we found that if $AX = B$ and if A^{-1} exists, then

$$X = A^{-1}B$$

Algebraic Solution

The inverse of A was found in Example 2 in the last section to be

$$A^{-1} = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

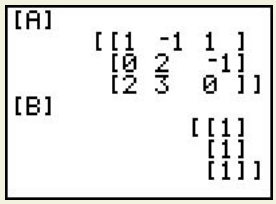
Thus,

$$\underset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{A^{-1}}{\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}} \underset{B}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 5 \\ -3 \\ -7 \end{bmatrix}$$

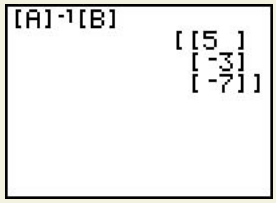
and we can conclude that $x_1 = 5, x_2 = -3$, and $x_3 = -7$. Check this result in system (1).

Graphing Calculator Solution

To solve this problem on a graphing utility, enter A and B (Fig. 1) and simply type $A^{-1}B$ (Fig. 2).



› Figure 1



› Figure 2

MATCHED PROBLEM 2

Use matrix inverse methods to solve the system (see Matched Problem 2 in the last section for the inverse of the coefficient matrix):

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 1 \\ -x_1 + x_2 &= 3 \\ x_1 + x_3 &= 2 \end{aligned}$$

At first glance, using matrix inverse methods seems to require the same amount of effort as using Gauss–Jordan elimination. In either case, row operations must be

applied to an augmented matrix involving the coefficients of the system. The advantage of the inverse matrix method becomes readily apparent when solving a number of systems with a common coefficient matrix and different constant terms.

EXAMPLE 3 Using Inverses to Solve Systems of Equations

Use matrix inverse methods to solve each of the following systems:

(A) $\begin{aligned} x_1 - x_2 + x_3 &= 3 \\ 2x_2 - x_3 &= 1 \\ 2x_1 + 3x_2 &= 4 \end{aligned}$ (B) $\begin{aligned} x_1 - x_2 + x_3 &= -5 \\ 2x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 &= -3 \end{aligned}$

SOLUTIONS

Notice that both systems have the same coefficient matrix A as system (1) in Example 2. Only the constant terms have been changed. Thus, we can use A^{-1} to solve these systems just as we did in Example 2.

(A) Algebraic Solution

$$\underset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{A^{-1}}{\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}} \underset{B}{\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}} = \begin{bmatrix} 8 \\ -4 \\ -9 \end{bmatrix}$$

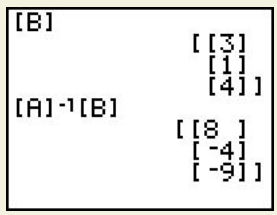
Thus, $x_1 = 8, x_2 = -4$, and $x_3 = -9$

(B) Algebraic Solution

$$\underset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{A^{-1}}{\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}} \underset{B}{\begin{bmatrix} -5 \\ 2 \\ -3 \end{bmatrix}} = \begin{bmatrix} -6 \\ 3 \\ 4 \end{bmatrix}$$

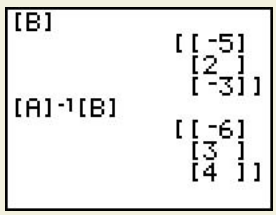
Thus, $x_1 = -6, x_2 = 3$, and $x_3 = 4$

(A) Graphing Calculator Solution



› Figure 3

(B) Graphing Calculator Solution



› Figure 4

MATCHED PROBLEM 3

Use matrix inverse methods to solve each of the following systems (see Matched Problem 2):

(A) $\begin{aligned} 3x_1 - x_2 + x_3 &= 3 \\ -x_1 + x_2 &= -3 \\ x_1 + x_3 &= 2 \end{aligned}$ (B) $\begin{aligned} 3x_1 - x_2 + x_3 &= -5 \\ -x_1 + x_2 &= 1 \\ x_1 + x_3 &= -4 \end{aligned}$

››› EXPLORE-DISCUSS 2

Use matrix inverse methods to solve each of the following systems, if possible, otherwise use Gauss–Jordan elimination. Describe the types of systems that can be solved by inverse methods and those that cannot. Are there any systems that cannot be solved by Gauss–Jordan elimination?

(A) $\begin{aligned} x_1 - x_2 &= 1 \\ x_1 + x_2 &= 7 \\ 3x_1 - x_2 &= 9 \end{aligned}$

(B) $\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ x_1 + x_2 - x_3 &= 7 \\ 3x_1 - x_2 + x_3 &= 9 \end{aligned}$

(C) $\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ x_1 + x_2 - x_3 &= 7 \\ 3x_1 - x_2 + x_3 &= 8 \end{aligned}$

(D) $\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ x_1 + x_2 - x_3 &= 7 \\ 3x_1 - x_2 + 2x_3 &= 8 \end{aligned}$

› USING INVERSE METHODS TO SOLVE SYSTEMS OF EQUATIONS

If the number of equations in a system equals the number of variables and the coefficient matrix has an inverse, then the system will always have a unique solution that can be found by using the inverse of the coefficient matrix to solve the corresponding matrix equation.

Matrix equation	Solution
$AX = B$	$X = A^{-1}B$

REMARK: What happens if the coefficient matrix does not have an inverse? In this case, it can be shown that the system does not have a unique solution and is either dependent or inconsistent. Gauss–Jordan elimination must be used to determine which is the case. Also, as we mentioned earlier, Gauss–Jordan elimination must always be used if the number of variables is not the same as the number of equations.

› Application

The application in Example 4 illustrates a practical use of the inverse method.

EXAMPLE 4 Investment Allocation

An investment adviser currently has two types of investments available for clients: an investment M that pays 10% per year and an investment N of higher risk that pays 20% per year. Clients may divide their investments between the two to achieve any total return desired between 10% and 20%. However, the higher the desired return,

the higher the risk. How should each client listed in the table invest to achieve the indicated return?

	Client			
	1	2	3	k
Total investment	\$20,000	\$50,000	\$10,000	k_1
Annual return desired	\$2,400	\$7,500	\$1,300	k_2
	(12%)	(15%)	(13%)	

SOLUTION

We first solve the problem for an arbitrary client k using inverses, and then apply the result to the three specific clients.

Let

$$x_1 = \text{Amount invested in } M$$
$$x_2 = \text{Amount invested in } N$$

Then

$$x_1 + x_2 = k_1 \quad \text{Total invested}$$
$$0.1x_1 + 0.2x_2 = k_2 \quad \text{Total annual return}$$

Write as a matrix equation:

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 0.1 & 0.2 \end{bmatrix}} \underset{X}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{B}{\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}}$$

If A^{-1} exists, then

$$X = A^{-1}B$$

To solve each client's investment problem on a graphing calculator, first we enter A (Fig. 5), then we enter the appropriate values for B and compute $A^{-1}B$ (Fig. 6).

```
[A]
        [[1   1  ]
         [.1  .2 ]]
```

› Figure 5

```
[B]
        [[20000]
         [2400 ]]
[A]⁻¹[B]
        [[16000]
         [4000 ]]
```

(a) Client 1

```
[B]
        [[50000]
         [7500 ]]
[A]⁻¹[B]
        [[25000]
         [25000]]
```

(b) Client 2

```
[B]
        [[10000]
         [1300 ]]
[A]⁻¹[B]
        [[7000]
         [3000]]
```

(c) Client 3

› Figure 6

From Figure 6, we see that Client 1 should invest \$16,000 in investment M and \$4,000 in investment N, Client 2 should invest \$25,000 in investment M and \$25,000

in investment N, and Client 3 should invest \$7,000 in investment M and \$3,000 in investment N.

MATCHED PROBLEM 4

Repeat Example 4 with investment M paying 8% and investment N paying 24%.

ANSWERS TO MATCHED PROBLEMS

1. $AX + C = B$

$$\begin{aligned}(AX + C) - C &= B - C\\ AX + (C - C) &= B - C\\ AX + 0 &= B - C\end{aligned}$$

$$AX = B - C$$

$$\begin{aligned}A^{-1}(AX) &= A^{-1}(B - C)\\ (A^{-1}A)X &= A^{-1}(B - C)\\ IX &= A^{-1}(B - C)\end{aligned}$$

$$X = A^{-1}(B - C)$$

2. $x_1 = 2, x_2 = 5, x_3 = 0$ **3.** (A) $x_1 = -2, x_2 = -5, x_3 = 4$ (B) $x_1 = 0, x_2 = 1, x_3 = -4$ **4.** $A^{-1} = \begin{bmatrix} 1.5 & -6.25 \\ -0.5 & 6.25 \end{bmatrix}$; Client 1: \$15,000 in M and \$5,000 in N; Client 2: \$28,125 in M and \$21,875 in N; Client 3: \$6,875 in M and \$3,125 in N

9-4 Exercises

1. Describe any differences between the addition properties of real numbers and the addition properties of matrices.

2. Describe any differences between the multiplication properties of real numbers and the multiplication properties of matrices.

3. Describe any differences between the distributive properties of real numbers and the distributive properties of matrices.

4. Explain how inverse matrices can be used to solve systems of linear equations by hand.

5. Explain how inverse matrices can be used to solve systems of linear equations on a graphing calculator.

6. If the coefficient matrix of a linear system is singular, does that mean that the system is inconsistent? Explain.

7. If the coefficient matrix of a linear system is singular, does that mean that the system is dependent? Explain.

8. How would you solve a linear system that has more variables than equations?

9. How would you solve a linear system that has fewer variables than equations?

10. How would you solve a linear system if the number of variables and the number of equations are equal?

Write Problems 11–14 as systems of linear equations without matrices.

11. $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

12. $\begin{bmatrix} -3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$

13. $\begin{bmatrix} -2 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}$

14. $\begin{bmatrix} 1 & -2 & 0 \\ -3 & 1 & -1 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$

Write each system in Problems 15–18 as a matrix equation of the form $AX = B$.

15. $\begin{aligned} 4x_1 - 3x_2 &= 2 \\ x_1 + 2x_2 &= 1 \end{aligned}$

16. $\begin{aligned} x_1 - 2x_2 &= 7 \\ -3x_1 + x_2 &= -3 \end{aligned}$

17. $\begin{aligned} x_1 - 2x_2 + x_3 &= -1 \\ -x_1 + x_2 &= 2 \\ 2x_1 + 3x_2 + x_3 &= -3 \end{aligned}$

18. $\begin{aligned} 2x_1 + 3x_3 &= 5 \\ x_1 - 2x_2 + x_3 &= -4 \\ -x_1 + 3x_2 &= 2 \end{aligned}$

In Problems 19–26, find x_1 and x_2.

19. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

20. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$

21. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

22. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

23. $\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$

25. $\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \end{bmatrix}$

26. $\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$

Write each system in Problems 27–34 as a matrix equation and solve using inverses.

27. $\begin{aligned} -x_1 - 2x_2 &= k_1 \\ 2x_1 + 5x_2 &= k_2 \end{aligned}$
(A) $k_1 = 2, k_2 = 5$
(B) $k_1 = -4, k_2 = 1$
(C) $k_1 = -3, k_2 = -2$

28. $\begin{aligned} 3x_1 - 4x_2 &= k_1 \\ -2x_1 + 3x_2 &= k_2 \end{aligned}$
(A) $k_1 = 3, k_2 = -1$
(B) $k_1 = 6, k_2 = 5$
(C) $k_1 = 0, k_2 = -4$

29. $\begin{aligned} -5x_1 + 7x_2 &= k_1 \\ 2x_1 - 3x_2 &= k_2 \end{aligned}$
(A) $k_1 = -5, k_2 = 1$
(B) $k_1 = 8, k_2 = -4$
(C) $k_1 = 6, k_2 = 0$

30. $\begin{aligned} 11x_1 + 4x_2 &= k_1 \\ 3x_1 + x_2 &= k_2 \end{aligned}$
(A) $k_1 = -2, k_2 = -3$
(B) $k_1 = -1, k_2 = 9$
(C) $k_1 = 4, k_2 = 5$

31. $\begin{aligned} x_1 - x_2 &= k_1 \\ -x_1 + x_2 - x_3 &= k_2 \\ -x_2 + x_3 &= k_3 \end{aligned}$
(A) $k_1 = 1, k_2 = 1, k_3 = 2$
(B) $k_1 = -1, k_2 = 0, k_3 = -4$
(C) $k_1 = 3, k_2 = -2, k_3 = 0$

32. $\begin{aligned} 2x_1 - x_2 &= k_1 \\ x_2 + x_3 &= k_2 \\ x_1 + x_3 &= k_3 \end{aligned}$
(A) $k_1 = -2, k_2 = 4, k_3 = -1$
(B) $k_1 = 2, k_2 = -3, k_3 = 1$
(C) $k_1 = -1, k_2 = 2, k_3 = -5$

33. $\begin{aligned} x_1 + 2x_2 + 5x_3 &= k_1 \\ 3x_1 + 5x_2 + 9x_3 &= k_2 \\ x_1 + x_2 - 2x_3 &= k_3 \end{aligned}$
(A) $k_1 = 0, k_2 = 1, k_3 = 4$
(B) $k_1 = 5, k_2 = -1, k_3 = 0$
(C) $k_1 = -6, k_2 = 0, k_3 = 2$

34. $\begin{aligned} x_1 - x_2 + x_3 &= k_1 \\ -2x_1 + 3x_2 + 2x_3 &= k_2 \\ 3x_1 - 3x_2 + 2x_3 &= k_3 \end{aligned}$
(A) $k_1 = 3, k_2 = -1, k_3 = 0$
(B) $k_1 = 0, k_2 = 4, k_3 = 5$
(C) $k_1 = -2, k_2 = 0, k_3 = 1$

In Problems 35–40, explain why the system cannot be solved by matrix inverse methods. Discuss methods that could be used and then solve the system.

35. $\begin{aligned} -2x_1 + 4x_2 &= -5 \\ 6x_1 - 12x_2 &= 15 \end{aligned}$

36. $\begin{aligned} -2x_1 + 4x_2 &= 5 \\ 6x_1 - 12x_2 &= 15 \end{aligned}$

37. $\begin{aligned} x_1 - 3x_2 - 2x_3 &= -1 \\ -2x_1 + 6x_2 + 4x_3 &= 3 \end{aligned}$

38. $\begin{aligned} x_1 - 3x_2 - 2x_3 &= -1 \\ -2x_1 + 7x_2 + 3x_3 &= 3 \end{aligned}$

39. $\begin{aligned} x_1 - 2x_2 + 3x_3 &= 1 \\ 2x_1 - 3x_2 - 2x_3 &= 3 \\ x_1 - x_2 - 5x_3 &= 2 \end{aligned}$

40. $\begin{aligned} x_1 - 2x_2 + 3x_3 &= 1 \\ 2x_1 - 3x_2 - 2x_3 &= 3 \\ x_1 - x_2 - 5x_3 &= 4 \end{aligned}$

For $n \times n$ matrices A and B and $n \times 1$ matrices C, D, and X, solve each matrix equation in Problems 41–46 for X. Assume all necessary inverses exist.

41. $AX = BX + C$

42. $AX + BX = C + D$

43. $X = AX + C$

44. $X + C = AX - BX$

45. $AX + C = 3X$

46. $AX + C = BX - 7X + D$

47. Use matrix inverse methods to solve the following system for the indicated values of k_1 and k_2.

$$\begin{aligned} x_1 + 2.001x_2 &= k_1 \\ x_1 + 2x_2 &= k_2 \end{aligned}$$

(A) $k_1 = 1, k_2 = 1$
(B) $k_1 = 1, k_2 = 0$
(C) $k_1 = 0, k_2 = 1$

Discuss the effect of small changes in the constant terms on the solution set of this system.

48. Repeat Problem 47 for the following system:

$$\begin{aligned} x_1 - 3.001x_2 &= k_1 \\ x_1 - 3x_2 &= k_2 \end{aligned}$$

APPLICATIONS

Solve Problems 49–55 using systems of equations and inverses.

49. **RESOURCE ALLOCATION** A concert hall has 10,000 seats. If tickets are \$4 and \$8, how many of each type of ticket should be sold (assuming all seats can be sold) to bring in each of the returns indicated in the table? Use decimals in computing the inverse.

	Concert		
	1	2	3
Tickets sold	10,000	10,000	10,000
Return required	\$56,000	\$60,000	\$68,000

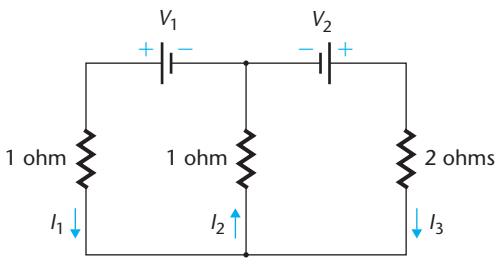
50. **PRODUCTION SCHEDULING** Labor and material costs for manufacturing two guitar models are given in the following table:

Guitar model	Labor cost	Material cost
A	\$30	\$20
B	\$40	\$30

If a total of \$3,000 a week is allowed for labor and material, how many of each model should be produced each week to exactly use each of the allocations of the \$3,000 indicated in the following table? Use decimals in computing the inverse.

	Weekly allocation		
	1	2	3
Labor	\$1,800	\$1,750	\$1,720
Material	\$1,200	\$1,250	\$1,280

51. **CIRCUIT ANALYSIS** A direct current electric circuit consisting of conductors (wires), resistors, and batteries is diagrammed in the figure.



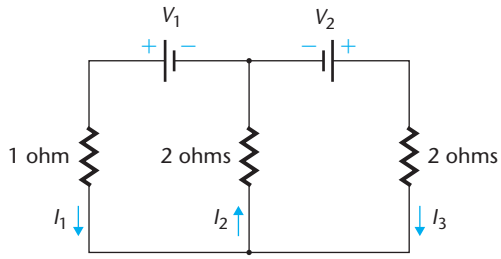
If I_1, I_2, and I_3 are the currents (in amperes) in the three branches of the circuit and V_1 and V_2 are the voltages (in volts) of the two batteries, then *Kirchhoff's* laws* can be used to show that the currents satisfy the following system of equations:

$$\begin{aligned} I_1 - I_2 + I_3 &= 0 \\ I_1 + I_2 &= V_1 \\ I_2 + 2I_3 &= V_2 \end{aligned}$$

Solve this system for:
(A) $V_1 = 10$ volts, $V_2 = 10$ volts
(B) $V_1 = 10$ volts, $V_2 = 15$ volts
(C) $V_1 = 15$ volts, $V_2 = 10$ volts

52. **CIRCUIT ANALYSIS** Repeat Problem 51 for the electric circuit shown in the figure.

$$\begin{aligned} I_1 - I_2 + I_3 &= 0 \\ I_1 + 2I_2 &= V_1 \\ 2I_2 + 2I_3 &= V_2 \end{aligned}$$



Solve Problems 53 and 54 two ways: by using matrix inverse methods and by using quadratic regression on a graphing calculator.

53. **GEOMETRY** The graph of $f(x) = ax^2 + bx + c$ passes through the points $(1, k_1)$, $(2, k_2)$, and $(3, k_3)$. Determine a, b, and c for:
(A) $k_1 = -2$, $k_2 = 1$, $k_3 = 6$
(B) $k_1 = 4$, $k_2 = 3$, $k_3 = -2$
(C) $k_1 = 8$, $k_2 = -5$, $k_3 = 4$

54. **GEOMETRY** Repeat Problem 53 if the graph passes through the points $(-1, k_1)$, $(0, k_2)$, and $(1, k_3)$.

*Gustav Kirchhoff (1824–1887), a German physicist, was among the first to apply theoretical mathematics to physics. He is best known for his development of certain properties of electric circuits, which are now known as **Kirchhoff's laws.**

55. DIETS A biologist has available two commercial food mixes with the following percentages of protein and fat:

Mix	Protein (%)	Fat (%)
A	20	2
B	10	6

How many ounces of each mix should be used to prepare each of the diets listed in the following table?

	Diet		
	1	2	3
Protein	20 oz	10 oz	10 oz
Fat	6 oz	4 oz	6 oz

9-5 Determinants

- Defining First-Order Determinants
- Evaluating Second-Order Determinants
- Evaluating Third-Order Determinants
- Evaluating Higher-Order Determinants

We have already studied various methods for solving systems of linear equations: graphing, substitution, elimination by addition, inverse matrices, and Gauss–Jordan elimination. Of these, Gauss–Jordan elimination is the most versatile, since it can be applied to systems of any size. In this section, we begin the development of another method that will lead to formulas that can be used to find the inverse of a matrix and the solution of a system of linear equations with the same number of equations and variables. This new method has many applications in geometry, physics, and engineering.

Defining First-Order Determinants

For any square matrix A, the **determinant** of A is a real number denoted by **det** (A) or $|A|$*. If A is a square matrix of order n, then det (A) is called a **determinant of order** n. If $A = [a_{11}]$ is a square matrix of order 1, then

$$\det(A) = a_{11}$$

is a **first-order determinant.** Now we proceed to define determinants of higher order.

*The absolute value notation will now have two interpretations: the absolute value of a real number or the determinant of a square matrix. These concepts are not the same. You must always interpret $|A|$ in terms of the context in which it is used.

Evaluating Second-Order Determinants

Given a second-order square matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the **second-order determinant** of A is

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \tag{1}$$

Formula (1) is easily remembered if you notice that the expression on the right is the product of the elements on **principal diagonal,** from upper left to lower right, minus the product of the elements on the **secondary diagonal,** from lower left to upper right.

EXAMPLE 1 Evaluating a Second-Order Determinant

Find $\det(A)$ for $A = \begin{bmatrix} -1 & 2 \\ -3 & -4 \end{bmatrix}$.

SOLUTIONS

Algebraic Solution

$$\begin{aligned} \det(A) &= \begin{vmatrix} -1 & 2 \\ -3 & -4 \end{vmatrix} \\ &= (-1)(-4) - (-3)(2) \\ &= 4 - (-6) \\ &= 10 \end{aligned}$$

Graphing Calculator Solution

Enter the given matrix and use the DET* command (Fig. 1).

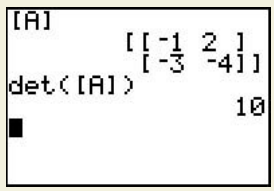


Figure 1

MATCHED PROBLEM 1

Find $\det(A)$ for $A = \begin{bmatrix} 3 & -5 \\ 4 & -2 \end{bmatrix}$.

››› CAUTION ›››

Remember that $A = \begin{bmatrix} 3 & -5 \\ 4 & -2 \end{bmatrix}$ is a matrix, but $\begin{vmatrix} 3 & -5 \\ 4 & -2 \end{vmatrix}$ represents a real number, the determinant of A. We will often refer to $\begin{vmatrix} 3 & -5 \\ 4 & -2 \end{vmatrix}$ as a determinant, and refer to the process of finding the real number it represents as "evaluating the determinant."

*On a TI-84, the DET command is located on the MATRIX/MATH menu.

› Evaluating Third-Order Determinants

Given the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, **the third-order determinant** of A is

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} + a_{21}a_{32}a_{13} - a_{21}a_{12}a_{33} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} \quad (2)$$

Don't panic! You don't need to memorize formula (2). After we introduce the ideas of *minor* and *cofactor* below, we will state a theorem that can be used to obtain the same result with much less trouble.

The **minor of an element** in a third-order determinant is a second-order determinant obtained by deleting the row and column that contains the element. For example, in the determinant in formula (2),

Deletions are usually done mentally.

$$\text{Minor of } a_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{32} - a_{31}a_{12}$$

$$\text{Minor of } a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{23} - a_{21}a_{13}$$

››› EXPLORE-DISCUSS 1

Write the minors of the other seven elements in the determinant in formula (2).

A quantity closely associated with the minor of an element is the **cofactor of an element** a_{ij} (from the ith row and jth column), which is defined as the product of the minor of a_{ij} and $(-1)^{i+j}$.

› DEFINITION 1 Cofactor

Cofactor of $a_{ij} = (-1)^{i+j}$ (Minor of a_{ij})

Thus, a cofactor of an element is nothing more than a signed minor. The sign is determined by raising -1 to a power that is the sum of the numbers indicating the

row and column in which the element appears. Note that $(-1)^{i+j}$ is 1 if $i + j$ is even and -1 if $i + j$ is odd. Thus, if we are given the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

then

$$\text{Cofactor of } a_{23} = (-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -(a_{11}a_{32} - a_{31}a_{12}) = -a_{11}a_{32} + a_{11}a_{12}$$

$$\text{Cofactor of } a_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$$

EXAMPLE 2 Finding Cofactors

Find the cofactors of -2 and 5 in the determinant

$$\begin{vmatrix} -2 & 0 & 3 \\ 1 & -6 & 5 \\ -1 & 2 & 0 \end{vmatrix}$$

SOLUTION

$$\text{Cofactor of } -2 = (-1)^{1+1}\begin{vmatrix} -6 & 5 \\ 2 & 0 \end{vmatrix} = \begin{vmatrix} -6 & 5 \\ 2 & 0 \end{vmatrix}$$

$$= (-6)(0) - (2)(5) = -10$$

$$\text{Cofactor of } 5 = (-1)^{2+3}\begin{vmatrix} -2 & 0 \\ -1 & 2 \end{vmatrix} = -\begin{vmatrix} -2 & 0 \\ -1 & 2 \end{vmatrix}$$

$$= -[(-2)(2) - (-1)(0)] = 4$$

MATCHED PROBLEM 2

Find the cofactors of 2 and 3 in the determinant in Example 2.

[*Note:* The sign in front of the minor, $(-1)^{i+j}$, can be determined rather mechanically by using a checkerboard pattern of $+$ and $-$ signs over the determinant, starting with $+$ in the upper left-hand corner:

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Use either the checkerboard or the exponent method—whichever is easier for you—to determine the sign in front of the minor.]

Now we are ready for the key theorem of this section, Theorem 1. This theorem provides us with an efficient step-by-step procedure, called an *algorithm,* for evaluating third-order determinants.

› THEOREM 1 Value of a Third-Order Determinant

The value of a determinant of order 3 is the sum of three products obtained by multiplying each element of any one row (or each element of any one column) by its cofactor.

To prove this theorem we must show that the expansions indicated by the theorem for any row or any column (six cases) produce the expression on the right of formula (2). Proofs of special cases of this theorem are left to the problems in the exercises for this section.

EXAMPLE 3 Evaluating a Third-Order Determinant

Evaluate $\begin{vmatrix} 2 & -2 & 0 \\ -3 & 1 & 2 \\ 1 & -3 & -1 \end{vmatrix}$

SOLUTIONS

Algebraic Solution

Expanding the first row, we have

$$\begin{vmatrix} 2 & -2 & 0 \\ -3 & 1 & 2 \\ 1 & -3 & -1 \end{vmatrix}$$

$$= a_{11}\begin{pmatrix}\text{Cofactor}\\ \text{of } a_{11}\end{pmatrix} + a_{12}\begin{pmatrix}\text{Cofactor}\\ \text{of } a_{12}\end{pmatrix} + a_{13}\begin{pmatrix}\text{Cofactor}\\ \text{of } a_{13}\end{pmatrix}$$

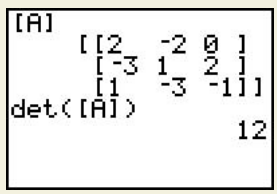
$$= 2\left[(-1)^{1+1}\begin{vmatrix} 1 & 2 \\ -3 & -1 \end{vmatrix}\right] + (-2)\left[(-1)^{1+2}\begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix}\right] + 0$$

$$= (2)(1)[(1)(-1) - (-3)(2)] + (-2)(-1)[(-3)(-1) - (1)(2)]$$

$$= (2)(5) + (2)(1) = 12$$

Graphing Calculator Solution

Enter the matrix and use the DET command (Fig. 2).



› Figure 2

MATCHED PROBLEM 3

Evaluate $\begin{vmatrix} 2 & 1 & -1 \\ -2 & -3 & 0 \\ -1 & 2 & 1 \end{vmatrix}$

Refer to Example 3. According to Theorem 1, we should get the same value if we expand any other row or column. Example 4 verifies this for the second column.

EXAMPLE 4 Expanding a Different Row or Column

Expand the second column to evaluate $\begin{vmatrix} 2 & -2 & 0 \\ -3 & 1 & 2 \\ 1 & -3 & -1 \end{vmatrix}$

SOLUTION

$$\begin{vmatrix} 2 & -2 & 0 \\ -3 & 1 & 2 \\ 1 & -3 & -1 \end{vmatrix} = a_{12}\left(\begin{matrix}\text{Cofactor} \\ \text{of } a_{12}\end{matrix}\right) + a_{22}\left(\begin{matrix}\text{Cofactor} \\ \text{of } a_{22}\end{matrix}\right) + a_{32}\left(\begin{matrix}\text{Cofactor} \\ \text{of } a_{32}\end{matrix}\right)$$

$$= (-2)\left[(-1)^{1+2}\begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix}\right] + (1)\left[(-1)^{2+2}\begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix}\right] + (-3)\left[(-1)^{3+2}\begin{vmatrix} 2 & 0 \\ -3 & 2 \end{vmatrix}\right]$$

$$= (-2)(-1)[(-3)(-1) - (1)(2)] + (1)(1)[(2)(-1) - (1)(0)] + (-3)(-1)[(2)(2) - (-3)(0)]$$

$$= (2)(1) + (1)(-2) + (3)(4) = 12$$

MATCHED PROBLEM 4

Expand a row or column different from the one you used to solve Matched Problem 3 to evaluate

$$\begin{vmatrix} 2 & 1 & -1 \\ -2 & -3 & 0 \\ -1 & 2 & 1 \end{vmatrix}$$

Evaluating Higher-Order Determinants

Theorem 1 and the definitions of minor and cofactor generalize completely for determinants of order higher than 3. These concepts are illustrated for a fourth-order determinant in Example 5.

EXAMPLE 5 Evaluating a Fourth-Order Determinant

Given the fourth-order determinant

$$\begin{vmatrix} 0 & -1 & 0 & 2 \\ -5 & -6 & 0 & -3 \\ 4 & 5 & -2 & 6 \\ 0 & 3 & 0 & -4 \end{vmatrix}$$

(A) Find the minor in determinant form of the element 3.
(B) Find the cofactor in determinant form of the element -5.
(C) Find the value of the fourth-order determinant.

SOLUTIONS

$$\text{(A) Minor of } 3 = \begin{vmatrix} 0 & 0 & 2 \\ -5 & 0 & -3 \\ 4 & -2 & 6 \end{vmatrix}$$

$$\text{(B) Cofactor of } -5 = (-1)^{2+1}\begin{vmatrix} -1 & 0 & 2 \\ 5 & -2 & 6 \\ 3 & 0 & -4 \end{vmatrix} = -\begin{vmatrix} -1 & 0 & 2 \\ 5 & -2 & 6 \\ 3 & 0 & -4 \end{vmatrix}$$

(C) Algebraic Solution

Generalizing Theorem 1, the value of this fourth-order determinant is the sum of four products obtained by multiplying each element of any one row (or each element of any one column) by its cofactor. The work involved in this evaluation is greatly reduced if we choose the row or column with the greatest number of zeros. Because column 3 has three zeros, we expand along this column:

$$\begin{vmatrix} 0 & -1 & 0 & 2 \\ -5 & -6 & 0 & -3 \\ 4 & 5 & -2 & 6 \\ 0 & 3 & 0 & -4 \end{vmatrix} = 0 + 0 + (-2)(-1)^{3+3}\begin{vmatrix} 0 & -1 & 2 \\ -5 & -6 & -3 \\ 0 & 3 & -4 \end{vmatrix} + 0$$

$$= (-2)\begin{vmatrix} 0 & -1 & 2 \\ -5 & -6 & -3 \\ 0 & 3 & -4 \end{vmatrix}$$

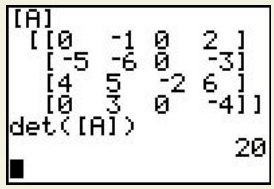
Expand this determinant along the first column.

$$= (-2)\left[0 + (-5)(-1)^{2+1}\begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix} + 0\right]$$

$$= (-2)(-5)(-1)(-2) = 20$$

Graphing Calculator Solution

Enter the matrix and use the DET command (Fig. 3).



› Figure 3

MATCHED PROBLEM 5

Repeat Example 5 for the following fourth-order determinant:

$$\begin{vmatrix} 0 & 4 & -2 & 0 \\ -3 & 3 & -1 & 2 \\ 0 & 6 & 0 & 0 \\ 5 & -6 & -5 & -4 \end{vmatrix}$$

››› EXPLORE-DISCUSS 2

Write a checkerboard pattern of + and − signs for a fourth-order determinant, and use it to determine the signs of the minors in Example 5.

REMARK: Where are determinants used? Many equations and formulas have particularly simple and compact representations in determinant form that are easily remembered. (See Problems 53–56 in the exercises for the next section). Also, in the last section of this chapter, we will see that the solutions to certain systems of equations can be expressed in terms of determinants. In addition, determinants are involved in theoretical work in advanced mathematics courses. For example, it can be shown that the inverse of a square matrix exists if and only if its determinant is not 0.

ANSWERS TO MATCHED PROBLEMS

1. 14 **2.** Cofactor of 2 = 13; cofactor of 3 = −4 **3.** 3 **4.** 3

5. (A) $\begin{vmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 5 & -5 & -4 \end{vmatrix}$ (B) $-\begin{vmatrix} 0 & 4 & 0 \\ -3 & 3 & 2 \\ 0 & 6 & 0 \end{vmatrix}$ (C) −24

9-5 Exercises

1. What is a determinant?

2. What is the order of a determinant?

3. Explain the difference between $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$.

4. Explain the difference between a matrix and a minor.

5. Explain the difference between a minor and a cofactor.

6. How do you evaluate a third-order determinant?

Evaluate each second-order determinant in Problems 7–12.

7. $\begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix}$ **8.** $\begin{vmatrix} 8 & -3 \\ 4 & 1 \end{vmatrix}$

9. $\begin{vmatrix} 3 & -7 \\ -5 & 6 \end{vmatrix}$ **10.** $\begin{vmatrix} 9 & -2 \\ 4 & 0 \end{vmatrix}$

11. $\begin{vmatrix} 4.3 & -1.2 \\ -5.1 & 3.7 \end{vmatrix}$ **12.** $\begin{vmatrix} -0.7 & -2.3 \\ 1.9 & -4.8 \end{vmatrix}$

Problems 13–20 pertain to the determinant below:

$$\begin{vmatrix} 5 & -1 & -3 \\ 3 & 4 & 6 \\ 0 & -2 & 8 \end{vmatrix}$$

Write the minor of each element given in Problems 13–16. Leave the answer in determinant form.

13. a_{11} **14.** a_{33} **15.** a_{23} **16.** a_{12}

Write the cofactor of each element given in Problems 17–20, and evaluate each.

17. a_{11} **18.** a_{33} **19.** a_{23} **20.** a_{12}

Evaluate Problems 21–26 using cofactors and using a graphing utility.

21. $\begin{vmatrix} 1 & 0 & 0 \\ -2 & 4 & 3 \\ 5 & -2 & 1 \end{vmatrix}$ **22.** $\begin{vmatrix} 2 & -3 & 5 \\ 0 & -3 & 1 \\ 0 & 6 & 2 \end{vmatrix}$

23. $\begin{vmatrix} 0 & 1 & 5 \\ 3 & -7 & 6 \\ 0 & -2 & -3 \end{vmatrix}$ **24.** $\begin{vmatrix} 4 & -2 & 0 \\ 9 & 5 & 4 \\ 1 & 2 & 0 \end{vmatrix}$

25. $\begin{vmatrix} -1 & 2 & -3 \\ -2 & 0 & -6 \\ 4 & -3 & 2 \end{vmatrix}$ **26.** $\begin{vmatrix} 0 & 2 & -1 \\ -6 & 3 & 1 \\ 7 & -9 & -2 \end{vmatrix}$

Given the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

write the cofactor in determinant form of each element in Problems 27–30.

27. a_{11} **28.** a_{44} **29.** a_{43} **30.** a_{23}

Evaluate each determinant in Problems 31–40 using cofactors and using a graphing utility.

31. $\begin{vmatrix} 3 & -2 & -8 \\ -2 & 0 & -3 \\ 1 & 0 & -4 \end{vmatrix}$ **32.** $\begin{vmatrix} 4 & -4 & 6 \\ 2 & 8 & -3 \\ 0 & -5 & 0 \end{vmatrix}$

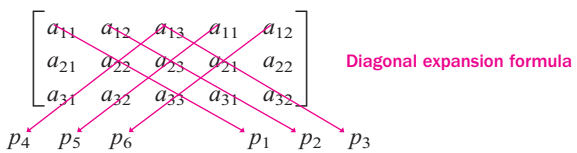
33. $\begin{vmatrix} 1 & 4 & 1 \\ 1 & 1 & -2 \\ 2 & 1 & -1 \end{vmatrix}$ **34.** $\begin{vmatrix} 3 & 2 & 1 \\ -1 & 5 & 1 \\ 2 & 3 & 1 \end{vmatrix}$

35. $\begin{vmatrix} 1 & 4 & 3 \\ 2 & 1 & 6 \\ 3 & -2 & 9 \end{vmatrix}$ **36.** $\begin{vmatrix} 4 & -6 & 3 \\ -1 & 4 & 1 \\ 5 & -6 & 3 \end{vmatrix}$

37. $\begin{vmatrix} 2 & 6 & 1 & 7 \\ 0 & 3 & 0 & 0 \\ 3 & 4 & 2 & 5 \\ 0 & 9 & 0 & 2 \end{vmatrix}$ **38.** $\begin{vmatrix} 0 & 1 & 0 & 1 \\ 2 & 4 & 7 & 6 \\ 0 & 3 & 0 & 1 \\ 0 & 6 & 2 & 5 \end{vmatrix}$

39. $\begin{vmatrix} -2 & 0 & 0 & 0 & 0 \\ 9 & -1 & 0 & 0 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ -1 & 4 & 2 & 2 & 0 \\ 7 & -2 & 3 & 5 & 5 \end{vmatrix}$ **40.** $\begin{vmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix}$

If A is a 3×3 matrix, $\det A$ can be evaluated by the following ***diagonal expansion.*** *Form a 3×5 matrix by augmenting A on the right with its first two columns, and compute the diagonal products $p_1, p_2, \ldots, p_6$ indicated by the arrows:*



The determinant of A is given by [compare with formula (2)]

$$\begin{aligned} \det A &= p_1 + p_2 + p_3 - p_4 - p_5 - p_6 \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \\ &\qquad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \end{aligned}$$

[Caution: The diagonal expansion procedure works only for 3×3 matrices. Do not apply it to matrices of any other size.]

Use the diagonal expansion formula to evaluate the determinants in Problems 41 and 42.

41. $\begin{vmatrix} 2 & 6 & -1 \\ 5 & 3 & -7 \\ -4 & -2 & 1 \end{vmatrix}$ **42.** $\begin{vmatrix} 4 & 1 & -5 \\ 1 & 2 & -6 \\ -3 & -1 & 7 \end{vmatrix}$

A square matrix is called an ***upper triangular matrix*** *if all elements below the principal diagonal are zero. In Problems 43–46, determine whether the statement is true or false. If true, explain why. If false, give a counterexample.*

43. If the determinant of an upper triangular matrix is 0, then the elements on the principal diagonal are all 0.

44. If A and B are upper triangular matrices, then $\det(A + B) = \det A + \det B$.

45. The determinant of an upper triangular matrix is the product of the elements on the principal diagonal.

46. If A and B are upper triangular matrices, then $\det(AB) = (\det A)(\det B)$.

In Problems 47–52, all the letters represent real numbers. Find an equation that each pair of determinants satisfies, and describe the relationship between the two determinants verbally.

47. $\begin{vmatrix} a & b \\ c & d \end{vmatrix}, \begin{vmatrix} c & d \\ a & b \end{vmatrix}$ **48.** $\begin{vmatrix} a & b \\ c & d \end{vmatrix}, \begin{vmatrix} b & a \\ d & c \end{vmatrix}$

49. $\begin{vmatrix} a & b \\ c & d \end{vmatrix}, \begin{vmatrix} ka & b \\ kc & d \end{vmatrix}$ **50.** $\begin{vmatrix} a & b \\ c & d \end{vmatrix}, \begin{vmatrix} a & b \\ kc & kd \end{vmatrix}$

51. $\begin{vmatrix} a & b \\ c & d \end{vmatrix}, \begin{vmatrix} kc + a & kd + b \\ c & d \end{vmatrix}$ **52.** $\begin{vmatrix} a & b \\ c & d \end{vmatrix}, \begin{vmatrix} a & ka + b \\ c & kc + d \end{vmatrix}$

53. Show that the expansion of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

by the first column is the same as its expansion by the third row.

54. Repeat Problem 53, using the second row and the third column.

55. If

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$$

show that $\det(AB) = (\det A)(\det B)$.

56. If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

show that $\det(AB) = (\det A)(\det B)$.

If A is an $n \times n$ matrix and I is the $n \times n$ identity matrix, then the function $f(x) = |xI - A|$ is called the ***characteristic polynomial*** *of A, and the zeros of $f(x)$ are called the* ***eigenvalues*** *of A. Characteristic polynomials and eigenvalues have many important applications that are discussed in more advanced treatments of matrices. In Problems 57–60, find the characteristic polynomial and the eigenvalues of each matrix.*

57. $\begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$

58. $\begin{bmatrix} 8 & -6 \\ 3 & -1 \end{bmatrix}$

59. $\begin{bmatrix} 4 & -4 & 0 \\ 2 & -2 & 0 \\ 4 & -8 & -4 \end{bmatrix}$

60. $\begin{bmatrix} -2 & 2 & 0 \\ -1 & 1 & 0 \\ -2 & 4 & 2 \end{bmatrix}$

9-6 Properties of Determinants

- Finding Additional Properties of Determinants
- Summarizing Properties of Determinants

In the solution of Example 5 in the last section, we noted that expansion along the row (or column) with the most zeros will reduce the work involved in the evaluation of a determinant. In this section, we will discuss more ways to reduce the work required to evaluate a determinant.

Finding Additional Properties of Determinants

We now state and discuss five general determinant properties in the form of theorems. Because the proofs for the general cases of these theorems are involved and notationally difficult, we will sketch only informal proofs for determinants of order 3. The theorems, however, apply to determinants of any order.

THEOREM 1 Multiplying a Row or Column by a Constant

If each element of any row (or column) of a determinant is multiplied by a constant k, the new determinant is k times the original.

PARTIAL PROOF: Let C_{ij} be the cofactor of a_{ij}. Then expanding by the first row, we have

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = ka_{11}C_{11} + ka_{12}C_{12} + ka_{13}C_{13}$$

$$= k(a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13})$$

$$= k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Theorem 1 also states that a factor common to all elements of a row (or column) can be taken out as a factor of the determinant.

EXAMPLE 1 Taking Out a Common Factor of a Column

Take out factors common to any row or any column:

$$\begin{vmatrix} 6 & 1 & 3 \\ -2 & 7 & -2 \\ 4 & 5 & 0 \end{vmatrix}$$

SOLUTION

$$\begin{vmatrix} 6 & 1 & 3 \\ -2 & 7 & -2 \\ 4 & 5 & 0 \end{vmatrix} = 2 \begin{vmatrix} 3 & 1 & 3 \\ -1 & 7 & -2 \\ 2 & 5 & 0 \end{vmatrix}$$

where 2 is a common factor of the first column.

MATCHED PROBLEM 1

Take out factors common to any row or any column:

$$\begin{vmatrix} 3 & 2 & 1 \\ 6 & 3 & -9 \\ 1 & 0 & -5 \end{vmatrix}$$

››› EXPLORE-DISCUSS 1

(A) How are $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and $\begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix}$ related?

(B) How are $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ and $\begin{vmatrix} ka & kb & kc \\ kd & ke & kf \\ kg & kh & ki \end{vmatrix}$ related?

› THEOREM 2 Row or Column of Zeros

If every element in a row (or column) is 0, the value of the determinant is 0.

Theorem 2 is an immediate consequence of Theorem 1, and its proof is left as an exercise. It is illustrated in the following example:

$$\begin{vmatrix} 3 & -2 & 5 \\ 0 & 0 & 0 \\ -1 & 4 & 9 \end{vmatrix} = 0$$

› THEOREM 3 Interchanging Rows or Columns

If two rows (or two columns) of a determinant are interchanged, the new determinant is the negative of the original.

A proof of Theorem 3 even for a determinant of order 3 is notationally involved. We suggest that you partially prove the theorem by direct expansion of the determinants before and after the interchange of two rows (or columns). The theorem is illustrated by the following example, where the second and third columns are interchanged:

$$\begin{vmatrix} 1 & 0 & 9 \\ -2 & 1 & 5 \\ 3 & 0 & 7 \end{vmatrix} = - \begin{vmatrix} 1 & 9 & 0 \\ -2 & 5 & 1 \\ 3 & 7 & 0 \end{vmatrix}$$

››› EXPLORE-DISCUSS 2

(A) What are the cofactors of each element in the first row of the following determinant? What is the value of the determinant?

$$\begin{vmatrix} a & b & c \\ d & e & f \\ d & e & f \end{vmatrix}$$

(B) What are the cofactors of each element in the second column of the following determinant? What is the value of the determinant?

$$\begin{vmatrix} a & b & a \\ d & e & d \\ g & h & g \end{vmatrix}$$

› THEOREM 4 Equal Rows or Columns

If the corresponding elements are equal in two rows (or columns), the value of the determinant is 0.

PROOF: The general proof of Theorem 4 follows directly from Theorem 3. If we start with a determinant D that has two rows (or columns) equal and we interchange the equal rows (or columns), the new determinant will be the same as the original. But by Theorem 3,

$$D = -D$$

hence,

$$2D = 0$$
$$D = 0$$

› THEOREM 5 Addition of Rows or Columns

If a multiple of any row (or column) of a determinant is added to any other row (or column), the value of the determinant is not changed.

PARTIAL PROOF: If, in a general third-order determinant, we add a k multiple of the second column to the first and then expand by the first column, we obtain (where C_{ij} is the cofactor of a_{ij} in the original determinant)

$$\begin{vmatrix} a_{11} + ka_{12} & a_{12} & a_{13} \\ a_{21} + ka_{22} & a_{22} & a_{23} \\ a_{31} + ka_{32} & a_{32} & a_{33} \end{vmatrix} = (a_{11} + ka_{12})C_{11} + (a_{21} + ka_{22})C_{21} + (a_{31} + ka_{32})C_{31}$$

$$= (a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}) + k(a_{12}C_{11} + a_{22}C_{21} + a_{32}C_{31})$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + k \begin{vmatrix} a_{12} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The determinant following k is 0 because the first and second columns are equal.

Note the similarity in the process described in Theorem 5 to that used to obtain row-equivalent matrices. We use this theorem to transform a determinant without 0 elements into one that contains a row or column with all elements 0 but one. The transformed determinant can then be easily expanded by this row (or column). An example best illustrates the process.

EXAMPLE 2 Evaluating a Determinant

Evaluate the determinant $\begin{vmatrix} 3 & -1 & 2 \\ -2 & 4 & -3 \\ 4 & -2 & 5 \end{vmatrix}$

SOLUTIONS

Algebraic Solution

We use Theorem 5 to obtain two zeros in the first row, and then expand the determinant by this row. To start, we replace the third column with the sum of it and 2 times the second column to obtain a 0 in the a_{13} position:

$$\begin{vmatrix} 3 & -1 & 2 \\ -2 & 4 & -3 \\ 4 & -2 & 5 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 0 \\ -2 & 4 & 5 \\ 4 & -2 & 1 \end{vmatrix} \qquad 2C_2 + C_1 \to C_1^*$$

Next, to obtain a 0 in the a_{11} position, we replace the first column with the sum of it and 3 times the second column:

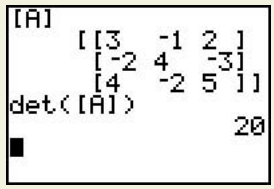
$$\begin{vmatrix} 3 & -1 & 0 \\ -2 & 4 & 5 \\ 4 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ 10 & 4 & 5 \\ -2 & -2 & 1 \end{vmatrix} \qquad 3C_2 + C_1 \to C_1$$

Now it is an easy matter to expand this last determinant by the first row to obtain

$$\begin{vmatrix} 0 & -1 & 0 \\ 10 & 4 & 5 \\ -2 & -2 & 1 \end{vmatrix} = 0 + (-1)\left[(-1)^{1+2}\begin{vmatrix} 10 & 5 \\ -2 & 1 \end{vmatrix}\right] + 0 = 20$$

Graphing Calculator Solution

Enter the matrix and use the DET command (Fig. 1).



› Figure 1

MATCHED PROBLEM 2

Evaluate the following determinant by first using Theorem 5 to obtain zeros in the a_{11} and a_{31} positions, and then expand by the first column.

$$\begin{vmatrix} 3 & 10 & -5 \\ 1 & 6 & -3 \\ 2 & 3 & 4 \end{vmatrix}$$

› Summarizing Properties of Determinants

We now summarize the five determinant properties discussed earlier in Table 1 for convenient reference. Although these properties hold for determinants of any order, for simplicity, we illustrate each property in terms of second-order determinants.

*C_1, C_2, and C_3 represent columns 1, 2, and 3, respectively.

Table 1 Summary of Determinant Properties

Property	Examples
1. If each element of any row (or column) of a determinant is multiplied by a constant k, the new determinant is k times the original.	$\begin{vmatrix} 2a & 2b \\ c & d \end{vmatrix} = 2\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ $3\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 3a & b \\ 3c & d \end{vmatrix}$
2. If every element in a row (or column) is 0, the value of the determinant is 0.	$\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0$ $\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} = 0$
3. If two rows (or two columns) of a determinant are interchanged, the new determinant is the negative of the original.	$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$ $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} b & a \\ d & c \end{vmatrix}$
4. If the corresponding elements are equal in two rows (or columns), the value of the determinant is 0.	$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$ $\begin{vmatrix} a & a \\ c & c \end{vmatrix} = 0$
5. If a multiple of any row (or column) of a determinant is added to any other row (or column), the value of the determinant is not changed.	$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c + ka & d + kb \end{vmatrix}$ $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a + kb & b \\ c + kd & d \end{vmatrix}$

ANSWERS TO MATCHED PROBLEMS

1. $3\begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 1 & 0 & -5 \end{vmatrix}$ **2.** 44

9-6 Exercises

1. If A is a square matrix, list the ways you can tell that $\det(A) = 0$ just by examining the rows and/or columns of A.

2. If A is a 2×2 matrix and k is a nonzero constant, how are $\det(kA)$ and $\det(A)$ related?

For each statement in Problems 3–12, identify the theorem from this section that justifies it. Do not evaluate.

3. $\begin{vmatrix} 16 & 8 \\ 0 & -1 \end{vmatrix} = 8\begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix}$ **4.** $\begin{vmatrix} 1 & -9 \\ 0 & -6 \end{vmatrix} = -3\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix}$

5. $-2\begin{vmatrix} 2 & 1 \\ -3 & 4 \end{vmatrix} = \begin{vmatrix} -4 & 1 \\ 6 & 4 \end{vmatrix}$ **6.** $4\begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} = \begin{vmatrix} -4 & 12 \\ 2 & 1 \end{vmatrix}$

7. $\begin{vmatrix} 3 & 0 \\ -2 & 0 \end{vmatrix} = 0$ **8.** $\begin{vmatrix} 5 & -7 \\ 0 & 0 \end{vmatrix} = 0$

9. $\begin{vmatrix} 5 & -1 \\ 8 & 0 \end{vmatrix} = -\begin{vmatrix} -1 & 5 \\ 0 & 8 \end{vmatrix}$ **10.** $\begin{vmatrix} 6 & 9 \\ 0 & 1 \end{vmatrix} = -\begin{vmatrix} 0 & 1 \\ 6 & 9 \end{vmatrix}$

11. $\begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 4-4 & 3-8 \\ 1 & 2 \end{vmatrix}$ **12.** $\begin{vmatrix} 3 & 2 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 3+4 & 2 \\ 5+2 & 1 \end{vmatrix}$

In Problems 13–16, Theorem 5 was used to transform the determinant on the left to that on the right. Replace each letter x with an appropriate numeral to complete the transformation.

13. $\begin{vmatrix} -1 & 3 \\ 2 & -4 \end{vmatrix} = \begin{vmatrix} -1 & x \\ 2 & 2 \end{vmatrix}$ **14.** $\begin{vmatrix} -1 & 3 \\ 5 & -2 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ x & 13 \end{vmatrix}$

15. $\begin{vmatrix} -1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 3 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 0 \\ 2 & 1 & 10 \\ 1 & 3 & x \end{vmatrix}$

16. $\begin{vmatrix} -1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 3 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 3 \\ 2 & x & 4 \\ 1 & 5 & 2 \end{vmatrix}$

Given that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 10$$

use the properties of determinants discussed in this section to evaluate each determinant in Problems 17–22.

17. $\begin{vmatrix} c & d \\ a & b \end{vmatrix}$ **18.** $\begin{vmatrix} 2a & 2b \\ c & d \end{vmatrix}$

19. $\begin{vmatrix} a+c & b+d \\ c & d \end{vmatrix}$ **20.** $\begin{vmatrix} a+b & b \\ c+d & d \end{vmatrix}$

21. $\begin{vmatrix} a & a-b \\ c & c-d \end{vmatrix}$ **22.** $\begin{vmatrix} a+c & b+d \\ -a & -b \end{vmatrix}$

In Problems 23–26, transform each determinant into one that contains a row (or column) with all elements 0 but one, if possible. Then expand the transformed determinant by this row (or column).

23. $\begin{vmatrix} -1 & 0 & 3 \\ 2 & 5 & 4 \\ 1 & 5 & 2 \end{vmatrix}$ **24.** $\begin{vmatrix} -1 & 2 & 0 \\ 2 & 1 & 10 \\ 1 & 3 & 5 \end{vmatrix}$

25. $\begin{vmatrix} 3 & 5 & 0 \\ 1 & 1 & -2 \\ 2 & 1 & -1 \end{vmatrix}$ **26.** $\begin{vmatrix} 2 & 0 & 1 \\ -1 & -3 & 4 \\ 1 & 2 & 3 \end{vmatrix}$

For each statement in Problems 27–32, identify the theorem from this section that justifies it.

27. $-2\begin{vmatrix} 1 & 0 & 2 \\ 3 & -2 & 4 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ -6 & 4 & -8 \\ 0 & 1 & 1 \end{vmatrix}$

28. $\begin{vmatrix} 8 & 0 & 1 \\ 12 & -1 & 0 \\ 4 & 3 & 2 \end{vmatrix} = 4\begin{vmatrix} 2 & 0 & 1 \\ 3 & -1 & 0 \\ 1 & 3 & 2 \end{vmatrix}$

29. $\begin{vmatrix} 1 & 2 & 0 \\ -1 & 3 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0$

30. $\begin{vmatrix} -2 & 5 & 13 \\ 1 & 7 & 12 \\ 0 & 8 & 15 \end{vmatrix} = -\begin{vmatrix} 5 & -2 & 13 \\ 7 & 1 & 12 \\ 8 & 0 & 15 \end{vmatrix}$

31. $\begin{vmatrix} 4 & 2 & -1 \\ 2 & 0 & 2 \\ -3 & 5 & -2 \end{vmatrix} = \begin{vmatrix} 4-4 & 2 & -1 \\ 2+8 & 0 & 2 \\ -3-8 & 5 & -2 \end{vmatrix}$

32. $\begin{vmatrix} 7 & 7 & 1 \\ -3 & -3 & 11 \\ 2 & 2 & 0 \end{vmatrix} = 0$

In Problems 33–36, Theorem 5 was used to transform the determinant on the left to that on the right. Replace each letter x and y with an appropriate numeral to complete the transformation.

33. $\begin{vmatrix} 2 & 1 & -1 \\ 3 & 4 & 1 \\ 1 & 2 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 \\ x & 5 & 1 \\ -3 & y & -2 \end{vmatrix}$

34. $\begin{vmatrix} 3 & -1 & 1 \\ -2 & 4 & 3 \\ 1 & 5 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ 10 & 4 & 7 \\ x & 5 & y \end{vmatrix}$

35. $\begin{vmatrix} 7 & 9 & 4 \\ 2 & 3 & 1 \\ 3 & 4 & -2 \end{vmatrix} = \begin{vmatrix} -1 & x & 0 \\ 2 & 3 & 1 \\ 7 & y & 0 \end{vmatrix}$

36. $\begin{vmatrix} 5 & 2 & 3 \\ 3 & 1 & 2 \\ -4 & -3 & 5 \end{vmatrix} = \begin{vmatrix} x & 0 & -1 \\ 3 & 1 & 2 \\ 5 & 0 & y \end{vmatrix}$

In Problems 37–44, transform each determinant into one that contains a row (or column) with all elements 0 but one, if possible. Then expand the transformed determinant by this row (or column).

37. $\begin{vmatrix} 1 & 5 & 3 \\ 4 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$ **38.** $\begin{vmatrix} -1 & 5 & 1 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{vmatrix}$

39. $\begin{vmatrix} 5 & 2 & -3 \\ -2 & 4 & 4 \\ 1 & -1 & 3 \end{vmatrix}$ **40.** $\begin{vmatrix} 5 & 3 & -6 \\ -1 & 1 & 4 \\ 4 & 3 & -6 \end{vmatrix}$

41. $\begin{vmatrix} 3 & -4 & 1 \\ 6 & -1 & 2 \\ 9 & 2 & 3 \end{vmatrix}$ **42.** $\begin{vmatrix} 2 & 3 & -1 \\ 5 & 4 & 7 \\ -4 & -6 & 2 \end{vmatrix}$

43. $\begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & -2 & 4 & 3 \\ 2 & 1 & 5 & 4 \\ 1 & 2 & 1 & 2 \end{vmatrix}$ **44.** $\begin{vmatrix} 2 & 3 & 1 & -1 \\ 3 & 1 & 2 & 1 \\ 0 & 5 & 4 & 0 \\ -1 & 2 & 3 & 0 \end{vmatrix}$

Transform each determinant in Problems 45 and 46 into one that contains a row (or column) with all elements 0 but one, if possible. Then expand the transformed determinant by this row (or column).

45. $\begin{vmatrix} 3 & 2 & 3 & 1 \\ 3 & -2 & 8 & 5 \\ 2 & 1 & 3 & 1 \\ 4 & 5 & 4 & -3 \end{vmatrix}$ **46.** $\begin{vmatrix} -1 & 4 & 2 & 1 \\ 5 & -1 & -3 & -1 \\ 2 & -1 & -2 & 3 \\ -3 & 3 & 3 & 3 \end{vmatrix}$

Problems 47–50 are representative cases of theorems discussed in this section. Use cofactor expansions to verify each statement directly, without reference to the theorem it represents.

47. $\begin{vmatrix} a & b & a \\ d & e & d \\ g & h & g \end{vmatrix} = 0$ **48.** $\begin{vmatrix} a & b & c \\ kd & ke & kf \\ g & h & i \end{vmatrix} = k \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

49. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}$

50. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + kc_1 & b_1 & c_1 \\ a_2 + kc_2 & b_2 & c_2 \\ a_3 + kc_3 & b_3 & c_3 \end{vmatrix}$

51. Without expanding, explain why (2, 5) and (−3, 4) satisfy the equation

$$\begin{vmatrix} x & y & 1 \\ 2 & 5 & 1 \\ -3 & 4 & 1 \end{vmatrix} = 0$$

52. Show that

$$\begin{vmatrix} x & y & 1 \\ 2 & 3 & 1 \\ -1 & 2 & 1 \end{vmatrix} = 0$$

is the equation of a line that passes through (2, 3) and (−1, 2).

53. Show that

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

is the equation of a line that passes through (x_1, y_1) and (x_2, y_2).

54. In analytic geometry it is shown that the area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is the absolute value of

$$\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Use this result to find the area of a triangle with vertices (−1, 4), (4, 8), and (1, 1).

55. What can we say about the three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) if the following equation is true?

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

[*Hint:* See Problem 54.]

56. If the three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are all on the same line, what can we say about the value of the determinant below?

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

9-7 Determinants and Cramer's Rule

- Solving Systems of Two Equations in Two Variables
- Solving Systems of Three Equations in Three Variables

Now let's see how determinants arise rather naturally in the process of solving systems of linear equations. We start by investigating two equations and two variables, and then extend our results to three equations and three variables.

Solving Systems of Two Equations in Two Variables

Instead of thinking of each system of linear equations in two variables as a different problem, let's see what happens when we attempt to solve the general system

$$a_{11}x + a_{12}y = k_1 \tag{1A}$$
$$a_{21}x + a_{22}y = k_2 \tag{1B}$$

once and for all, in terms of the unspecified real constants $a_{11}, a_{12}, a_{21}, a_{22}, k_1$, and k_2.

We proceed by multiplying equations (1A) and (1B) by suitable constants so that when the resulting equations are added, left side to left side and right side to right side, one of the variables drops out. Suppose we choose to eliminate y. What constant should we use to make the coefficients of y the same except for the signs? Multiply equation (1A) by a_{22} and (1B) by $-a_{12}$; then add:

a_{22}(1A): $a_{11}a_{22}x + a_{12}a_{22}y = k_1a_{22}$

$-a_{12}$(1B): $-a_{21}a_{12}x - a_{12}a_{22}y = -k_2a_{12}$

$a_{11}a_{22}x - a_{21}a_{12}x + 0y = k_1a_{22} - k_2a_{12}$ y is eliminated.

$(a_{11}a_{22} - a_{21}a_{12})x = k_1a_{22} - k_2a_{12}$ Solve for x.

$$x = \frac{k_1a_{22} - k_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \qquad a_{11}a_{22} - a_{21}a_{12} \neq 0$$

What do the numerator and denominator remind you of? From your experience with determinants in the preceding two sections, you might recognize these expressions as

$$x = \frac{\begin{vmatrix} k_1 & a_{12} \\ k_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Similarly, starting with system (1A) and (1B) and eliminating x (this is left as an exercise), we obtain

$$y = \frac{\begin{vmatrix} a_{11} & k_1 \\ a_{21} & k_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

These results are summarized in Theorem 1, **Cramer's rule,** which is named after the Swiss mathematician Gabriel Cramer (1704–1752).

› THEOREM 1 Cramer's Rule for Two Equations and Two Variables

Given the system

$$\begin{aligned} a_{11}x + a_{12}y &= k_1 \\ a_{21}x + a_{22}y &= k_2 \end{aligned} \quad \text{with} \quad D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$$

then

$$x = \frac{\begin{vmatrix} k_1 & a_{12} \\ k_2 & a_{22} \end{vmatrix}}{D} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_{11} & k_1 \\ a_{21} & k_2 \end{vmatrix}}{D}$$

The determinant D is called the *coefficient determinant*. If $D \neq 0$, then the system has exactly one solution, which is given by Cramer's rule. If, on the other hand, $D = 0$, then it can be shown that the system is either inconsistent and has no solutions or is dependent and has an infinite number of solutions. We must use other methods to determine the exact nature of the solutions when $D = 0$.

EXAMPLE 1 Solving a System with Cramer's Rule

Solve using Cramer's rule:
$$\begin{aligned} 3x - 5y &= 2 \\ -4x + 3y &= -1 \end{aligned}$$

SOLUTIONS

Algebraic Solution

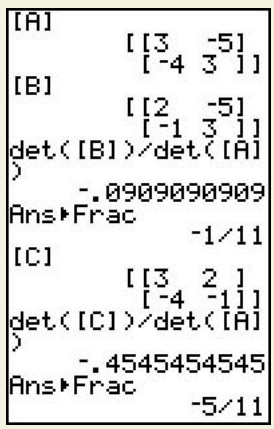
$$D = \begin{vmatrix} 3 & -5 \\ -4 & 3 \end{vmatrix} = -11$$

$$x = \frac{\begin{vmatrix} 2 & -5 \\ -1 & 3 \end{vmatrix}}{-11} = -\frac{1}{11}$$

$$y = \frac{\begin{vmatrix} 3 & 2 \\ -4 & -1 \end{vmatrix}}{-11} = -\frac{5}{11}$$

Graphing Calculator Solution

Store the coefficient matrix in A, the matrix with the constants in the first column in B, and the matrix with the constants in the second column in C. Then use the DET command to apply Cramer's rule (Fig. 1).



› Figure 1

MATCHED PROBLEM 1

Solve using Cramer's rule:
$$\begin{aligned} 3x + 2y &= -4 \\ -4x + 3y &= -10 \end{aligned}$$

››› EXPLORE-DISCUSS 1

Recall that a system of linear equations must have zero, one, or an infinite number of solutions. Discuss the number of solutions for the system

$$\begin{aligned} ax + 3y &= b \\ 4x + 2y &= 8 \end{aligned}$$

where a and b are real numbers. Use Cramer's rule where appropriate and Gauss–Jordan elimination otherwise.

› Solving Systems of Three Equations in Three Variables

Cramer's rule can be generalized completely for any size linear system that has the *same number of variables as equations*. However, it cannot be used to solve systems where the number of variables is not equal to the number of equations. In Theorem 2 we state without proof Cramer's rule for three equations and three variables.

› THEOREM 2 Cramer's Rule for Three Equations and Three Variables

Given the system

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= k_1 \\ a_{21}x + a_{22}y + a_{23}z &= k_2 \\ a_{31}x + a_{32}y + a_{33}z &= k_3 \end{aligned} \quad \text{with} \quad D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

then

$$x = \frac{\begin{vmatrix} k_1 & a_{12} & a_{13} \\ k_2 & a_{22} & a_{23} \\ k_3 & a_{32} & a_{33} \end{vmatrix}}{D} \qquad y = \frac{\begin{vmatrix} a_{11} & k_1 & a_{13} \\ a_{21} & k_2 & a_{23} \\ a_{31} & k_3 & a_{33} \end{vmatrix}}{D} \qquad z = \frac{\begin{vmatrix} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \\ a_{31} & a_{32} & k_3 \end{vmatrix}}{D}$$

You can easily remember these determinant formulas for x, y, and z if you observe the following:

1. Determinant D is formed from the coefficients of x, y, and z, keeping the same relative position in the determinant as found in the system of equations.
2. Determinant D appears in the denominators for x, y, and z.
3. The numerator for x can be obtained from D by replacing the coefficients of x (a_{11}, a_{21}, and a_{31}) with the constants k_1, k_2, and k_3, respectively. Similar statements can be made for the numerators for y and z.

EXAMPLE 2 Solving a System with Cramer's Rule

Solve using Cramer's rule:
$$\begin{aligned} x + y &= 2 \\ 3y - z &= -4 \\ x + z &= 3 \end{aligned}$$

SOLUTIONS

Algebraic Solution

$$D = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 2$$

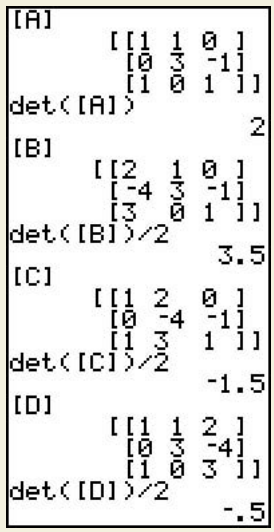
$$x = \frac{\begin{vmatrix} 2 & 1 & 0 \\ -4 & 3 & -1 \\ 3 & 0 & 1 \end{vmatrix}}{2} = \frac{7}{2}$$

$$y = \frac{\begin{vmatrix} 1 & 2 & 0 \\ 0 & -4 & -1 \\ 1 & 3 & 1 \end{vmatrix}}{2} = -\frac{3}{2}$$

$$z = \frac{\begin{vmatrix} 1 & 1 & 2 \\ 0 & 3 & -4 \\ 1 & 0 & 3 \end{vmatrix}}{2} = -\frac{1}{2}$$

Graphing Calculator Solution

Store the coefficient matrix in A, and the matrix with the constants replacing the coefficients of x, y, and z in B, C, and D, respectively. Then use the DET command to apply Cramer's rule (Fig. 2).



› Figure 2

MATCHED PROBLEM 2

Solve using Cramer's rule:
$$\begin{aligned} 3x - z &= 5 \\ x - y + z &= 0 \\ x + y &= 1 \end{aligned}$$

In practice, Cramer's rule is rarely used to solve systems of order higher than 2 or 3 by hand, because more efficient methods are available using computer methods. However, Cramer's rule is a valuable tool in more advanced theoretical and applied mathematics.

ANSWERS TO MATCHED PROBLEMS

1. $x = \frac{8}{17}, y = -\frac{46}{17}$ **2.** $x = \frac{6}{5}, y = -\frac{1}{5}, z = -\frac{7}{5}$

9-7 Exercises

1. If A is the 2×2 coefficient matrix for a linear system and $\det(A) = 0$, what can you conclude about the solution set for the system?

2. Can you use Cramer's rule to solve a linear system with a 3×2 coefficient matrix? Explain.

3. Can you use Cramer's rule to solve a linear system with a 2×3 coefficient matrix? Explain.

4. List all the possible solution methods for linear systems that we have discussed in Chapters 8 and 9. Which is your favorite and why?

Solve Problems 5–12 using Cramer's rule.

5. $\begin{aligned} x + 2y &= 1 \\ x + 3y &= -1 \end{aligned}$

6. $\begin{aligned} x + 2y &= 3 \\ x + 3y &= 5 \end{aligned}$

7. $\begin{aligned} 2x + y &= 1 \\ 5x + 3y &= 2 \end{aligned}$

8. $\begin{aligned} x + 3y &= 1 \\ 2x + 8y &= 0 \end{aligned}$

9. $\begin{aligned} 2x - y &= -3 \\ -x + 3y &= 3 \end{aligned}$

10. $\begin{aligned} -3x + 2y &= 1 \\ 2x - 3y &= -3 \end{aligned}$

11. $\begin{aligned} 4x - 3y &= 4 \\ 3x + 2y &= -2 \end{aligned}$

12. $\begin{aligned} 5x + 2y &= -1 \\ 2x - 3y &= 2 \end{aligned}$

Solve Problems 13–16 to two significant digits using Cramer's rule.

13. $\begin{aligned} 0.9925x - 0.9659y &= 0 \\ 0.1219x + 0.2588y &= 2{,}500 \end{aligned}$

14. $\begin{aligned} 0.9877x - 0.9744y &= 0 \\ 0.1564x + 0.2250y &= 1{,}900 \end{aligned}$

15. $\begin{aligned} 0.9954x - 0.9942y &= 0 \\ 0.0958x + 0.1080y &= 155 \end{aligned}$

16. $\begin{aligned} 0.9973x - 0.9957y &= 0 \\ 0.0732x + 0.0924y &= 112 \end{aligned}$

Solve Problems 17–24 using Cramer's rule:

17. $\begin{aligned} x + y &= 0 \\ 2y + z &= -5 \\ -x + z &= -3 \end{aligned}$

18. $\begin{aligned} x + y &= -4 \\ 2y + z &= 0 \\ -x + z &= 5 \end{aligned}$

19. $\begin{aligned} x + y &= 1 \\ 2y + z &= 0 \\ -y + z &= 1 \end{aligned}$

20. $\begin{aligned} x + 3y &= -3 \\ 2y + z &= 3 \\ -x + 3z &= 7 \end{aligned}$

21. $\begin{aligned} 3y + z &= -1 \\ x + 2z &= 3 \\ x - 3y &= -2 \end{aligned}$

22. $\begin{aligned} x - z &= 3 \\ 2x - y &= -3 \\ x + y + z &= 1 \end{aligned}$

23. $\begin{aligned} 2y - z &= -3 \\ x - y - z &= 2 \\ x - y + 2z &= 4 \end{aligned}$

24. $\begin{aligned} 2x + y &= 2 \\ x - y + z &= -1 \\ x + y + z &= 2 \end{aligned}$

Discuss the number of solutions for the systems in Problems 25 and 26 where a and b are real numbers. Use Cramer's rule where appropriate and Gauss–Jordan elimination otherwise.

25. $\begin{aligned} ax + 3y &= b \\ 2x + 4y &= 5 \end{aligned}$

26. $\begin{aligned} 2x + ay &= b \\ 3x + 4y &= 7 \end{aligned}$

In Problems 27 and 28, use Cramer's rule to solve for x only.

27. $\begin{aligned} 2x - 3y + z &= -3 \\ -4x + 3y + 2z &= -11 \\ x - y - z &= 3 \end{aligned}$

28. $\begin{aligned} x + 4y - 3z &= 25 \\ 3x + y - z &= 2 \\ -4x + y + 2z &= 1 \end{aligned}$

In Problems 29 and 30, use Cramer's rule to solve for y only.

29. $\begin{aligned} 12x - 14y + 11z &= 5 \\ 15x + 7y - 9z &= -13 \\ 5x - 3y + 2z &= 0 \end{aligned}$

30. $\begin{aligned} 2x - y + 4z &= 15 \\ -x + y + 2z &= 5 \\ 3x + 4y - 2z &= 4 \end{aligned}$

In Problems 31 and 32, use Cramer's rule to solve for z only.

31.
$$\begin{aligned} 3x - 4y + 5z &= 18 \\ -9x + 8y + 7z &= -13 \\ 5x - 7y + 10z &= 33 \end{aligned}$$

32.
$$\begin{aligned} 13x + 11y + 10z &= 2 \\ 10x + 8y + 7z &= 1 \\ 8x + 5y + 4z &= 4 \end{aligned}$$

It is clear that $x = 0, y = 0, z = 0$ is a solution to each of the systems given in Problems 33 and 34. Use Cramer's rule to determine whether this solution is unique. [Hint: If $D \neq 0$, what can you conclude? If $D = 0$, what can you conclude?]

33.
$$\begin{aligned} x - 4y + 9z &= 0 \\ 4x - y + 6z &= 0 \\ x - y + 3z &= 0 \end{aligned}$$

34.
$$\begin{aligned} 3x - y + 3z &= 0 \\ 5x + 5y - 9z &= 0 \\ -2x + y - 3z &= 0 \end{aligned}$$

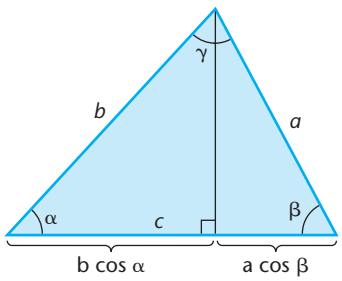
35. Prove Theorem 1 for y.

36. (Omit this problem if you have not studied trigonometry.) The angles α, β, and γ and the sides a, b, and c of a triangle (see the figure) satisfy

$$\begin{aligned} c &= b\cos\alpha + a\cos\beta \\ b &= c\cos\alpha + a\cos\gamma \\ a &= c\cos\beta + b\cos\gamma \end{aligned}$$

Use Cramer's rule to express $\cos\alpha$ in terms of a, b, and c, thereby deriving the familiar law of cosines from trigonometry:

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$



APPLICATIONS

37. **REVENUE ANALYSIS** A supermarket sells two brands of coffee: brand A at $\$p$ per pound and brand B at $\$q$ per pound. The daily demand equations for brands A and B are, respectively,

$$\begin{aligned} x &= 200 - 6p + 4q \\ y &= 300 + 2p - 3q \end{aligned} \tag{1}$$

(both in pounds). The daily revenue R is given by

$$R = xp + yq$$

(A) To analyze the effect of price changes on the daily revenue, an economist wants to express the daily revenue R in terms of p and q only. Use system (1) to eliminate x and y in the equation for R, thus expressing the daily revenue in terms of p and q.
(B) To analyze the effect of changes in demand on the daily revenue, the economist now wants to express the daily revenue in terms of x and y only. Use Cramer's rule to solve system (1) for p and q in terms of x and y and then express the daily revenue R in terms of x and y.

38. **REVENUE ANALYSIS** A company manufactures ten-speed and three-speed bicycles. The weekly demand equations are

$$\begin{aligned} p &= 230 - 10x + 5y \\ q &= 130 + 4x - 4y \end{aligned} \tag{2}$$

where $\$p$ is the price of a ten-speed bicycle, $\$q$ is the price of a three-speed bicycle, x is the weekly demand for ten-speed bicycles, and y is the weekly demand for three-speed bicycles. The weekly revenue R is given by

$$R = xp + yq$$

(A) Use system (2) to express the daily revenue in terms of x and y only.
(B) Use Cramer's rule to solve system (2) for x and y in terms of p and q, and then express the daily revenue R in terms of p and q only.

CHAPTER 9 Review

9-1 Systems of Linear Equations: Gauss–Jordan Elimination

The method of solution using elimination by addition can be transformed into a more efficient method for larger-scale systems by the introduction of an *augmented matrix*. A **matrix** is a rectangular array of numbers written within brackets. Each number in a matrix is called an **element** of the matrix. If a matrix has m rows and n columns, it is called an $\boldsymbol{m \times n}$ **matrix** (read "m by n matrix"). The expression $m \times n$ is called the **size** of the matrix, and the numbers m and n are called the **dimensions** of the matrix. A matrix with n rows and n columns is called a **square matrix of order n.** A matrix with only one column is called a **column matrix,** and a matrix with only one row is called a **row matrix.** The **position** of an element in a matrix is the row and column containing the element. This is usually denoted using **double subscript notation** a_{ij}, where i is the row and j is the column containing the element a_{ij}. The **principal diagonal** of a matrix A consists of the

elements a_{ij}, $i = 1, 2, \ldots, n$. Rather than using x, y, and z to denote variables, we will use subscript notation x_1, x_2, and x_3.

Related to the system

$$\begin{aligned} x_1 + 5x_2 - 3x_3 &= 4 \\ 6x_1 \qquad - 4x_3 &= 1 \\ -2x_1 + 3x_2 + 4x_3 &= 7 \end{aligned}$$

are the following matrices:

Coefficient matrix

$$\begin{bmatrix} 1 & 5 & -3 \\ 6 & 0 & -4 \\ -2 & 3 & 4 \end{bmatrix}$$

Constant matrix

$$\begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix}$$

Augmented coefficient matrix

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & 4 \\ 6 & 0 & -4 & 1 \\ -2 & 3 & 4 & 7 \end{array}\right]$$

Two augmented matrices are **row-equivalent,** denoted by the symbol $\sim$ between the two matrices, if they are augmented matrices of equivalent systems of equations. An augmented matrix is transformed into a row-equivalent matrix if any of the following **row operations** is performed:

1. Two rows are interchanged.

2. A row is multiplied by a nonzero constant.

3. A constant multiple of another row is added to a given row.

The following symbols are used to describe these row operations:

1. $R_i \leftrightarrow R_j$ means "interchange row i with row j."

2. $kR_i \rightarrow R_i$ means "multiply row i by the constant k."

3. $kR_j + R_i \rightarrow R_i$ means "multiply row j by the constant k and add to R_i."

Our objective is to start with the augmented matrix of a linear system and transform it using row operations into a simple form where the solution can be read by inspection. The simple form, called the **reduced form,** is achieved if:

1. Each row consisting entirely of 0's is below any row having at least one nonzero element.

2. The leftmost nonzero element in each row is 1.

3. The column containing the leftmost 1 of a given row has 0's above and below the 1.

4. The leftmost 1 in any row is to the right of the leftmost 1 in the preceding row.

A **reduced system** is a system of linear equations that corresponds to a reduced augmented matrix. When a reduced system has more variables than equations and contains no contradictions, the system is dependent and has infinitely many solutions.

The **Gauss–Jordan elimination** procedure for solving a system of linear equations is given in step-by-step form as follows:

Step 1. Choose the leftmost nonzero column, and use appropriate row operations to get a 1 at the top.

Step 2. Use multiples of the row containing the 1 from step 1 to get zeros in all remaining places in the column containing this 1.

Step 3. Repeat step 1 with the **submatrix** formed by (mentally) deleting the row used in step 2 and all rows above this row.

Step 4. Repeat step 2 with the **entire matrix,** including the mentally deleted rows. Continue this process until the entire matrix is in reduced form.

If at any point in the preceding process we obtain a row with all 0's to the left of the vertical line and a nonzero number n to the right, we can stop, since we have a contradiction: $0 = n, n \neq 0$. We can then conclude that the system has no solution. If this does not happen and we obtain an augmented matrix in reduced form without any contradictions, the solution can be read by inspection.

9-2 Matrix Operations

Two matrices are **equal** if they are the same size and their corresponding elements are equal. The **sum of two matrices** of the same size is a matrix with elements that are the sums of the corresponding elements of the two given matrices. Matrix addition is **commutative** and **associative.** A matrix with all zero elements is called the **zero matrix.** The **negative of a matrix M,** denoted $-M$, is a matrix with elements that are the negatives of the elements in M. If A and B are matrices of the same size, then we define **subtraction** as follows: $A - B = A + (-B)$. The **product of a number k and a matrix M,** denoted by kM, is a matrix formed by multiplying each element of M by k. The **product** of a $1 \times n$ row matrix and an $n \times 1$ column matrix is a 1×1 matrix given by

$$\underset{1 \times n}{[a_1\, a_2 \ldots a_n]} \underset{n \times 1}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}} = \underset{1 \times 1}{[a_1 b_1 + a_2 b_2 + \cdots + a_n b_n]}$$

If A is an $m \times p$ matrix and B is a $p \times n$ matrix, then the **matrix product** of A and B, denoted AB, is an $m \times n$ matrix whose element in the ith row and jth column is the real number obtained from the product of the ith row of A and the jth column of B. If the number of columns in A does not equal the number of rows in B, then the matrix product AB is **not defined. Matrix multiplication is not commutative,** and the **zero property does not hold for matrix multiplication.** That is, for matrices A and B, the matrix product AB can be zero without either A or B being the zero matrix.

9-3 Inverse of a Square Matrix

The **identity matrix for multiplication** for the set of all **square matrices of order *n*** is the square matrix of order n, denoted by I, with ones along the **principal diagonal** (from upper left corner to lower right corner) and zeros elsewhere. If A is a square matrix of order n and I is the identity matrix of order n, then

$$IA = AI = A$$

If A is a square matrix of order n and if there exists a matrix A^{-1} (read "A inverse") such that

$$A^{-1}A = AA^{-1} = I$$

then A^{-1} is called the **multiplicative inverse of *A*** or, more simply, the **inverse of *A*.** If the augmented matrix $[A|I]$ is transformed by row operations into $[I|B]$, then the resulting matrix B is A^{-1}. If, however, we obtain all zeros in one or more rows to the left of the vertical line, then A^{-1} does not exist and A is called a **singular matrix.**

9-4 Matrix Equations and Systems of Linear Equations

The following properties of matrices are fundamental to the process of solving matrix equations. Assuming all products and sums are defined for the indicated matrices A, B, C, I, and 0, then:

Addition Properties

Associative:	$(A + B) + C = A + (B + C)$
Commutative:	$A + B = B + A$
Additive Identity:	$A + 0 = 0 + A = A$
Additive Inverse:	$A + (-A) = (-A) + A = 0$

Multiplication Properties

Associative Property:	$A(BC) = (AB)C$
Multiplicative Identity:	$AI = IA = A$
Multiplicative Inverse:	If A is a square matrix and A^{-1} exists, then $AA^{-1} = A^{-1}A = I$.

Combined Properties

Left Distributive:	$A(B + C) = AB + AC$
Right Distributive:	$(B + C)A = BA + CA$

Equality

Addition:	If $A = B$, then $A + C = B + C$.
Left Multiplication:	If $A = B$, then $CA = CB$.
Right Multiplication:	If $A = B$, then $AC = BC$.

A system of linear equations with the same number of variables as equations such as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= k_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= k_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= k_3 \end{aligned}$$

can be written as the matrix equation

$$\overset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} \overset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \overset{B}{\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}}$$

If the inverse of A exists, then the matrix equation has a unique solution given by

$$X = A^{-1}B$$

After multiplying B by A^{-1} from the left, it is easy to read the solution to the original system of equations.

9-5 Determinants

Associated with each square matrix A is a real number called the **determinant** of the matrix. The determinant of A is denoted by **det *A*,** or simply by writing the array of elements in A using vertical lines in place of square brackets. For example,

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

A **determinant of order *n*** is a determinant with n rows and n columns.

The **value of a second-order determinant** is the real number given by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

The **value of a third-order determinant** is the sum of three products obtained by multiplying each element of any one row (or each element of any one column) by its cofactor. The **cofactor of an element** a_{ij} (from the ith row and jth column) is the product of the minor of a_{ij} and $(-1)^{i+j}$. The **minor of an element** a_{ij} is the determinant remaining after deleting the ith row and jth column. A similar process can be used to evaluate determinants of order higher than 3.

9-6 Properties of Determinants

The use of the following five determinant properties can greatly reduce the effort in evaluating determinants of order 3 or greater:

1. If each element of any row (or column) of a determinant is multiplied by a constant k, the new determinant is k times the original.

$$\begin{vmatrix} 2a & 2b \\ c & d \end{vmatrix} = 2\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$3\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 3a & b \\ 3c & d \end{vmatrix}$$

2. If every element in a row (or column) is 0, the value of the determinant is 0.

$$\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} = 0$$

3. If two rows (or two columns) of a determinant are interchanged, the new determinant is the negative of the original.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} b & a \\ d & c \end{vmatrix}$$

4. If the corresponding elements are equal in two rows (or columns), the value of the determinant is 0.

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$$

$$\begin{vmatrix} a & a \\ c & c \end{vmatrix} = 0$$

5. If a multiple of any row (or column) of a determinant is added to any other row (or column), the value of the determinant is not changed.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c + ka & d + kb \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a + kb & b \\ c + kd & d \end{vmatrix}$$

9-7 Determinants and Cramer's Rule

Systems of equations having the same number of variables as equations can also be solved using determinants and Cramer's rule. **Cramer's rule for three equations and three variables** is as follows: Given the system

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= k_1 \\ a_{21}x + a_{22}y + a_{23}z &= k_2 \\ a_{31}x + a_{32}y + a_{33}z &= k_3 \end{aligned} \quad \text{with} \quad D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

then

$$x = \frac{\begin{vmatrix} k_1 & a_{12} & a_{13} \\ k_2 & a_{22} & a_{23} \\ k_3 & a_{32} & a_{33} \end{vmatrix}}{D} \quad y = \frac{\begin{vmatrix} a_{11} & k_1 & a_{13} \\ a_{21} & k_2 & a_{23} \\ a_{31} & k_3 & a_{33} \end{vmatrix}}{D} \quad z = \frac{\begin{vmatrix} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \\ a_{31} & a_{32} & k_3 \end{vmatrix}}{D}$$

Cramer's rule can be generalized completely for any size linear system that has the same number of variables as equations. The formulas are easily remembered if you observe the following:

1. Determinant D is formed from the coefficients of x, y, and z, keeping the same relative position in the determinant as found in the system of equations.

2. Determinant D appears in the denominators for x, y, and z.

3. The numerator for x can be obtained from D by replacing the coefficients of x (a_{11}, a_{21}, and a_{31}) with the constants k_1, k_2, and k_3, respectively. Similar statements can be made for the numerators for y and z.

Cramer's rule is rarely used to solve systems of order higher than 3 by hand, because more efficient methods are available. Cramer's rule, however, is a valuable tool in more advanced theoretical and applied mathematics.

CHAPTER 9 Review Exercises

Work through all the problems in this chapter review and check answers in the back of the book. Answers to all review problems are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

Perform each of the row operations indicated in Problems 1–3 on the following augmented matrix:

$$\left[\begin{array}{cc|c} 1 & -4 & 5 \\ 3 & -6 & 12 \end{array}\right]$$

1. $R_1 \leftrightarrow R_2$ **2.** $\frac{1}{3}R_2 \rightarrow R_2$ **3.** $(-3)R_1 + R_2 \rightarrow R_2$

In Problems 4–6, write the linear system corresponding to each reduced augmented matrix and solve.

4. $\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -7 \end{array}\right]$ **5.** $\left[\begin{array}{cc|c} 1 & -1 & 4 \\ 0 & 0 & 1 \end{array}\right]$

6. $\left[\begin{array}{cc|c} 1 & -1 & 4 \\ 0 & 0 & 0 \end{array}\right]$

In Problems 7–15, perform the operations that are defined, given the following matrices:

$$A = \begin{bmatrix} 4 & -2 \\ 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 5 \\ -4 & 6 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 4 \end{bmatrix} \quad D = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

7. AB **8.** CD **9.** CB

10. AD **11.** $A + B$ **12.** $C + D$

13. $A + C$ **14.** $2A - 5B$ **15.** $CA + C$

16. Find the inverse of

$$A = \begin{bmatrix} 4 & 7 \\ -1 & -2 \end{bmatrix}$$

Show that $A^{-1}A = I$.

17. Write the system

$$\begin{aligned} 3x_1 + 2x_2 &= k_1 \\ 4x_1 + 3x_2 &= k_2 \end{aligned}$$

as a matrix equation, and solve using matrix inverse methods for:

(A) $k_1 = 3, k_2 = 5$

(B) $k_1 = 7, k_2 = 10$

(C) $k_1 = 4, k_2 = 2$

Evaluate the determinants in Problems 18 and 19.

18. $\begin{vmatrix} 2 & -3 \\ -5 & -1 \end{vmatrix}$ **19.** $\begin{vmatrix} 2 & 3 & -4 \\ 0 & 5 & 0 \\ 1 & -4 & -2 \end{vmatrix}$

20. Solve the system using Cramer's rule:

$$\begin{aligned} 3x - 2y &= 8 \\ x + 3y &= -1 \end{aligned}$$

21. Use properties of determinants to find each of the following, given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2$$

(A) $\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix}$ (B) $\begin{vmatrix} a & 3b & c \\ d & 3e & f \\ g & 3h & i \end{vmatrix}$

(C) $\begin{vmatrix} a & b & a+b+c \\ d & e & d+e+f \\ g & h & g+h+i \end{vmatrix}$

22. Use Gauss–Jordan elimination to solve the system

$$\begin{aligned} x_1 - x_2 &= 4 \\ 2x_1 + x_2 &= 2 \end{aligned}$$

Then write the linear system represented by each augmented matrix in your solution, and solve each of these systems graphically. Discuss the relationship between the solutions of these systems.

Solve Problems 23–28 using Gauss–Jordan elimination.

23. $\begin{aligned} 3x_1 + 2x_2 &= 3 \\ x_1 + 3x_2 &= 8 \end{aligned}$

24. $\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - x_3 &= -2 \\ x_2 + 2x_3 &= 4 \end{aligned}$

25. $\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 3x_2 + 4x_3 &= 3 \\ x_1 + 2x_2 + x_3 &= 3 \end{aligned}$

26. $\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= -3 \\ 3x_1 + 5x_2 &= -1 \end{aligned}$

27. $\begin{aligned} x_1 - 2x_2 &= 1 \\ 2x_1 - x_2 &= 0 \\ x_1 - 3x_2 &= -2 \end{aligned}$

28. $\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ 3x_1 - x_2 + 2x_3 &= -3 \end{aligned}$

In Problems 29–34, perform the operations that are defined, given the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ -3 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 6 \\ 0 \\ -4 \end{bmatrix} \quad C = [2 \quad 4 \quad -1]$$

$$D = \begin{bmatrix} 7 & 0 & -5 \\ 0 & 8 & -2 \end{bmatrix} \quad E = \begin{bmatrix} 9 & -3 \\ -6 & 2 \end{bmatrix}$$

29. AD **30.** DA **31.** BC

32. CB **33.** DE **34.** ED

35. Find the inverse of

$$A = \begin{bmatrix} 1 & 0 & 4 \\ -2 & 1 & 0 \\ 4 & -1 & 4 \end{bmatrix}$$

Show that $AA^{-1} = I$.

36. Write the system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= k_1 \\ 2x_1 + 3x_2 + 4x_3 &= k_2 \\ x_1 + 2x_2 + x_3 &= k_3 \end{aligned}$$

as a matrix equation, and solve using matrix inverse methods for:

(A) $k_1 = 1, k_2 = 3, k_3 = 3$

(B) $k_1 = 0, k_2 = 0, k_3 = -2$

(C) $k_1 = -3, k_2 = -4, k_3 = 1$

Evaluate the determinants in Problems 37 and 38.

37. $\begin{vmatrix} -\frac{1}{4} & \frac{3}{2} \\ \frac{1}{2} & \frac{2}{3} \end{vmatrix}$ **38.** $\begin{vmatrix} 2 & -1 & 1 \\ -3 & 5 & 2 \\ 1 & -2 & 4 \end{vmatrix}$

39. Solve for y only using Cramer's rule:

$$\begin{aligned} x - 2y + z &= -6 \\ y - z &= 4 \\ 2x + 2y + z &= 2 \end{aligned}$$

(Find the numerator and denominator first; then reduce.)

40. Discuss the number of solutions for a system of n equations in n variables if the coefficient matrix:

(A) Has an inverse (B) Does not have an inverse

41. If A is a nonzero square matrix of order n satisfying $A^2 = 0$, can A^{-1} exist? Explain.

42. Discuss the number of solutions for the system corresponding to the reduced form shown below if

(A) $m \neq 0$ (B) $m = 0$ and $n \neq 0$

(C) $m = 0$ and $n = 0$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & m & n \end{array}\right]$$

43. For $n \times n$ matrices A and C and $n \times 1$ column matrices B and X, solve for X assuming all necessary inverses exist:

$$AX - B = CX$$

44. Find the inverse of

$$A = \begin{bmatrix} 4 & 5 & 6 \\ 4 & 5 & -6 \\ 1 & 1 & 1 \end{bmatrix}$$

Show that $A^{-1}A = I$.

45. Clear the decimals in the system

$$\begin{aligned} 0.04x_1 + 0.05x_2 + 0.06x_3 &= 360 \\ 0.04x_1 + 0.05x_2 - 0.06x_3 &= 120 \\ x_1 + x_2 + x_3 &= 7{,}000 \end{aligned}$$

by multiplying the first two equations by 100. Then write the resulting system as a matrix equation and solve using the inverse found in Problem 44.

46. $\begin{vmatrix} -1 & 4 & 1 & 1 \\ 5 & -1 & 2 & -1 \\ 2 & -1 & 0 & 3 \\ -3 & 3 & 0 & 3 \end{vmatrix} = ?$

47. Show that

$$\begin{vmatrix} u & v \\ w & x \end{vmatrix} = \begin{vmatrix} u + kv & v \\ w + kx & x \end{vmatrix}$$

48. Explain why the points (1, 2) and (−1, 5) must satisfy the equation

$$\begin{vmatrix} x & y & 1 \\ 1 & 2 & 1 \\ -1 & 5 & 1 \end{vmatrix} = 0$$

Describe the set of all points that satisfy this equation.

APPLICATIONS

49. DIET A laboratory assistant wishes to obtain a food mix that contains, among other things, 27 grams of protein, 5.4 grams of fat, and 19 grams of moisture. He has available mixes A, B, and C with the compositions listed in the table. How many grams of each mix should be used to get the desired diet mix? Set up a system of equations and solve using Gauss–Jordan elimination.

Mix	Protein (%)	Fat (%)	Moisture (%)
A	30	3	10
B	20	5	20
C	10	4	10

50. PUZZLE A piggy bank contains 30 coins worth $1.90.

(A) If the bank contains only nickels and dimes, how many coins of each type does it contain?

(B) If the bank contains nickels, dimes, and quarters, how many coins of each type does it contain?

51. RESOURCE ALLOCATION A Colorado mining company operates mines at Big Bend and Saw Pit. The Big Bend mine produces ore that is 5% nickel and 7% copper. The Saw Pit mine produces ore that is 3% nickel and 4% copper. How many tons of ore should be produced at each mine to obtain the amounts of nickel and copper listed in the table? Set up a matrix equation and solve using matrix inverses.

	Nickel	Copper
(A)	3.6 tons	5 tons
(B)	3 tons	4.1 tons
(C)	3.2 tons	4.4 tons

52. LABOR COSTS A company with manufacturing plants in North Carolina and South Carolina has labor-hour and wage requirements for the manufacturing of computer desks and printer stands as given in matrices L and H:

$$L = \begin{array}{c} \\ \text{Desk} \\ \text{Stand} \end{array} \begin{array}{c} \begin{array}{ccc} \multicolumn{3}{c}{\text{Labor-hour requirements}} \\ \text{Fabricating dept.} & \text{Assembly dept.} & \text{Packaging dept.} \end{array} \\ \begin{bmatrix} 1.7\text{ h} & 2.4\text{ h} & 0.8\text{ h} \\ 0.9\text{ h} & 1.8\text{ h} & 0.6\text{ h} \end{bmatrix} \end{array}$$

$$H = \begin{array}{c} \\ \text{Fabricating department} \\ \text{Assembly department} \\ \text{Packaging department} \end{array} \begin{array}{c} \begin{array}{cc} \multicolumn{2}{c}{\text{Hourly wages}} \\ \text{North Carolina plant} & \text{South Carolina plant} \end{array} \\ \begin{bmatrix} \$11.50 & \$10.00 \\ \$9.50 & \$8.50 \\ \$5.00 & \$4.50 \end{bmatrix} \end{array}$$

(A) Find the labor cost for producing one printer stand at the South Carolina plant.

(B) Discuss possible interpretations of the elements in the matrix products HL and LH.

(C) If either of the products HL or LH has a meaningful interpretation, find the product and label its rows and columns.

53. **LABOR COSTS** The monthly production of computer desks and printer stands for the company in Problem 52 for the months of January and February is given in matrices J and F:

$$J = \begin{matrix} \\ \text{Desks} \\ \text{Stands} \end{matrix} \begin{matrix} \begin{matrix} \text{January production} \\ \text{North Carolina plant} \end{matrix} & \begin{matrix} \\ \text{South Carolina plant} \end{matrix} \\ \begin{bmatrix} 1{,}500 & 1{,}650 \\ 850 & 700 \end{bmatrix} \end{matrix}$$

$$F = \begin{matrix} \\ \text{Desks} \\ \text{Stands} \end{matrix} \begin{matrix} \begin{matrix} \text{February production} \\ \text{North Carolina plant} \end{matrix} & \begin{matrix} \\ \text{South Carolina plant} \end{matrix} \\ \begin{bmatrix} 1{,}700 & 1{,}810 \\ 930 & 740 \end{bmatrix} \end{matrix}$$

(A) Find the average monthly production for the months of January and February.

(B) Find the increase in production from January to February.

(C) Find $J\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and interpret.

54. **CRYPTOGRAPHY** The following message was encoded with the matrix B shown. Decode the message:

25 8 26 24 25 33 21 41 48 41 30 50
21 32 41 52 52 79

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

CHAPTER 9

››› GROUP ACTIVITY Using Matrices to Find Cost, Revenue, and Profit

A toy distributor purchases model train components from various suppliers and packages these components in three different ready-to-run train sets: the Limited, the Empire, and the Comet. The components used in each set are listed in Table 1. For convenience, the total labor time (in minutes) required to prepare a set for shipping is included as a component.

Table 1 Product Components

	Train sets		
Components	**Limited**	**Empire**	**Comet**
Locomotives	1	1	2
Cars	5	6	8
Track pieces	20	24	32
Track switches	1	2	4
Power pack	1	1	1
Labor (min)	15	18	24

The current costs of the components are given in Table 2, and the distributor's selling prices for the sets are given in Table 3.

Table 2 Component Costs

Components	Cost per unit ($)
Locomotive	12.52
Car	1.43
Track piece	0.25
Track switch	2.29
Power pack	12.54
Labor (per min)	0.15

Table 3 Selling Prices

Set	Price
Limited	$54.60
Empire	$62.28
Comet	$81.15

The distributor has just received the order shown in Table 4 from a retail toy store.

Table 4 Customer Order

Set	Quantity
Limited	48
Empire	24
Comet	12

The distributor wants to store the information in each table in a matrix and use matrix operations to find the following information:

1. The inventory (parts and labor) required to fill the order
2. The cost (parts and labor) of filling the order
3. The revenue (sales) received from the customer
4. The profit realized on the order

(A) Use a single letter to designate the matrix representing each table, and write matrix expressions in terms of these letters that will provide the required information. Discuss the size of the matrix you must use to represent each table so that all the pertinent matrix operations are defined.

(B) Evaluate the matrix expressions in part A.

Shortly after filling the order in Table 4, a supplier informs the distributor that the cars and locomotives used in these train sets are no longer available. The distributor currently has 30 locomotives and 134 cars in stock.

(C) How many train sets of each type can the distributor produce using all the available locomotives and cars? Assume that the distributor has unlimited quantities of the other components used in these sets.

(D) How much profit will the distributor make if all these sets are sold? If there is more than one way to use all the available locomotives and cars, which one will produce the largest profit?

CHAPTERS 8–9 Cumulative Review

Work through all the problems in this cumulative review and check answers in the back of the book. Answers to all review problems are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

1. Solve using substitution or elimination by addition:

$$\begin{aligned} 3x - 5y &= 11 \\ 2x + 3y &= 1 \end{aligned}$$

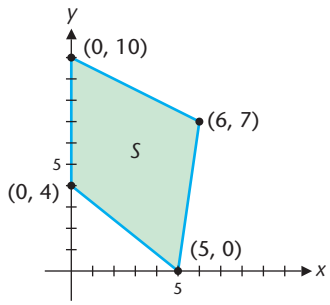
2. Solve by graphing:
$$\begin{aligned} 2x - y &= -4 \\ 3x + y &= -1 \end{aligned}$$

3. Solve by substitution or elimination by addition:

$$\begin{aligned} -6x + 3y &= 2 \\ 2x - y &= 1 \end{aligned}$$

4. Solve by graphing:
$$\begin{aligned} 3x + 5y &\le 15 \\ x, y &\ge 0 \end{aligned}$$

5. Find the maximum and minimum value of $z = 2x + 3y$ over the feasible region S:



6. Perform the operations that are defined, given the following matrices:

$$M = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 2 \end{bmatrix} \qquad Q = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

(A) $M - 2N$ (B) $P + Q$ (C) PQ

(D) MN (E) PN (F) QM

7. Evaluate: $\begin{vmatrix} 0 & 2 & 0 \\ 1 & 3 & 2 \\ -1 & 4 & 3 \end{vmatrix}$

8. Write the linear system corresponding to each augmented matrix and solve:

(A) $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -4 \end{array}\right]$ (B) $\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 0 & 0 \end{array}\right]$

(C) $\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 0 & 1 \end{array}\right]$

9. Given the system:
$$\begin{aligned} x_1 + x_2 &= 3 \\ -x_1 + x_2 &= 5 \end{aligned}$$

(A) Write the augmented matrix for the system.

(B) Transform the augmented matrix into reduced form.

(C) Write the solution to the system.

10. Given the system:
$$\begin{aligned} x_1 - 3x_2 &= k_1 \\ 2x_1 - 5x_2 &= k_2 \end{aligned}$$

(A) Write the system as a matrix equation of the form $AX = B$.

(B) Find the inverse of the coefficient matrix A.

(C) Use A^{-1} to find the solution for $k_1 = -2$ and $k_2 = 1$.

(D) Use A^{-1} to find the solution for $k_1 = 1$ and $k_2 = -2$.

11. Given the system:
$$\begin{aligned} 2x - 3y &= 1 \\ 4x - 5y &= 2 \end{aligned}$$

(A) Find the determinant of the coefficient matrix.

(B) Solve the system using Cramer's rule.

12. Use Gauss–Jordan elimination to solve the system

$$\begin{aligned} x_1 + 3x_2 &= 10 \\ 2x_1 - x_2 &= -1 \end{aligned}$$

Then write the linear system represented by each augmented matrix in your solution, and solve each of these systems graphically. Discuss the relationship between the solutions of these systems.

13. Solve graphically to two decimal places:

$$\begin{aligned} -2x + 3y &= 7 \\ 3x + 4y &= 18 \end{aligned}$$

Solve Problems 14–16 using Gauss–Jordan elimination.

14. $\begin{aligned} x_1 + 2x_2 - x_3 &= 3 \\ x_2 + x_3 &= -2 \\ 2x_1 + 3x_2 + x_3 &= 0 \end{aligned}$ **15.** $\begin{aligned} x_1 + x_2 - x_3 &= 2 \\ 4x_2 + 6x_3 &= -1 \\ 6x_2 + 9x_3 &= 0 \end{aligned}$

16. $\begin{aligned} x_1 - 2x_2 + x_3 &= 1 \\ 3x_1 - 2x_2 - x_3 &= -5 \end{aligned}$

17. Given $M = [1 \quad 2 \quad -1]$ and $N = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. Find:

(A) MN (B) NM

18. Given

$$L = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \quad M = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & 1 \end{bmatrix} \quad N = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

Find, if defined: (A) $LM - 2N$ (B) $ML + N$

19. Graph the solution region and indicate whether the solution region is bounded or unbounded. Find the coordinates of each corner point.

$$\begin{aligned} 3x + 2y &\geq 12 \\ x + 2y &\geq 8 \\ x, y &\geq 0 \end{aligned}$$

20. Solve the linear programming problem:

$$\begin{aligned} \text{Maximize} \quad & z = 4x + 9y \\ \text{Subject to} \quad & x + 2y \leq 14 \\ & 2x + y \leq 16 \\ & x, y \geq 0 \end{aligned}$$

21. Given the system: $\begin{aligned} x_1 + 4x_2 + 2x_3 &= k_1 \\ 2x_1 + 6x_2 + 3x_3 &= k_2 \\ 2x_1 + 5x_2 + 2x_3 &= k_3 \end{aligned}$

(A) Write the system as a matrix equation in the form $AX = B$.

(B) Find the inverse of the coefficient matrix A.

(C) Use A^{-1} to solve the system when $k_1 = -1$, $k_2 = 2$, and $k_3 = 1$.

(D) Use A^{-1} to solve the system when $k_1 = 2$, $k_2 = 0$, and $k_3 = -1$.

22. Given the system: $\begin{aligned} x + 2y - z &= 1 \\ 2x + 8y + z &= -2 \\ -x + 3y + 5z &= 2 \end{aligned}$

(A) Evaluate the coefficient determinant D.

(B) Solve for z using Cramer's rule.

23. Discuss the number of solutions for the system corresponding to the reduced form shown below if

(A) $m = 0$ and $n = 0$ (B) $m = 0$ and $n \neq 0$

(C) $m \neq 0$

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & m & n \end{array}\right]$$

24. If a square matrix A satisfies the equation $A^2 = A$, find A. Assume that A^{-1} exists.

25. Which of the following augmented matrices are in reduced form?

$$L = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad M = \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$N = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right] \quad P = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & -2 \\ 0 & 0 & 1 & 3 & 1 \end{array}\right]$$

26. Show that

$$k\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} ka & b \\ kc & d \end{vmatrix}$$

27. Show that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c + ka & d + kb \end{vmatrix}$$

28. If $M = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and $\det M \neq 0$, show that

$$M^{-1} = \frac{1}{\det M}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Recall that a square matrix is called ***upper triangular*** *if all elements below the principal diagonal are zero, and it is called* ***diagonal*** *if all elements not on the principal diagonal are zero. A square matrix is called* ***lower triangular*** *if all elements above the principal diagonal are zero. In Problems 29–36, determine whether the statement is true or false. If true, explain why. If false, give a counterexample.*

29. The sum of two upper triangular matrices is upper triangular.

30. The product of two lower triangular matrices is lower triangular.

31. The sum of an upper triangular matrix and a lower triangular matrix is a diagonal matrix.

32. The product of an upper triangular matrix and a lower triangular matrix is a diagonal matrix.

33. A matrix that is both upper triangular and lower triangular is a diagonal matrix.

34. If a diagonal matrix has no zero elements on the principal diagonal, then it has an inverse.

35. The determinant of a diagonal matrix is the product of the elements on the principal diagonal.

36. The determinant of a lower triangular matrix is the product of the elements on the principal diagonal.

APPLICATIONS

37. **FINANCE** An investor has $12,000 to invest. If part is invested at 8% and the rest in a higher-risk investment at 14%, how much should be invested at each rate to produce the same yield as if all had been invested at 10%?

38. **DIET** In an experiment involving mice, a zoologist needs a food mix that contains, among other things, 23 grams of protein, 6.2 grams of fat, and 16 grams of moisture. She has on hand mixes of the following compositions: Mix A contains 20% protein, 2% fat, and 15% moisture; mix B contains 10% protein, 6% fat, and 10% moisture; and mix C contains 15% protein, 5% fat, and 5% moisture. How many grams of each mix should be used to get the desired diet mix?

39. **PURCHASING** A soft-drink distributor has budgeted $300,000 for the purchase of 12 new delivery trucks. If a model A truck costs $18,000, a model B truck costs $22,000, and a model C truck costs $30,000, how many trucks of each model should the distributor purchase to use exactly all the budgeted funds?

40. **MANUFACTURING** A manufacturer makes two types of day packs, a standard model and a deluxe model. Each standard model requires 0.5 labor-hour from the fabricating department and 0.3 labor-hour from the sewing department. Each deluxe model requires 0.5 labor-hour from the fabricating department and 0.6 labor-hour from the sewing department. The maximum number of labor-hours available per week in the fabricating department and the sewing department are 300 and 240, respectively.

(A) If the profit on a standard day pack is $8 and the profit on a deluxe day pack is $12, how many of each type of pack should be manufactured each day to realize a maximum profit? What is the maximum profit?

(B) Discuss the effect on the production schedule and the maximum profit if the profit on a standard day pack decreases by $3 and the profit on a deluxe day pack increases by $3.

(C) Discuss the effect on the production schedule and the maximum profit if the profit on a standard day pack increases by $3 and the profit on a deluxe day pack decreases by $3.

41. **AVERAGING TESTS** A teacher has given four tests to a class of five students and stored the results in the following matrix:

	Tests 1	2	3	4
Ann	78	84	81	86
Bob	91	65	84	92
Carol	95	90	92	91
Dan	75	82	87	91
Eric	83	88	81	76

$= M$

Discuss methods of matrix multiplication that the teacher can use to obtain the indicated information in parts A–C below. In each case, state the matrices to be used and then perform the necessary multiplications.

(A) The average on all four tests for each student, assuming that all four tests are given equal weight

(B) The average on all four tests for each student, assuming that the first three tests are given equal weight and the fourth is given twice this weight

(C) The class average on each of the four tests

Sequences, Induction, and Probability

THE lists

$$1, 4, 9, 16, 25, 36, 49, 64, \ldots$$

and

$$3, 6, 3, 1, 4, 2, 1, 4, \ldots$$

are examples of sequences. In the first sequence, a pattern is noticeable: You probably recognize it as the sequence of perfect squares. Its terms are increasing, and as we will see, the differences between terms form a clear pattern. You probably don't recognize the second sequence because the terms don't suggest an obvious pattern. In fact, we obtained the second sequence by recording the results of repeatedly tossing a single die. Sequences, and the related concept of *series,* are useful tools in almost all areas of mathematics. In this chapter, they will play roles in the development of several topics: a method of proof called *mathematical induction,* techniques for counting, and probability.

CHAPTER

10

OUTLINE

10-1 Sequences and Series

10-2 Mathematical Induction

10-3 Arithmetic and Geometric Sequences

10-4 The Multiplication Principle, Permutations, and Combinations

10-5 Sample Spaces and Probability

10-6 The Binomial Formula

Chapter 10 Review

Chapter 10 Group Activity: Sequences Specified by Recursion Formulas

10-1 Sequences and Series

- Defining Sequences
- Defining Series

In this section, we introduce special notation and formulas for representing and generating sequences and sums of sequences.

Defining Sequences

Consider the following list of numbers: 1, 3, 5, 7, 9, ... This is an example of a **sequence,** which can be defined informally as a list of numbers in a specific order. This particular sequence is the sequence of positive odd integers.

Now consider the function f given by

$$f(n) = 2n - 1 \tag{1}$$

where the domain of f is $\{1, 2, 3, \ldots\}$ (that is, the set of natural numbers N). Note that

$$\begin{aligned} f(1) &= 2(1) - 1 = 1 \\ f(2) &= 2(2) - 1 = 3 \\ f(3) &= 2(3) - 1 = 5 \\ &\vdots \qquad\quad \vdots \end{aligned}$$

The outputs of the function f form the same list of odd positive integers that we started with above. This provides an alternative (and more precise) definition of sequence: A **sequence** is a function whose domain is a set of successive integers.

While the function f above is a perfectly good way to describe a sequence, a special notation for describing sequences with formulas has evolved over the years. Our first order of business should be to become familiar with this notation.

To start, the range value $f(n)$ is usually symbolized more compactly with a symbol such as a_n. So in place of equation (1) we write

$$a_n = 2n - 1$$

The domain is understood to be the set of natural numbers N unless otherwise specified. The elements in the range are called **terms of the sequence:** a_1 is the first term, a_2 the second term, and a_n the nth term, or the **general term:**

$$\begin{aligned} a_1 &= 2(1) - 1 = 1 && \text{First term} \\ a_2 &= 2(2) - 1 = 3 && \text{Second term} \\ a_3 &= 2(3) - 1 = 5 && \text{Third term} \\ &\vdots \qquad\quad \vdots \\ a_n &= 2n - 1 && \text{General, or } n\text{th, term} \end{aligned}$$

The ordered list of elements

$$1, 3, 5, \ldots, 2n - 1, \ldots$$

in which the terms of a sequence are written in their natural order with respect to the domain values matches our informal definition of sequence. A sequence is also sometimes represented in the abbreviated form $\{a_n\}$, where a symbol for the nth term is placed between braces. For example, we can refer to the sequence

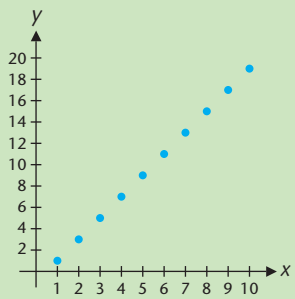
$$1, 3, 5, \ldots, 2n - 1, \ldots$$

as the sequence $\{2n - 1\}$.

If a sequence contains finitely many terms, it is called a **finite sequence.** If a sequence contains infinitely many terms, it is called an **infinite sequence.** The sequence $\{2n - 1\}$ above is an example of an infinite sequence.

››› EXPLORE-DISCUSS 1

The sequence $\{2n - 1\}$ is a function whose domain is the set of natural numbers, and so it may be graphed in the same way as any function whose domain and range are sets of real numbers (Fig. 1).



› **Figure 1** Graph of $\{2n - 1\}$.

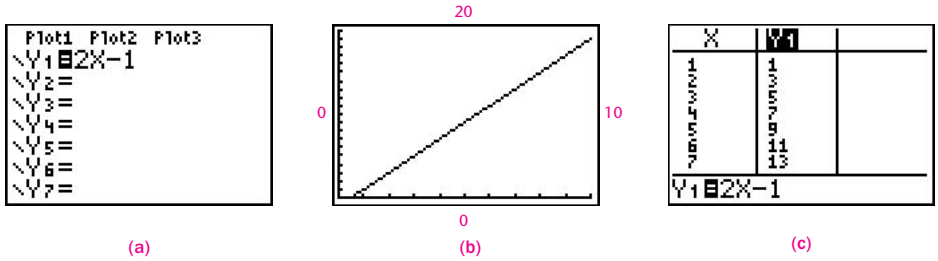
(A) Explain why the graph of the sequence $\{2n - 1\}$ is not continuous.

(B) Explain why the points on the graph of $\{2n - 1\}$ lie on a line. Find an equation for that line.

(C) Graph the sequence $\left\{\dfrac{2n^2 - n + 1}{n}\right\}$. How are the graphs of $\{2n - 1\}$ and $\left\{\dfrac{2n^2 - n + 1}{n}\right\}$ related?

There are several different ways a graphing utility can be used in the study of sequences. One approach is illustrated in Explore-Discuss 1. Figure 2(a) shows the sequence $\{2n - 1\}$ entered as a function in the equation editor of a graphing calculator. This produces a continuous graph [Fig. 2(b)] that contains the points in the graph of

the sequence (Fig. 1). Figure 2(c) shows the points on the graph of the sequence displayed in a table.

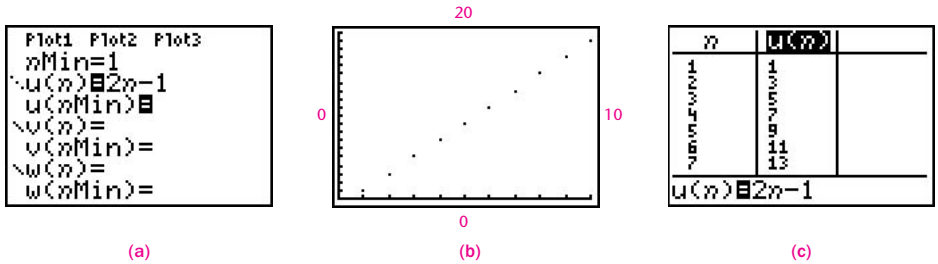


(a) (b) (c)

› Figure 2

The problem with this approach is that the graph contains a lot of points that don't belong to the sequence. But it is still useful, especially for computing the terms of a sequence quickly.

Many graphing calculators have a **sequence mode** designed specifically to graph sequences. When in sequence mode, the equation editor is set up to input sequences. Figure 3(a) shows the sequence $\{2n - 1\}$ entered in the sequence editor, Figure 3(b) shows the graph of this sequence, and Figure 3(c) displays the points on the graph in a table.



(a) (b) (c)

› Figure 3

Examining graphs and displaying values are very helpful activities when working with sequences. You should consult the manual for your graphing calculator to see which of these methods works on your graphing calculator.

Some sequences are specified by a **recursion formula**—that is, a formula that defines each term in terms of one or more preceding terms. The sequence we have chosen to illustrate a recursion formula is a very famous sequence in the history of mathematics called the **Fibonacci sequence.** It is named after the most celebrated mathematician of the thirteenth century, Leonardo Fibonacci from Italy (1180?–1250?).

EXAMPLE 1 Fibonacci Sequence

List the first seven terms of the sequence specified by

$$a_1 = 1$$
$$a_2 = 1$$
$$a_n = a_{n-2} + a_{n-1} \qquad n \geq 3$$

SOLUTIONS

Algebraic Solution

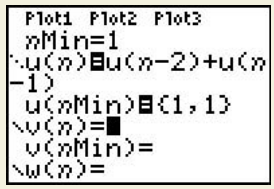
The first two terms are provided for us. The recursion formula indicates that every term from the third term on (represented by a_n) is obtained by adding the two previous terms (a_{n-1} and a_{n-2}).

$$
\begin{aligned}
a_1 &= 1 \\
a_2 &= 1 \\
a_3 &= a_1 + a_2 = 1 + 1 = 2 \\
a_4 &= a_2 + a_3 = 1 + 2 = 3 \\
a_5 &= a_3 + a_4 = 2 + 3 = 5 \\
a_6 &= a_4 + a_5 = 3 + 5 = 8 \\
a_7 &= a_5 + a_6 = 5 + 8 = 13
\end{aligned}
$$

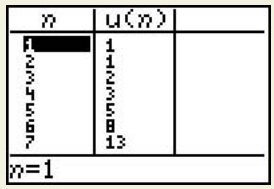
*

Graphing Calculator Solution

The sequence mode on a TI-84 can be used to graph and compute values for recursive sequences. The syntax for entering the sequence is shown in Figure 4(a). The recursion formula is specified by $u(n) = u(n - 2) + u(n - 1)$, and the first two terms are specified by $u(\text{nMin}) = \{1, 1\}$. The resulting table of terms is in Figure 4(b).



(a)



(b)

› Figure 4

MATCHED PROBLEM 1

List the first seven terms of the sequence specified by

$$
\begin{aligned}
a_1 &= 1 \\
a_2 &= 1 \\
a_n &= a_{n-2} - a_{n-1} \qquad n \geq 3
\end{aligned}
$$

››› EXPLORE-DISCUSS 2

A multiple-choice test question asked for the next term in the sequence:

$$1, 3, 9, \ldots$$

and gave the following choices:

(A) 16 **(B)** 19 **(C)** 27

Which is the correct answer?

Compare the first four terms of the following sequences:

(A) $a_n = 3^{n-1}$ **(B)** $b_n = 1 + 2(n - 1)^2$ **(C)** $c_n = 8n + \dfrac{12}{n} - 19$

Now which of the choices appears to be correct?

Now that we've seen how to use a formula for a sequence to find the first few terms, we might wonder if the reverse is possible. Given the first few terms, can we find a

*The dashed "think boxes" are used to enclose steps that may be performed mentally.

formula for a sequence? In general, the answer is no. Not every sequence has a pattern, as the sequence 3, 6, 3, 1, 4, 2, 1, 4, . . . in the chapter introduction shows. Also, Explore-Discuss 2 illustrates that many different sequences can start with the same terms. Actually, any finite number of beginning terms cannot specify one particular sequence.

But we can still try to find a formula (of many possible) that defines a sequence when given the first few terms. If we can observe a simple pattern in the given terms, then we may be able to find a general term that will produce the pattern. The next example illustrates this approach.

EXAMPLE 2 Finding the General Term of a Sequence

Find the general term of a sequence whose first four terms are

(A) 5, 6, 7, 8, . . . (B) 2, −4, 8, −16, . . .

SOLUTIONS

(A) Because these terms are consecutive integers, one solution is $a_n = n, n \geq 5$. If we want the domain of the sequence to be all natural numbers, then another solution is $b_n = n + 4$.

CHECK

n	1	2	3	4
$n + 4$	5	6	7	8

(B) Each of these terms can be written as the product of a power of 2 and a power of −1:

$$\begin{aligned} 2 &= (-1)^2 2^1 \\ -4 &= (-1)^3 2^2 \\ 8 &= (-1)^4 2^3 \\ -16 &= (-1)^5 2^4 \end{aligned}$$

If we choose the domain to be all natural numbers, then a solution is

$$a_n = (-1)^{n+1} 2^n$$

CHECK

n	1	2	3	4
$(-1)^{n+1}2^n$	2	−4	8	−16

MATCHED PROBLEM 2

Find the general term of a sequence whose first four terms are

(A) 2, 4, 6, 8, . . . (B) 1, $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}$, . . .

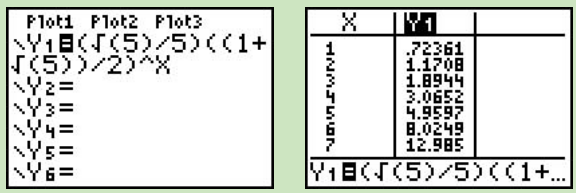
››› CAUTION ›››

After finding a formula for the general term of a sequence, you should always check your answer by substituting several values into your formula to make sure that they generate the beginning terms you were given.

In general, there is usually more than one way to represent the nth term of a given sequence. This was seen in the solution of Example 2, part A. However, unless stated to the contrary, we will choose a formula for which the domain of the sequence is the set of natural numbers N.

››› EXPLORE-DISCUSS 3

The sequence with general term $b_n = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n$ is closely related to the Fibonacci sequence. Compute the first 20 terms of both sequences and discuss the relationship. [The first seven values of b_n are shown in Fig. 5(b)].



(a) (b)

› Figure 5

› Defining Series

If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a sequence, then the expression

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is called a *series*. A **series** is the result of adding the terms of a sequence. If the sequence is finite, the corresponding series is a **finite series.** If the sequence is infinite, the corresponding series is an **infinite series.** For example,

$1, 2, 4, 8, 16$ Finite sequence

$1 + 2 + 4 + 8 + 16$ Finite series

We will restrict our discussion to finite series in this section.

Series are often represented in a compact form called **summation notation** using the symbol Σ, which is a stylized version of the Greek letter sigma. Consider the following examples:

$$\sum_{k=1}^{4} a_k = a_1 + a_2 + a_3 + a_4 \qquad \text{Domain is } \{k : k = 1, 2, 3, 4\}.$$

$$\sum_{k=3}^{7} b_k = b_3 + b_4 + b_5 + b_6 + b_7 \qquad \text{Domain is } \{k : k = 3, 4, 5, 6, 7\}.$$

$$\sum_{k=0}^{n} c_k = c_0 + c_1 + c_2 + \cdots + c_n \qquad \text{Domain is the set of integers } k \text{ satisfying } 0 \le k \le n.$$

The terms on the right are obtained from the expression on the left by successively replacing the **summing index** k with integers, starting with the first number indicated below Σ and ending with the number that appears above Σ. Thus, for example, if we are given the sequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}$$

the corresponding series is

$$\sum_{k=1}^{n} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$$

EXAMPLE 3 Writing the Terms of a Series

Write without summation notation: $\displaystyle\sum_{k=1}^{5} \frac{k-1}{k}$

SOLUTION

Substitute the values 1, 2, 3, 4, and 5 for k, then add the results.

$$\sum_{k=1}^{5} \frac{k-1}{k} = \frac{1-1}{1} + \frac{2-1}{2} + \frac{3-1}{3} + \frac{4-1}{4} + \frac{5-1}{5}$$

$$= 0 + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5}$$

MATCHED PROBLEM 3

Write without summation notation: $\displaystyle\sum_{k=0}^{5} \frac{(-1)^k}{2k+1}$

If the terms of a series are alternately positive and negative, it is called an **alternating series.** For successive integer values of k, the expression $(-1)^k$ is alternately -1 (for odd k's) and $+1$ (for even k's). The same is true for $(-1)^{k+1}$, the difference being the sign of the first term. This observation is helpful in writing an alternating series using summation notation.

EXAMPLE 4 Writing a Series in Summation Notation

Write the following series using summation notation:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$$

(A) Start the summing index at $k = 1$.

(B) Start the summing index at $k = 0$.

SOLUTIONS

(A) $(-1)^{k+1}$ provides the alternation of sign, with the term for $k = 1$ positive. The expression $1/k$ provides the rest of each term. We can write the series as

$$\sum_{k=1}^{6} (-1)^{k+1} \frac{1}{k}$$

You can check this result by substituting $k = 1, \ldots, 6$.

(B) $(-1)^k$ provides the alternation of sign, this time with the term for $k = 0$ positive. The expression $1/(k + 1)$ provides the rest of each term. We can write the series as

$$\sum_{k=0}^{5} (-1)^k \frac{1}{k+1}$$

Again, you can check by substituting $k = 0, \ldots, 5$.

MATCHED PROBLEM 4

Write the following series using summation notation:

$$1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81}$$

(A) Start with $k = 1$. (B) Start with $k = 0$.

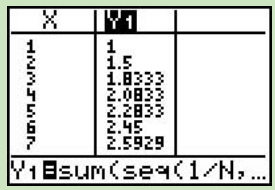
››› EXPLORE-DISCUSS 4

(A) Find the smallest number of terms of the infinite series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

that, when added together, give a number greater than 3.

(B) Enter the function $y_1 = \text{sum}(\text{seq}(1/N,N,1,X))$ in the equation editor of a graphing utility* and examine a table (Fig. 6) to find the smallest number of terms of the infinite series in part A that, when added together, give a number greater than 4.



› Figure 6

(C) Find the smallest number of terms of the infinite series

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots$$

that, when added together, give a number greater than 0.99; greater than 0.999. Will the sum ever exceed 1? Explain.

ANSWERS TO MATCHED PROBLEMS

1. 1, 1, 0, 1, −1, 2, −3 **2.** (A) $a_n = 2n$ (B) $a_n = (-1)^{n+1}\left(\frac{1}{2}\right)^{n-1}$

3. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11}$ **4.** (A) $\sum_{k=1}^{5}(-1)^{k+1}\left(\frac{2}{3}\right)^{k-1}$ (B) $\sum_{k=0}^{4}(-1)^{k}\left(\frac{2}{3}\right)^{k}$

*On a TI-84, sum is found on the LIST menu (2^{nd}-STAT) under "MATH", and seq is on the LIST menu under "OPS".

10-1 Exercises

1. Write an informal definition of sequence, then write a definition in terms of functions.

2. Write a definition of series.

3. Explain why the following statement is not true: The general term of the sequence 1, 3, 7, . . . is $2^n - 1$.

4. Explain why at least one term must be provided when defining a sequence recursively.

Write the first four terms for each sequence in Problems 5–10.

5. $a_n = n - 2$ **6.** $a_n = n + 3$ **7.** $a_n = \dfrac{n-1}{n+1}$

8. $a_n = \left(1 + \dfrac{1}{n}\right)^n$ **9.** $a_n = (-2)^{n+1}$ **10.** $a_n = \dfrac{(-1)^{n+1}}{n^2}$

11. Write the eighth term in the sequence in Problem 5.

12. Write the tenth term in the sequence in Problem 6.

13. Write the one-hundredth term in the sequence in Problem 7.

14. Write the two-hundredth term in the sequence in Problem 8.

In Problems 15–20, write each series in expanded form without summation notation.

15. $\sum_{k=1}^{5} k$ **16.** $\sum_{k=1}^{4} k^2$ **17.** $\sum_{k=1}^{3} \dfrac{1}{10^k}$

18. $\sum_{k=1}^{5} \left(\dfrac{1}{3}\right)^k$ **19.** $\sum_{k=1}^{4} (-1)^k$ **20.** $\sum_{k=1}^{6} (-1)^{k-1} k$

Write the first five terms of each sequence in Problems 21–30.

21. $a_n = (-1)^{n+1} n^2$ **22.** $a_n = (-1)^{n+1}\left(\dfrac{1}{2^n}\right)$

23. $a_n = \dfrac{1}{3}\left(1 - \dfrac{1}{10^n}\right)$ **24.** $a_n = n[1 - (-1)^n]$

25. $a_n = (-\frac{1}{2})^{n-1}$ **26.** $a_n = (-\frac{3}{2})^{n-1}$

27. $a_1 = 7; a_n = a_{n-1} - 4, n \geq 2$

28. $a_1 = 3; a_n = a_{n-1} + 5, n \geq 2$

29. $a_1 = 4; a_n = \frac{1}{4}a_{n-1}, n \geq 2$

30. $a_1 = 2; a_n = 2a_{n-1}, n \geq 2$

In Problems 31–34, write the first seven terms of each sequence. Use a graphing calculator to check your answers.

31. $a_1 = 1, a_2 = 2, a_n = a_{n-2} + 2a_{n-1}, n \geq 3$

32. $a_1 = 1, a_2 = -1, a_n = a_{n-2} - a_{n-1}, n \geq 3$

33. $a_1 = -1, a_2 = 2, a_n = 2a_{n-2} + a_{n-1}, n \geq 3$

34. $a_1 = 2, a_2 = 1, a_n = -a_{n-2} + a_{n-1}, n \geq 3$

In Problems 35–46, find the general term of a sequence whose first four terms are given.

35. 4, 5, 6, 7, . . . **36.** −2, −1, 0, 1, . . .

37. 3, 6, 9, 12, . . . **38.** −2, −4, −6, −8, . . .

39. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ **40.** $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots$

41. 1, −1, 1, −1, . . . **42.** 1, −2, 3, −4, . . .

43. −2, 4, −8, 16, . . . **44.** 1, −3, 5, −7, . . .

45. $x, \dfrac{x^2}{2}, \dfrac{x^3}{3}, \dfrac{x^4}{4}, \ldots$ **46.** $x, -x^3, x^5, -x^7, \ldots$

In Problems 47–52, find general terms for two different sequences with the property that both sequences have identical terms for n = 1, 2, and 3. [Hint: Graph the points (n, a_n) and apply quadratic regression to find one formula.]

47. 1, 2, 4, . . . **48.** 1, 4, 16, . . . **49.** 1, 8, 27, . . .

50. 1, 16, 81, . . . **51.** −3, 6, −9, . . . **52.** 4, −8, 12, . . .

In Problems 53–56, use a graphing calculator to graph the first 20 terms of each sequence.

53. $a_n = 1/n$ **54.** $a_n = 2 + \pi n$

55. $a_n = (-0.9)^n$ **56.** $a_1 = -1, a_n = \frac{2}{3}a_{n-1} + \frac{1}{2}$

In Problems 57–62, write each series in expanded form without summation notation.

57. $\sum_{k=1}^{4} \dfrac{(-2)^{k+1}}{k}$ **58.** $\sum_{k=1}^{5} (-1)^{k+1}(2k-1)^2$

59. $\sum_{k=1}^{3} \dfrac{1}{k} x^{k+1}$ **60.** $\sum_{k=1}^{5} x^{k-1}$

61. $\sum_{k=1}^{5} \dfrac{(-1)^{k+1}}{k} x^k$ **62.** $\sum_{k=0}^{4} \dfrac{(-1)^k x^{2k+1}}{2k+1}$

In Problems 63–70, write each series using summation notation with the summing index k starting at k = 1.

63. $1^2 + 2^2 + 3^2 + 4^2$ **64.** $2 + 3 + 4 + 5 + 6$

65. $\dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \dfrac{1}{2^4} + \dfrac{1}{2^5}$ **66.** $1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4}$

67. $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$

68. $2 + \frac{3}{2} + \frac{4}{3} + \cdots + \frac{n+1}{n}$

69. $1 - 4 + 9 - \cdots + (-1)^{n+1}n^2$

70. $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \cdots + \frac{(-1)^{n+1}}{2^n}$

The sequence

$$a_n = \frac{a_{n-1}^2 + M}{2a_{n-1}} \qquad n \geq 2, M \text{ a positive real number}$$

can be used to find $\sqrt{M}$ to any decimal-place accuracy desired. To start the sequence, you can choose a_1 to be any positive real numbers. Problems 71 and 72 are related to this sequence.

71. (A) Find the first four terms of the sequence

$$a_1 = 3 \qquad a_n = \frac{a_{n-1}^2 + 2}{2a_{n-1}} \qquad n \geq 2$$

(B) Compare the terms with $\sqrt{2}$ from a calculator.
(C) Repeat parts A and B letting a_1 be any other positive number, say 1.

72. (A) Find the first four terms of the sequence

$$a_1 = 2 \qquad a_n = \frac{a_{n-1}^2 + 5}{2a_{n-1}} \qquad n \geq 2$$

(B) Find $\sqrt{5}$ with a calculator, and compare with the results of part A.
(C) Repeat parts A and B letting a_1 be any other positive number, say 3.

73. Let $\{a_n\}$ denote the Fibonacci sequence and let $\{b_n\}$ denote the sequence defined by $b_1 = 1, b_2 = 3$, and $b_n = b_{n-1} + b_{n-2}$ for $n \geq 3$. Compute 10 terms of the sequence $\{c_n\}$, where $c_n = b_n/a_n$. Describe the terms of $\{c_n\}$ for large values of n.

74. Define sequences $\{u_n\}$ and $\{v_n\}$ by $u_1 = 1$, $v_1 = 0$, $u_n = u_{n-1} + v_{n-1}$ and $v_n = u_{n-1}$ for $n \geq 2$. Find the first 10 terms of each sequence, and explain their relationship to the Fibonacci sequence.

In calculus, it can be shown that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \approx 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

where the larger n is, the better the approximation. Problems 75 and 76 refer to this series. Note that n!, read "n factorial," is defined by $0! = 1$ and $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$ for $n \in N$.

75. Approximate $e^{0.2}$ using the first five terms of the series. Compare this approximation with your calculator evaluation of $e^{0.2}$.

76. Approximate $e^{-0.5}$ using the first five terms of the series. Compare this approximation with your calculator evaluation of $e^{-0.5}$.

77. Show that $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$

78. Show that $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

APPLICATIONS

79. PHYSICS Suppose that a rubber ball is dropped from a height of 20 feet. If it bounces 10 times, with each bounce going half as high as the one before, the heights of these bounces can be described by the sequence $a_n = 10(\frac{1}{2})^{n-1}$ $(1 \leq n \leq 10)$.
(A) How high is the fifth bounce? The tenth?
(B) Find the value of the series $\sum_{n=1}^{10} a_n$. What does this number represent?

80. PHYSICS A bungee jumper dives off a bridge that is 300 feet above the ground. He bounces back 100 feet on the first bounce, then continue to bounce nine more times before coming to rest, with each bounce 1/3 as high as the previous. The heights of these bounces can be described by the sequence $a_n = 100(\frac{1}{3})^{n-1}$ $(1 \leq n \leq 10)$.
(A) How high is the fifth bounce? The tenth?
(B) Find the value of the series $\sum_{n=1}^{10} a_n$. What does this number represent?

81. SALARY INCREMENT Suppose that you are offered a job with a starting annual salary of \$40,000 and annual increases of 4% of the current salary.
(A) Write out the first six terms of a sequence a_n whose terms describe your salary in the first 6 years on this job.
(B) Write the general term of the sequence in part A.
(C) Find the value of the series $\sum_{n=1}^{6} a_n$. What does this number represent?

82. SALARY INCREMENT A marketing firm is advertising entry-level positions with a starting annual salary of \$24,000 and annual increments of 3% of the current salary.
(A) Write out the first six terms of a sequence a_n whose terms describe the salary for this position in the first 6 years on this job.
(B) Write the general term of the sequence in part A.
(C) Find the value of the series $\sum_{n=1}^{6} a_n$. What does this number represent?

10-2 Mathematical Induction

- Using Counterexamples
- Using Mathematical Induction
- Additional Examples of Mathematical Induction
- Three Famous Problems

Many of the most important facts and formulas in this book have been stated as theorems. But a theorem is not a theorem until it has been proved, and proving theorems is one of the most challenging tasks in mathematics. There is a big difference between being pretty sure that a statement is true, and *proving* that statement. Let's look at an example.

Suppose that we are interested in the sum of the first n consecutive odd integers, where n is a positive integer. We can begin by writing the sums for the first few values of n to see if we can observe a pattern:

$$\begin{aligned} 1 &= 1 && n = 1\\ 1 + 3 &= 4 && n = 2\\ 1 + 3 + 5 &= 9 && n = 3\\ 1 + 3 + 5 + 7 &= 16 && n = 4\\ 1 + 3 + 5 + 7 + 9 &= 25 && n = 5 \end{aligned}$$

Is there any pattern to the sums 1, 4, 9, 16, and 25? You most likely noticed that each is a perfect square and, in fact, each is the square of the number of terms in the sum. Thus, the following *conjecture** seems reasonable:

CONJECTURE *P*: For each positive integer n,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

(Recall that the general term $2n - 1$ was used to list the odd positive integers in the last section.)

At this point, you may be pretty sure that our conjecture is true. You might even look at the previous five calculations and think that we have proved our conjecture. But in actuality, all we have proved is that the conjecture is true for $n = 1, 2, 3, 4$, and 5. We are trying to prove that it is true for *every* positive integer, not just those five! With that in mind, continuing to check the conjecture for specific n's like 6, 7, 8, . . . is pointless: You can keep trying for the rest of your life, but you will never be able to check *every* positive integer. Instead, in this section, we will use a much more powerful tool called *mathematical induction* to prove conjectures. Before we learn about this method of proof, we first consider how to prove that a conjecture is false.

*A **conjecture** is a statement that is believed to be true, but has not been proved.

› Using Counterexamples

Consider the following conjecture:

CONJECTURE Q: For each positive integer n, the number $n^2 - n + 41$ is a prime number.

Since the conjecture states that this fact is true for *every* positive integer n, if we can find even one positive integer n for which it is false, then the conjecture will be proved false.

A single case or example for which a conjecture fails is called a **counterexample.** We checked the conjecture for a few particular cases in Table 1. From the table, it certainly appears that conjecture Q has a good chance of being true. You may want to check a few more cases. If you persist, you will find that conjecture Q is true for n up to 40.

Most students would guess that the statement is always true long before getting to $n = 41$. But then something interesting happens at $n = 41$:

$$41^2 - 41 + 41 = 41^2$$

which is not prime. Because $n = 41$ provides a counterexample, conjecture Q is false. Here we see the danger of generalizing without proof from a few special cases, even if that "few" is 40 cases!

This example was discovered by Euler (1701–1783), the same mathematician that introduced the number e as the base of the natural exponential function.

Table 1

n	$n^2 - n + 41$	Prime?
1	41	Yes
2	43	Yes
3	47	Yes
4	53	Yes
5	61	Yes

EXAMPLE 1 Finding a Counterexample

Prove that the following conjecture is false by finding a counterexample: For every positive integer $n \geq 2$, at least half of the positive integers less than or equal to n are prime.

SOLUTION

We will check the conjecture for positive integer values of n starting at 2.

n	Primes less than or equal to n	Fraction of positive integers less than or equal to n that are prime	True or false
2	2	1/2	True
3	2, 3	2/3	True
4	2, 3	2/4	True
5	2, 3, 5	3/5	True
6	2, 3, 5	3/6	True
7	2, 3, 5, 7	4/7	True
8	2, 3, 5, 7	4/8	True
9	2, 3, 5, 7	4/9	False

Since $n = 9$ provides a counterexample, the conjecture is false. ◉

MATCHED PROBLEM 1

Prove that the following conjecture is false by finding a counterexample: For every positive integer n, the last digit of n^3 is less than 9.

Using Mathematical Induction

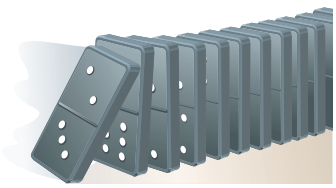
To begin our study of proving conjectures, we will state the **principle of mathematical induction,** which forms the basis for all of our work in this section.

THEOREM 1 Principle of Mathematical Induction

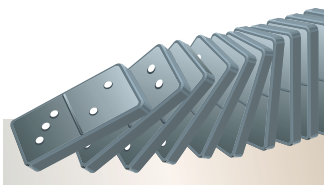
Let P_n be a statement associated with each positive integer n, and suppose the following conditions are satisfied:

1. P_1 is true.
2. For any positive integer k, if P_k is true, then P_{k+1} is also true.

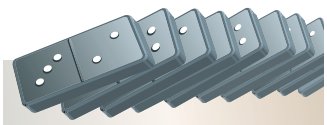
Then the statement P_n is true for all positive integers n.



Condition 1: The first domino can be pushed over.
(a)



Condition 2: If the kth domino falls, then so does the $(k + 1)$st.
(b)



Conclusion: All the dominoes will fall.
(c)

Figure 1 Interpreting mathematical induction.

Theorem 1 must be read very carefully. At first glance, it seems to say that if we assume a statement is true, then it is true. But that is not the case at all. If the two conditions in Theorem 1 are satisfied, then we can reason as follows:

P_1 is true.	Condition 1
P_2 is true, because P_1 is true.	Condition 2
P_3 is true, because P_2 is true.	Condition 2
P_4 is true, because P_3 is true.	Condition 2
$\vdots$	$\vdots$

Because this chain of implications never ends, we will eventually reach P_n for any positive integer n.

This is *not* the same as checking each case separately: The truth of *any* case follows from knowing that the previous one is true once we have established condition 2.

To help visualize this process, picture a row of dominoes that goes on forever (Fig. 1) and interpret the conditions in Theorem 1 as follows: Condition 1 says that the first domino can be pushed over. Condition 2 says that when *any* domino falls, the one after it does as well. Together, these two conditions imply that all the dominoes must eventually fall.

We will illustrate the process of proof by mathematical induction by returning to our conjecture P_n from the beginning of the section.

EXAMPLE 2 Proving a Conjecture Using Induction

Prove that for all positive integers n,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

SOLUTION

State P_n:

$$P_n: \quad 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

CONDITION 1

Show that P_1 is true.

$$P_1: \quad 1 = 1^2$$

CONDITION 2

Show that if P_k is true, then P_{k+1} must be true.

It's a good idea to always write out both P_k and P_{k+1} at the beginning of this step to see what we can use, and what we need to prove.

P_k: $1 + 3 + 5 + \cdots + (2k - 1) = k^2$ We assume this is a true statement.

P_{k+1}: $1 + 3 + 5 + \cdots + (2k - 1) + [2(k + 1) - 1] = (k + 1)^2$ We need to show that this is also true.

Note that P_{k+1} can be simplified a bit:

$$P_{k+1}: \quad 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2$$

We will perform algebraic operations on the equation P_k (which we know is true) with a goal of obtaining P_{k+1}. Note that the left side of P_{k+1} is the left side of P_k plus the addition term $2k + 1$.

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2 \quad \text{Add } 2k + 1 \text{ to both sides.}$$

$$1 + 3 + 5 + \cdots + (2k - 1) + \mathbf{(2k + 1)} = k^2 + \mathbf{2k + 1} \quad \text{Factor the right side.}$$

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2 \quad \text{This is } P_{k+1}!$$

P_{k+1} was obtained by adding the same number to both sides of P_k, so if P_k is true, then P_{k+1} must be as well.

CONCLUSION

Both conditions of Theorem 1 are satisfied, so P_n is true for all positive integers n. ◉

MATCHED PROBLEM 2

Prove that for all positive integers n

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}$$

Additional Examples of Mathematical Induction

Now we will consider some additional examples of proof by induction. The first is another summation formula. Mathematical induction is the primary tool for proving that formulas of this type are true.

EXAMPLE 3 **Proving a Summation Formula**

Prove that for all positive integers n

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

PROOF

State P_n:

$$P_n: \quad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

CONDITION 1

Show that P_1 is true.

$$P_1: \quad \frac{1}{2} = \frac{2^1 - 1}{2^1}$$

$$= \frac{1}{2} \quad \text{True}$$

CONDITION 2

Show that if P_k is true, then P_{k+1} is true.

$$P_k: \quad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} = \frac{2^k - 1}{2^k}$$ We assume P_k is true.

$$P_{k+1}: \quad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}}$$ We must show that P_{k+1} follows from P_k.

We will start with the true statement P_k, and add $1/2^{k+1}$ to both sides to match the left side of P_{k+1}.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^k}=\frac{2^k-1}{2^k}$$

Add $\frac{1}{2^{k+1}}$ to both sides.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=\frac{2^k-1}{2^k}+\frac{1}{2^{k+1}}$$

Get common denominator of 2^{k+1} on the right side.

$$=\frac{2^k-1}{2^k}\cdot\frac{2}{2}+\frac{1}{2^{k+1}}$$

Add fractions.

$$=\frac{2^{k+1}-2+1}{2^{k+1}}$$

Simplify numerator.

$$=\frac{2^{k+1}-1}{2^{k+1}}$$

So,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=\frac{2^{k+1}-1}{2^{k+1}} \qquad P_{k+1}$$

and we have shown that if P_k is true, then P_{k+1} is true.

CONCLUSION

Both conditions in Theorem 1 are satisfied, so, P_n is true for all positive integers n.

MATCHED PROBLEM 3

Prove that for all positive integers n.

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots+\frac{1}{3^n}=\frac{1}{2}\left(\frac{3^n-1}{3^n}\right)$$

Example 4 provides a proof of a law of exponents that previously we had to assume was true. First we redefine a^n for n a positive integer, using a recursion formula:

› DEFINITION 1 Recursive Definition of a^n

For n a positive integer

(i) $a^1 = a$ (ii) $a^{n+1} = a^n a \quad n > 1$

EXAMPLE 4 Proving a Law of Exponents

Prove that $(xy)^n = x^n y^n$ for all positive integers n.

PROOF

State P_n:

$$P_n\colon (xy)^n = x^n y^n$$

CONDITION 1

Show that P_1 is true.

$$\begin{aligned} (xy)^1 &= xy && \text{Definition 1, part i} \\ &= x^1 y^1 && \text{Definition 1, part i} \end{aligned}$$

P_1 is true.

CONDITION 2

Show that if P_k is true, then P_{k+1} is true.

$$\begin{aligned} P_k\colon \quad (xy)^k &= x^k y^k && \text{Assume } P_k \text{ is true.} \\ P_{k+1}\colon \quad (xy)^{k+1} &= x^{k+1} y^{k+1} && \text{Show that } P_{k+1} \text{ follows from } P_k. \end{aligned}$$

Here we start with the left side of P_{k+1} and use P_k to find the right side of P_{k+1}. Using part ii of Definition 1,

$$\begin{aligned} (xy)^{k+1} &= (xy)^k (xy)^1 && \text{Use } P_k\colon (xy)^k = x^k y^k. \\ &= x^k y^k xy && \text{Use commutative and associative laws.} \\ &= (x^k x)(y^k y) && \text{Definition 1, part ii} \\ &= x^{k+1} y^{k+1} && P_{k+1} \end{aligned}$$

So, $(xy)^{k+1} = x^{k+1} y^{k+1}$, and we have shown that if P_k is true, then P_{k+1} is true.

CONCLUSION

Both conditions in Theorem 1 are satisfied, so, P_n is true for all positive integers n. ◉

MATCHED PROBLEM 4

Prove that $(x/y)^n = x^n/y^n$ for all positive integers n.

◉

Example 5 deals with factors of integers. Before we start, recall that an integer p is *divisible* by an integer q if $p = qr$ for some integer r.

EXAMPLE 5 **Proving a Divisibility Property**

Prove that $4^{2n} - 1$ is divisible by 5 for all positive integers n.

PROOF

If $4^{2n} - 1$ is divisible by 5, it means that $4^{2n} - 1$ can be written as 5 times an integer, so we state P_n as:

$$P_n: 4^{2n} - 1 = 5r \quad \text{for some integer } r$$

CONDITION 1

Show that P_1 is true.

$$P_1: 4^2 - 1 = 15 = 5 \cdot 3$$

Thus, P_1 is true.

CONDITION 2

Show that if P_k is true, then P_{k+1} is true.

$$\begin{aligned} P_k&: 4^{2k} - 1 = 5r \quad \text{for some integer } r && \text{Assume } P_k \text{ is true.}\\ P_{k+1}&: 4^{2(k+1)} - 1 = 5s \quad \text{for some integer } s && \text{Show that } P_{k+1} \text{ must follow.}\end{aligned}$$

If we expand the exponent on the left side of P_{k+1}, we get

$$4^{2(k+1)} - 1 = 4^{2k+2} - 1$$

This suggests that multiplying the 4^{2k} on the left side of P_k by 4^2 might be a good start in obtaining P_{k+1}.

$$\begin{aligned} 4^{2k} - 1 &= 5r && \text{Multiply both sides by } 4^2.\\ 4^2(4^{2k} - 1) &= 4^2(5r) && \text{Simplify.}\\ 4^{2k+2} - 16 &= 80r && \text{Add 15 to both sides to obtain the left side of } P_{k+1}.\\ 4^{2(k+1)} - 1 &= 80r + 15 && \text{Factor out 5.}\\ &= 5(16r + 3) && 16r + 3 \text{ is an integer.}\end{aligned}$$

So,

$$4^{2(k+1)} - 1 = 5s \quad P_{k+1}$$

where $s = 16r + 3$ is an integer, and we have shown that if P_k is true, then P_{k+1} is true.

CONCLUSION

Both conditions in Theorem 1 are satisfied, so, P_n is true for all positive integers n. ◉

MATCHED PROBLEM 5

Prove that $8^n - 1$ is divisible by 7 for all positive integers n.

In some cases, a conjecture may be true only for $n \geq m$, for some positive integer m, rather than for all $n \geq 0$. For example, see Problems 39 and 40 in Exercises 10.2. The principle of mathematical induction can be extended to cover cases like this as follows:

› THEOREM 2 Extended Principle of Mathematical Induction

Let m be a positive integer, let P_n be a statement associated with each integer $n \geq m$, and suppose the following conditions are satisfied:

1. P_m is true.
2. For any integer $k \geq m$, if P_k is true, then P_{k+1} is also true.

Then the statement P_n is true for all integers $n \geq m$.

The only difference is that we start at m, rather than 1.

› Three Famous Problems

The problem of determining whether a certain statement about the positive integers is true may be extremely difficult. Proofs can require remarkable insight and ingenuity and the development of techniques far more advanced than mathematical induction. Consider, for example, the famous problems of proving the following statements:

1. **Lagrange's Four Square Theorem, 1772:** Each positive integer can be expressed as the sum of four or fewer squares of positive integers.
2. **Fermat's Last Theorem, 1637:** For $n > 2$, $x^n + y^n = z^n$ does not have solutions in the natural numbers.
3. **Goldbach's Conjecture, 1742:** Every positive even integer greater than 2 is the sum of two prime numbers.

The first statement was considered by the early Greeks and finally proved in 1772 by Lagrange. Fermat's last theorem was stated without proof by Pierre de Fermat in 1637 and defied proof by some of the world's greatest mathematical minds for over 350 years. It was finally proved in a 200-page paper by Professor Andrew Wiles of Princeton University in 1993. To this date no one has been able to prove or disprove Goldbach's conjecture.

››› EXPLORE-DISCUSS 1

(A) Explain the difference between a theorem and a conjecture.

(B) Fermat's last theorem was identified by this name for over a hundred years prior to 1993. Explain why this was an inappropriate name, and suggest one that would have been more accurate.

ANSWERS TO MATCHED PROBLEMS

1. The last digit of $9^3 = 729$ is greater than 8.

2. Sketch of proof.

P_n: $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$

Condition 1. $1 = \dfrac{1(1+1)}{2}$. P_1 is true.

Condition 2. Show that if P_k is true, then P_{k+1} is true.

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \qquad P_k$$

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{(k+1)(k+2)}{2} \qquad P_{k+1}$$

Conclusion: P_n is true for all positive integers n.

3. Sketch of proof.

P_n: $\dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \cdots + \dfrac{1}{3^n} = \dfrac{1}{2}\left(\dfrac{3^n - 1}{3^n}\right)$

Condition 1. $\dfrac{1}{3} = \dfrac{1}{2}\left(\dfrac{3^1 - 1}{3^1}\right)$ P_1 is true.

Condition 2. Show that if P_k is true, then P_{k+1} is true.

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^k} = \frac{1}{2}\left(\frac{3^k - 1}{3^k}\right) \qquad P_k$$

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^k} + \frac{1}{3^{k+1}} = \frac{1}{2}\left(\frac{3^k - 1}{3^k}\right) + \frac{1}{3^{k+1}}$$

$$= \frac{3^k - 1}{2 \cdot 3^k} \cdot \frac{3}{3} + \frac{1}{3^{k+1}} \cdot \frac{2}{2}$$

$$= \frac{3^{k+1} - 3}{2 \cdot 3^{k+1}} + \frac{2}{2 \cdot 3^{k+1}}$$

$$= \frac{1}{2}\left(\frac{3^{k+1} - 1}{3^{k+1}}\right) \qquad P_{k+1}$$

Conclusion: P_n is true for all positive integers n.

4. Sketch of proof. P_n: $\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$

Condition 1. $\left(\dfrac{x}{y}\right)^1 = \dfrac{x}{y} = \dfrac{x^1}{y^1}$. P_1 is true.

Condition 2. Show that if P_k is true, then P_{k+1} is true.

$$\left(\frac{x}{y}\right)^{k+1} = \left(\frac{x}{y}\right)^k \left(\frac{x}{y}\right) = \frac{x^k}{y^k}\left(\frac{x}{y}\right) = \frac{x^k x}{y^k y} = \frac{x^{k+1}}{y^{k+1}}$$

Conclusion: P_n is true for all positive integers n.

5. Sketch of proof. P_n: $8^n - 1 = 7r$ for some integer r

Condition 1. $8^1 - 1 = 7 = 7 \cdot 1$. P_1 is true.

Condition 2. Show that if P_k is true, then P_{k+1} is true.

$8^k - 1 = 7r$ P_k

$8(8^k - 1) = 8(7r)$

$8^{k+1} - 1 = 56r + 7 = 7(8r + 1) = 7s$ P_{k+1}

Conclusion: P_n is true for all positive integers n.

10-2 Exercises

1. Explain how falling dominos can be applied to understanding mathematical induction.

2. In the section introduction, we showed that the sum of n consecutive odd integers starting with 1 is equal to n^2 for $n = 1, 2, 3, 4$, and 5. Explain why this does not prove that fact true for all natural numbers n.

In Problems 3–6, find the first positive integer n that causes the statement to fail.

3. $(3 + 5)^n = 3^n + 5^n$ **4.** $2n + 5 < 20$

5. $n^2 = 3n - 2$ **6.** $n^3 + 11n = 6n^2 + 6$

In Problems 7–12, (A) verify each statement for n = 1, 2, and 3; (B) write P_k and P_{k+1} for each statement P_n; and (C) use mathematical induction to prove that each P_n is true for all positive integers n.

7. P_n: $2 + 6 + 10 + \cdots + (4n - 2) = 2n^2$

8. P_n: $4 + 8 + 12 + \cdots + 4n = 2n(n + 1)$

9. P_n: $a^5a^n = a^{5+n}$ **10.** P_n: $(a^5)^n = a^{5n}$

11. P_n: $9^n - 1$ is divisible by 4

12. P_n: $4^n - 1$ is divisible by 3

In Problems 13–16, prove the statement is false by finding a counterexample.

13. If $n > 2$, then any polynomial of degree n has at least one real zero.

14. Any positive integer $n > 7$ can be written as the sum of three or fewer squares of positive integers.

15. If n is a positive integer, then there is at least one prime number p such that $n < p < n + 6$.

16. If a, b, c, and d are positive integers such that $a^2 + b^2 = c^2 + d^2$, then $a = c$ or $a = d$.

In Problems 17–32, use mathematical induction to prove each statement is true for all positive integers n, unless restricted otherwise.

17. $2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$

18. $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \cdots + \dfrac{1}{2^n} = 1 - \left(\dfrac{1}{2}\right)^n$

19. $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \frac{1}{3}(4n^3 - n)$

20. $1 + 8 + 16 + \cdots + 8(n - 1) = (2n - 1)^2;\ n > 1$

21. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n + 1)(2n + 1)}{6}$

22. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \dfrac{n(n + 1)(n + 2)}{3}$

23. $\dfrac{a^n}{a^3} = a^{n-3};\ n > 3$ **24.** $\dfrac{a^5}{a^n} = \dfrac{1}{a^{n-5}};\ n > 5$

25. $a^ma^n = a^{m+n}$; $m, n \in N$ [*Hint:* Choose m as an arbitrary element of N, and then use induction on n.]

26. $(a^n)^m = a^{mn}$; $m, n \in N$

27. $x^n - 1$ is divisible by $x - 1, x \neq 1$ [*Hint:* Recall that divisible means that $x^n - 1 = (x - 1)Q(x)$ for some polynomial $Q(x)$.]

28. $x^n - y^n$ is divisible by $x - y$; $x \neq y$

29. $x^{2n} - 1$ is divisible by $x - 1$; $x \neq 1$

30. $x^{2n} - 1$ is divisible by $x + 1$; $x \neq -1$

31. $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$ [*Hint:* See Matched Problem 2 following Example 2.]

32. $$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots + \frac{1}{n(n + 1)(n + 2)} = \frac{n(n + 3)}{4(n + 1)(n + 2)}$$

In Problems 33–36, suggest a formula for each expression, and prove your conjecture using mathematical induction, $n \in N$.

33. $2 + 4 + 6 + \cdots + 2n$

34. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n + 1)}$

35. The number of lines determined by n points in a plane, no three of which are collinear.

36. The number of diagonals in a polygon with n sides.

Prove Problems 37–40 true for all integers n as specified.

37. $a > 1$ implies that $a^n > 1$; for every natural number n

38. $0 < a < 1$ implies that $0 < a^n < 1$; for every natural number n

39. $n^2 > 2n$; for every natural number n $n \geq 3$

40. $2^n > n^2$; for every natural number n $n \geq 5$

41. Prove or disprove the generalization of the following two facts:

$$3^2 + 4^2 = 5^2$$
$$3^3 + 4^3 + 5^3 = 6^3$$

42. Prove or disprove: $n^2 + 21n + 1$ is a prime number for all natural numbers n.

If $\{a_n\}$ and $\{b_n\}$ are two sequences, we write $\{a_n\} = \{b_n\}$ if and only if $a_n = b_n$, for every natural number n. In Problems 43–46, use mathematical induction to show that $\{a_n\} = \{b_n\}$.

43. $a_1 = 1, a_n = a_{n-1} + 2; b_n = 2n - 1$

44. $a_1 = 2, a_n = a_{n-1} + 2; b_n = 2n$

45. $a_1 = 2, a_n = 2^2 a_{n-1}; b_n = 2^{2n-1}$

46. $a_1 = 2, a_n = 3a_{n-1}; b_n = 2 \cdot 3^{n-1}$

10-3 Arithmetic and Geometric Sequences

- Arithmetic and Geometric Sequences
- Developing *n*th-Term Formulas
- Developing Sum Formulas for Finite Arithmetic Series
- Developing Sum Formulas for Finite Geometric Series
- Developing a Sum Formula for Infinite Geometric Series

For most sequences, it is difficult to add up an arbitrary number of terms of the sequence without adding the terms one at a time. In this section we, will study two special types of sequences, *arithmetic sequences* and *geometric sequences*. One of the things that make them special is that we can develop formulas for the sum of the corresponding series.

Arithmetic and Geometric Sequences

Consider the sequence defined by the general term $a_n = 5 + 2(n - 1)$, $n \geq 1$. The first five terms are 5, 7, 9, 11, and 13. It's not hard to see that after starting at 5, every term is obtained by adding 2 to the previous term. This is an example of an arithmetic sequence.

DEFINITION 1 Arithmetic Sequence

A sequence

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

is called an **arithmetic sequence,** or **arithmetic progression,** if there exists a constant d, called the **common difference,** such that

$$a_n - a_{n-1} = d$$

That is,

$$a_n = a_{n-1} + d \qquad \text{for every } n > 1$$

In short, a sequence is arithmetic when every term is obtained by adding some fixed number to the previous term. This fixed number is called the *common difference,* and is usually represented by the letter d.

Now consider the sequence with general term $a_n = 5(\frac{1}{2})^{n-1}$. The first five terms are 5, 10, 20, 40, and 80. It also starts at 5, but this time every term is obtained by *multiplying* the previous term by 2. This is an example of a geometric sequence.

› DEFINITION 2 Geometric Sequence

A sequence

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

is called a **geometric sequence,** or **geometric progression,** if there exists a nonzero constant r, called the **common ratio,** such that

$$\frac{a_n}{a_{n-1}} = r$$

That is,

$$a_n = ra_{n-1} \qquad \text{for every } n > 1$$

In short, a sequence is geometric when every term is obtained by multiplying the previous term by some fixed number. This fixed number is called the *common ratio,* and is usually represented by the letter r.

››› EXPLORE-DISCUSS 1

(A) Graph the arithmetic sequence 5, 7, 9, ...
Describe the graph of any arithmetic sequence with common difference 2.

(B) Graph the geometric sequence 5, 10, 20, ...
Describe the graph of any geometric sequence with common ratio 2.

EXAMPLE 1 Recognizing Arithmetic and Geometric Sequences

Which of the following could be the first four terms of an arithmetic sequence? Of a geometric sequence?

(A) 1, 2, 3, 5, . . .
(B) $-1, 3, -9, 27, \ldots$
(C) 3, 3, 3, 3, . . .
(D) 10, 8.5, 7, 5.5, . . .

SOLUTIONS

(A) To check if a sequence could be arithmetic, find the differences between successive terms:

$$2 - 1 = 1; \qquad 3 - 2 = 1; \qquad 5 - 3 = 2$$

There is no common difference, so the sequence is not arithmetic.

To check if a sequence could be geometric, divide each term by the previous:

$$\frac{2}{1} = 2; \qquad \frac{3}{2} = 1.5$$

There is no common ratio, so the sequence is not geometric.

(B) The sequence is geometric with common ratio -3, but it is not arithmetic.

(C) The sequence is arithmetic with common difference 0 and it is also geometric with common ratio 1.

(D) The sequence is arithmetic with common difference -1.5, but it is not geometric.

MATCHED PROBLEM 1

Which of the following could be the first four terms of an arithmetic sequence? Of a geometric sequence?

(A) 8, 2, 0.5, 0.125, . . . (B) $-7, -2, 3, 8, \ldots$ (C) 1, 5, 25, 100, . . .

Developing *n*th-Term Formulas

If $\{a_n\}$ is an arithmetic sequence with common difference d, then

$$a_2 = a_1 + d \qquad \text{Definition of arithmetic sequence}$$
$$a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d$$
$$a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + 3d$$

This suggests Theorem 1, which can be proved by mathematical induction (see Problem 69 in Exercises 10.3).

THEOREM 1 The *n*th Term of an Arithmetic Sequence

$$a_n = a_1 + (n - 1)d \qquad \text{for every } n > 1$$

Similarly, if $\{a_n\}$ is a geometric sequence with common ratio r, then

$$\begin{aligned} a_2 &= a_1 r && \text{Definition of geometric sequence} \\ a_3 &= a_2 r = (a_1 r) r = a_1 r^2 \\ a_4 &= a_3 r = a_1 r^3 = (a_1 r^2) r = a_1 r^3 \end{aligned}$$

This suggests Theorem 2, which can also be proved by mathematical induction (see Problem 75 in the exercises).

› THEOREM 2 The *n*th Term of a Geometric Sequence

$$a_n = a_1 r^{n-1} \qquad \text{for every } n > 1$$

EXAMPLE 2 Finding Terms in Arithmetic and Geometric Sequences

(A) If the first and tenth terms of an arithmetic sequence are 3 and 30, respectively, find the fiftieth term of the sequence.

(B) If the first and tenth terms of a geometric sequence are 1 and 4, find the seventeenth term to three decimal places.

SOLUTIONS

(A) First use Theorem 1 with $a_1 = 3$ and $a_{10} = 30$ to find d:

$$\begin{aligned} a_n &= a_1 + (n-1)d && \text{Substitute } n = 10. \\ a_{10} &= a_1 + (10-1)d && \text{Substitute } a_{10} = 30, a_1 = 3. \\ 30 &= 3 + 9d && \text{Solve for } d. \\ d &= 3 \end{aligned}$$

Now find a_{50}:

$$\begin{aligned} a_{50} &= a_1 + (50-1)3 && \text{Substitute } a_1 = 3. \\ &= 3 + 49 \cdot 3 && \text{Simplify.} \\ &= 150 \end{aligned}$$

(B) First let $n = 10, a_1 = 1, a_{10} = 4$ and use Theorem 2 to find r.

$$\begin{aligned} a_n &= a_1 r^{n-1} && \text{Substitute } n = 10, a_1 = 1, a_{10} = 4. \\ 4 &= 1 r^{10-1} && \text{Solve for } r. \\ r &= 4^{1/9} \end{aligned}$$

Now use Theorem 2 again, this time with $n = 17$.

$$a_{17} = a_1 r^{16} = 1(4^{1/9})^{16} = 4^{16/9} \approx 11.758$$

MATCHED PROBLEM 2

(A) If the first and fifteenth terms of an arithmetic sequence are -5 and 23, respectively, find the seventy-third term of the sequence.

(B) Find the eighth term of the geometric sequence $\frac{1}{64}, -\frac{1}{32}, \frac{1}{16}, \ldots$.

Developing Sum Formulas for Finite Arithmetic Series

If $a_1, a_2, a_3, \ldots, a_n$ is a finite arithmetic sequence, then the corresponding series $a_1 + a_2 + a_3 + \cdots + a_n$ is called an *arithmetic series*. We will derive two simple and very useful formulas for the sum of an arithmetic series.

Throughout the remainder of the section, we will use the symbol S_n to represent the sum of the first n terms of a sequence. For example, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$, and in general $S_n = a_1 + a_2 + a_3 + \cdots + a_n$. This is often referred to as the *nth partial sum* of a series.

Let d be the common difference of the arithmetic sequence $a_1, a_2, a_3, \ldots, a_n$. Then

$$S_n = \overset{a_1}{a_1} + \overset{a_2}{(a_1 + d)} + \overset{a_3}{(a_1 + 2d)} + \cdots + \overset{a_{n-1}}{[a_1 + (n-2)d]} + \overset{a_n}{[a_1 + (n-1)d]} \quad (1)$$

If we write the same sum in the opposite order, we get

$$S_n = [a_1 + (n-1)d] + [a_1 + (n-2)d] + \cdots + (a_1 + 2d) + (a_1 + d) + a_1 \quad (2)$$

Now if we add the first terms of equations (1) and (2), we get

$$a_1 + [a_1 + (n-1)d] = 2a_1 + (n-1)d$$

Adding the second terms of equations (1) and (2),

$$(a_1 + d) + [a_1 + (n-2)d] = 2a_1 + (n-1)d$$

In fact, we get the same result for each matching pair of terms from equations (1) and (2), so if we add these two entire equations, the right side will be n identical copies of the number $2a_1 + (n-1)d$, which is n times $2a_1 + (n-1)d$. The left side will simply be $S_n + S_n$, or $2S_n$.

$$2S_n = n[2a_1 + (n-1)d] \qquad \text{Divide both sides by 2.}$$

$$S_n = \frac{n}{2}[2a_1 + (n-1)d]$$

We have developed Theorem 3.

› THEOREM 3 Sum of an Arithmetic Series—First Form

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$$

We can obtain a second useful formula for the sum with the help of Theorem 1.

$$\begin{aligned} S_n &= \frac{n}{2}[2a_1 + (n - 1)d] && \text{Rewrite } 2a_1 \text{ as } a_1 + a_1. \\ &= \frac{n}{2}[a_1 + a_1 + (n - 1)d] && a_1 + (n - 1)d = a_n \text{ (Theorem 1)} \\ &= \frac{n}{2}(a_1 + a_n) \end{aligned}$$

› THEOREM 4 Sum of an Arithmetic Series—Second Form

$$S_n = \frac{n}{2}(a_1 + a_n)$$

The first sum formula can also be proved using mathematical induction (see Problem 70 in the exercises).

EXAMPLE 3 Finding the Sum of an Arithmetic Series

Find the sum of the first 26 terms of an arithmetic series if the first term is -7 and $d = 3$.

SOLUTIONS

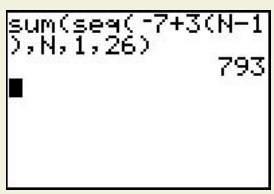
Algebraic Solution

Let $n = 26$, $a_1 = -7$, $d = 3$, and use Theorem 3.

$$\begin{aligned} S_n &= \frac{n}{2}[2a_1 + (n - 1)d] && \text{Substitute } n = 26, a_1 = -7, \text{ and } d = 3. \\ S_{26} &= \tfrac{26}{2}[2(-7) + (26 - 1)3] && \text{Simplify.} \\ &= 793 \end{aligned}$$

Graphing Calculator Solution

By Theorem 1, the general term of the series is $a_n = -7 + 3(n - 1)$, which we can sum using the SUM and SEQ commands (Fig. 1). (See Explore-Discuss 4 in the section before last.)



› Figure 1

MATCHED PROBLEM 3

Find the sum of the first 52 terms of an arithmetic series if the first term is 23 and $d = -2$.

EXAMPLE 4 Finding the Sum of an Arithmetic Series

Find the sum of all the odd numbers between 51 and 99, inclusive.

SOLUTIONS

Algebraic Solution

The odd numbers from 51 to 99 form an arithmetic sequence with common difference 2.

First, use $a_1 = 51$, $a_n = 99$, and Theorem 1 to find n:

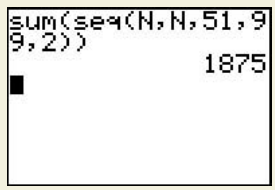
$$a_n = a_1 + (n - 1)d \qquad \text{Substitute } a_n = 99,\ a_1 = 51, \text{ and } d = 2.$$
$$99 = 51 + (n - 1)2 \qquad \text{Solve for } n.$$
$$n = 25$$

Now use Theorem 4 to find S_{25}:

$$S_n = \frac{n}{2}(a_1 + a_n) \qquad \text{Substitute } n = 25,\ a_1 = 51, \text{ and } a_n = 99.$$
$$S_{25} = \tfrac{25}{2}(51 + 99) \qquad \text{Simplify.}$$
$$= 1{,}875$$

Graphing Calculator Solution

We can again sum using the SUM and SEQ commands. We sum the sequence with nth term equal to n, and sum over values from 51 to 99, and increment by 2 (Fig. 2).



› Figure 2

MATCHED PROBLEM 4

Find the sum of all the even numbers between -22 and 52, inclusive.

EXAMPLE 5 Prize Money

A 16-team bowling league has $8,000 to be awarded as prize money. If the last-place team is awarded $275 in prize money and the award increases by the same amount for each successive finishing place, how much will the first-place team receive?

SOLUTION

If a_1 is the award for the first-place team, a_2 is the award for the second-place team, and so on, then the prize money awards form an arithmetic sequence with $n = 16$, $a_{16} = 275$, and $S_{16} = 8{,}000$. We can use Theorem 4 to find a_1.

$$S_n = \frac{n}{2}(a_1 + a_n) \qquad \text{Substitute } n = 16,\ S_{16} = 8{,}000, \text{ and } a_{16} = 275.$$

$$8{,}000 = \tfrac{16}{2}(a_1 + 275) \qquad \text{Solve for } a_1.$$

$$a_1 = 725$$

The first-place team receives \$725.

MATCHED PROBLEM 5

Refer to Example 5. How much prize money is awarded to the second-place team?

Developing Sum Formulas for Finite Geometric Series

If $a_1, a_2, a_3, \ldots, a_n$ is a finite geometric sequence, then the corresponding series $a_1 + a_2 + a_3 + \cdots + a_n$ is called a *geometric series*. As with arithmetic series, we can derive two simple and very useful formulas for the sum of a geometric series. Let r be the common ratio of the geometric sequence $a_1, a_2, a_3, \ldots, a_n$ and let S_n again denote the sum of the series $a_1 + a_2 + a_3 + \cdots + a_n$. Then

$$S_n = a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-2} + a_1r^{n-1} \tag{3}$$

If we multiply both sides of this equation by r (you'll see why in a bit), we get

$$rS_n = a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1} + a_1r^n \tag{4}$$

Now we will subtract equation (4) from equation (3):

$$\begin{aligned} S_n &= a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-2} + a_1r^{n-1} \\ -rS_n &= \qquad a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-2} + a_1r^{n-1} + a_1r^n \\ \hline S_n - rS_n &= a_1 - a_1r^n \end{aligned}$$

Most of the terms on the right cancelled! Now we can factor the left side and solve for S_n:

$$S_n - rS_n = a_1 - a_1r^n \qquad \text{Factor out } S_n \text{ on the left side.}$$

$$S_n(1 - r) = a_1 - a_1r^n \qquad \text{Divide both sides by } 1 - r \text{ (for } r \neq 1).$$

$$S_n = \frac{a_1 - a_1r^n}{1 - r}$$

We have developed Theorem 5.

› THEOREM 5 Sum of a Geometric Series—First Form

$$S_n = \frac{a_1 - a_1 r^n}{1 - r} \qquad r \neq 1$$

We can develop an alternative form using Theorem 2.

$$a_n = a_1 r^{n-1}$$

Multiply both sides by r.

$$ra_n = a_1 r^n$$

› THEOREM 6 Sum of a Geometric Series—Second Form

$$S_n = \frac{a_1 - ra_n}{1 - r} \qquad r \neq 1$$

The proof of the first sum formula (Theorem 5) by mathematical induction is left as an exercise (see Problem 76 in the exercises), as is the case where $r = 1$ (Problem 52).

EXAMPLE 6 Finding the Sum of a Geometric Series

Find the sum of the first 20 terms of a geometric series if the first term is 1 and $r = 2$.

SOLUTIONS

Algebraic Solution

Let $n = 20$, $a_1 = 1$, $r = 2$, and use Theorem 5.

$$S_n = \frac{a_1 - a_1 r^n}{1 - r}$$

$$= \frac{1 - 1 \cdot 2^{20}}{1 - 2} = 1{,}048{,}575$$

Graphing Calculator Solution

The general term of the series is $a_n = a_1 r^{n-1} = 2^{n-1}$. We can use the SUM and SEQ commands to sum the first 20 terms (Fig. 3).

› Figure 3

MATCHED PROBLEM 6

Find the sum, to two decimal places, of the first 14 terms of a geometric series if the first term is $\frac{1}{64}$ and $r = -2$.

Developing a Sum Formula for Infinite Geometric Series

We now turn to a topic that most students find nonintuitive, maybe even bizarre: adding up all of the terms of an infinite series. It seems obvious that the result should be infinite. Or does it?

Consider an infinite geometric series with $a_1 = 5$ and $r = \frac{1}{2}$. We will approach the problem of finding the sum by looking at what happens to the sum S_n as n increases. To answer that question, it is helpful to first write the sum formula from Theorem 5 in a different form by splitting the fraction into two terms. (This is the opposite of subtracting fractions with a common denominator.)

$$S_n = \frac{a_1 - a_1 r^n}{1 - r} = \frac{a_1}{1 - r} - \frac{a_1 r^n}{1 - r} \tag{5}$$

For $a_1 = 5$ and $r = \frac{1}{2}$,

$$S_n = \frac{5}{1 - 1/2} - \frac{5(1/2)^n}{1 - 1/2} = 10 - 10\left(\frac{1}{2}\right)^n$$

Let's find the sum for some values of n:

$$S_2 = 10 - 10\left(\frac{1}{4}\right) = 7.5$$

$$S_4 = 10 - 10\left(\frac{1}{16}\right) = 9.375$$

$$S_{10} = 10 - 10\left(\frac{1}{1{,}024}\right) \approx 9.99023$$

$$S_{20} = 10 - 10\left(\frac{1}{1{,}048{,}576}\right) \approx 9.99999$$

It appears that $(\frac{1}{2})^n$ becomes smaller and smaller as n increases and that the sum gets closer and closer to 10.

In general, it is possible to show that, if $|r| < 1$, then r^n will get closer and closer to 0 as n increases. In symbols, $r^n \to 0$ as $n \to \infty$. When this happens, the term

$$\frac{a_1 r^n}{1 - r}$$

in equation (5) will approach 0 as n increases, and S_n will approach

$$\frac{a_1}{1 - r}$$

In other words, if $|r| < 1$, then S_n can be made as close to

$$\frac{a_1}{1 - r}$$

as we wish by choosing n sufficiently large. The result is that we can we define the **sum of an infinite geometric series** by the following formula:

› DEFINITION 3 Sum of an Infinite Geometric Series

$$S_\infty = \frac{a_1}{1 - r} \qquad |r| < 1$$

When an infinite series has a sum, we say that it *converges*. If $|r| \geq 1$ an infinite geometric series has no sum. We say that the series *diverges* in this case. (See Problems 79 and 80.)

EXAMPLE 7 Expressing a Repeating Decimal as a Fraction

Represent the repeating decimal $0.454\,545 \cdots = 0.\overline{45}$ as the quotient of two integers. Recall that a repeating decimal corresponds to a rational number, and that any rational number can be represented as the quotient of two integers.

SOLUTION

$$\begin{aligned} 0.\overline{45} &= 0.45 + 0.0045 + 0.000\,045 + \cdots \\ &= \frac{45}{100} + \frac{45}{10{,}000} + \frac{45}{1{,}000{,}000} + \cdots \end{aligned}$$

The right side of the equation is an infinite geometric series with $a_1 = \frac{45}{100}$ and $r = \frac{1}{100}$. Thus,

$$S_\infty = \frac{a_1}{1 - r} = \frac{\frac{45}{100}}{1 - \frac{1}{100}} = \frac{0.45}{0.99} = \frac{5}{11}$$

Hence, $0.\overline{45}$ and $\frac{5}{11}$ define the same rational number. You can check the result by dividing 5 by 11.

MATCHED PROBLEM 7

Repeat Example 7 for $0.818\,181 \cdots = 0.\overline{81}$.

EXAMPLE 8 Economy Stimulation

A state government uses proceeds from a lottery to provide a tax rebate for property owners. Suppose an individual receives a $500 rebate and spends 80% of this, and each of the recipients of the money spent by this individual also spends 80% of what he or

she receives, and this process continues without end. According to the **multiplier doctrine** in economics, the effect of the original \$500 tax rebate on the economy is multiplied many times. What is the total amount spent if the process continues as indicated?

SOLUTION

The individual receives \$500 and spends 0.8(500) = \$400. The recipients of this \$400 spend 0.8(400) = \$320, the recipients of this \$320 spend 0.8(320) = \$256, and so on. The total spending generated by the \$500 rebate is

$$400 + 320 + 256 + \cdots = 400 + 0.8(400) + (0.8)^2(400) + \cdots$$

which we recognize as an infinite geometric series with $a_1 = 400$ and $r = 0.8$. According to Definition 3, the total amount spent is

$$S_\infty = \frac{a_1}{1 - r} = \frac{400}{1 - 0.8} = \frac{400}{0.2} = \$2{,}000$$

MATCHED PROBLEM 8

Repeat Example 8 if the tax rebate is \$1,000 and the percentage spent by all recipients is 90%.

››› EXPLORE-DISCUSS 2

(A) Find an infinite geometric series with $a_1 = 10$ whose sum is 1,000.

(B) Find an infinite geometric series with $a_1 = 10$ whose sum is 6.

(C) Suppose that an infinite geometric series with $a_1 = 10$ has a sum. Explain why that sum must be greater than 5.

ANSWERS TO MATCHED PROBLEMS

1. (A) The sequence is geometric with $r = \frac{1}{4}$, but not arithmetic.
(B) The sequence is arithmetic with $d = 5$, but not geometric.
(C) The sequence is neither arithmetic nor geometric.
2. (A) 139 (B) −2 **3.** −1,456 **4.** 570 **5.** \$695 **6.** −85.33
7. $\frac{9}{11}$ **8.** \$9,000

10-3 Exercises

1. Given the first five terms of a sequence, explain how you would decide if the sequence is arithmetic.

2. Given the first five terms of a sequence, explain how you would decide if the sequence is geometric.

3. Explain what the symbol S_n is used to represent in the study of series.

4. Explain why an infinite arithmetic series cannot have a sum.

In Problems 5 and 6, determine whether the following can be the first three terms of an arithmetic or geometric sequence, and, if so, find the common difference or common ratio and the next two terms of the sequence.

5. (A) $-11, -16, -21, \ldots$ (B) $2, -4, 8, \ldots$
(C) $1, 4, 9, \ldots$ (D) $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \ldots$

6. (A) $5, 20, 100, \ldots$ (B) $-5, -5, -5, \ldots$
(C) $7, 6.5, 6, \ldots$ (D) $512, 256, 128, \ldots$

In Problems 7–14, let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be an arithmetic sequence. Find the indicated quantities.

7. $a_1 = -5, d = 4; a_2 = ?, a_3 = ?, a_4 = ?$

8. $a_1 = -18, d = 3; a_2 = ?, a_3 = ?, a_4 = ?$

9. $a_1 = -3, d = 5; a_{15} = ?, S_{11} = ?$

10. $a_1 = 3, d = 4; a_{22} = ?, S_{21} = ?$

11. $a_1 = 1, a_2 = 5; S_{21} = ?$

12. $a_1 = 5, a_2 = 11; S_{11} = ?$

13. $a_1 = 7, a_2 = 5; a_{15} = ?$

14. $a_1 = -3, d = -4; a_{10} = ?$

In Problems 15–20, let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be a geometric sequence. Find each of the indicated quantities.

15. $a_1 = -6, r = -\frac{1}{2}; a_2 = ?, a_3 = ?, a_4 = ?$

16. $a_1 = 12, r = \frac{2}{3}; a_2 = ?, a_3 = ?, a_4 = ?$

17. $a_1 = 81, r = \frac{1}{3}; a_{10} = ?$

18. $a_1 = 64, r = \frac{1}{2}; a_{13} = ?$

19. $a_1 = 3, a_7 = 2{,}187, r = 3; S_7 = ?$

20. $a_1 = 1, a_7 = 729, r = -3; S_7 = ?$

In Problems 21–28, let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be an arithmetic sequence. Find the indicated quantities.

21. $a_1 = 3, a_{20} = 117; d = ?, a_{101} = ?$

22. $a_1 = 7, a_8 = 28; d = ?, a_{25} = ?$

23. $a_1 = -12, a_{40} = 22; S_{40} = ?$

24. $a_1 = 24, a_{24} = -28; S_{24} = ?$

25. $a_1 = \frac{1}{3}, a_2 = \frac{1}{2}; a_{11} = ?, S_{11} = ?$

26. $a_1 = \frac{1}{6}, a_2 = \frac{1}{4}; a_{19} = ?, S_{19} = ?$

27. $a_3 = 13, a_{10} = 55; a_1 = ?$

28. $a_9 = -12, a_{13} = 3; a_1 = ?$

In Problems 29–34, let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be a geometric sequence. Find each of the indicated quantities.

29. $a_1 = 100, a_6 = 1; r = ?$ **30.** $a_1 = 10, a_{10} = 30; r = ?$

31. $a_1 = 5, r = -2; S_{10} = ?$ **32.** $a_1 = 3, r = 2; S_{10} = ?$

33. $a_1 = 9, a_4 = \frac{8}{3}; a_2 = ?, a_3 = ?$

34. $a_1 = 12, a_4 = -\frac{4}{9}; a_2 = ?, a_3 = ?$

35. $S_{51} = \sum_{k=1}^{51}(3k + 3) = ?$ **36.** $S_{40} = \sum_{k=1}^{40}(2k - 3) = ?$

37. $S_7 = \sum_{k=1}^{7}(-3)^{k-1} = ?$ **38.** $S_7 = \sum_{k=1}^{7}3^k = ?$

39. Find $g(1) + g(2) + g(3) + \cdots + g(51)$ if $g(t) = 5 - t$.

40. Find $f(1) + f(2) + f(3) + \cdots + f(20)$ if $f(x) = 2x - 5$.

41. Find $g(1) + g(2) + \cdots + g(10)$ if $g(x) = (\frac{1}{2})^x$.

42. Find $f(1) + f(2) + \cdots + f(10)$ if $f(x) = 2^x$.

43. Find the sum of all the even integers between 21 and 135.

44. Find the sum of all the odd integers between 100 and 500.

45. Show that the sum of the first n odd natural numbers is n^2, using appropriate formulas from this section.

46. Show that the sum of the first n even natural numbers is $n + n^2$, using appropriate formulas from this section.

47. Find a positive number x so that $-2 + x - 6$ is a three-term geometric series.

48. Find a positive number x so that $6 + x + 8$ is a three-term geometric series.

49. For a given sequence in which $a_1 = -3$ and $a_n = a_{n-1} + 3, n > 1$, find a_n in terms of n.

50. For the sequence in Problem 49, find $S_n = \sum_{k=1}^{n}a_k$ in terms of n.

51. Find a formula for S_n for an arithmetic sequence with $d = 0$.

52. Find a formula for S_n for a geometric sequence with $r = 1$.

In Problems 53–56, find the least positive integer n such that $a_n < b_n$ by graphing the sequences $\{a_n\}$ and $\{b_n\}$ with a graphing utility. Check your answer by using a graphing calculator to display both sequences in table form.

53. $a_n = 5 + 8n, b_n = 1.1^n$

54. $a_n = 96 + 47n, b_n = 8(1.5)^n$

55. $a_n = 1{,}000(0.99)^n, b_n = 2n + 1$

56. $a_n = 500 - n, b_n = 1.05^n$

In Problems 57–62, find the sum of each infinite geometric series that has a sum.

57. $3 + 1 + \frac{1}{3} + \cdots$

58. $16 + 4 + 1 + \cdots$

59. $2 + 4 + 6 + \cdots$

60. $4 + 6 + 9 + \cdots$

61. $2 - \frac{1}{2} + \frac{1}{8} - \cdots$

62. $21 - 3 + \frac{3}{7} - \cdots$

In Problems 63–68, represent each repeating decimal as the quotient of two integers.

63. $0.\overline{7} = 0.7777\cdots$

64. $0.\overline{5} = 0.5555\cdots$

65. $0.\overline{54} = 0.545\,454\cdots$

66. $0.\overline{27} = 0.272\,727\cdots$

67. $3.\overline{216} = 3.216\,216\,216\cdots$

68. $5.\overline{63} = 5.636\,363\cdots$

69. Prove, using mathematical induction, that if $\{a_n\}$ is an arithmetic sequence, then

$$a_n = a_1 + (n - 1)d \qquad \text{for every } n > 1$$

70. Prove, using mathematical induction, that if $\{a_n\}$ is an arithmetic sequence, then

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$$

71. If in a given sequence, $a_1 = -2$ and $a_n = -3a_{n-1}, n > 1$, find a_n in terms of n.

72. For the sequence in Problem 71, find $S_n = \sum_{k=1}^{n} a_k$ in terms of n.

73. Show that $(x^2 + xy + y^2), (z^2 + xz + x^2)$, and $(y^2 + yz + z^2)$ are consecutive terms of an arithmetic progression if x, y, and z form an arithmetic progression. (From U.S.S.R. Mathematical Olympiads, 1955–1956, Grade 9.)

74. Take 121 terms of each arithmetic progression 2, 7, 12, . . . and 2, 5, 8, ... How many numbers will there be in common? (From U.S.S.R. Mathematical Olympiads, 1955–1956, Grade 9.)

75. Prove, using mathematical induction, that if $\{a_n\}$ is a geometric sequence, then

$$a_n = a_1 r^{n-1} \qquad n \in N$$

76. Prove, using mathematical induction, that if $\{a_n\}$ is a geometric sequence, then

$$S_n = \frac{a_1 - a_1 r^n}{1 - r} \qquad n \in N, r \neq 1$$

77. Given the system of equations

$$ax + by = c$$
$$dx + ey = f$$

where a, b, c, d, e, f is any arithmetic progression with a nonzero constant difference, show that the system has a unique solution.

78. The sum of the first and fourth terms of an arithmetic sequence is 2, and the sum of their squares is 20. Find the sum of the first eight terms of the sequence.

79. For a geometric series with $a_1 = 1$ and $r = 2$, find S_5, S_{10}, S_{15}, and S_{20}. What do you suspect happens to S_n as n approaches infinity for a geometric series with $r > 1$?

80. For a geometric series with $a_1 = 1$ and $r = -2$, find S_5, S_{10}, S_{15}, and S_{20}. What do you suspect happens to S_n as n approaches infinity for a geometric series with $r < -1$?

APPLICATIONS

81. **BUSINESS** In investigating different job opportunities, you find that firm A will start you at \$25,000 per year and guarantee you a raise of \$1,200 each year whereas firm B will start you at \$28,000 per year but will guarantee you a raise of only \$800 each year. Over a period of 15 years, how much would you receive from each firm?

82. **BUSINESS** In Problem 81, what would be your annual salary at each firm for the tenth year?

83. **ECONOMICS** The government, through a subsidy program, distributes \$1,000,000. If we assume that each individual or agency spends 0.8 of what is received, and 0.8 of this is spent, and so on, how much total increase in spending results from this government action?

84. **ECONOMICS** Because of reduced taxes, an individual has an extra \$600 in spendable income. If we assume that the individual spends 70% of this on consumer goods, that the producers of these goods in turn spend 70% of what they receive on consumer goods, and that this process continues indefinitely, what is the total amount spent on consumer goods?

85. **BUSINESS** If \$$P$ is invested at r% compounded annually, the amount A present after n years forms a geometric sequence with a common ratio $1 + r$. Write a formula for the amount present after n years. How long will it take a sum of money P to double if invested at 6% interest compounded annually?

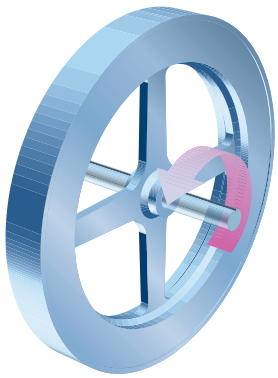
86. **POPULATION GROWTH** If a population of A_0 people grows at the constant rate of r% per year, the population after t years forms a geometric sequence with a common ratio $1 + r$. Write

a formula for the total population after t years. If the world's population is increasing at the rate of 2% per year, how long will it take to double?

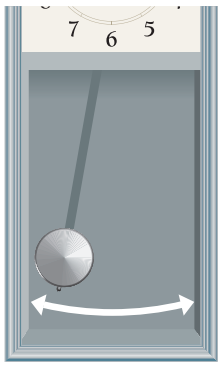
87. **FINANCE** Eleven years ago an investment earned \$7,000 for the year. Last year the investment earned \$14,000. If the earnings from the investment have increased the same amount each year, what is the yearly increase and how much income has accrued from the investment over the past 11 years?

88. **AIR TEMPERATURE** As dry air moves upward, it expands. In so doing, it cools at the rate of about 5°F for each 1,000-foot rise. This is known as the **adiabatic process.**
(A) Temperatures at altitudes that are multiples of 1,000 feet form what kind of a sequence?
(B) If the ground temperature is 80°F, write a formula for the temperature T_n in terms of n, if n is in thousands of feet.

89. **ENGINEERING** A rotating flywheel coming to rest rotates 300 revolutions the first minute (see the figure). If in each subsequent minute it rotates two-thirds as many times as in the preceding minute, how many revolutions will the wheel make before coming to rest?



90. **PHYSICS** The first swing of a bob on a pendulum is 10 inches. If on each subsequent swing it travels 0.9 as far as on the preceding swing, how far will the bob travel before coming to rest?



91. **FOOD CHAIN** A plant is eaten by an insect, an insect by a trout, a trout by a salmon, a salmon by a bear, and the bear is eaten by you. If only 20% of the energy is transformed from one stage to the next, how many calories must be supplied by plant food to provide you with 2,000 calories from the bear meat?

92. **GENEALOGY** If there are 30 years in a generation, how many direct ancestors did each of us have 600 years ago? By *direct* ancestors we mean parents, grandparents, great-grandparents, and so on.

93. **PHYSICS** An object falling from rest in a vacuum near the surface of the Earth falls 16 feet during the first second, 48 feet during the second second, 80 feet during the third second, and so on.
(A) How far will the object fall during the eleventh second?
(B) How far will the object fall in 11 seconds?
(C) How far will the object fall in t seconds?

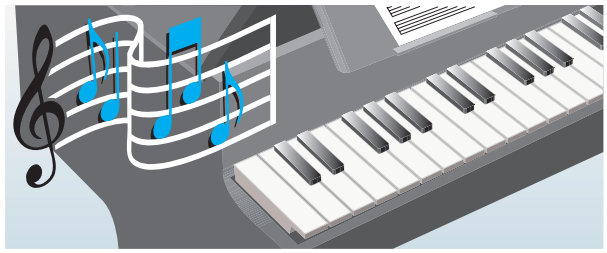
94. **PHYSICS** In Problem 93, how far will the object fall during:
(A) The twentieth second? (B) The tth second?

95. **BACTERIA GROWTH** A single cholera bacterium divides every $\frac{1}{2}$ hour to produce two complete cholera bacteria. If we start with a colony of A_0 bacteria, how many bacteria will we have in t hours, assuming adequate food supply?

96. **CELL DIVISION** One leukemic cell injected into a healthy mouse will divide into two cells in about $\frac{1}{2}$ day. At the end of the day these two cells will divide again, with the doubling process continuing each $\frac{1}{2}$ day until there are 1 billion cells, at which time the mouse dies. On which day after the experiment is started does this happen?

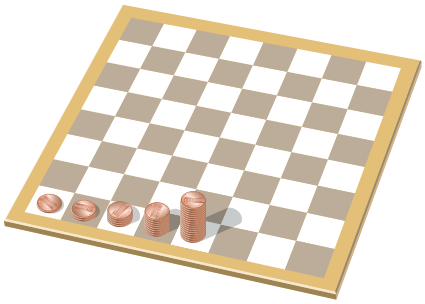
97. **ASTRONOMY** Ever since the time of the Greek astronomer Hipparchus, second century B.C., the brightness of stars has been measured in terms of magnitude. The brightest stars, excluding the sun, are classed as magnitude 1, and the dimmest visible to the eye are classed as magnitude 6. In 1856, the English astronomer N. R. Pogson showed that first-magnitude stars are 100 times brighter than sixth-magnitude stars. If the ratio of brightness between consecutive magnitudes is constant, find this ratio. [*Hint:* If b_n is the brightness of an nth-magnitude star, find r for the geometric progression $b_1, b_2, b_3, \ldots$, given $b_1 = 100b_6$.]

98. **MUSIC** The notes on a piano, as measured in cycles per second, form a geometric progression.
(A) If A is 400 cycles per second and A′, 12 notes higher, is 800 cycles per second, find the constant ratio r.
(B) Find the cycles per second for C, three notes higher than A.



99. **PUZZLE** If you place 1¢ on the first square of a chessboard, 2¢ on the second square, 4¢ on the third, and so on, continuing

to double the amount until all 64 squares are covered, how much money will be on the sixty-fourth square? How much money will there be on the whole board?



100. PUZZLE If a sheet of very thin paper 0.001-inch thick is torn in half, and each half is again torn in half, and this process is repeated for a total of 32 times, how high will the stack of paper be if the pieces are placed one on top of the other? Give the answer to the nearest mile.

101. ATMOSPHERIC PRESSURE If atmospheric pressure decreases roughly by a factor of 10 for each 10-mile increase in altitude up to 60 miles, and if the pressure is 15 pounds per square inch at sea level, what will the pressure be 40 miles up?

102. ZENO'S PARADOX Visualize a hypothetical 440-yard oval racetrack that has tapes stretched across the track at the halfway point and at each point that marks the halfway point of each remaining distance thereafter. A runner running around the track has to break the first tape before the second, the second before the third, and so on. From this point of view it appears that he will never finish the race. This famous paradox is attributed to the Greek philosopher Zeno (495–435 B.C.). If we assume the runner runs at 440 yards per minute, the times between tape breakings form an infinite geometric progression. What is the sum of this progression?

103. GEOMETRY If the midpoints of the sides of an equilateral triangle are joined by straight lines, the new figure will be an equilateral triangle with a perimeter equal to half the original. If we start with an equilateral triangle with perimeter 1 and form a sequence of "nested" equilateral triangles proceeding as described, what will be the total perimeter of all the triangles that can be formed in this way?

104. PHOTOGRAPHY The shutter speeds and f-stops on a camera are given as follows:

Shutter speeds: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{15}, \frac{1}{30}, \frac{1}{60}, \frac{1}{125}, \frac{1}{250}, \frac{1}{500}$

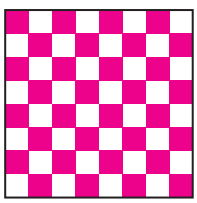
f-stops: 1.4, 2, 2.8, 4, 5.6, 8, 11, 16, 22

These are very close to being geometric progressions. Estimate their common ratios.

105. GEOMETRY We know that the sum of the interior angles of a triangle is 180°. Show that the sums of the interior angles of polygons with 3, 4, 5, 6, . . . sides form an arithmetic sequence. Find the sum of the interior angles for a 21-sided polygon.

10-4 The Multiplication Principle, Permutations, and Combinations

- Counting With the Multiplication Principle
- Using Factorials
- Counting Permutations
- Counting Combinations



› Figure 1

This section introduces some new mathematical tools that are usually referred to as *counting techniques*. In general, a **counting technique** is a mathematical method of determining the number of objects in a set without actually enumerating the objects in the set as 1, 2, 3, ... For example, we can count the number of squares in a checkerboard (Fig. 1) by counting 1, 2, 3, . . . , 64. This is enumeration. Or we can note that there are eight rows with eight squares in each row. Thus, the total number of squares must be $8 \times 8 = 64$. This is a very simple counting technique.

Now consider the problem of assigning telephone numbers. How many different seven-digit telephone numbers can be formed? As we will soon see, the answer is $10^7 = 10{,}000{,}000$, a number that is much too large to obtain by counting them all individually. Thus, counting techniques are essential tools for determining the number of objects in a set if that number is very large. The techniques developed in this section will be applied to a brief introduction to probability theory in the next section, and to a famous algebraic formula in the last section of this chapter.

› Counting With the Multiplication Principle

The multiplication principle is one of the most basic and useful counting techniques. We will start with an example.

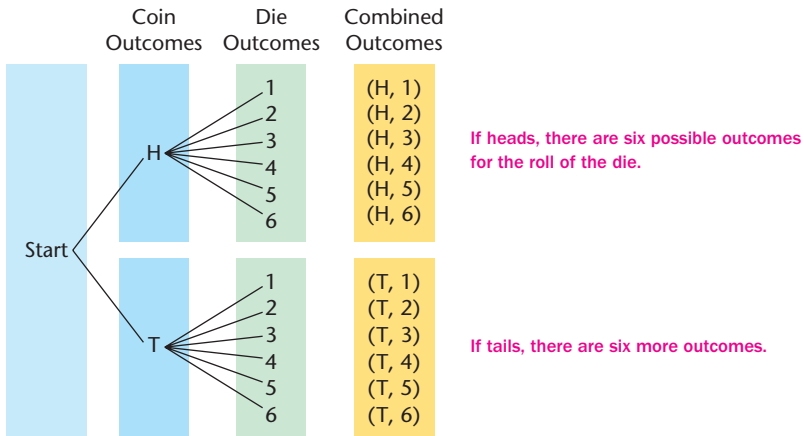
EXAMPLE 1 **Combined Outcomes**

Suppose we flip a coin and then throw a single die (Fig. 2). What are the possible combined outcomes?

› **Figure 2** Coin and die outcomes.

SOLUTION

To solve this problem, we use a **tree diagram:**



There are 12 possible combined outcomes—two ways in which the coin can come up followed by six ways in which the die can come up.

MATCHED PROBLEM 1

Use a tree diagram to determine the number of possible outcomes of throwing a single die followed by flipping a coin.

Now suppose you are asked, "From the 26 letters in the alphabet, how many ways can 3 letters appear in a row on a license plate if no letter is repeated?" To try to count the possibilities using a tree diagram would be extremely time-consuming, to say the least. The following **multiplication principle,** also called the **fundamental counting principle,** enables us to solve this problem easily. In addition, it forms the basis for several other counting techniques developed later in this section.

› THE MULTIPLICATION PRINCIPLE

1. If two operations O_1 and O_2 are performed in order with N_1 possible outcomes for the first operation and N_2 possible outcomes for the second operation, then there are

$$N_1 \cdot N_2$$

possible combined outcomes of the first operation followed by the second.

2. In general, if n operations $O_1, O_2, \ldots, O_n$ are performed in order, with possible number of outcomes $N_1, N_2, \ldots, N_n$, respectively, then there are

$$N_1 \cdot N_2 \cdot \cdots \cdot N_n$$

possible combined outcomes of the operations performed in the given order.

In Example 1, we see that there are two possible outcomes from the first operation of flipping a coin and six possible outcomes from the second operation of throwing a die. Hence, by the multiplication principle, there are $2 \cdot 6 = 12$ possible combined outcomes of flipping a coin followed by throwing a die. This is, of course, the same answer we found using a tree diagram. Now try to use the multiplication principle to solve Matched Problem 1.

To answer the license plate question, we reason as follows: There are 26 ways the first letter can be chosen. After a first letter is chosen, 25 letters remain; hence there are 25 ways a second letter can be chosen. And after 2 letters are chosen, there are 24 ways a third letter can be chosen. Hence, using the multiplication principle, there are $26 \cdot 25 \cdot 24 = 15{,}600$ possible ways 3 letters can be chosen from the alphabet without allowing any letter to repeat. By not allowing any letter to repeat, earlier selections affect the choice of subsequent selections. If we allow letters to repeat, then earlier selections do not affect the choice in subsequent selections, and there are 26 possible choices for each of the 3 letters. So, if we allow letters to repeat, there are $26 \cdot 26 \cdot 26 = 26^3 = 17{,}576$

possible ways the 3 letters can be chosen from the alphabet. (Now it's really easy to see why a tree diagram would be a bad idea for this problem!)

EXAMPLE 2 Computer-Generated Tests

Many universities and colleges are now using computer-assisted testing procedures. Suppose a screening test is to consist of five questions, and a computer stores five equivalent questions for the first test question, eight equivalent questions for the second, six for the third, five for the fourth, and ten for the fifth. How many different five-question tests can the computer select? Two tests are considered different if they differ in one or more questions.

SOLUTION

O_1:	Select the first question	N_1:	five ways
O_2:	Select the second question	N_2:	eight ways
O_3:	Select the third question	N_3:	six ways
O_4:	Select the fourth question	N_4:	five ways
O_5:	Select the fifth question	N_5:	ten ways

Using the multiplication principle, we find that the computer can generate

$$5 \cdot 8 \cdot 6 \cdot 5 \cdot 10 = 12{,}000 \text{ different tests}$$

MATCHED PROBLEM 2

Each question on a multiple-choice test has five choices. If there are five such questions on a test, how many different response sheets are possible if only one choice is marked for each question?

EXAMPLE 3 Counting Code Words

How many three-letter code words are possible using the first eight letters of the alphabet if:

(A) No letter can be repeated? (B) Letters can be repeated?

(C) Adjacent letters cannot be alike?

SOLUTIONS

(A) No letter can be repeated.

O_1:	Select first letter	N_1:	eight ways	
O_2:	Select second letter	N_2:	seven ways	Because one letter has been used
O_3:	Select third letter	N_3:	six ways	Because two letters have been used

There are

$$8 \cdot 7 \cdot 6 = 336 \text{ possible code words}$$

(B) Letters can be repeated.

O_1: Select first letter N_1: eight ways

O_2: Select second letter N_2: eight ways Repeats are allowed.

O_3: Select third letter N_3: eight ways Repeats are allowed.

There are

$$8 \cdot 8 \cdot 8 = 8^3 = 512 \text{ possible code words}$$

(C) Adjacent letters cannot be alike.

O_1: Select first letter N_1: eight ways

O_2: Select second letter N_2: seven ways Cannot be the same as the first

O_3: Select third letter N_3: seven ways Cannot be the same as the second, but can be the same as the first

There are

$$8 \cdot 7 \cdot 7 = 392 \text{ possible code words}$$

MATCHED PROBLEM 3

How many four-letter code words are possible using the first ten letters of the alphabet under the three conditions stated in Example 3?

››› EXPLORE-DISCUSS 1

The postal service of a developing country is choosing a five-character postal code consisting of letters (of the English alphabet) and digits. At least half a million postal codes must be accommodated. Which format would you recommend to make the codes easy to remember?

The multiplication principle can be used to develop two additional counting techniques that are extremely useful in more complicated counting problems. Both of these methods use the factorial function, which we introduce next.

› Using Factorials

For n a natural number, ***n* factorial**—denoted by $n!$—is the product of the first n natural numbers. **Zero factorial** is defined to be 1.

› DEFINITION 1 *n* Factorial

For n a natural number

$$n! = n(n-1)\cdot\cdots\cdot 2\cdot 1$$
$$1! = 1$$
$$0! = 1$$

It is also useful to note that

› THEOREM 1 Recursion Formula for *n* Factorial

$$n! = n\cdot(n-1)!$$

EXAMPLE 4 Evaluating Factorials

Calculate the value of each expression.

(A) $4!$ (B) $5!$ (C) $\dfrac{7!}{6!}$ (D) $\dfrac{8!}{5!}$ (E) $\dfrac{9!}{6!3!}$

SOLUTIONS

(A) $4! = 4\cdot 3! = 4\cdot 3\cdot 2! = 4\cdot 3\cdot 2\cdot 1! = 4\cdot 3\cdot 2\cdot 1 = 24$

(B) $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$

(C) $\dfrac{7!}{6!} = \dfrac{7\cdot \cancel{6!}}{\cancel{6!}} = 7$ or $\dfrac{7!}{6!} = \dfrac{7\cdot \cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{\cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}} = 7$

(D) $\dfrac{8!}{5!} = \dfrac{8\cdot 7\cdot 6\cdot \cancel{5!}}{\cancel{5!}} = 336$

(E) $\dfrac{9!}{6!3!} = \dfrac{\overset{3}{\cancel{9}}\cdot \overset{4}{\cancel{8}}\cdot 7\cdot \cancel{6!}}{\cancel{6!}\,\cancel{3}\cdot 2\cdot 1} = 84$

Look carefully at the two alternative solutions in part C. Try part B of Matched Problem 4 both ways, and decide which method you understand best.

MATCHED PROBLEM 4

Calculate the value of each expression

(A) $6!$ (B) $\dfrac{6!}{5!}$ (C) $\dfrac{9!}{6!}$ (D) $\dfrac{10!}{7!3!}$

››› CAUTION ›››

When reducing fractions involving factorials, don't confuse the single integer n with the symbol $n!$, which represents the product of n consecutive integers. You cannot "cancel" the integers and ignore the factorial symbol.

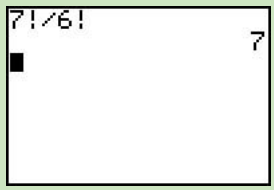
$$\frac{6!}{3!} \neq 2! \qquad \frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!} = 6 \cdot 5 \cdot 4 = 120$$

››› EXPLORE-DISCUSS 2

(A) Using one of the methods in Example 4, part C, calculate each of the following:

$$\frac{3!}{2!} \quad \frac{8!}{7!} \quad \frac{20!}{19!} \quad \frac{50!}{49!}$$

(B) Check your answers using a graphing calculator, as in Fig. 3.*



› Figure 3

(C) Based on your answers, make a conjecture on a formula for simplifying the expression $\frac{(n+1)!}{n!}$.

(D) Prove your conjecture using mathematical induction.

It is interesting and useful to look at how fast the expression $n!$ grows as n increases.

*The factorial symbol ! and related symbols can be found under the MATH-PROB menus on a TI-84 or TI-86.

››› EXPLORE-DISCUSS 3

(A) Fill in Table 1

Table 1

n	n^2	2^n	$n!$
2			
3			
4			
5			
10			
15			

Discuss how $n!$ grows as n increases, compared to n^2 and 2^n.

(B) If $n!$ is too large for a calculator to store and display, you will get an error message. Find the lowest integer value of n for which your calculator cannot evaluate $n!$.

› Counting Permutations

Suppose four pictures are to be arranged from left to right on one wall of an art gallery. How many arrangements are possible? Using the multiplication principle, there are four ways of selecting the first picture. After the first picture is selected, there are three ways of selecting the second picture. After the first two pictures are selected, there are two ways of selecting the third picture. And after the first three pictures are selected, there is only one way to select the fourth. So, the number of arrangements possible for the four pictures is

$$4 \cdot 3 \cdot 2 \cdot 1 = 4! \quad \text{or} \quad 24$$

In general, we refer to a particular arrangement, or *ordering,* of n objects without repetition as a **permutation** of the n objects. How many permutations of n objects are there? From the reasoning above, there are n ways in which the first object can be chosen, there are $n - 1$ ways in which the second object can be chosen, and so on. Applying the multiplication principle, we have discovered Theorem 2.

› THEOREM 2 Permutations of n Objects

The number of permutations of n objects, denoted by $P_{n,n}$, is given by

$$P_{n,n} = n \cdot (n - 1) \cdot \cdots \cdot 1 = n!$$

Now suppose the director of the art gallery decides to use only two of the four available pictures on the wall, arranged from left to right. How many arrangements of two pictures can be formed from the four possible choices? There are four ways the first picture can be selected. After selecting the first picture, there are three ways the second picture can be selected. Using the multiplication principle, the number of arrangements of two pictures from four pictures, denoted by $P_{4,2}$, is given by

$$P_{4,2} = 4 \cdot 3 = 12$$

In terms of factorials, if we multiply $4 \cdot 3$ by 1 in the form 2!/2!, we have

$$P_{4,2} = 4 \cdot 3 = \frac{4 \cdot 3 \cdot \mathbf{2!}}{\mathbf{2!}} = \frac{4!}{2!}$$

This last form gives $P_{4,2}$ in terms of factorials, which is useful in some cases.

When r objects are chosen from a larger set of n objects in a specific order, we call that a **permutation of n objects taken r at a time.** We use the symbol $P_{n,r}$ to denote such a permutation. Applying the same line of reasoning that we did to the art gallery question above, we can show that

$$P_{n,r} = \underset{\substack{\text{1st}\\ \text{object}}}{n}\underset{\substack{\text{2nd}\\ \text{object}}}{(n-1)}\underset{\substack{\text{3rd}\\ \text{object}}}{(n-2)}\cdots\underset{\substack{r\text{th}\\ \text{object}}}{(n-r+1)}$$

Multiplying the right side of this equation by 1 in the form $(n-r)!/(n-r)!$, we can obtain an equivalent factorial form for $P_{n,r}$:

$$P_{n,r} = n(n-1)(n-2)\cdot\cdots\cdot(n-r+1)\frac{(n-r)!}{(n-r)!}$$

But

$$n(n-1)(n-2)\cdot\cdots\cdot(n-r+1)(n-r)! = n!$$

So we get

$$P_{n,r} = \frac{n!}{(n-r)!}$$

› THEOREM 3 Permutation of n Objects Taken r at a Time

The number of permutations of n objects taken r at a time is given by

$$P_{n,r} = \underbrace{n(n-1)(n-2)\cdot\cdots\cdot(n-r+1)}_{r \text{ factors}}$$

or

$$P_{n,r} = \frac{n!}{(n-r)!} \qquad 0 \le r \le n$$

Note that if $r = n$, then the number of permutations of n objects taken n at a time is

$$P_{n,n} = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n! \qquad \text{Recall, } 0! = 1.$$

which agrees with Theorem 2, as it should.

The permutation symbol $P_{n,r}$ *also can be denoted by* P_r^n, ${}_nP_r$, or $P(n, r)$. Many calculators use ${}_nP_r$ to denote the function that evaluates the permutation symbol.

EXAMPLE 5 Evaluating $P_{n,r}$

Find the number of permutations of 25 objects taken

(A) Two at a time (B) Four at a time (C) Eight at a time

SOLUTION

Algebraic Solution

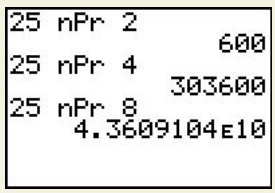
$$\text{(A) } P_{25,2} = \frac{25!}{(25-2)!} = \frac{25!}{23!} = 25 \cdot 24 = 600$$

$$\text{(B) } P_{25,4} = \frac{25!}{(25-4)!} = \frac{25!}{21!} = 25 \cdot 24 \cdot 23 \cdot 22 = 303{,}600$$

$$\text{(C) } P_{25,8} = \frac{25!}{(25-8)!} = \frac{25!}{17!} = 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 \cdot 18$$
$$= 4.3609104 \times 10^{10}$$

Graphing Calculator Solution

On the TI-84, $P_{n,r}$ is denoted by nPr, and can be found on the MATH-PROB menu (Fig. 4).



› Figure 4

MATCHED PROBLEM 5

Find the number of permutations of 30 objects taken

(A) Two at a time (B) Four at a time (C) Six at a time

EXAMPLE 6 Selecting Officers

From a committee of eight people, in how many ways can we choose a chair and a vice-chair, assuming one person cannot hold more than one position?

SOLUTION

Algebraic Solution

We are actually asking for the number of permutations of eight objects taken two at a time—that is, $P_{8,2}$:

$$P_{8,2} = \frac{8!}{(8-2)!} = \frac{8!}{6!} = \frac{8 \cdot 7 \cdot 6!}{6!} = 56$$

Graphing Calculator Solution

Figure 5 shows the solution on a graphing calculator.

› Figure 5

MATCHED PROBLEM 6

From a committee of 10 people, in how many ways can we choose a chair, vice-chair, and secretary, assuming one person cannot hold more than one position?

› Counting Combinations

Now suppose that an art museum owns eight paintings by a given artist and another art museum wishes to borrow three of these paintings for a special show. How many ways can three paintings be selected for shipment out of the eight available? The key difference between this question and earlier permutation problems is that here the order of the items selected doesn't matter. What we are actually interested in is how many subsets of three objects can be formed from a set of eight objects. We call such a subset a **combination** of eight objects taken three at a time. The total number of combinations is denoted by the symbol

$$C_{8,3} \quad \text{or} \quad \binom{8}{3}$$

This is often read as "eight choose three" to find the number of combinations of eight objects taken three at a time, $C_{8,3}$, we will make use of the formula for $P_{n,r}$ and the multiplication principle. We know that the number of permutations of eight objects taken three at a time is given by $P_{8,3}$, and we have a formula for computing this quantity. Now suppose we think of $P_{8,3}$ in terms of two operations:

O_1: Select a subset of three objects (paintings)
N_1: $C_{8,3}$ ways This is what we want to find.
O_2: Arrange the subset in a given order
N_2: 3! ways Theorem 2

The combined operation, O_1 followed by O_2, produces a permutation of eight objects taken three at a time. Thus,

$$P_{8,3} = C_{8,3} \cdot 3! \quad \text{Multiplication principle}$$

To find $C_{8,3}$, we replace $P_{8,3}$ in the preceding equation with $8!/(8 - 3)!$ (Theorem 3) and solve for $C_{8,3}$:

$$\frac{8!}{(8-3)!} = C_{8,3} \cdot 3!$$

$$C_{8,3} = \frac{8!}{3!(8-3)!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3 \cdot 2 \cdot 1 \cdot 5!} = 56$$

So, the museum can make 56 different selections of three paintings from the eight available.

When r objects are chosen from a larger set of n objects in no particular order, we call that a **combination of a set of n objects taken r at a time.** We use the symbol $C_{n,r}$ to denote such a combination. Using the line of reasoning in the preceding art museum problem, we can deduce that $P_{n,r} = C_{n,r} \cdot r!$, and solve for $C_{n,r}$.

$$P_{n,r} = C_{n,r} \cdot r! \quad \text{Divide both sides by } r!$$

$$C_{n,r} = \frac{P_{n,r}}{r!} \quad P_{n,r} = \frac{n!}{(n-r)!}$$

$$= \frac{n!}{r!(n-r)!}$$

› THEOREM 4 Combination of n Objects Taken r at a Time

The number of combinations of n objects taken r at a time is given by

$$C_{n,r} = \binom{n}{r} = \frac{P_{n,r}}{r!} = \frac{n!}{r!(n-r)!} \qquad 0 \le r \le n$$

The combination symbols $C_{n,r}$ and $\binom{n}{r}$ also can be denoted by C_r^n, ${}_nC_r$, or $C(n, r)$.

EXAMPLE 7 Evaluating $C_{n,r}$

Find the number of combinations of 25 objects taken

(A) Two at a time (B) Four at a time (C) Eight at a time

SOLUTION

Algebraic Solution

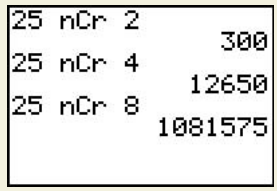
$$\text{(A) } C_{25,2} = \frac{25!}{2!(25-2)!} = \frac{25!}{2!23!} = \frac{25 \cdot 24}{2} = 300$$

$$\text{(B) } C_{25,4} = \frac{25!}{4!(25-4)!} = \frac{25!}{4!21!} = \frac{25 \cdot 24 \cdot 23 \cdot 22}{4 \cdot 3 \cdot 2 \cdot 1} = 12{,}650$$

$$\text{(C) } C_{25,8} = \frac{25!}{8!(25-8)!} = \frac{25!}{8!17!} = \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 \cdot 18}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 1{,}081{,}575$$

Graphing Calculator Solution

On the TI-84, $C_{n,r}$ is denoted by nCr, and can be found on the MATH-PROB menu (Fig. 6).



› Figure 6

MATCHED PROBLEM 7

Find the number of combinations of 30 objects taken

(A) Two at a time (B) Four at a time (C) Six at a time

EXAMPLE 8 Selecting Subcommittees

From a committee of eight people, in how many ways can we choose a subcommittee of two people?

SOLUTIONS

Algebraic Solution

Notice how this example differs from Example 6, where we wanted to know how many ways a chair and a vice-chair can be chosen from a committee of eight people. In Example 6, ordering matters. In choosing a subcommittee of two people, the ordering does not matter. Thus, we are actually asking for the number of combinations of eight objects taken two at a time. The number is given by

$$C_{8,2} = \binom{8}{2} = \frac{8!}{2!(8-2)!} = \frac{8 \cdot 7 \cdot 6!}{2 \cdot 1 \cdot 6!} = 28$$

Graphing Calculator Solution

Figure 7 shows the solution on a graphing calculator.

› Figure 7

MATCHED PROBLEM 8

How many subcommittees of three people can be chosen from a committee of eight people?

In a permutation, order matters. In a combination, order does not matter.
To decide whether a permutation or a combination is needed, decide whether rearranging the collection or listing makes a difference. If so, use permutations. If no, use combinations.

››› EXPLORE-DISCUSS 4

Each of the following is a selection without repetition. Would you consider the selection to be a combination? A permutation? Discuss your reasoning.

(A) A student checks out three books from the library.

(B) A baseball manager names his starting lineup.

(C) The newly elected president names his cabinet members.

(D) The president selects a delegation of three cabinet members to attend the funeral of a head of state.

(E) An orchestra conductor chooses three pieces of music for a symphony program.

› **Figure 8** A standard deck of cards.

A **standard deck** of 52 cards (Fig. 8) has four 13-card suits: diamonds, hearts, clubs, and spades. Each 13-card suit contains cards numbered from 2 to 10, a jack, a queen, a king, and an ace. The jack, queen, and king are called **face cards.** Depending on the game, the ace may be counted as the lowest and/or the highest card in the suit. Example 9, as well as other examples and exercises in this chapter, refer to this standard deck.

EXAMPLE 9 Counting Card Hands

Out of a standard 52-card deck, how many 5-card hands will have three aces and two kings?

SOLUTION

O_1: Choose three aces out of four possible — Order is not important.
N_1: $C_{4,3}$
O_2: Choose two kings out of four possible — Order is not important.
N_2: $C_{4,2}$

Using the multiplication principle, we have

$$\text{Number of hands} = C_{4,3} \cdot C_{4,2} = 4 \cdot 6 = 24$$

MATCHED PROBLEM 9

From a standard 52-card deck, how many 5-card hands will have three hearts and two spades?

EXAMPLE 10 **Counting Serial Numbers**

Serial numbers for a product are to be made using two letters followed by three numbers. If the letters are to be taken from the first eight letters of the alphabet with no repeats and the numbers from the 10 digits 0 through 9 with no repeats, how many serial numbers are possible?

SOLUTION

O_1: Choose two letters out of eight available — Order is important.
N_1: $P_{8,2}$
O_2: Choose three numbers out of ten available — Order is important.
N_2: $P_{10,3}$

Using the multiplication principle, we have

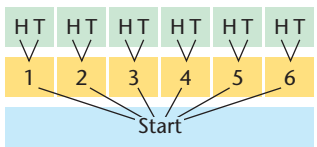
$$\text{Number of serial numbers} = P_{8,2} \cdot P_{10,3} = 40{,}320$$

MATCHED PROBLEM 10

Repeat Example 10 under the same conditions, except the serial numbers are now to have three letters followed by two digits with no repeats.

ANSWERS TO MATCHED PROBLEMS

1.



2. 5^5, or 3,125 **3.** (A) $10 \cdot 9 \cdot 8 \cdot 7 = 5{,}040$ (B) $10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$
(C) $10 \cdot 9 \cdot 9 \cdot 9 = 7{,}290$ **4.** (A) 720 (B) 6 (C) 504 (D) 120

5. (A) 870 (B) 657,720 (C) 427,518,000 **6.** $P_{10,3} = \dfrac{10!}{(10-3)!} = 720$

7. (A) 435 (B) 27,405 (C) 593,775 **8.** $C_{8,3} = \dfrac{8!}{3!(8-3)!} = 56$

9. $C_{13,3} \cdot C_{13,2} = 22{,}308$ **10.** $P_{8,3} \cdot P_{10,2} = 30{,}240$

10-4 Exercises

1. Explain the difference between enumeration and a counting technique.

2. Explain the difference between permutations and combinations.

3. What does the multiplication principle say, and how do we use it in counting techniques?

4. Write a sentence explaining what is represented by the symbol $P_{n,r}$. Do the same for $C_{n,r}$.

5. Explain why the following attempt at simplifying a fraction is incorrect:

$$\frac{8!}{2!} = 4!$$

6. Explain why $P_{n,n} = n!$. (*Hint:* Use the multiplication principle.)

Evaluate Problems 7–18.

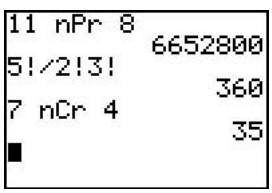
7. $9!$ **8.** $10!$ **9.** $11!$

10. $12!$ **11.** $\dfrac{11!}{8!}$ **12.** $\dfrac{14!}{12!}$

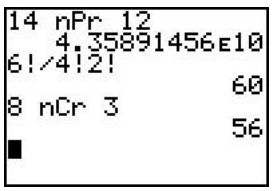
13. $\dfrac{5!}{2!3!}$ **14.** $\dfrac{6!}{4!2!}$ **15.** $\dfrac{7!}{4!(7-4)!}$

16. $\dfrac{8!}{3!(8-3)!}$ **17.** $\dfrac{7!}{7!(7-7)!}$ **18.** $\dfrac{8!}{0!(8-0)!}$

19. The figure shows calculator solutions to Problems 11, 13, and 15. Check these answers. If any are incorrect, explain why and find a correct calculator solution.



20. The figure shows calculator solutions to Problems 12, 14, and 16. Check these answers. If any are incorrect, explain why and find a correct calculator solution.



Evaluate Problems 21–28.

21. $P_{5,3}$ **22.** $P_{4,2}$ **23.** $P_{52,4}$

24. $P_{52,2}$ **25.** $C_{5,3}$ **26.** $C_{4,2}$

27. $C_{52,4}$ **28.** $C_{52,2}$

In Problems 29 and 30, would you consider the selection to be a combination or a permutation? Explain your reasoning.

29. (A) The recently elected chief executive officer (CEO) of a company named three new vice-presidents, of marketing, research, and manufacturing.
(B) The CEO selected three of her vice-presidents to attend the dedication ceremony of a new plant.

30. (A) An individual rented four videos from a rental store to watch over a weekend.
(B) The same individual did some holiday shopping by buying four videos, one for his father, one for his mother, one for his younger sister, and one for his older brother.

31. In a horse race, how many different finishes among the first three places are possible for a 10-horse race? Exclude ties.

32. In a long-distance foot race, how many different finishes among the first five places are possible for a 50-person race? Exclude ties.

33. How many ways can a subcommittee of three people be selected from a committee of seven people? How many ways

can a president, vice-president, and secretary be chosen from a committee of seven people?

34. Suppose nine cards are numbered with the nine digits from 1 to 9. A three-card hand is dealt, one card at a time. How many hands are possible where:
(A) Order is taken into consideration?
(B) Order is not taken into consideration?

35. There are 10 teams in a league. If each team is to play every other team exactly once, how many games must be scheduled?

36. Given seven points, no three of which are on a straight line, how many lines can be drawn joining two points at a time?

37. How many four-letter code words are possible from the first six letters of the alphabet, with no letter repeated? Allowing letters to repeat?

38. How many five-letter code words are possible from the first seven letters of the alphabet, with no letter repeated? Allowing letters to repeat?

39. A combination lock has five wheels, each labeled with the 10 digits from 0 to 9. How many opening combinations of five numbers are possible, assuming no digit is repeated? Assuming digits can be repeated?

40. A small combination lock on a suitcase has three wheels, each labeled with digits from 0 to 9. How many opening combinations of three numbers are possible, assuming no digit is repeated? Assuming digits can be repeated?

41. From a standard 52-card deck, how many 5-card hands will have all hearts?

42. From a standard 52-card deck, how many 5-card hands will have all face cards? All face cards, but no kings? Consider only jacks, queens, and kings to be face cards.

43. How many different license plates are possible if each contains three letters followed by three digits? How many of these license plates contain no repeated letters and no repeated digits?

44. How may five-digit zip codes are possible? How many of these codes contain no repeated digits?

45. From a standard 52-card deck, how many 7-card hands have exactly five spades and two hearts?

46. From a standard 52-card deck, how many 5-card hands will have two clubs and three hearts?

47. A particular new car model is available with five choices of color, three choices of transmission, four types of interior, and two types of engine. How many different variations of this model car are possible?

48. A deli serves sandwiches with the following options: three kinds of bread, five kinds of meat, and lettuce or sprouts. How many different sandwiches are possible, assuming one item is used out of each category?

49. A catering service offers eight appetizers, 10 main courses, and seven desserts. A banquet chairperson is to select three appetizers, four main courses, and two desserts for a banquet. How many ways can this be done?

50. Three research departments have 12, 15, and 18 members, respectively. If each department is to select a delegate and an alternate to represent the department at a conference, how many ways can this be done?

51. A sporting goods store has 12 pairs of ski gloves of 12 different brands thrown loosely in a bin. The gloves are all the same size. In how many ways can a left-hand glove and a right-hand glove be selected that do not match relative to brand?

52. A sporting goods store has six pairs of running shoes of six different styles thrown loosely in a basket. The shoes are all the same size. In how many ways can a left shoe and a right shoe be selected that do not match?

53. A 5-card hand is dealt from a standard 52-card deck. Which is more likely: the hand contains exactly one king or the hand contains no hearts?

54. A 10-card hand is dealt from a standard 52-card deck. Which is more likely: all cards in the hand are red or the hand contains all four aces?

55. How many ways can two people be seated in a row of five chairs? Three people? Four people? Five people?

56. Each of two countries sends five delegates to a negotiating conference. A rectangular table is used with five chairs on each long side. If each country is assigned a long side of the table, how many seating arrangements are possible? [*Hint:* Operation 1 is assigning a long side of the table to each country.]

57. A basketball team has five distinct positions. Out of eight players, how many starting teams are possible if
(A) The distinct positions are taken into consideration?
(B) The distinct positions are not taken into consideration?
(C) The distinct positions are not taken into consideration, but either Mike or Ken, but not both, must start?

58. How many committees of four people are possible from a group of nine people if
(A) There are no restrictions?
(B) Both Juan and Mary must be on the committee?
(C) Either Juan or Mary, but not both, must be on the committee?

59. Eight distinct points are selected on the circumference of a circle.
(A) How many chords can be drawn by joining the points in all possible ways?
(B) How many triangles can be drawn using these eight points as vertices?
(C) How many quadrilaterals can be drawn using these eight points as vertices?

60. Five distinct points are selected on the circumference of a circle.
(A) How many chords can be drawn by joining the points in all possible ways?
(B) How many triangles can be drawn using these five points as vertices?

61. (A) Use a graphing calculator to display the sequences $P_{10,0}, P_{10,1}, \ldots, P_{10,10}$ and $0!, 1! \ldots, 10!$ in table form, and show that $P_{10,r} \geq r!$ for $r = 0, 1, \ldots, 10$.
(B) Find all values of r such that $P_{10,r} = r!$
(C) Explain why $P_{n,r} \geq r!$ whenever $0 \leq r \leq n$.

62. (A) How are the sequences $\frac{P_{10,0}}{0!}, \frac{P_{10,1}}{1!}, \ldots, \frac{P_{10,10}}{10!}$ and $C_{10,1}, \ldots, C_{10,10}$ related?
(B) Use a graphing calculator to graph each sequence and confirm the relationship of part A.

10-5 Sample Spaces and Probability

- Sample Spaces and Events
- Finding the Probability of an Event
- Making Equally Likely Assumptions
- Finding or Approximating Empirical Probability

This section provides an introduction to probability. It's going to need to be a relatively brief one, because probability is a topic to which entire books and courses are devoted. Probability involves many subtle notions, and care must be taken at the beginning to understand the fundamental concepts on which the subject is based. Our development of probability, because of space limitations, must be somewhat informal. More formal and precise treatments can be found in books on probability.

Sample Spaces and Events

Our first step in constructing a mathematical model for probability studies is to describe the type of experiments on which probability studies are based. Some types of experiments do not yield the same results, no matter how carefully they are repeated under the same conditions. These experiments are called **random experiments.** Some standard examples of random experiments are flipping coins, rolling dice, observing the frequency of defective items from an assembly line, or observing the frequency of deaths in a certain age group.

Probability theory is a branch of mathematics that has been developed to deal with outcomes of random experiments. In the work that follows, the word **experiment** will be used to mean a random experiment.

The outcomes of experiments are typically described in terms of *sample spaces* and *events.* Our second step in constructing a mathematical model for probability studies is to define these two terms.

Consider the experiment, "A single six-sided die is rolled." What outcomes might we observe? There are any number of questions we might ask: How many dots are facing up? Is that number even? Is it divisible by 3? The list of possible outcomes appears endless. In general, there is no unique method of analyzing all possible outcomes of an experiment. Therefore, before conducting an experiment, it is important to decide just what outcomes are of interest.

In the die experiment, suppose we limit our interest to the number of dots facing up when the die comes to rest. Having decided what to observe, we can make a list of outcomes of the experiment, called *simple events*. An event is a *simple event* if there is exactly one outcome of the experiment that corresponds to that event. In this case, each of the possible rolls of one die, from one through six, is a simple event because exactly one outcome of the experiment matches each event. The set of all simple events for an experiment is called a **sample space** for the experiment. The sample space S we have chosen for the die-rolling experiment is

$$S = \{1, 2, 3, 4, 5, 6\}$$

Now consider the outcome, "The number of dots facing up is an even number." This outcome is not a simple event, because there are three different outcomes of the experiment (a roll of 2, 4, or 6) that will match this event. In other words, whenever an element in the subset of S defined by

$$E = \{2, 4, 6\}$$

occurs, this event has occurred. When an event corresponds to more than one outcome from a sample space, it is called a *compound event*. In general, we can write the following definition:

› DEFINITION 1 Event

Given a sample space S for an experiment, we define an **event E** to be any subset of S. If an event E has only one element in it, it is called a **simple event.** If event E has more than one element, it is called a **compound event.** We say that **an event E occurs** if any of the simple events in E occurs.

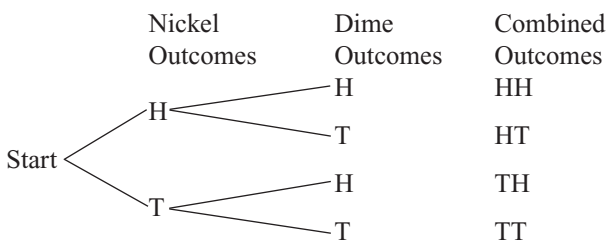
EXAMPLE 1 Choosing a Sample Space

A nickel and a dime are tossed. How can we identify a sample space for this experiment?

SOLUTIONS

There are a number of possibilities, depending on our interest. We will consider three.

(A) If we are interested in whether each coin falls heads (H) or tails (T), then, using a tree diagram, we can easily determine an appropriate sample space for the experiment:



Our sample space in this case is

$$S_1 = \{HH, HT, TH, TT\}$$

and there are four simple events in the sample space.

(B) If we are interested only in the number of heads that appear on a single toss of the two coins, then we can let

$$S_2 = \{0, 1, 2\}$$

and there are three simple events in the sample space.

(C) If we are interested in whether the coins match (M) or don't match (D), then we can let

$$S_3 = \{M, D\}$$

and there are only two simple events in the sample space.

MATCHED PROBLEM 1

An experiment consists of recording the boy–girl composition of families with two children.

(A) What is an appropriate sample space if we are interested in the sex of each child in the order of their births? Draw a tree diagram.

(B) What is an appropriate sample space if we are interested only in the number of girls in a family?

(C) What is an appropriate sample space if we are interested only in whether the sexes are alike (A) or different (D)?

(D) What is an appropriate sample space for all three interests expressed above?

In Example 1, sample space S_1 contains more information than either S_2 or S_3. If we know which outcome has occurred in S_1, then we know which outcome has occurred in S_2 and S_3. However, the reverse is not true. In this sense, we say that S_1 is a more **fundamental sample space** than either S_2 or S_3.

Important Remark: There is no one correct sample space for a given experiment. When specifying a sample space for an experiment, we include as much detail as necessary to answer *all* questions of interest regarding the outcomes of the experiment. If in doubt, include more elements in the sample space rather than fewer.

Now let's return to the two-coin problem in Example 1 and the sample space

$$S_1 = \{HH, HT, TH, TT\}$$

Suppose we are interested in the outcome, "Exactly 1 head is up." Looking at S_1, we find that it occurs if either of the two simple events HT or TH occurs.* So, to say that the event, "Exactly 1 head is up" occurs is the same as saying the experiment has an outcome in the set

$$E = \{\text{HT, TH}\}$$

This is a subset of the sample space S_1. The event E is a compound event.

EXAMPLE 2 Rolling Two Dice

Consider an experiment of rolling two dice. A convenient sample space that we can use to answer a variety of questions in this section is shown in Figure 1. Let S be the set of all ordered pairs listed in the figure. Note that the simple event (3, 2) is to be distinguished from the simple event (2, 3). The former indicates a 3 turned up on the first die and a 2 on the second, whereas the latter indicates a 2 turned up on the first die and a 3 on the second. What is the event that corresponds to each of the following outcomes?

(A) A sum of 7 turns up.
(B) A sum of 11 turns up.
(C) A sum less than 4 turns up.
(D) A sum of 12 turns up.

SECOND DIE

FIRST DIE						
	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

› **Figure 1** A sample space for rolling two dice.

SOLUTIONS

(A) By "A sum of 7 turns up," we mean that the sum of all dots on both turned-up faces is 7. There are six different pairs whose sum is 7. This outcome corresponds to the event

$$\{(6, 1), (5, 2), (4, 3), (3, 4), (2, 5), (1, 6)\}$$

*Technically, we should write {HT} and {TH}, because there is a logical distinction between an element of a set and a subset consisting of only that element. But we will just keep this in mind and drop the braces for simple events to simplify the notation.

(B) There are only two pairs that add to 11, so "A sum of 11 turns up" corresponds to the event

$$\{(6, 5), (5, 6)\}$$

(C) "A sum less than 4 turns up" corresponds to the event

$$\{(1, 1), (2, 1), (1, 2)\}$$

(D) "A sum of 12 turns up" corresponds to the event

$$\{(6, 6)\}$$

MATCHED PROBLEM 2

Refer to the sample space in Example 2 (Fig. 1). What is the event that corresponds to each of the following outcomes?

(A) A sum of 5 turns up.

(B) A sum that is a prime number greater than 7 turns up.

Informally, we often use the terms *event* and *outcome of an experiment* interchangeably. In Example 2 we might say "the event 'A sum of 11 turns up'" in place of "the outcome 'A sum of 11 turns up,'" or even write

$$E = \text{A sum of 11 turns up} = \{(6, 5), (5, 6)\}$$

Technically speaking, an event is the mathematical counterpart of an outcome of an experiment, but it won't hurt us to treat them as synonymous.

Finding the Probability of an Event

At its simplest, the word probability refers to a ratio, or fraction, that indicates the likelihood of an event occurring. Over a 7-day period, the number of days that end with the letter y is 7, so the probability of a randomly selected day ending in y is 7/7, or 1. Using the same reasoning, the probability of a randomly selected day beginning with T is 2/7, and the probability of it having an x in it is 0/7.

These examples illustrate some important aspects of probability. The probability of an event is a number between 0 and 1 inclusive. This number is an indicator of how likely an event is to occur. An event that must occur has probability 1, and an event that cannot occur has probability 0.

The probabilities just discussed are examples of *empirical probabilities,* which we will discuss later in the section. This means that they are based on the observed outcomes of an experiment, in this case observing the letters in the 7 days of the week. Much of our study of probability will focus on *theoretical probability,* where we use reasonable assumptions about the likelihood of an event to assign probabilities to simple events.

These assignments are generally based on the expected or actual percentage of times a simple event occurs when an experiment is repeated a large number of times. Assignments based on this principle are called *reasonable.*

For example, think about the experiment of flipping a single coin, with sample space

$$S = \{\text{H}, \text{T}\}$$

If a coin appears to be fair, we are inclined to assign probabilities to the simple events in S as follows:

$$P(\text{H}) = \tfrac{1}{2} \qquad \text{and} \qquad P(\text{T}) = \tfrac{1}{2}$$

These assignments are based on reasoning that, because there are two ways a coin can land, in the long run a head will turn up half the time and a tail will turn up half the time. There are two key things that make this assignment of probability a good one:

1. The probability of each event is between 0 and 1 inclusive.
2. The sum of all probabilities (which represents the probability of *something* happening when flipping one coin) is 1.

These two conditions enable us to write a general definition of exactly what we mean by probability.

› DEFINITION 2 Probabilities for Simple Events

Given a sample space

$$S = \{e_1, e_2, \ldots, e_n\}$$

with n simple events, to each simple event e_i we assign a real number, denoted by $P(e_i)$, that is called the **probability of the event e_i.** These numbers may be assigned in any manner as long as the following two conditions are satisfied:

1. $0 \le P(e_i) \le 1$ The probability of any event is between 0 and 1 inclusive.
2. $P(e_1) + P(e_2) + \cdots + P(e_n) = 1$ The sum of the probabilities of all simple events in the sample space is 1.

Any probability assignment that meets conditions 1 and 2 is said to be an **acceptable probability assignment.**

››› CAUTION ›››

Students are often confused by the part of the definition that says probabilities can be assigned "in any manner." There is a big difference between an *acceptable* probability assignment and a *reasonable* one. See the following for further discussion.

For the simple coin toss experiment above, there are other acceptable probability assignments. Maybe after flipping a coin 1,000 times we find that the head turns up 376 times and the tail turns up 624 times. With this result, we might suspect that the coin is not fair and assign the simple events in the sample space S the probabilities

$$P(\text{H}) = .376 \qquad \text{and} \qquad P(\text{T}) = .624$$

This is also an acceptable assignment. But the probability assignment

$$P(\text{H}) = 1 \qquad \text{and} \qquad P(\text{T}) = 0$$

though acceptable according to Definition 2, is not reasonable, unless the coin has two heads. The assignment

$$P(\text{H}) = .6 \qquad \text{and} \qquad P(\text{T}) = .8$$

is not acceptable, because $.6 + .8 = 1.4$, which violates condition 2 in Definition 2. In probability studies, the 0 to the left of the decimal is usually omitted. So, we write .8 and not 0.8.

It is important to keep in mind that out of the infinitely many possible acceptable probability assignments to simple events in a sample space, we are generally inclined to choose one assignment over the others based on reasoning or experimental results.

Given an acceptable probability assignment for simple events in a sample space S, how do we define the probability of an arbitrary event E associated with S?

› DEFINITION 3 The Probability of an Event E

Given an acceptable probability assignment for the simple events in a sample space S, we define the **probability of an arbitrary event E,** denoted by $P(E)$, as follows:

1. If E is the empty set, then $P(E) = 0$.
2. If E is a simple event, then $P(E)$ has already been assigned.
3. If E is a compound event, then $P(E)$ is the sum of the probabilities of all the simple events in E.
4. If E is the entire sample space S, then $P(E) = P(S) = 1$. This is a special case of 3.

This definition is most easily understood by working through an example.

EXAMPLE 3 Finding Probabilities of Events

Let's return to Example 1, the tossing of a nickel and dime, and the sample space

$$S = \{\text{HH, HT, TH, TT}\}$$

Because there are four simple outcomes and the coins are assumed to be fair, it appears that each outcome should occur in the long run 25% of the time. Let's assign the same probability of $\frac{1}{4}$ to each simple event in S:

Simple event, e_i	HH	HT	TH	TT
$P(e_i)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

This is an acceptable assignment according to Definition 2 and a reasonable assignment for ideal coins that are perfectly balanced or coins close to ideal.

(A) What is the probability of the coin landing on heads exactly once?
(B) What is the probability of the coin landing on heads at least once?
(C) What is the probability of the coin landing on heads or tails at least once?
(D) What is the probability of the coin landing on heads three times?

SOLUTIONS

(A) There are two outcomes in which heads comes up exactly once, so

$$E_1 = \text{one coin comes up heads} = \{\text{HT, TH}\}$$

Because E_1 is a compound event, we use item 3 in Definition 3 and find $P(E_1)$ by adding the probabilities of the simple events in E_1. Thus,

$$P(E_1) = P(\text{HT}) + P(\text{TH}) = \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}$$

(B) $E_2 = \text{At least one coin comes up heads} = \{\text{HH, HT, TH}\}$

$$P(E_2) = P(\text{HH}) + P(\text{HT}) + P(\text{TH})$$
$$= \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{3}{4}$$

(C) $E_3 = \text{Landing on heads or tails at least once} = \{\text{HH, HT, TH, TT}\} = S$

$P(E_3) = P(S) = 1$ $\quad \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$

(D) $E_3 = \text{Three tosses come up heads} = \varnothing$ $\quad$ Empty set

$P(\varnothing) = 0$

› STEPS FOR FINDING PROBABILITIES OF EVENTS

Step 1. Set up an appropriate sample space S for the experiment.

Step 2. Assign acceptable probabilities to the simple events in S.

Step 3. To obtain the probability of an arbitrary event E, add the probabilities of the simple events in E.

The function P defined in steps 2 and 3 is called a **probability function.** The domain of this function is all possible events in the sample space S, and the range is a set of real numbers between 0 and 1, inclusive.

MATCHED PROBLEM 3

Return to Matched Problem 1, recording the boy–girl composition of families with two children and the sample space

$$S = \{\text{BB, BG, GB, GG}\}$$

Statistics from the U.S. Census Bureau indicate that an acceptable and reasonable probability for this sample space is

Simple event, e_i	BB	BG	GB	GG
$P(e_i)$	.26	.25	.25	.24

Find the probabilities of the following events:

(A) E_1 = Having at least one girl in the family
(B) E_2 = Having at most one girl in the family
(C) E_3 = Having two children of the same sex in the family

››› EXPLORE-DISCUSS 1

(A) For each question in Example 3, compute the probability by dividing the number of outcomes in the sample space that satisfy the given event by the total number of outcomes. Do the results agree with the probabilities computed using Definition 3?

(B) Would this method work if the coins were not fair, and heads was more likely to come up than tails? Discuss.

› Making Equally Likely Assumptions

In tossing a nickel and dime (Example 3), we assigned the same probability, $\frac{1}{4}$, to each simple event in the sample space $S = \{\text{HH, HT, TH, TT}\}$. By assigning the same probability to each simple event in S, we are actually making the assumption that each simple event is as likely to occur as any other. We refer to this as an **equally likely assumption.**

DEFINITION 4 The Probability of a Simple Event Under an Equally Likely Assumption

If, in a sample space

$$S = \{e_1, e_2, \ldots, e_n\}$$

with n elements, we assume each simple event e_i is as likely to occur as any other, then we assign the probability $1/n$ to each. That is,

$$P(e_i) = \frac{1}{n}$$

Under an equally likely assumption, we can develop a very simple and useful formula for finding probabilities of arbitrary events associated with a sample space S, as in Explore-Discuss 1. Consider the following example.

If a single die is rolled and we assume each face is as likely to come up as any other, then for the sample space

$$S = \{1, 2, 3, 4, 5, 6\}$$

we assign a probability of $\frac{1}{6}$ to each simple event, because there are six simple events. Then according to Definition 3, the probability of

$$E = \text{Rolling a prime number} = \{2, 3, 5\}$$

is

$$P(E) = P(2) + P(3) + P(5) = \tfrac{1}{6} + \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{3}{6} = \tfrac{1}{2}$$

Notice that we could have obtained the same probability by dividing the number of outcomes that satisfy the event E, 3, by the total number of possible outcomes, 6. This is an example of how to use Theorem 1.

THEOREM 1 The Probability of an Arbitrary Event Under an Equally Likely Assumption

If we assume each simple event in sample space S is as likely to occur as any other, then the probability of an arbitrary event E in S is given by

$$P(E) = \frac{\text{Number of elements in } E}{\text{Number of elements in } S} = \frac{n(E)}{n(S)}$$

EXAMPLE 4 Finding Probabilities of Events

If in rolling two dice we assume each simple event in the sample space shown in Figure 1 on p. 929 is as likely as any other, find the probabilities of the following events:

(A) E_1 = A sum of 7 turns up
(B) E_2 = A sum of 11 turns up
(C) E_3 = A sum less than 4 turns up
(D) E_4 = A sum of 12 turns up

SOLUTIONS

Referring to Figure 1, we see that:

(A) $P(E_1) = \dfrac{n(E_1)}{n(S)} = \dfrac{6}{36} = \dfrac{1}{6}$

(B) $P(E_2) = \dfrac{n(E_2)}{n(S)} = \dfrac{2}{36} = \dfrac{1}{18}$

(C) $P(E_3) = \dfrac{n(E_3)}{n(S)} = \dfrac{3}{36} = \dfrac{1}{12}$

(D) $P(E_4) = \dfrac{n(E_4)}{n(S)} = \dfrac{1}{36}$

MATCHED PROBLEM 4

Under the conditions in Example 4, find the probabilities of the following events:

(A) E_5 = A sum of 5 turns up
(B) E_6 = A sum that is a prime number greater than 7 turns up

››› EXPLORE-DISCUSS 2

A box contains four red balls and seven green balls. A ball is drawn at random and then, without replacing the first ball, a second ball is drawn. Discuss whether the equally likely assumption would be appropriate for the sample space $S = \{RR, RG, GR, GG\}$.

We now turn to some examples that make use of the counting techniques developed in the last section.

EXAMPLE 5 Drawing Cards

In drawing 5 cards from a 52-card deck without replacement, what is the probability of getting five spades?

SOLUTION

Let the sample space S be the set of all 5-card hands from a 52-card deck. Because the order in a hand does not matter, $n(S) = C_{52,5}$. The event E we seek is the set of all 5-card hands from 13 spades

Again, the order does not matter and $n(E) = C_{13,5}$. Thus, assuming each 5-card hand is as likely as any other,

$$P(E) = \frac{n(E)}{n(S)} = \frac{C_{13,5}}{C_{52,5}} = \frac{13!/5!8!}{52!/5!47!} = \frac{13!}{5!8!} \cdot \frac{5!47!}{52!} \approx .0005$$

MATCHED PROBLEM 5

In drawing 7 cards from a 52-card deck without replacement, what is the probability of getting seven hearts?

EXAMPLE 6 Selecting Committees

The board of regents of a university is made up of 12 men and 16 women. If a committee of six is chosen at random, what is the probability that it will contain three men and three women?

SOLUTION

Let S = The set of all 6-person committees out of 28 people. Order doesn't matter, so

$$n(S) = C_{28,6}$$

Let E be the set of all 6-person committees with 3 men and 3 women. To find $n(E)$, we use the multiplication principle and the following two operations:

O_1:	Select 3 men out of the 12 available	N_1:	$C_{12,3}$
O_2:	Select 3 women out of the 16 available	N_2:	$C_{16,3}$

So,

$$n(E) = C_{12,3} \cdot C_{16,3}$$

and

$$P(E) = \frac{n(E)}{n(S)} = \frac{C_{12,3} \cdot C_{16,3}}{C_{28,6}} \approx .327$$

MATCHED PROBLEM 6

What is the probability that the committee in Example 6 will have four men and two women?

Finding or Approximating Empirical Probability

In the earlier examples in this section, we made a reasonable assumption about an experiment and used deductive reasoning to assign probabilities. For example, it is reasonable to assume that an ordinary coin will come up heads about as often as it will come up tails. As mentioned earlier, probabilities determined in this manner are called **theoretical probabilities.** No experiments are ever conducted. But what if the theoretical probabilities are not obvious? Then we can assign probabilities to simple events based on the results of actual experiments. Probabilities determined from the results of actually performing an experiment are called **empirical probabilities.** As an experiment is repeated over and over, the percentage of times an event occurs may get closer and closer to a single fixed number. If so, this single fixed number is generally called the **actual probability** of the event.

EXPLORE-DISCUSS 3

Like a coin, a thumbtack tossed into the air will land in one of two positions, point up or point down [Fig. 2(a)]. Unlike a coin, we would not expect both events to occur with the same frequency. Indeed, the frequencies of landing point up and point down may well vary from one thumbtack to another [Fig. 2(b)]. Find two thumbtacks of different sizes and guess which one is likely to land point up more frequently. Then toss each tack 100 times and record the number of times each lands point up. Did the experiment confirm your initial guess?

(a) Point up or point down (b) Two different tacks

Figure 2

Suppose when tossing one of the thumbtacks in Explore-Discuss 3, we observe that the tack lands point up 43 times and point down 57 times. Based on this experiment, it seems reasonable to say that for this particular thumbtack

$$P(\text{Point up}) = \frac{43}{100} = .43$$

$$P(\text{Point down}) = \frac{57}{100} = .57$$

Probability assignments based on the results of repeated trials of an experiment are called **approximate empirical probabilities.**

In general, if we conduct an experiment n times and an event E occurs with **frequency** $f(E)$, then the ratio $f(E)/n$ is called the **relative frequency** of the occurrence of event E in n trials. It is reasonable to believe that more and more trials will result

in the relative frequency providing a more accurate estimate of the probability of an event occurring. That is, we expect $f(E)/n$ to approach some number as n gets larger and larger. If there is such a number, we will call it the *empirical probability* of E, and denote it $P(E)$. Of course, for any particular n, the relative frequency $f(E)/n$ is generally only approximately equal to $P(E)$. However, as n increases, we expect the approximation to improve.

› DEFINITION 5 Empirical Probability

If $f(E)$ is the frequency of event E in n trials, then the **empirical probability** of E is

$$P(E) \approx \frac{\text{Frequency of occurrence of } E}{\text{Total number of trials}} = \frac{f(E)}{n}$$

If we can also deduce theoretical probabilities for an experiment, then we expect the approximate empirical probabilities to approach the theoretical probabilities. If this does not happen, then we should be skeptical of the manner in which the theoretical probabilities were assigned.

Suppose that the theoretical probability of an event E in some experiment is $\frac{1}{2}$. If we perform that experiment 50 times, how many times would you expect E to occur? If you said 25, then you understand the concept of *expected frequency*.

› DEFINITION 6 Expected Frequency

If $P(E)$ is the theoretical probability of an event E, and the experiment is performed n times, the **expected frequency** of E is given by $n \cdot P(E)$.

EXAMPLE 7 Finding Approximate Empirical and Theoretical Probabilities

Two coins are tossed 500 times with the following frequencies of outcomes:

Two heads: 121
One head: 262
Zero heads: 117

(A) Compute the approximate empirical probability for each outcome.
(B) Compute the theoretical probability for each outcome.
(C) Compute the expected frequency for each outcome.

SOLUTIONS

(A) $P(\text{two heads}) \approx \dfrac{121}{500} = .242$

$P(\text{one heads}) \approx \dfrac{262}{500} = .524$

$P(\text{zero heads}) \approx \dfrac{117}{500} = .234$

(B) A sample space of equally likely simple events is $S = \{\text{HH, HT, TH, TT}\}$. Let

$$E_1 = \text{two heads} = \{\text{HH}\}$$
$$E_2 = \text{one head} = \{\text{HT, TH}\}$$
$$E_3 = \text{zero heads} = \{\text{TT}\}$$

Then

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{1}{4} = .25$$
$$P(E_2) = \frac{n(E_2)}{n(S)} = \frac{2}{4} = .50$$
$$P(E_3) = \frac{n(E_3)}{n(S)} = \frac{1}{4} = .25$$

(C) The expected frequencies are

$$E_1\text{: } 500(.25) = 125$$
$$E_2\text{: } 500(.5) = 250$$
$$E_3\text{: } 500(.25) = 125$$

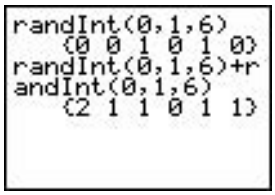
The actual frequencies obtained from performing the experiment are reasonably close to the expected frequencies. Increasing the number of trials of the experiment would likely produce even better approximations.

MATCHED PROBLEM 7

One die is rolled 500 times with the following frequencies of outcomes:

Outcome	1	2	3	4	5	6
Frequency	89	83	77	91	72	88

(A) Compute the approximate empirical probability for each outcome.
(B) Compute the theoretical probability for each outcome.
(C) Compute the expected frequency for each outcome.

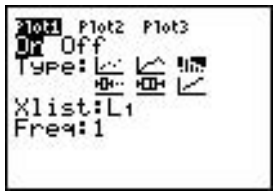


› **Figure 3** Using a random number generator.

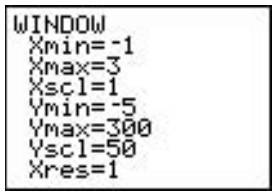
Tossing two coins 500 times is certainly a tedious task and we did not do this to generate the data in Example 7. Instead, we used a random number generator on a graphing calculator to simulate this experiment. Specifically, we used the command **randInt(*i*,*k*,*n*)*** on a Texas Instruments TI-84, which generates a random sequence of n integers between i and k, inclusively. If we let 0 represent tails and 1 represent heads, then a random sequence of 0s and 1s can be used to represent the outcomes of repeated tosses of one coin (see the first two lines of Fig. 3). In this case, in six tosses, we obtained two heads and four tails.

To simulate tossing two coins, we simply add together two similar statements, as shown in lines three through five of Figure 3. We see that in these six tosses zero heads occurred once, one head occurred four times, and two heads occurred once. Of course, to obtain meaningful results, we need to toss the coins many more times. Figure 4(a) shows a command that will simulate 500 tosses of two coins. To determine the frequency of each outcome, we construct a histogram [Figs. 4(b) and 4(c)] and use the TRACE command to determine the following frequencies [Figs. 5(a), 5(b), and 5(c)].

(a) Generating the random numbers

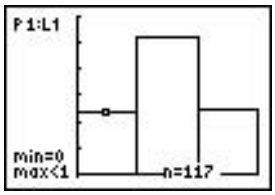


(b) Setting up the histogram

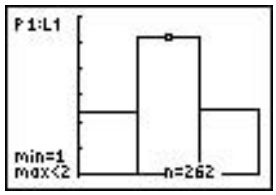


(c) Selecting the window variables

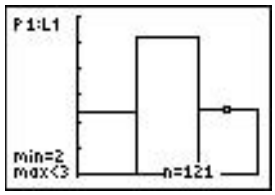
› **Figure 4** Simulating 500 tosses of two coins.



(a) 0 heads: 117



(b) 1 head: 262



(c) 2 heads: 121

› **Figure 5** Results of the simulation.

If you perform the same simulation on your graphing utility, you are not likely to get exactly the same results. But the approximate empirical probabilities you obtain will most likely be close to the theoretical probabilities.

*On the TI-84, randInt is on the MATH-PRB menu.

››› EXPLORE-DISCUSS 4

This discussion assumes that your graphing utility has the ability to generate and manipulate sequences of random integers.

(A) As an alternative to using the histogram in Figure 5 to count the outcomes of the sequence of random integers in Figure 4(a), enter the following function and evaluate it for $x = 0$, 1, and 2:

$$y_1 = \text{sum}(\text{seq}(L_1(I) = X,I,1,\dim(L_1)))$$

(B) Simulate the experiment of rolling a single die and compare your empirical results with the results in Matched Problem 7.

EXAMPLE 8 Empirical Probabilities for an Insurance Company

An insurance company selected 1,000 drivers at random in a particular city to determine a relationship between age and accidents. The data obtained are listed in Table 1. Compute the approximate empirical probabilities of the following events for a driver chosen at random in the city:

(A) E_1: being under 20 years old *and* having exactly three accidents in 1 year
(B) E_2: being 30–39 years old *and* having one or more accidents in 1 year
(C) E_3: having no accidents in 1 year
(D) E_4: being under 20 years old *or* having exactly three accidents in 1 year

Table 1

	Accidents in 1 year				
Age	0	1	2	3	Over 3
Under 20	50	62	53	35	20
20–29	64	93	67	40	36
30–39	82	68	32	14	4
40–49	38	32	20	7	3
Over 49	43	50	35	28	24

SOLUTIONS

(A) $P(E_1) \approx \dfrac{35}{1{,}000} = .035$

(B) $P(E_2) \approx \dfrac{68 + 32 + 14 + 4}{1{,}000} = .118$

(C) $P(E_3) \approx \dfrac{50 + 64 + 82 + 38 + 43}{1{,}000} = .277$

(D) $P(E_4) \approx \dfrac{50 + 62 + 53 + 35 + 20 + 40 + 14 + 7 + 28}{1{,}000} = .309$

Notice that in this type of problem, which is typical of many realistic problems, approximate empirical probabilities are the only type we can compute.

MATCHED PROBLEM 8

Referring to Table 1 in Example 8, compute the approximate empirical probabilities of the following events for a driver chosen at random in the city:

(A) E_1: being under 20 years old with no accidents in 1 year

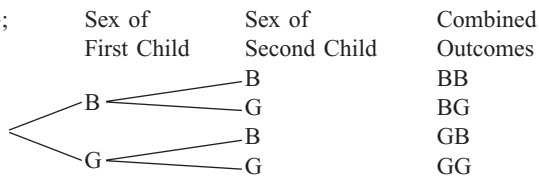
(B) E_2: being 20–29 years old and having fewer than two accidents in 1 year

(C) E_3: not being over 49 years old

Approximate empirical probabilities are often used to test theoretical probabilities, since equally likely assumptions may not be justified in reality. In addition to this use, there are many situations in which it is either very difficult or impossible to compute the theoretical probabilities for given events. For example, insurance companies use past experience to establish approximate empirical probabilities to predict future accident rates; baseball teams use batting averages, which are approximate empirical probabilities based on past experience, to predict the future performance of a player; and pollsters use approximate empirical probabilities to predict outcomes of elections.

ANSWERS TO MATCHED PROBLEMS

1. (A) $S_1 = \{BB, BG, GB, GG\}$;



(B) $S_2 = \{0, 1, 2\}$ (C) $S_3 = \{A, D\}$ (D) The sample space in part A.

2. (A) $\{(4, 1), (3, 2), (2, 3), (1, 4)\}$ (B) $\{(6, 5), (5, 6)\}$

3. (A) .74 (B) .76 (C) .5 **4.** (A) $P(E_5) = \frac{1}{9}$ (B) $P(E_6) = \frac{1}{18}$

5. $C_{13,7}/C_{52,7} \approx .000013$ **6.** $C_{12,4} \cdot C_{16,2}/C_{28,6} \approx .158$

7. (A) $P(E_1) \approx .178$, $P(E_2) \approx .166$, $P(E_3) \approx .154$, $P(E_4) \approx .182$, $P(E_5) \approx .144$, $P(E_6) \approx .176$ (B) $\frac{1}{6} \approx .167$ for each (C) 83.3 for each

8. (A) $P(E_1) \approx .05$ (B) $P(E_2) \approx .157$ (C) $P(E_3) \approx .82$

10-5 Exercises

1. Write definitions in your own words for experiment, event, and sample space in the context of this section.

2. Explain how the probability of an event occurring can be simply described in terms of fractions.

3. How can you distinguish between simple and compound events?

4. Explain the difference between empirical and theoretical probabilities.

5. Explain why equally likely assumptions are important in computing theoretical probabilities.

6. If you needed to predict the likely outcome of an experiment, would you rather base your prediction on a theoretical probability or an empirical probability? Explain your reasoning.

7. How would you interpret $P(E) = 1$?

8. How would you interpret $P(E) = 0$?

9. A spinner can land on four different colors: red (R), green (G), yellow (Y), and blue (B). If we do not assume each color is as likely to turn up as any other, which of the following probability assignments have to be rejected, and why?
(A) $P(R) = .15, P(G) = -.35, P(Y) = .50, P(B) = .70$
(B) $P(R) = .32, P(G) = .28, P(Y) = .24, P(B) = .30$
(C) $P(R) = .26, P(G) = .14, P(Y) = .30, P(B) = .30$

10. Under the probability assignments in Problem 9, part C, what is the probability that the spinner will not land on blue?

11. Under the probability assignments in Problem 9, part C, what is the probability that the spinner will land on red or yellow?

12. Under the probability assignments in Problem 9, part C, what is the probability that the spinner will not land on red or yellow?

13. A ski jumper has jumped over 300 feet in 25 out of 250 jumps. What is the approximate empirical probability of the next jump being over 300 feet?

14. In a certain city there are 4,000 youths between 16 and 20 years old who drive cars. If 560 of them were involved in accidents last year, what is the approximate empirical probability of a youth in this age group being involved in an accident this year?

15. Out of 420 times at bat, a baseball player gets 189 hits. What is the approximate empirical probability that the player will get a hit next time at bat?

16. In a medical experiment, a new drug is found to help 2,400 out of 3,000 people. If a doctor prescribes the drug for a particular patient, what is the approximate empirical probability that the patient will be helped?

In Problems 17–20, decide whether the description allows you to write a theoretical or empirical probability.

17. It has rained on April 12 in 8 of the last 12 years.

18. A roulette ball is equally likely to land in each of the 38 spaces on the wheel.

19. There are 1,000 possible combinations in a pick three lottery game.

20. A survey at an intersection indicates that 27 of 150 cars passing by in a 1-hour span were white in color.

21. A small combination lock on a suitcase has three wheels, each labeled with the 10 digits from 0 to 9. If an opening combination is a particular sequence of three digits with no repeats, what is the probability of a person guessing the right combination?

22. A combination lock has five wheels, each labeled with the 10 digits from 0 to 9. If an opening combination is a particular sequence of five digits with no repeats, what is the probability of a person guessing the right combination?

Problems 23–28 involve an experiment consisting of dealing 5 cards from a standard 52-card deck. In Problems 23–26, what is the probability of being dealt:

23. Five black cards

24. Five hearts

25. Five face cards if an ace is considered to be a face card.

26. Five nonface cards if an ace is considered to be a one and not a face.

27. If we are interested in the number of aces in a 5-card hand, would $S = \{0, 1, 2, 3, 4\}$ be an acceptable sample space? Would it be an equally-likely sample space? Explain.

28. If we are interested in the number of black cards in a 5-card hand, would $S = \{0, 1, 2, 3, 4, 5\}$ be an acceptable sample space? Would it be an equally-likely sample space? Explain.

29. If four-digit numbers less than 5,000 are randomly formed from the digits 1, 3, 5, 7, and 9, what is the probability of forming a number divisible by 5? Digits may be repeated; for example, 1,355 is acceptable.

30. If code words of four letters are generated at random using the letters A, B, C, D, E, and F, what is the probability of forming a word without a vowel in it? Letters may be repeated.

31. Suppose five thank-you notes are written and five envelopes are addressed. Accidentally, the notes are randomly inserted into the envelopes and mailed without checking the addresses. What is the probability that all five notes will be inserted into the correct envelopes?

32. Suppose six people check their coats in a checkroom. If all claim checks are lost and the six coats are randomly returned, what is the probability that all six people will get their own coats back?

An experiment consists of rolling two fair dice and adding the dots on the two sides facing up. Using the sample space shown in Figure 1 (p. 929) and assuming each simple event is as likely as any other, find the probabilities of the sums of dots indicated in Problems 33–48.

33. Sum is 2.

34. Sum is 10.

35. Sum is 6.

36. Sum is 8.

37. Sum is less than 5.

38. Sum is greater than 8.

39. Sum is not 7 or 11.

40. Sum is not 2, 4, or 6.

41. Sum is 1.

42. Sum is not 13.

43. Sum is divisible by 3.

44. Sum is divisible by 4.

45. Sum is 7 or 11 (a "natural").

46. Sum is 2, 3, or 12 ("craps").

47. Sum is divisible by 2 or 3.

48. Sum is divisible by 2 and 3.

49. Five thousand people work in a large auto plant. An individual is selected at random and his or her birthday (month and day, not year) is recorded. Set up an appropriate sample space for this experiment and assign acceptable probabilities to the simple events.

50. In a hotly contested three-way race for governor of Minnesota, the leading candidates are running neck-and-neck while the third candidate is receiving half the support of either of the others. Registered voters are chosen at random and are asked for which of the three they are most likely to vote. Set up an appropriate sample space for the random survey experiment and assign acceptable probabilities to the simple events.

51. A pair of dice is rolled 500 times with the following frequencies:

Sum	2	3	4	5	6	7	8	9	10	11	12
Frequency	11	35	44	50	71	89	72	52	36	26	14

(A) Compute the approximate empirical probability for each outcome.
(B) Compute the theoretical probability for each outcome, assuming fair dice.
(C) Compute the expected frequency of each outcome.
(D) Describe how a random number generator could be used to simulate this experiment. If your graphing utility has a random number generator, use it to simulate 500 tosses of a pair of dice and compare your results with part C.

52. Three coins are flipped 500 times with the following frequencies of outcomes:

Three heads: 58 Two heads: 198
One head: 190 Zero heads: 54

(A) Compute the approximate empirical probability for each outcome.
(B) Compute the theoretical probability for each outcome, assuming fair coins.
(C) Compute the expected frequency of each outcome.
(D) Describe how a random number generator could be used to simulate this experiment. If your graphing utility has a random number generator, use it to simulate 500 tosses of three coins and compare your results with part C.

53. (A) Is it possible to get 29 heads in 30 flips of a fair coin? Explain.
(B) If you flip a coin 50 times and get 42 heads, would you suspect that the coin was unfair? Why or why not? If you suspect an unfair coin, what empirical probabilities would you assign to the simple events of the sample space?

54. (A) Is it possible to get nine double sixes in 12 rolls of a pair of fair dice? Explain.
(B) If you roll a pair of dice 40 times and get 14 double sixes, would you suspect that the dice were unfair? Why or why not? If you suspect loaded dice, what empirical probability would you assign to the event of rolling a double six?

An experiment consists of tossing three fair coins, but one of the three coins has a head on both sides. Compute the probabilities of obtaining the indicated results in Problems 55–60.

55. One head

56. Two heads

57. Three heads

58. Zero heads

59. More than one head

60. More than one tail

An experiment consists of rolling two fair dice and adding the dots on the two sides facing up. Each die has one dot on two opposite faces, two dots on two opposite faces, and three dots on two opposite faces. Compute the probabilities of obtaining the indicated sums in Problems 61–68.

61. 2

62. 3

63. 4 **64.** 5

65. 6 **66.** 7

67. An odd sum **68.** An even sum

An experiment consists of dealing 5 cards from a standard 52-card deck. In Problems 69–76, what is the probability of being dealt the following cards?

69. Five cards, jacks through aces

70. Five cards, 2 through 10

71. Four aces

72. Four of a kind

73. Straight flush, ace high; that is, 10, jack, queen, king, ace in one suit

74. Straight flush, starting with 2; that is, 2, 3, 4, 5, 6 in one suit

75. Two aces and three queens

76. Two kings and three aces

APPLICATIONS

77. MARKET ANALYSIS A company selected 1,000 households at random and surveyed them to determine a relationship between income level and the number of television sets in a home. The information gathered is listed in the table:

	Televisions per household				
Yearly income ($)	0	1	2	3	Above 3
Less than 12,000	0	40	51	11	0
12,000–19,999	0	70	80	15	1
20,000–39,999	2	112	130	80	12
40,000–59,999	10	90	80	60	21
60,000 or more	30	32	28	25	20

Compute the approximate empirical probabilities:
(A) Of a household earning $12,000–$19,999 per year *and* owning exactly three television sets
(B) Of a household earning $20,000–$39,999 per year *and* owning more than one television set
(C) Of a household earning $60,000 or more per year *or* owning more than three television sets
(D) Of a household not owning zero television sets

78. MARKET ANALYSIS Use the sample results in Problem 77 to compute the approximate empirical probabilities:
(A) Of a household earning $40,000–$59,999 per year *and* owning zero television sets
(B) Of a household earning $12,000–$39,999 per year *and* owning more than two television sets
(C) Of a household earning less than $20,000 per year *or* owning exactly two television sets
(D) Of a household not owning more than three television sets

10-6 The Binomial Formula

- Using Pascal's Triangle
- The Binomial Formula
- Proof of the Binomial Formula

In a surprising number of areas in math, it turns out to be useful to expand expressions of the form $(a + b)^n$, where n is a natural number. This is known as a *binomial expansion*. Expanding a binomial is pretty straightforward for small values of n, but gets hard very quickly as n increases. The good news is that it turns out that the coefficients in such an expansion are related to counting techniques that we have already learned about.

Using Pascal's Triangle

Let's begin by expanding $(a + b)^n$ for the first few values of n. We include $n = 0$, which is not a natural number, for reasons of completeness that will become apparent later.

$$\begin{aligned}(a + b)^0 &= 1\\ (a + b)^1 &= a + b\\ (a + b)^2 &= (a + b)(a + b) = a^2 + 2ab + b^2\\ (a + b)^3 &= (a + b)(a + b)(a + b) = a^3 + 3a^2b + 3ab^2 + b^3\end{aligned} \tag{1}$$

EXPLORE-DISCUSS 1

Based on the expansions in equations (1), how many terms would you expect $(a + b)^n$ to have? What is the first term? What is the last term?

CAUTION

Be aware of the fact that exponents cannot be "distributed." That is, $(a + b)^n \neq a^n + b^n$!

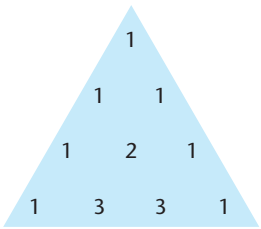


Figure 1 Pascal's triangle.

Now let's examine just the coefficients of the expansions in equations (1) arranged in a form that is usually referred to as **Pascal's triangle** (Fig. 1).

EXPLORE-DISCUSS 2

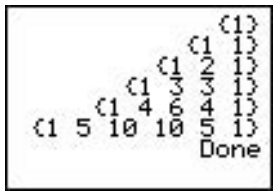
Refer to Figure 1.

(A) How is the middle element in the third row related to the elements in the row above it?

(B) How are the two inner elements in the fourth row related to the elements in the row above them?

(C) Based on your observations in parts A and B, make a conjecture about the fifth and sixth rows. Check your conjecture by expanding $(a + b)^4$ and $(a + b)^5$ using multiplication.

[*Hint:* Begin by multiplying $(a + b)$ by the last result in equations (1).]



› Figure 2

As indicated in Explore-Discuss 2, every element other than the first and last in any row of Pascal's triangle is found by adding the two elements diagonally above it.

Many students find Pascal's triangle a useful tool for determining the coefficients in the expansion of $(a + b)^n$, especially for small values of n. Figure 2 contains output from a program called PASCAL.* You should recognize the output in the table—it is the first six lines of Pascal's triangle.

EXAMPLE 1 **Using Pascal's Triangle**

Use Pascal's triangle to expand $(x + y)^4$.

SOLUTION

Equations (1) indicate that the coefficients come from the fourth row of Pascal's triangle (Fig. 2). The power of x starts at 4 and decreases to zero, while the power of y starts at zero and increases to 4.

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

MATCHED PROBLEM 1

Use Pascal's triangle to expand $(t + z)^5$.

The major drawback of using this triangle, whether it is generated by hand or on a graphing calculator, is that to find the elements in a given row, you have to write out all the rows that came before it. It would be useful to find a formula that gives the coefficients for a binomial expansion for a particular value of n directly. Fortunately, such a formula exists—the combination formula $C_{n,r}$ introduced in the section before last.

› The Binomial Formula

When working with binomial expansions, it is customary to use our alternate notation for the combination formula from the section before last.

› DEFINITION 1 Combination Formula

For nonnegative integers r and n, $0 \le r \le n$,

$$\binom{n}{r} = C_{n,r} = \frac{n!}{r!(n - r)!}$$

*Programs for TI-84 and TI-86 graphing calculators can be found at the website for this book.

Theorem 1 establishes that the coefficients in a binomial expansion can always be expressed in terms of the combination formula. This is both a very important theoretical result and a very practical tool. As we will see, using this theorem in conjunction with a graphing utility provides a very efficient method for expanding binomials.

› THEOREM 1 Binomial Formula

For n a positive integer

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

We defer the proof of Theorem 1 until the end of this section. Because the values of the combination formula are the coefficients in a binomial expansion, it is natural to call them **binomial coefficients.**

The use of the formula will be a lot clearer after working an example.

EXAMPLE 2 Using the Binomial Formula

Use the binomial formula to expand $(x + y)^6$.

SOLUTION

Compute the Coefficients

We will use the combination command on a graphing calculator to compute the coefficients. Enter $y_1 = 6\ nCr\ x$ and use the TABLE command. See Figure 3.

X	Y1
0	1
1	6
2	15
3	20
4	15
5	6
6	1

Y1■6 nCr X

› Figure 3

Write the Expansion

$$\begin{aligned}(x + y)^6 &= \sum_{k=0}^{6} \binom{6}{k} x^{6-k} y^k \\ &= \binom{6}{0}x^6 + \binom{6}{1}x^5y + \binom{6}{2}x^4y^2 + \binom{6}{3}x^3y^3 \\ &\quad + \binom{6}{4}x^2y^4 + \binom{6}{5}xy^5 + \binom{6}{6}y^6 \\ &= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6\end{aligned}$$

MATCHED PROBLEM 2

Use the binomial formula to expand $(x + 1)^5$.

EXAMPLE 3 Using the Binomial Formula

Use the binomial formula to expand $(3p - 2q)^4$.

SOLUTION

Note that to match the form of Theorem 1, we will write $3p - 2q$ as $3p + (-2q)$. Computing the coefficients is complicated by the coefficients of p and q, so we'll begin by writing out the expansion, and simplifying a bit:

$$\begin{aligned}(3p - 2q)^4 &= [(3p) + (-2q)]^4 && \text{Use Theorem 1.}\\ &= \sum_{k=0}^{4} \binom{4}{k} (3p)^{4-k}(-2q)^k && \text{Expand parentheses.}\\ &= \sum_{k=0}^{4} \binom{4}{k} 3^{4-k}(-2)^k p^{4-k} q^k\end{aligned}$$

Now we can see that the coefficients we need are $\binom{4}{k} 3^{4-k}(-2)^k$ for $k = 0, 1, 2, 3,$ and 4. We compute them by entering $y_1 = (4\ nCr\ x)3\ \wedge\ (4 - k)(-2)\ \wedge\ k$ and using the TABLE command (Fig. 4).

X	Y1
0	81
1	-216
2	216
3	-96
4	16
5	0
6	0

Y1■(4 nCr X)3^(…

› Figure 4 $y_1 = C_{4,x}3^{4-x}(-2)^x$.

$$(3p - 2q)^4 = 81p^4 - 216p^3q + 216p^2q^2 - 96pq^3 + 16q^4$$

MATCHED PROBLEM 3

Use the binomial formula to expand $(2m - 5n)^3$.

››› EXPLORE-DISCUSS 3

(A) Compute each term and also the sum of the alternating series

$$\binom{6}{0} - \binom{6}{1} + \binom{6}{2} - \cdots + \binom{6}{6}.$$

(B) What result about an alternating series can be deduced by letting $a = 1$ and $b = -1$ in the binomial formula?

EXAMPLE 4 Using the Binomial Formula

Find the term containing x^9 in the expansion of $(x + 3)^{14}$.

SOLUTION

In the expansion

$$(x + 3)^{14} = \sum_{k=0}^{14} \binom{14}{k} x^{14-k} 3^k$$

the exponent of x is 9 when $k = 5$. So, the term containing x^9 is

$$\binom{14}{5} x^9 3^5 = (2{,}002)(243)x^9 = 486{,}486x^9$$

MATCHED PROBLEM 4

Find the term containing y^8 in the expansion of $(2 + y)^{14}$.

EXAMPLE 5 Using the Binomial Formula

If the terms in the expansion of $(x - 2)^{20}$ are arranged in decreasing powers of x, find the fourth term and the sixteenth term.

SOLUTION

According to Theorem 1, in the expansion of $(a + b)^n$, the exponent of b in the rth term is $r - 1$ and the exponent of a is $n - (r - 1)$. Using $r = 4$ and $r = 16$, we get

Fourth term:

$$\begin{aligned}\binom{20}{3} x^{17}(-2)^3 &= \frac{20 \cdot 19 \cdot 18}{3 \cdot 2 \cdot 1} x^{17}(-8) \\ &= -9{,}120x^{17}\end{aligned}$$

Sixteenth term:

$$\begin{aligned}\binom{20}{15} x^5(-2)^{15} &= \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} x^5(-32.768) \\ &= -508{,}035{,}072x^5\end{aligned}$$

MATCHED PROBLEM 5

If the terms in the expansion of $(u - 1)^{18}$ are arranged in decreasing powers of u, find the fifth term and the twelfth term.

› Proof of the Binomial Formula

We will now prove that the binomial formula holds for all natural numbers n using mathematical induction.

PROOF: State the conjecture.

$$P_n: \quad (a+b)^n = \sum_{i}^{n} \binom{n}{i} a^{n-j}b^j$$

PART 1: Show that P_1 is true.

$$\sum_{j=0}^{1} \binom{1}{j} a^{1-j}b^j = \binom{1}{0}ab^0 + \binom{1}{1}a^0b = a + b = (a+b)^1$$

P_1 is true.

PART 2: Show that if P_k is true, then P_{k+1} is true.

$$P_k: \quad (a+b)^k = \sum_{j=0}^{k} \binom{k}{j} a^{k-j}b^j \qquad \text{Assume } P_k \text{ is true.}$$

$$P_{k+1}: \quad (a+b)^{k+1} = \sum_{j=0}^{k+1} \binom{k+1}{j} a^{k+1-j}b^j \qquad \text{Show } P_{k+1} \text{ is true.}$$

We begin by multiplying both sides of P_k by $(a + b)$:

$$(a+b)^k(a+b) = \left[\sum_{j=0}^{k} \binom{k}{j} a^{k-j}b^j\right](a+b)$$

The left side of this equation is the left side of P_{k+1}. Now we multiply out the right side of the equation and try to obtain the right side of P_{k+1}:

$$\begin{aligned}
(a+b)^{k+1} &= \left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k}b^k\right](a+b) && \text{Distribute.}\\
&= \left[\binom{k}{0}a^{k+1} + \binom{k}{1}a^kb + \binom{k}{2}a^{k-1}b^2 + \cdots + \binom{k}{k}ab^k\right]\\
&\quad + \left[\binom{k}{0}a^kb + \binom{k}{1}a^{k-1}b^2 + \cdots + \binom{k}{k-1}ab^k + \binom{k}{k}b^{k+1}\right] && \text{Group like terms.}\\
&= \binom{k}{0}a^{k+1} + \left[\binom{k}{0} + \binom{k}{1}\right]a^kb + \left[\binom{k}{1} + \binom{k}{2}\right]a^{k-1}b^2 + \cdots\\
&\quad + \left[\binom{k}{k-1} + \binom{k}{k}\right]ab^k + \binom{k}{k}b^{k+1}
\end{aligned}$$

We will need to use the following facts (the proofs are left as exercises; see Problems 67–69):

$$\binom{k}{r-1} + \binom{k}{r} = \binom{k+1}{r} \qquad \binom{k}{0} = \binom{k+1}{0} \qquad \binom{k}{k} = \binom{k+1}{k+1}$$

Now we can rewrite the right side as

$$\binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^k b + \binom{k+1}{2}a^{k-1}b^2 + \cdots + \binom{k+1}{k}ab^k + \binom{k+1}{k+1}b^{k+1} = \sum_{j=0}^{k+1}\binom{k+1}{j}a^{k+1-j}b^j$$

Because the right side of the last equation is the right side of P_{k+1}, we have shown that P_{k+1} follows from P_k.

CONCLUSION: P_n is true. That is, the binomial formula holds for all positive integers n.

ANSWERS TO MATCHED PROBLEMS

1. $t^5 + 5t^4z + 10t^3z^2 + 10t^2z^3 + 5tz^4 + z^5$ **2.** $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$
3. $8m^3 - 60m^2n + 150mn^2 - 125n^3$ **4.** 192, $192y^8$ **5.** $3{,}060u^{14}$; $-31{,}824u^7$

10-6 Exercises

1. Explain how to draw Pascal's triangle.

2. How can Pascal's triangle be used to expand $(a + b)^4$?

3. According to the binomial formula, are there any natural numbers n for which $(a + b)^n = a^n + b^n$? Explain.

4. How is the combination $\binom{n}{r}$ related to Pascal's triangle?

In Problems 5–8, expand each expression using Pascal's triangle.

5. $(a + b)^4$ **6.** $(x + y)^3$ **7.** $(x + 2)^5$ **8.** $(y + 1)^5$

In Problems 9–16, use Pascal's triangle to evaluate each expression. [Hint: According to the binomial formula, $\binom{n}{k}$ is the coefficient of the kth term of $(a + b)^n$.]

9. $\binom{5}{3}$ **10.** $\binom{6}{4}$ **11.** $\binom{4}{2}$ **12.** $\binom{7}{5}$ **13.** $C_{6,3}$

14. $C_{5,3}$ **15.** $C_{7,4}$ **16.** $C_{4,3}$

In Problems 17–24, use a graphing utility to evaluate each expression.

17. $\binom{9}{3}$ **18.** $\binom{10}{6}$ **19.** $\binom{12}{10}$ **20.** $\binom{13}{8}$

21. $\binom{17}{13}$ **22.** $\binom{20}{16}$ **23.** $\binom{50}{4}$ **24.** $\binom{50}{45}$

In Problems 25–36, expand the binomial using the binomial formula.

25. $(m + n)^3$ **26.** $(x + 2)^3$ **27.** $(2x - 3y)^3$

28. $(3u + 2v)^3$ **29.** $(x - 2)^4$ **30.** $(x - y)^4$

31. $(m + 3n)^4$ **32.** $(3p - q)^4$ **33.** $(2x - y)^5$

34. $(2x - 1)^5$ **35.** $(m + 2n)^6$ **36.** $(2x - y)^6$

In Problems 37–46, find the term of the binomial expansion containing the given power of x.

37. $(x + 1)^7$; x^4 **38.** $(x + 1)^8$; x^5 **39.** $(2x - 1)^{11}$; x^6

40. $(3x + 1)^{12}$; x^7 **41.** $(2x + 3)^{18}$; x^{14} **42.** $(3x - 2)^{17}$; x^5

43. $(x^2 - 1)^6$; x^8 **44.** $(x^2 - 1)^9$; x^7 **45.** $(x^2 + 1)^9$; x^{11}

46. $(x^2 + 1)^{10}$; x^{14}

In Problems 47–54, find the indicated term in each expansion if the terms of the expansion are arranged in decreasing powers of the first term in the binomial.

47. $(u + v)^{15}$; seventh term **48.** $(a + b)^{12}$; fifth term

49. $(2m + n)^{12}$; eleventh term **50.** $(x + 2y)^{20}$; third term

51. $[(w/2) - 2]^{12}$; seventh term **52.** $(x - 3)^{10}$; fourth term

53. $(3x - 2y)^8$; sixth term **54.** $(2p - 3q)^7$; fourth term

In Problems 55–58, use the binomial formula to expand and simplify the difference quotient

$$\frac{f(x+h)-f(x)}{h}$$

for the indicated function f. Discuss the behavior of the simplified form as h approaches 0.

55. $f(x) = x^3$ **56.** $f(x) = x^4$

57. $f(x) = x^5$ **58.** $f(x) = x^6$

In Problems 59–62, use a graphing utility to graph each sequence and to display it in table form.

59. Find the number of terms of the sequence

$$\binom{20}{0}, \binom{20}{1}, \binom{20}{2}, \ldots, \binom{20}{20}$$

that are greater than one-half of the largest term.

60. Find the number of terms of the sequence

$$\binom{40}{0}, \binom{40}{1}, \binom{40}{2}, \ldots, \binom{40}{40}$$

that are greater than one-half of the largest term.

61. (A) Find the largest term of the sequence $a_0, a_1, a_2, \ldots, a_{10}$ to three decimal places, where

$$a_k = \binom{10}{k}(0.6)^{10-k}(0.4)^k$$

(B) According to the binomial formula, what is the sum of the series $a_0 + a_1 + a_2 + \cdots + a_{10}$?

62. (A) Find the largest term of the sequence $a_0, a_1, a_2, \ldots, a_{10}$ to three decimal places, where

$$a_k = \binom{10}{k}(0.3)^{10-k}(0.7)^k$$

(B) According to the binomial formula, what is the sum of the series $a_0 + a_1 + a_2 + \cdots + a_{10}$?

63. Evaluate $(1.01)^{10}$ to four decimal places, using the binomial formula. [*Hint:* Let $1.01 = 1 + 0.01$.]

64. Evaluate $(0.99)^6$ to four decimal places, using the binomial formula.

65. Show that: $\binom{n}{r} = \binom{n}{n-r}$ **66.** Show that: $\binom{n}{0} = \binom{n}{n}$

67. Show that: $\binom{k}{r-1} + \binom{k}{r} = \binom{k+1}{r}$

68. Show that: $\binom{k}{0} = \binom{k+1}{0}$

69. Show that: $\binom{k}{k} = \binom{k+1}{k+1}$

70. Show that: $\binom{n}{r}$ is given by the recursion formula

$$\binom{n}{r} = \frac{n-r+1}{r}\binom{n}{r-1}$$

where $\binom{n}{0} = 1$.

71. Write $2^n = (1+1)^n$ and expand, using the binomial formula to obtain

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

72. Write $0 = (1-1)^n$ and expand, using the binomial formula, to obtain

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n}$$

CHAPTER 10 Review

10-1 Sequences and Series

A **sequence** is a list of numbers in a specific order. More formally, a sequence is a function with the domain a set of successive integers. The symbol a_n, called the ***n*th term,** or **general term,** represents the range value associated with the domain value n. Unless specified otherwise, the domain is understood to be the set of natural numbers. Given a short list of numbers, there can be many sequences that have those numbers as its first few terms. Still, we can often find a general term that matches the first few terms of a sequence.

A **finite sequence** has a finite domain, and an **infinite sequence** has an infinite domain. A **recursion formula** defines

each term of a sequence in terms of one or more of the preceding terms. For example, the **Fibonacci sequence** is defined by $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$, where $a_1 = a_2 = 1$. If $a_1, a_2, \ldots, a_n, \ldots$ is a sequence, then the expression $a_1 + a_2 + \cdots + a_n + \cdots$ is called a **series.** A finite sequence produces a **finite series,** and an infinite sequence produces an **infinite series.** Series can be represented using the summation notation

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$$

where k is called the **summing index.** If the terms in the series are alternately positive and negative, the series is called an **alternating series.**

10-2 Mathematical Induction

A wide variety of statements can be proven using the **principle of mathematical induction:** Let P_n be a statement associated with each positive integer n and suppose the following conditions are satisfied:

1. P_1 is true.

2. For any positive integer k, if P_k is true, then P_{k+1} is also true.

Then the statement P_n is true for all positive integers n.

It's important to remember that a result cannot be proved by showing that it is true for any amount of specific natural numbers. To prove that a result is false, you only need to find a single **counterexample;** that is, one specific case for which the result fails.

To use mathematical induction to prove statements involving laws of exponents, it is convenient to state a **recursive definition of a^n:**

$$a^1 = a \quad \text{and} \quad a^{n+1} = a^n a \quad \text{for any integer } n > 1$$

To deal with conjectures that may be true only for $n \geq m$, where m is a positive integer, we use the **extended principle of mathematical induction:** Let m be a positive integer, let P_n be a statement associated with each integer $n \geq m$, and suppose the following conditions are satisfied:

1. P_m is true.

2. For any integer $k \geq m$, if P_k is true, then P_{k+1} is also true.

Then the statement P_n is true for all integers $n \geq m$.

10-3 Arithmetic and Geometric Sequences

A sequence is called an **arithmetic sequence,** or **arithmetic progression,** if there exists a constant d, called the **common difference,** such that

$$a_n - a_{n-1} = d \quad \text{or} \quad a_n = a_{n-1} + d \quad \text{for every } n > 1$$

That is, each term is obtained by adding the same fixed number to the previous term.

The following formulas are useful when working with arithmetic sequences and their corresponding series: (Recall that the symbol S_n represents the sum of the first n terms of a sequence.)

$$a_n = a_1 + (n-1)d \qquad \textbf{nth-Term Formula}$$

$$S_n = \frac{n}{2}[2a_1 + (n-1)d] \qquad \textbf{Sum Formula—First Form}$$

$$S_n = \frac{n}{2}(a_1 + a_n) \qquad \textbf{Sum Formula—Second Form}$$

A sequence is called a **geometric sequence,** or a **geometric progression,** if there exists a nonzero constant r, called the **common ratio,** such that

$$\frac{a_n}{a_{n-1}} = r \quad \text{or} \quad a_n = ra_{n-1} \quad \text{for every } n > 1$$

That is, each term is obtained by multiplying the previous term by the same fixed number.

The following formulas are useful when working with geometric sequences and their corresponding series:

$$a_n = a_1 r^{n-1} \qquad \textbf{nth-Term Formula}$$

$$S_n = \frac{a_1 - a_1 r^n}{1 - r} \quad r \neq 1 \qquad \textbf{Sum Formula—First Form}$$

$$S_n = \frac{a_1 - ra_n}{1 - r} \quad r \neq 1 \qquad \textbf{Sum Formula—Second Form}$$

$$S_\infty = \frac{a_1}{1 - r} \quad |r| < 1 \qquad \textbf{Sum of an Infinite Geometric Series}$$

10-4 Multiplication Principle, Permutations, and Combinations

A **counting technique** is a process for finding the number of objects in a given set without counting them one at a time.

Given a sequence of operations, **tree diagrams** are often used to list all the possible combined outcomes. To count the number of combined outcomes without actually listing them, we use the **multiplication principle** (also called the **fundamental counting principle**):

1. If operations O_1 and O_2 are performed in order with N_1 possible outcomes for the first operation and N_2 possible outcomes for the second operation, then there are

$$N_1 \cdot N_2$$

possible outcomes of the first operation followed by the second.

2. In general, if n operations $O_1, O_2, \ldots, O_n$ are performed in order, with possible number of outcomes $N_1, N_2, \ldots, N_n$ respectively, then there are

$$N_1 \cdot N_2 \cdot \cdots \cdot N_n$$

possible combined outcomes of the operations performed in the given order.

For any natural number n, ***n!*** (read **"*n* factorial"**) is defined by $n! = n \cdot (n-1) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$ and $0!$ is defined to be 1.

A particular arrangement or ordering of n objects without repetition is called a **permutation.** The number of permutations of n objects is given by

$$P_{n,n} = n \cdot (n-1) \cdot \cdots \cdot 1 = n!$$

A ***permutation of n* objects taken *r* at a time** is an arrangement of the r objects in a specific order. The number of permutations of n objects taken r at a time is given by

$$P_{n,r} = \frac{n!}{(n-r)!} \qquad 0 \le r \le n$$

A **combination of *n* objects taken *r* at a time** is an r-element subset of the n objects. The number of combinations of n objects taken r at a time is given by

$$C_{n,r} = \binom{n}{r} = \frac{P_{n,r}}{r!} = \frac{n!}{r!(n-r)!} \qquad 0 \le r \le n$$

In a permutation, order is important. In a combination, order is not important.

10-5 Sample Spaces and Probability

The outcomes of an experiment are called **simple events** if one and only one of these results will occur in each trial of the experiment. The set of all simple events is called the **sample space.** Any subset of the sample space is called an **event.** An event is a **simple event** if it has only one element in it and a **compound event** if it has more than one element in it. We say that **an event *E* occurs** if any of the simple events in E occurs. A sample space S_1 is **more fundamental** than a second sample space S_2 if knowledge of which event occurs in S_1 tells us which event in S_2 occurs, but not conversely.

Given a sample space $S = \{e_1, e_2, \ldots, e_n\}$ with n simple events, to each simple event e_i we assign a real number, denoted by $P(e_1)$, that is called the **probability of the event e_i** and satisfies:

1. $0 \le P(e_1) \le 1$

2. $P(e_1) + P(e_2) + \cdots + P(e_n) = 1$

Informally, probability is a number between zero and 1 inclusive that represents the likelihood of an event occurring. Higher probabilities indicate events that are more likely to occur. Any probability assignment that meets conditions 1 and 2 is said to be an **acceptable probability assignment.**

Given an acceptable probability assignment for the simple events in a sample space S, the **probability of an arbitrary event *E*** is defined as follows:

1. If E is the empty set, then $P(E) = 0$.

2. If E is a simple event, then $P(E)$ has already been assigned.

3. If E is a compound event, then $P(E)$ is the sum of the probabilities of all the simple events in E.

4. If E is the entire sample space S, then $P(E) = P(S) = 1$.

If each of the simple events in a sample space $S = \{e_1, e_2, \ldots, e_n\}$ with n simple events is **equally likely** to occur, then we assign the probability $1/n$ to each. If E is any event in S, then

$$P(E) = \frac{\text{Number of elements in } E}{\text{Number of elements in } S} = \frac{n(E)}{n(S)}$$

When reasonable assumptions about the likelihood of an event are used to assign probabilities, we call these **theoretical probabilities.** If we conduct an experiment n times and event E occurs with **frequency** $f(E)$, then the ratio $f(E)/n$ is called the **relative frequency** of the occurrence of event E in n trials. As n increases, $f(E)/n$ usually approaches a number that is called the **empirical probability** $P(E)$. Thus, $f(E)/n$ is used as an **approximate empirical probability** for $P(E)$.

If $P(E)$ is the theoretical probability of an event E and the experiment is performed n times, then the **expected frequency** of the occurrence of E is $n \cdot P(E)$.

The command **randInt(*i*,*k*,*n*)** on a Texas Instruments TI-84 generates a random sequence of n integers between i and k, inclusively, that can be used to simulate repeated trials of experiments.

10-6 Binomial Formula

Pascal's triangle is a triangular array of coefficients for the expansion of the binomial $(a+b)^n$, where n is a positive integer. An alternative notation for the combination formula is

$$\binom{n}{r} = C_{n,r} = \frac{n!}{r!(n-r)!}$$

For n a positive integer, the **binomial formula** is

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

The numbers $\binom{n}{k}$, $0 \le k \le n$, are called **binomial coefficients.**

CHAPTER 10 Review Exercises

Work through all the problems in this chapter review and check answers in the back of the book. Answers to all review problems are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

1. Determine whether each of the following can be the first three terms of a geometric sequence, an arithmetic sequence, or neither.

(A) 16, −8, 4, . . . (B) 5, 7, 9, . . .

(C) −8, −5, −2, . . . (D) 2, 3, 5, . . .

(E) −1, 2, −4, . . .

In Problems 2–5:
(A) Write the first four terms of each sequence.
(B) Find a_{10}. (C) Find S_{10}.

2. $a_n = 2n + 3$ **3.** $a_n = 32(\frac{1}{2})^n$

4. $a_1 = -8$; $a_n = a_{n-1} + 3$, $n \geq 2$

5. $a_1 = -1$; $a_n = (-2)a_{n-1}$, $n \geq 2$

6. Find S_∞ in Problem 3.

Evaluate Problems 7–10.

7. $6!$ **8.** $\dfrac{22!}{19!}$

9. $\dfrac{7!}{2!(7-2)!}$ **10.** $C_{6,2}$ and $P_{6,2}$

11. A single die is rolled and a coin is flipped. How many combined outcomes are possible? Solve

(A) By using a tree diagram

(B) By using the multiplication principle

12. How many seating arrangements are possible with six people and six chairs in a row? Solve by using the multiplication principle.

13. Solve Problem 12 using permutations or combinations, whichever is applicable.

14. In a single deal of 5 cards from a standard 52-card deck, what is the probability of being dealt five clubs?

15. Betty and Bill are members of a 15-person ski club. If the president and treasurer are selected by lottery, what is the probability that Betty will be president and Bill will be treasurer? A person cannot hold more than one office.

16. A drug has side effects for 50 out of 1,000 people in a test. What is the approximate empirical probability that a person using the drug will have side effects?

Verify Problems 17–19 for $n = 1$, 2, and 3.

17. P_n: $5 + 7 + 9 + \cdots + (2n + 3) = n^2 + 4n$

18. P_n: $2 + 4 + 8 + \cdots + 2^n = 2^{n+1} - 2$

19. P_n: $49^n - 1$ is divisible by 6

In Problems 20–22, write P_k and P_{k+1}.

20. For P_n in Problem 17 **21.** For P_n in Problem 18

22. For P_n in Problem 19

23. Either prove the statement is true or prove it is false by finding a counterexample: If n is a positive integer, then the sum of the series $1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}$ is less than 4.

Write Problems 24 and 25 without summation notation, and find the sum.

24. $S_{10} = \displaystyle\sum_{k=1}^{10} (2k - 8)$ **25.** $S_7 = \displaystyle\sum_{k=1}^{7} \frac{16}{2^k}$

26. Find S_∞ for $27 - 18 + 12 + \cdots$

27. Write

$$S_n = \frac{1}{3} - \frac{1}{9} + \frac{1}{27} + \cdots + \frac{(-1)^{n+1}}{3^n}$$

using summation notation, and find S_∞.

28. Someone tells you that the following approximate empirical probabilities apply to the sample space $\{e_1, e_2, e_3, e_4\}$: $P(e_1) \approx .1$, $P(e_2) \approx -.2$, $P(e_3) \approx .6$, $P(e_4) \approx 2$. There are three reasons why P cannot be a probability function. Name them.

29. Six distinct points are selected on the circumference of a circle. How many triangles can be formed using these points as vertices?

30. In an arithmetic sequence, $a_1 = 13$ and $a_7 = 31$. Find the common difference d and the fifth term a_5.

31. How many three-letter code words are possible using the first eight letters of the alphabet if no letter can be repeated? If letters can be repeated? If adjacent letters cannot be alike?

32. Two coins are flipped 1,000 times with the following frequencies:

Two heads: 210

One head: 480

Zero heads: 310

(A) Compute the empirical probability for each outcome.

(B) Compute the theoretical probability for each outcome.

(C) Using the theoretical probabilities computed in part B, compute the expected frequency of each outcome, assuming fair coins.

33. From a standard deck of 52 cards, what is the probability of obtaining a 5-card hand:

(A) Of all diamonds?

(B) Of three diamonds and two spades?

Write answers in terms of $C_{n,r}$ or $P_{n,r}$, as appropriate. Do not evaluate.

34. A group of 10 people includes one married couple. If four people are selected at random, what is the probability that the married couple is selected?

35. A spinning device has three numbers, 1, 2, 3, each as likely to turn up as the other. If the device is spun twice, what is the probability that:

(A) The same number turns up both times?

(B) The sum of the numbers turning up is 5?

36. Use the formula for the sum of an infinite geometric series to write $0.727\,272\cdots = 0.\overline{72}$ as the quotient of two integers.

37. Solve the following problems using $P_{n,r}$ or $C_{n,r}$, as appropriate:

(A) How many three-digit opening combinations are possible on a combination lock with six digits if the digits cannot be repeated?

(B) Suppose five tennis players have made the finals. If each of the five players is to play every other player exactly once, how many games must be scheduled?

Evaluate Problems 38–40.

38. $\dfrac{20!}{18!(20-18)!}$ **39.** $\binom{16}{12}$ **40.** $\binom{11}{11}$

41. Expand $(x-y)^5$ using the binomial formula.

42. Find the term containing x^6 in the expansion of $(x+2)^9$.

43. If the terms in the expansion of $(2x-y)^{12}$ are arranged in descending powers of x, find the tenth term.

Prove each statement in Problems 44–46 for all natural numbers, using mathematical induction.

44. P_n in Problem 17 **45.** P_n in Problem 18

46. P_n in Problem 19

In Problems 47 and 48, find the smallest positive integer n such that $a_n < b_n$ by graphing the sequences $\{a_n\}$ and $\{b_n\}$ with a graphing utility. Check your answer by using a graphing utility to display both sequences in table form.

47. $a_n = C_{50,n}$, $b_n = 3^n$

48. $a_1 = 100$, $a_n = 0.99a_{n-1} + 5$, $b_n = 9 + 7n$

49. How many different families with five children are possible, excluding multiple births, where the sex of each child in the order of their birth is taken into consideration? How many families are possible if the order pattern is not taken into account?

50. A free-falling body travels $g/2$ feet in the first second, $3g/2$ feet during the next second, $5g/2$ feet the next, and so on. Find the distance fallen during the twenty-fifth second and the total distance fallen from the start to the end of the twenty-fifth second.

51. How many ways can two people be seated in a row of four chairs?

52. Expand $(x+i)^6$, where i is the imaginary unit, using the binomial formula.

53. If three people are selected from a group of seven men and three women, what is the probability that at least one woman is selected?

54. Three fair coins are tossed 1,000 times with the following frequencies of outcomes:

Number of heads	0	1	2	3
Frequency	120	360	350	170

(A) What is the approximate empirical probability of obtaining two heads?

(B) What is the theoretical probability of obtaining two heads?

(C) What is the expected frequency of obtaining two heads?

Prove that each statement in Problems 55–59 holds for all positive integers, using mathematical induction.

55. $\displaystyle\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^n$

56. $x^{2n} - y^{2n}$ is divisible by $x - y$, $x \neq y$

57. $\dfrac{a^n}{a^m} = a^{n-m}$; $n > m$, n, m positive integers

58. $\{a_n\} = \{b_n\}$, where $a_n = a_{n-1} + 2$, $a_1 = -3$, $b_n = -5 + 2n$

59. $(1!)1 + (2!)2 + (3!)3 + \cdots + (n!)n = (n + 1)! - 1$
(From U.S.S.R. Mathematical Olympiads, 1955–1956, Grade 10.)

APPLICATIONS

60. **LOAN REPAYMENT** You borrow \$7,200 and agree to pay 1% of the unpaid balance each month for interest. If you decide to pay an additional \$300 each month to reduce the unpaid balance, how much interest will you pay over the 24 months it will take to repay this loan?

61. **ECONOMICS** Due to reduced taxes, an individual has an extra \$2,400 in spendable income. If we assume that the individual spends 75% of this on consumer goods, and the producers of those consumer goods in turn spend 75% on consumer goods, and that this process continues indefinitely, what is the total amount (to the nearest dollar) spent on consumer goods?

62. **COMPOUND INTEREST** If \$500 is invested at 6% compounded annually, the amount A present after n years forms a geometric sequence with common ratio $1 + 0.06 = 1.06$. Use a geometric sequence formula to find the amount A in the account (to the nearest cent) after 10 years. After 20 years.

63. **TRANSPORTATION** A distribution center A wishes to distribute its products to five different retail stores, B, C, D, E, and F, in a city. How many different route plans can be constructed so that a single truck can start from A, deliver to each store exactly once, and then return to the center?

64. **MARKET ANALYSIS** A DVD company selected 1,000 persons at random and surveyed them to determine a relationship between age of purchaser and annual DVD purchases. The results are given in the table.

	DVD's purchased annually				
Age	**0**	**1**	**2**	**Above 2**	**Totals**
Under 12	60	70	30	10	170
12–18	30	100	100	60	290
19–25	70	110	120	30	330
Over 25	100	50	40	20	210
Totals	260	330	290	120	1,000

Find the empirical probability that a person selected at random

(A) Is over 25 *and* buys exactly two DVD's annually.

(B) Is 12–18 years old *and* buys more than one DVD annually.

(C) Is 12–18 years old *or* buys more than one DVD annually.

65. **QUALITY CONTROL** Twelve precision parts, including two that are substandard, are sent to an assembly plant. The plant manager selects four at random and will return the whole shipment if one or more of the sample are found to be substandard. What is the probability that the shipment will be returned?

CHAPTER 10

››› GROUP ACTIVITY Sequences Specified by Recursion Formulas

The recursion formula $a_n = 5a_{n-1} - 6a_{n-2}$, together with the initial values $a_1 = 4$ and $a_2 = 14$, specifies the sequence $\{a_n\}$ whose first several terms are 4, 14, 46, 146, 454, 1394, ... The sequence $\{a_n\}$ is neither arithmetic nor geometric. Nevertheless, because it satisfies a simple recursion formula, it is possible to obtain an nth-term formula for $\{a_n\}$ that is analogous to the nth-term formulas for arithmetic and geometric sequences. Such an nth-term formula is valuable because it enables us to estimate a term of a sequence without computing all the preceding terms.

If the geometric sequence $\{r^n\}$ satisfies the recursion formula above, then $r^n = 5r^{n-1} - 6r^{n-2}$. Dividing both sides by r^{n-2} leads to the quadratic equation $r^2 - 5r + 6 = 0$, whose solutions are $r = 2$ and $r = 3$. Now it is easy to check that the geometric sequences $\{2^n\} = 2, 4, 8, 16, \ldots$ and $\{3^n\} = 3, 9, 27, 81, \ldots$ satisfy the

recursion formula. Therefore, any sequence of the form $\{u2^n + v3^n\}$, where u and v are constants, will satisfy the same recursion formula.

We now find u and v so that the first two terms of $\{u2^n + v3^n\}$, are $a_1 = 4$ and $a_2 = 14$. Letting $n = 1$ and $n = 2$ we see that u and v must satisfy the following linear system:

$$\begin{aligned} 2u + 3v &= 4 \\ 4u + 9v &= 14 \end{aligned}$$

Solving the system gives $u = -1$ and $v = 2$. Therefore, an nth-term formula for the original sequence is $a_n = (-1)2^n + (2)3^n$.

Note that the nth-term formula was obtained by solving a quadratic equation and a system of two linear equations in two variables.

(A) Compute $(-1)2^n + (2)3^n$ for $n = 1, 2, \ldots, 6$, and compare with the terms of $\{a_n\}$.
(B) Estimate the one-hundredth term of $\{a_n\}$.
(C) Show that any sequence of the form $\{u2^n + v3^n\}$, where u and v are constants, satisfies the recursion formula $a_n = 5a_{n-1} - 6a_{n-2}$.
(D) Find an nth-term formula for the sequence $\{b_n\}$ that is specified by $b_1 = 5$, $b_2 = 55$, $b_n = 3b_{n-1} + 4b_{n-2}$.
(E) Find an nth-term formula for the Fibonacci sequence.
(F) Find an nth-term formula for the sequence $\{c_n\}$ that is specified by $c_1 = -3$, $c_2 = 15$, $c_3 = 99$, and $c_n = 6c_{n-1} - 3c_{n-2} - 10c_{n-3}$. (Because the recursion formula involves the three terms that precede c_n, our method will involve the solution of a cubic equation and a system of three linear equations in three variables.)

Additional Topics in Analytic Geometry

ANALYTIC geometry is the study of geometric objects using algebraic techniques. René Descartes (1596–1650), the French philosopher-mathematician, is generally recognized as the founder of the subject. In Chapter 2, we used analytic geometry to obtain equations of lines. In this chapter, we take a similar approach to the study of parabolas, ellipses, and hyperbolas. Each of these geometric objects is a *conic section,* that is, the intersection of a plane and a cone. We will derive equations for the conic sections, solve systems involving equations of conic sections, and explore a wealth of applications in architecture, communications, engineering, medicine, optics, and space science.

CHAPTER

11

OUTLINE

11-1 Conic Sections; Parabola

11-2 Ellipse

11-3 Hyperbola

11-4 Translation and Rotation of Axes

11-5 Systems of Nonlinear Equations

Chapter 11 Review

Chapter 11 Group Activity: Focal Chords

Cumulative Review Chapters 10 and 11

11-1 Conic Sections; Parabola

- Conic Sections
- Defining a Parabola
- Drawing a Parabola
- Standard Equations of Parabolas and Their Graphs
- Applications

In this section, we introduce the general concept of a conic section and then discuss the particular conic section called a *parabola*. In the next two sections, we will discuss two other conic sections called *ellipses* and *hyperbolas.*

Conic Sections

In Section 2-1 we found that the graph of a first-degree equation in two variables,

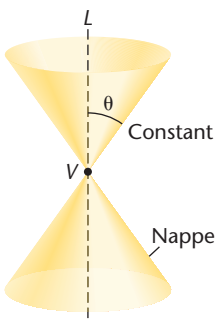
$$Ax + By = C \tag{1}$$

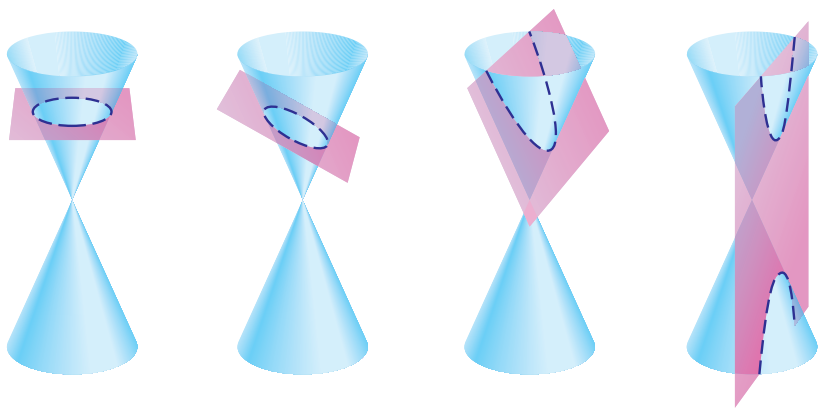
where A and B are not both 0, is a straight line, and every straight line in a rectangular coordinate system has an equation of this form. What kind of graph will a second-degree equation in two variables,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{2}$$

where A, B, and C are not all 0, yield for different sets of values of the coefficients? The graphs of equation (2) for various choices of the coefficients are plane curves obtainable by intersecting a cone* with a plane, as shown in Figure 1. These curves are called **conic sections.**

Figure 1 Conic sections.





*Starting with a fixed line L and a fixed point V on L, the surface formed by all straight lines through V making a constant angle θ with L is called a **right circular cone.** The fixed line L is called the **axis** of the cone, and V is its **vertex.** The two parts of the cone separated by the vertex are called **nappes.**

If a plane cuts clear through one nappe, then the intersection curve is called a **circle** if the plane is perpendicular to the axis and an **ellipse** if the plane is not perpendicular to the axis. If a plane cuts only one nappe, but does not cut clear through, then the intersection curve is called a **parabola.** Finally, if a plane cuts through both nappes, but not through the vertex, the resulting intersection curve is called a **hyperbola.** A plane passing through the vertex of the cone produces a **degenerate conic**—a point, a line, or a pair of lines.

Conic sections are very useful and are readily observed in your immediate surroundings: wheels (circle), the path of water from a garden hose (parabola), some serving platters (ellipses), and the shadow on a wall from a light surrounded by a cylindrical or conical lamp shade (hyperbola) are some examples (Fig. 2). We will discuss many applications of conics throughout the remainder of this chapter.

› **Figure 2** Examples of conics.

Wheel (circle) (a) | Water from garden hose (parabola) (b) | Serving platter (ellipse) (c) | Lamp light shadow (hyperbola) (d)

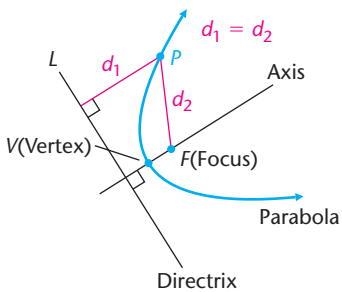
A definition of a conic section that does not depend on the coordinates of points in any coordinate system is called a **coordinate-free definition.** In Appendix A, Section A-3 we gave a coordinate-free definition of a circle and developed its standard equation in a rectangular coordinate system. In this and the next two sections, we will give coordinate-free definitions of a parabola, ellipse, and hyperbola, and we will develop standard equations for each of these conics in a rectangular coordinate system.

› Defining a Parabola

The following definition of a parabola does not depend on the coordinates of points in any coordinate system:

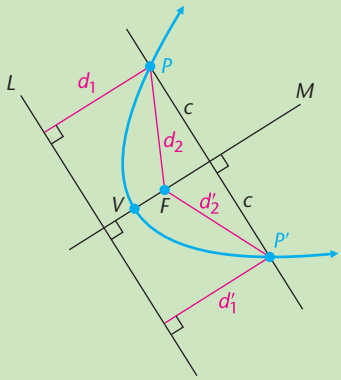
› DEFINITION 1 Parabola

A **parabola** is the set of all points in a plane equidistant from a fixed point F and a fixed line L in the plane. The fixed point F is called the **focus,** and the fixed line L is called the **directrix.** A line through the focus perpendicular to the directrix is called the **axis,** and the point on the axis halfway between the directrix and focus is called the **vertex.**



››› EXPLORE-DISCUSS 1

In a plane, the reflection of a point P through a line M is the point P' such that line M is the perpendicular bisector of the segment PP'. The figure shown here can be used to verify that the graph of a parabola is symmetric with respect to line M.



(A) Use the figure to show that $d_2 = d_2'$.

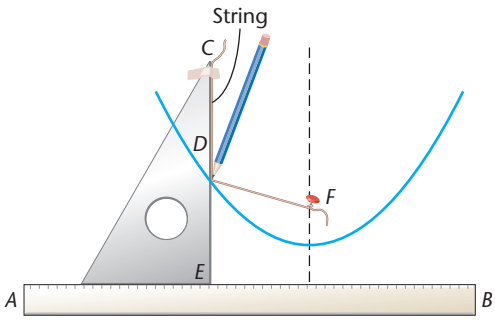
(B) Use the figure and part A to show that $d_1 = d_1'$.

Can you now conclude that the graph of a parabola is, in fact, symmetric with respect to its axis of symmetry? Explain.

› Drawing a Parabola

Using Definition 1, we can draw a parabola with fairly simple equipment—a straightedge, a right-angle drawing triangle, a piece of string, a thumbtack, and a pencil. Referring to Figure 3, tape the straightedge along the line AB and place the thumbtack above the line AB. Place one leg of the triangle along the straightedge as indicated, then take a piece of string the same length as the other leg, tie one end to the thumbtack, and fasten the other end with tape at C on the triangle. Now press the string to the edge of the triangle, and keeping the string taut, slide the triangle along the straightedge. Because DE will always equal DF, the resulting curve will be part of a parabola with directrix AB lying along the straightedge and focus F at the thumbtack.

› **Figure 3** Drawing a parabola.



››› EXPLORE-DISCUSS 2

The line through the focus F that is perpendicular to the axis of a parabola intersects the parabola in two points G and H. Explain why the distance from G to H is twice the distance from F to the directrix of the parabola.

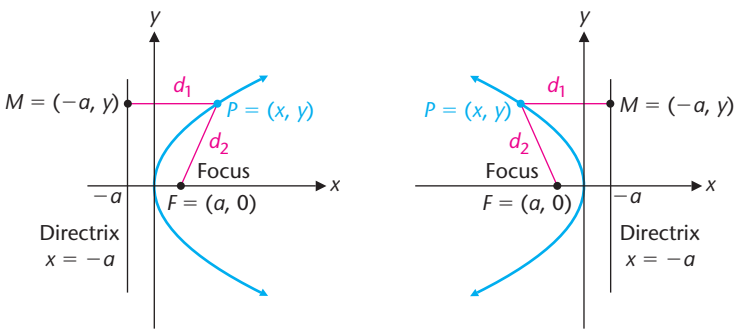
Standard Equations and Their Graphs

Using the definition of a parabola and the distance-between-two-points formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{3}$$

we can derive simple standard equations for a parabola located in a rectangular coordinate system with its vertex at the origin and its axis along a coordinate axis. We start with the axis of the parabola along the x axis and the focus at $F = (a, 0)$. We locate the parabola in a coordinate system as in Figure 4 and label key lines and points. This is an important step in finding an equation of a geometric figure in a coordinate system. Note that the parabola opens to the right if $a > 0$ and to the left if $a < 0$. The vertex is at the origin, the directrix is $x = -a$, and the coordinates of M are $(-a, y)$.

Figure 4 Parabola with vertex at the origin and axis of symmetry the x axis.



$a > 0$. focus on positive x axis
(a)

$a < 0$. focus on negative x axis
(b)

The point $P = (x, y)$ is a point on the parabola if and only if

$$\begin{aligned}
d_1 &= d_2 \\
d(P, M) &= d(P, F) \\
\sqrt{(x + a)^2 + (y - y)^2} &= \sqrt{(x - a)^2 + (y - 0)^2} && \text{Use equation (3).} \\
(x + a)^2 &= (x - a)^2 + y^2 && \text{Square both sides.} \\
x^2 + 2ax + a^2 &= x^2 - 2ax + a^2 + y^2 && \text{Simplify.} \\
y^2 &= 4ax
\end{aligned} \tag{4}$$

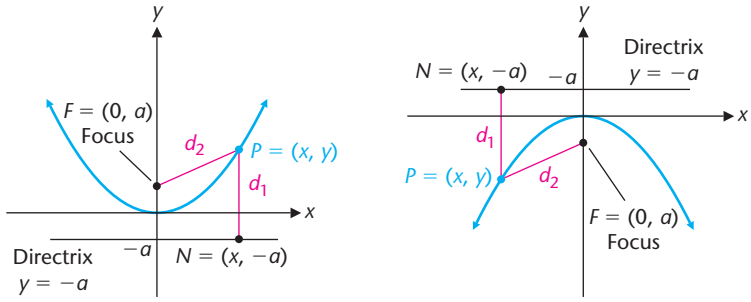
Equation (4) is the standard equation of a parabola with vertex at the origin, axis of symmetry the x axis, and focus at $(a, 0)$.

By a similar derivation (see Problem 51 in the exercises), the standard equation of a parabola with vertex at the origin, axis of symmetry the y axis, and focus at $(0, a)$ is given by equation (5).

$$x^2 = 4ay \tag{5}$$

Looking at Figure 5, note that the parabola opens upward if $a > 0$ and downward if $a < 0$.

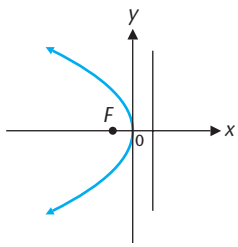
› **Figure 5** Parabola with vertex at the origin and axis of symmetry the y axis.



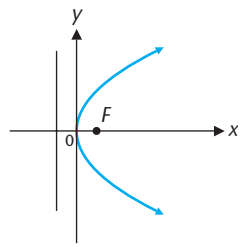
We summarize these results for easy reference in Theorem 1.

› **THEOREM 1 Standard Equations of a Parabola with Vertex at (0, 0)**

1. $y^2 = 4ax$
Vertex: (0, 0)
Focus: $(a, 0)$
Directrix: $x = -a$
Symmetric with respect to the x axis
Axis of symmetry the x axis

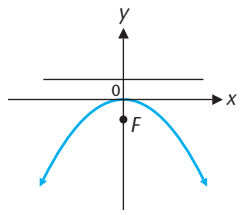


$a < 0$ (opens left)

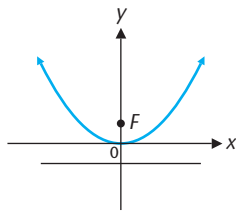


$a > 0$ (opens right)

2. $x^2 = 4ay$
Vertex: (0, 0)
Focus: $(0, a)$
Directrix: $y = -a$
Symmetric with respect to the y axis
Axis of symmetry the y axis



$a < 0$ (opens down)



$a > 0$ (opens up)

EXAMPLE 1 Graphing a Parabola

Locate the focus and directrix and sketch the graph of $y^2 = 16x$.

SOLUTIONS

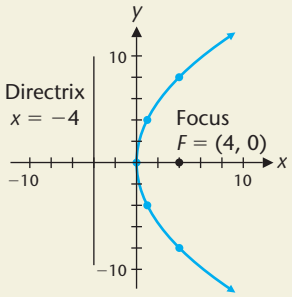
The equation $y^2 = 16x$ has the form $y^2 = 4ax$ with $4a = 16$, so $a = 4$. Therefore, the focus is (4, 0) and the directrix is the line $x = -4$.

Hand-Drawn Solution

To sketch the graph, we choose some values of x that make the right side of the equation a perfect square and solve for y.

x	0	1	4
y	0	± 4	± 8

Note that x must be greater than or equal to 0 for y to be a real number. Then we plot the resulting points. Because $a > 0$, the parabola opens to the right (Fig. 6).



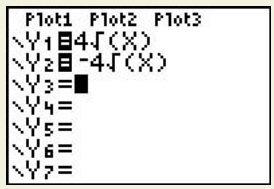
› Figure 6

Graphical Solution

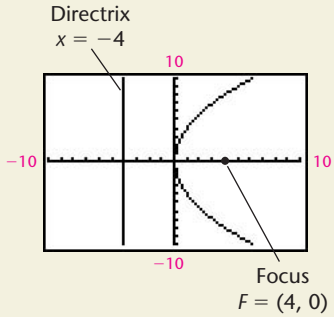
To graph $y^2 = 16x$ on a graphing calculator, we solve this equation for y.

$$y^2 = 16x \qquad \text{Take the square root of both sides.}$$
$$y = \pm 4\sqrt{x}$$

This results in two functions, $y = 4\sqrt{x}$ and $y = -4\sqrt{x}$. Entering these functions in a graphing utility (Fig. 7) and graphing in a standard viewing window produces the graph of the parabola (Fig. 8).



› Figure 7



› Figure 8

MATCHED PROBLEM 1

Graph $y^2 = -8x$, and locate the focus and directrix.

››› CAUTION ›››

A common error in making a quick sketch of $y^2 = 4ax$ or $x^2 = 4ay$ is to sketch the first with the y axis as its axis of symmetry and the second with the x axis as its axis of symmetry. The graph of $y^2 = 4ax$ is symmetric with respect to the x axis, and the graph of $x^2 = 4ay$ is symmetric with respect to the y axis, as a quick symmetry check will reveal.

EXAMPLE 2 Finding the Equation of a Parabola

(A) Find the equation of a parabola having the origin as its vertex, the y axis as its axis of symmetry, and $(-10, -5)$ on its graph.

(B) Find the coordinates of its focus and the equation of its directrix.

SOLUTIONS

(A) Because the axis of symmetry of the parabola is the y axis, the parabola has an equation of the form $x^2 = 4ay$. Because $(-10, -5)$ is on the graph, we have

$$\begin{aligned} x^2 &= 4ay && \text{Substitute } x = -10 \text{ and } y = -5. \\ (-10)^2 &= 4a(-5) && \text{Simplify.} \\ 100 &= -20a && \text{Divide both sides by } -20. \\ a &= -5 \end{aligned}$$

Therefore the equation of the parabola is

$$\begin{aligned} x^2 &= 4(-5)y \\ x^2 &= -20y \end{aligned}$$

(B) Focus: $F = (0, a) = (0, -5)$
Directrix: $y = -a$
$y = 5$

MATCHED PROBLEM 2

(A) Find the equation of a parabola having the origin as its vertex, the x axis as its axis of symmetry, and $(4, -8)$ on its graph.

(B) Find the coordinates of its focus and the equation of its directrix.

Remark
By the graph transformations of Section 1-4, the graph of

$$y = \frac{(x-h)^2}{4a} + k$$

is the same as the graph of

$$y = \frac{x^2}{4a}$$

shifted h units to the right and k units upward. Solving each equation for the square, we see that the graph of $(x-h)^2 = 4a(y-k)$ is the same as the graph of $x^2 = 4ay$ shifted h units to the right and k units upward. So $(x-h)^2 = 4a(y-k)$ is the standard equation for a parabola with vertex (h, k) and axis of symmetry $x = h$. Similarly, $(y-k)^2 = 4a(x-h)$ is the standard equation for a parabola with vertex (h, k) and axis of symmetry $y = k$. In applications of parabolas, we normally choose a coordinate system so that the vertex of the parabola is the origin and the axis of symmetry is one of the coordinate axes. With such a choice, the equation of the parabola will have one of the standard forms of Theorem 1.

Applications

Parabolic forms are frequently encountered in the physical world. Suspension bridges, arch bridges, microphones, symphony shells, satellite antennas, radio and optical telescopes, radar equipment, solar furnaces, and searchlights are only a few of many items that use parabolic forms in their design.

Figure 9(a) illustrates a parabolic reflector used in all reflecting telescopes—from 3- to 6-inch home types to the 200-inch research instrument on Mount Palomar in California. Parallel light rays from distant celestial bodies are reflected to the focus off a parabolic mirror. If the light source is the sun, then the parallel rays are focused at F and we have a solar furnace. Temperatures of over 6,000°C have been achieved by such furnaces. If we locate a light source at F, then the rays in Figure 9(a) reverse, and we have a spotlight or a searchlight. Automobile headlights can use parabolic reflectors with special lenses over the light to diffuse the rays into useful patterns.

Figure 9(b) shows a suspension bridge, such as the Golden Gate Bridge in San Francisco. The suspension cable is a parabola. It is interesting to note that a free-hanging cable, such as a telephone line, does not form a parabola. It forms another curve called a *catenary*.

Parabolic reflector (a) | Suspension bridge (b) | Arch bridge (c)

› **Figure 9** Uses of parabolic forms.

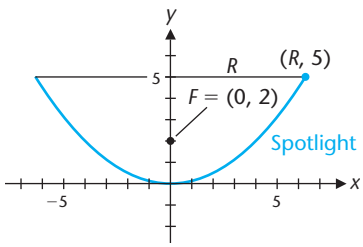
Figure 9(c) shows a concrete arch bridge. If all the loads on the arch are to be compression loads (concrete works very well under compression), then using physics and advanced mathematics, it can be shown that the arch must be parabolic.

EXAMPLE 3 Parabolic Reflector

A *paraboloid* is formed by revolving a parabola about its axis of symmetry. A spotlight in the form of a paraboloid 5 inches deep has its focus 2 inches from the vertex. Find, to one decimal place, the radius R of the opening of the spotlight.

SOLUTION

Step 1. Locate a parabolic cross section containing the axis of symmetry in a rectangular coordinate system, and label all known parts and parts to be found. This is a very important step and can be done in infinitely many ways. We can make things simpler for ourselves by locating the vertex at the origin and choosing a coordinate axis as the axis of symmetry. We choose the y axis as the axis of symmetry of the parabola with the parabola opening upward (Fig. 10).



› **Figure 10**

Step 2. Find the equation of the parabola in the figure. Because the parabola has the y axis as its axis of symmetry and the vertex at the origin, the equation is of the form

$$x^2 = 4ay$$

We are given $F = (0, a) = (0, 2)$; thus, $a = 2$, and the equation of the parabola is

$$x^2 = 8y$$

Step 3. Use the equation found in step 2 to find the radius R of the opening. Because $(R, 5)$ is on the parabola, we have

$$R^2 = 8(5)$$
$$R = \sqrt{40} \approx 6.3 \text{ inches}$$

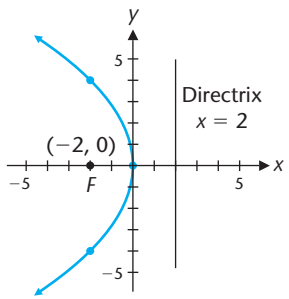
MATCHED PROBLEM 3

Repeat Example 3 with a paraboloid 12 inches deep and a focus 9 inches from the vertex.

ANSWERS TO MATCHED PROBLEMS

1. Focus: $(-2, 0)$
Directrix: $x = 2$

x	0	-2
y	0	± 4



2. (A) $y^2 = 16x$ (B) Focus: $(4, 0)$; Directrix: $x = -4$ **3.** $R = 20.8$ inches

11-1 Exercises

1. Use the geometric objects cone and plane to explain the difference between a circle and an ellipse.

2. Use the geometric objects cone and plane to explain why a parabola is a single curve, while a hyperbola consists of two separate curves.

3. What is a degenerate conic?

4. What is a parabolic mirror?

5. What happens when light rays parallel to the axis of a parabolic mirror hit the mirror?

6. What happens when light rays emitted from the focus of a parabolic mirror hit the mirror?

In Problems 7–18, graph each equation, and locate the focus and directrix.

7. $y^2 = 4x$ **8.** $y^2 = 8x$

9. $x^2 = 8y$ **10.** $x^2 = 4y$

11. $y^2 = -12x$ **12.** $y^2 = -4x$

13. $x^2 = -4y$ **14.** $x^2 = -8y$

15. $y^2 = -20x$ **16.** $x^2 = -24y$

17. $x^2 = 10y$ **18.** $y^2 = 6x$

Find the coordinates to two decimal places of the focus for each parabola in Problems 19–24.

19. $y^2 = 39x$ **20.** $x^2 = 58y$

21. $x^2 = -105y$ **22.** $y^2 = -93x$

23. $y^2 = -77x$ **24.** $x^2 = -205y$

In Problems 25–32, find the equation of a parabola with vertex at the origin, axis of symmetry the x or y axis, and

25. Directrix $y = -3$ **26.** Directrix $y = 4$

27. Focus $(0, -7)$ **28.** Focus $(0, 5)$

29. Directrix $x = 6$ **30.** Directrix $x = -9$

31. Focus $(2, 0)$ **32.** Focus $(-4, 0)$

In Problems 33–38, find the equation of the parabola having its vertex at the origin, its axis of symmetry as indicated, and passing through the indicated point.

33. y axis; $(4, 2)$ **34.** x axis; $(4, 8)$

35. x axis; $(-3, 6)$ **36.** y axis; $(-5, 10)$

37. y axis; $(-6, -9)$ **38.** x axis; $(-6, -12)$

In Problems 39–42, find the first-quadrant points of intersection for each pair of parabolas to three decimal places.

39. $x^2 = 4y$, $y^2 = 4x$ **40.** $y^2 = 3x$, $x^2 = 3y$

41. $y^2 = 6x$, $x^2 = 5y$ **42.** $x^2 = 7y$, $y^2 = 2x$

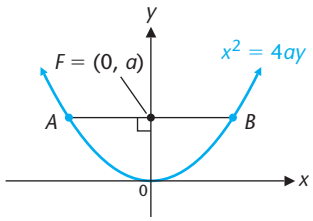
43. Consider the parabola with equation $x^2 = 4ay$.

(A) How many lines through $(0, 0)$ intersect the parabola in exactly one point? Find their equations.

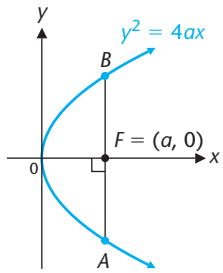
(B) Find the coordinates of all points of intersection of the parabola with the line through $(0, 0)$ having slope $m \neq 0$.

44. Find the coordinates of all points of intersection of the parabola with equation $x^2 = 4ay$ and the parabola with equation $y^2 = 4bx$.

45. The line segment AB through the focus in the figure is called a **focal chord** of the parabola. Find the coordinates of A and B.



46. The line segment AB through the focus in the figure is called a **focal chord** of the parabola. Find the coordinates of A and B.



In Problems 47–50, use the definition of a parabola and the distance formula to find the equation of a parabola with

47. Directrix $y = -4$ and focus $(2, 2)$

48. Directrix $y = 2$ and focus $(-3, 6)$

49. Directrix $x = 2$ and focus $(6, -4)$

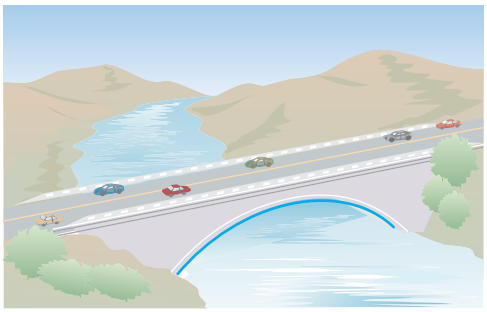
50. Directrix $x = -3$ and focus $(1, 4)$

51. Use the definition of a parabola and the distance formula to derive the equation of a parabola with focus $F = (0, a)$ and directrix $y = -a$ for $a \neq 0$.

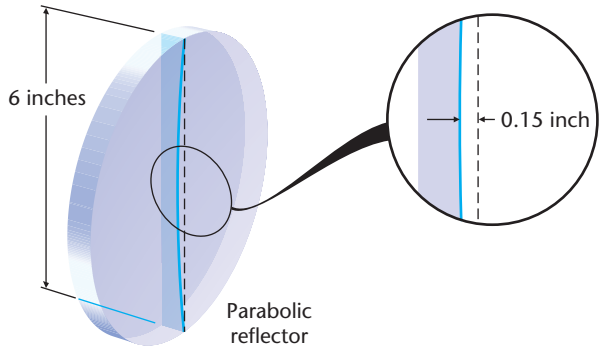
52. Let F be a fixed point and let L be a fixed line in the plane that contains F. Describe the set of all points in the plane that are equidistant from F and L.

APPLICATIONS

53. ENGINEERING The parabolic arch in the concrete bridge in the figure must have a clearance of 50 feet above the water and span a distance of 200 feet. Find the equation of the parabola after inserting a coordinate system with the origin at the vertex of the parabola and the vertical y axis (pointing upward) along the axis of symmetry of the parabola.

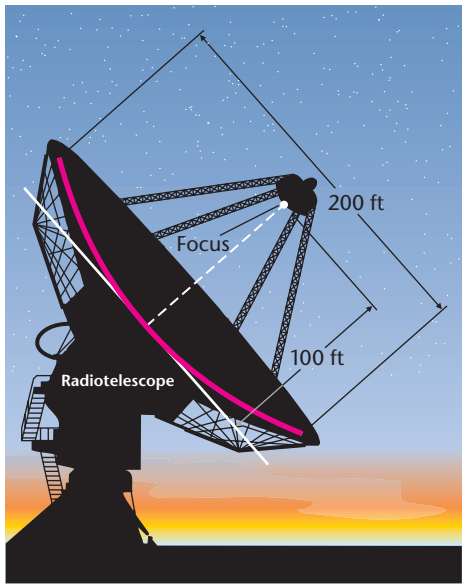


54. ASTRONOMY The cross section of a parabolic reflector with 6-inch diameter is ground so that its vertex is 0.15 inch below the rim (see the figure).



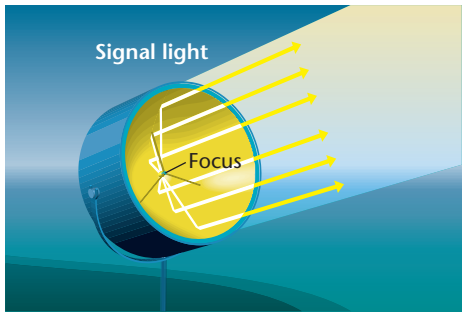
(A) Find the equation of the parabola after inserting an xy coordinate system with the vertex at the origin and the y axis (pointing upward) the axis of symmetry of the parabola.
(B) How far is the focus from the vertex?

55. SPACE SCIENCE A designer of a 200-foot-diameter parabolic electromagnetic antenna for tracking space probes wants to place the focus 100 feet above the vertex (see the figure).



(A) Find the equation of the parabola using the axis of symmetry of the parabola as the y axis (up positive) and vertex at the origin.
(B) Determine the depth of the parabolic reflector.

56. SIGNAL LIGHT A signal light on a ship is a spotlight with parallel reflected light rays (see the figure). Suppose the parabolic reflector is 12 inches in diameter and the light source is located at the focus, which is 1.5 inches from the vertex.



(A) Find the equation of the parabola using the axis of symmetry of the parabola as the x axis (right positive) and vertex at the origin.
(B) Determine the depth of the parabolic reflector.

11-2 Ellipse

- Defining an Ellipse
- Drawing an Ellipse
- Standard Equations of Ellipses and Their Graphs
- Applications

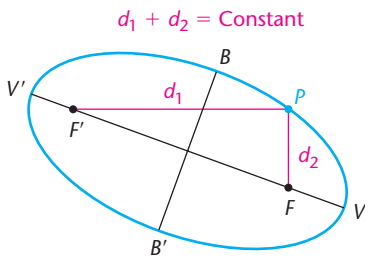
We start our discussion of the ellipse with a coordinate-free definition. Using this definition, we show how an ellipse can be drawn and we derive standard equations for ellipses specially located in a rectangular coordinate system.

Defining an Ellipse

The following is a coordinate-free definition of an ellipse:

DEFINITION 1 Ellipse

An **ellipse** is the set of all points P in a plane such that the sum of the distances from P to two distinct fixed points in the plane is constant (the constant is required to be greater than the distance between the two fixed points). Each of the fixed points, F' and F, is called a **focus,** and together they are called **foci.** Referring to the figure, the line segment $V'V$ through the foci is the **major axis.** The perpendicular bisector $B'B$ of the major axis is the **minor axis.** Each end of the major axis, V' and V, is called a **vertex.** The midpoint of the line segment $F'F$ is called the **center** of the ellipse.

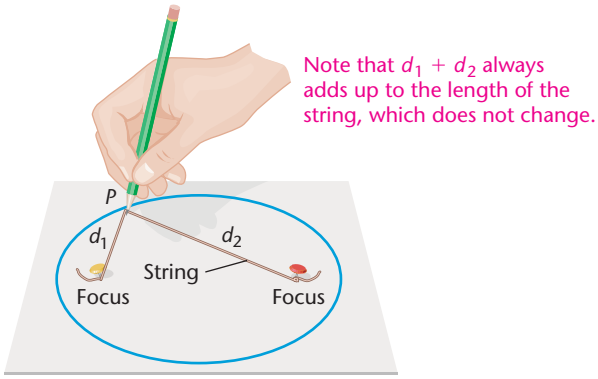


Drawing an Ellipse

An ellipse is easy to draw. All you need is a piece of string, two thumbtacks, and a pencil or pen (Fig. 1). Place the two thumbtacks in a piece of cardboard. These form the foci of the ellipse. Take a piece of string longer than the distance between the two thumbtacks—this represents the constant in the definition—and tie each end to a thumbtack.

Finally, catch the tip of a pencil under the string and move it while keeping the string taut. The resulting figure is by definition an ellipse. Ellipses of different shapes result, depending on the placement of thumbtacks and the length of the string joining them.

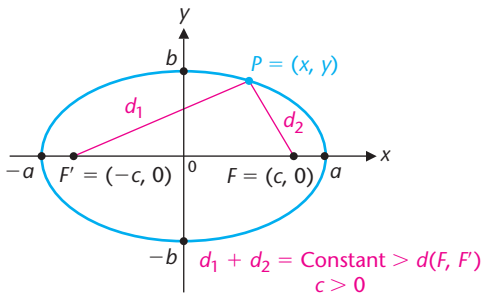
› **Figure 1** Drawing an ellipse.



› Standard Equations of Ellipses and Their Graphs

Using the definition of an ellipse and the distance-between-two-points formula, we can derive standard equations for an ellipse located in a rectangular coordinate system. We start by placing an ellipse in the coordinate system with the foci on the x axis at $F' = (-c, 0)$ and $F = (c, 0)$ with $c > 0$ (Fig. 2). By Definition 1 the constant sum $d_1 + d_2$ is required to be greater than $2c$ (the distance between F and F'). Therefore, the ellipse intersects the x axis at points $V' = (-a, 0)$ and $V = (a, 0)$ with $a > c > 0$, and it intersects the y axis at points $B' = (-b, 0)$ and $B = (b, 0)$ with $b > 0$.

› **Figure 2** Ellipse with foci on x axis.



Study Figure 2: Note first that if $P = (a, 0)$, then $d_1 + d_2 = 2a$. (Why?) Therefore, the constant sum $d_1 + d_2$ is equal to the distance between the vertices. Second, if $P = (0, b)$, then $d_1 = d_2 = a$ and $a^2 = b^2 + c^2$ by the Pythagorean theorem; in particular, $a > b$.

Referring again to Figure 2, the point $P = (x, y)$ is on the ellipse if and only if

$$d_1 + d_2 = 2a$$

Using the distance formula for d_1 and d_2, eliminating radicals, and simplifying (see Problem 41 in the exercises), we obtain the equation of the ellipse pictured in Figure 2:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

By similar reasoning (see Problem 42 in Exercises 11-2) we obtain the equation of an ellipse centered at the origin with foci on the y axis. Both cases are summarized in Theorem 1.

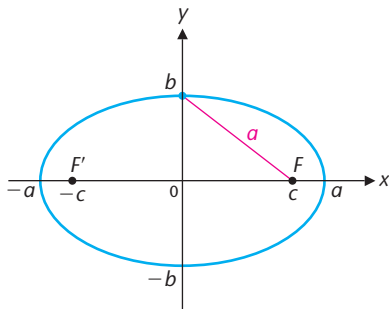
› THEOREM 1 Standard Equations of an Ellipse with Center at (0, 0)

1. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \qquad a > b > 0$

x intercepts: $\pm a$ (vertices)
y intercepts: $\pm b$
Foci: $F' = (-c, 0), F = (c, 0)$

$$c^2 = a^2 - b^2$$

Major axis length $= 2a$
Minor axis length $= 2b$

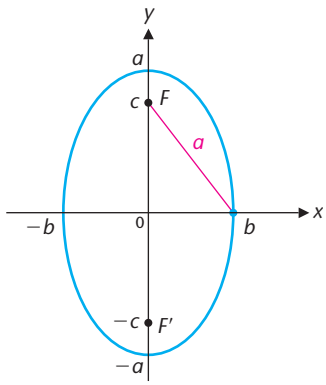


2. $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1 \qquad a > b > 0$

x intercepts: $\pm b$
y intercepts: $\pm a$ (vertices)
Foci: $F' = (0, -c), F = (0, c)$

$$c^2 = a^2 - b^2$$

Major axis length $= 2a$
Minor axis length $= 2b$



[*Note:* Both graphs are symmetric with respect to the x axis, y axis, and origin. Also, the major axis is always longer than the minor axis.]

››› EXPLORE-DISCUSS 1

The line through a focus F of an ellipse that is perpendicular to the major axis intersects the ellipse in two points G and H. For each of the two standard equations of an ellipse with center (0, 0), find an expression in terms of a and b for the distance from G to H.

EXAMPLE 1 Graphing an Ellipse

Find the coordinates of the foci, find the lengths of the major and minor axes, and graph the following equation:

$$9x^2 + 16y^2 = 144$$

SOLUTIONS

First, write the equation in standard form by dividing both sides by 144 and determine a and b:

$$9x^2 + 16y^2 = 144 \qquad \text{Divide both sides by 144.}$$

$$\frac{9x^2}{144} + \frac{16y^2}{144} = \frac{144}{144}* \qquad \text{Simplify.}$$

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

$$a = 4 \quad \text{and} \quad b = 3$$

x intercepts: ± 4 Major axis length: $2(4) = 8$

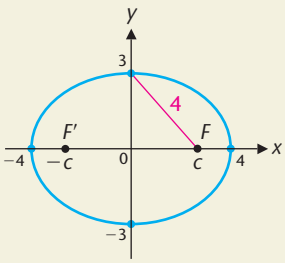
y intercepts: ± 3 Minor axis length: $2(3) = 6$

$$\text{Foci: } c^2 = a^2 - b^2 \qquad \text{Substitute } a = 4 \text{ and } b = 3.$$
$$= 16 - 9$$
$$= 7$$
$$c = \sqrt{7} \qquad c \text{ must be positive.}$$

Thus, the foci are $F' = (-\sqrt{7}, 0)$ and $F = (\sqrt{7}, 0)$.

Hand-Drawn Solution

Plot the foci and intercepts and sketch the ellipse (Fig. 3)



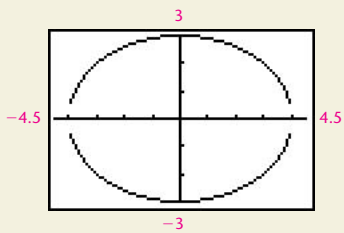
› Figure 3

Graphical Solution

Solve the original equation for y:

$$9x^2 + 16y^2 = 144 \qquad \text{Subtract } 9x^2 \text{ from both sides.}$$
$$y^2 = (144 - 9x^2)/16 \qquad \text{Then divide both sides by 16.}$$
$$y = \pm\sqrt{(144 - 9x^2)/16} \qquad \text{Take the square root of both sides.}$$

This produces the two functions whose graphs are shown in Figure 4. Notice that we used a squared viewing window to avoid distorting the shape of the ellipse. Also note the gaps in the graph at ± 4. This is due to the relatively low resolution of a graphing utility screen.



› Figure 4

*The dashed boxes "think boxes" are used to enclose steps that may be performed mentally.

MATCHED PROBLEM 1

Find the coordinates of the foci, find the lengths of the major and minor axes, and graph the following equation:

$$x^2 + 4y^2 = 4$$

EXAMPLE 2 Graphing an Ellipse

Find the coordinates of the foci, find the lengths of the major and minor axes, and graph the following equation:

$$2x^2 + y^2 = 10$$

SOLUTION

First, write the equation is standard form by dividing both sides by 10 and determine a and b:

$$2x^2 + y^2 = 10 \qquad \text{Divide both sides by 10.}$$

$$\frac{2x^2}{10} + \frac{y^2}{10} = \frac{10}{10} \qquad \text{Simplify.}$$

$$\frac{x^2}{5} + \frac{y^2}{10} = 1$$

$$a = \sqrt{10} \quad \text{and} \quad b = \sqrt{5}$$

y intercepts: $\pm\sqrt{10} \approx \pm 3.16$ Major axis length: $2\sqrt{10} \approx 6.32$

x intercepts: $\pm\sqrt{5} \approx \pm 2.24$ Minor axis length: $2\sqrt{5} \approx 4.47$

$$\text{Foci: } c^2 = a^2 - b^2 \qquad \text{Substitute } a = \sqrt{10},\ b = \sqrt{5}.$$

$$= 10 - 5$$

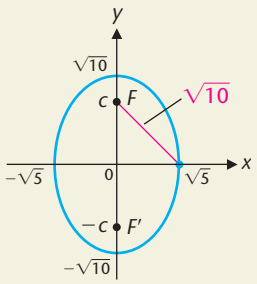
$$= 5$$

$$c = \sqrt{5} \qquad c \text{ must be positive.}$$

Thus, the foci are $F' = (0, -\sqrt{5})$ and $F = (0, \sqrt{5})$.

Hand-Drawn Solution

Plot the foci and intercepts and sketch the ellipse (Fig. 5).



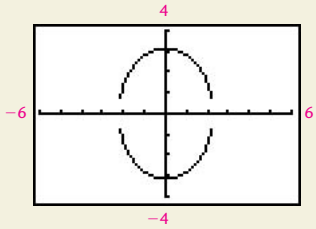
› Figure 5

Graphical Solution

Solve for y:

$$\begin{aligned} 2x^2 + y^2 &= 10 && \text{Subtract } 2x^2 \text{ from both sides.} \\ y^2 &= 10 - 2x^2 && \text{Take the square root of both sides.} \\ y &= \pm\sqrt{10 - 2x^2} \end{aligned}$$

Graph $y_1 = \sqrt{10 - 2x^2}$ and $y_2 = -\sqrt{10 - 2x^2}$ in a squared viewing window (Fig. 6).



› Figure 6

MATCHED PROBLEM 2

Find the coordinates of the foci, find the lengths of the major and minor axes, and graph the following equation:

$$3x^2 + y^2 = 18$$

EXAMPLE 3 Finding the Equation of an Ellipse

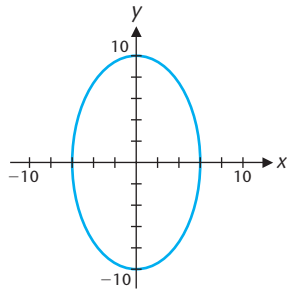
Find an equation of an ellipse in the form

$$\frac{x^2}{M} + \frac{y^2}{N} = 1 \qquad M, N > 0$$

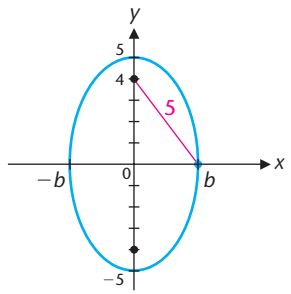
if the center is at the origin, the major axis is along the y axis, and

(A) Length of major axis $= 20$
Length of minor axis $= 12$

(B) Length of major axis $= 10$
Distance of foci from center $= 4$



› Figure 7



› Figure 8

Remark
Using graph transformations from Section 1-4, the graphs of

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \qquad a>b>0 \tag{1}$$

and

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1 \qquad a>b>0 \tag{2}$$

are the same as the graphs of Theorem 1 shifted h units to the right and k units upward. Equations (1) and (2) are therefore the standard forms for equations of ellipses with centers (h, k) and major axes parallel to the x axis or y axis, respectively. In applications of ellipses we normally choose a coordinate system so that the center of the ellipse is the origin and the major axis lies on one of the coordinate axes. With such a choice, the equation of the ellipse will have one of the standard forms of Theorem 1.

SOLUTIONS

(A) Compute x and y intercepts and make a rough sketch of the ellipse, as shown in Figure 7.

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

$$a=\frac{20}{2}=10 \qquad b=\frac{12}{2}=6$$

$$\frac{x^2}{36}+\frac{y^2}{100}=1$$

(B) Make a rough sketch of the ellipse, as shown in Figure 8; locate the foci and y intercepts, then determine the x intercepts using the fact that $a^2=b^2+c^2$:

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

$$a=\frac{10}{2}=5 \qquad b^2=5^2-4^2=25-16=9$$

$$b=3$$

$$\frac{x^2}{9}+\frac{y^2}{25}=1$$

MATCHED PROBLEM 3

Find an equation of an ellipse in the form

$$\frac{x^2}{M}+\frac{y^2}{N}=1 \qquad M, N>0$$

if the center is at the origin, the major axis is along the x axis, and

(A) Length of major axis $=50$
Length of minor axis $=30$

(B) Length of minor axis $=16$
Distance of foci from center $=6$

››› EXPLORE-DISCUSS 2

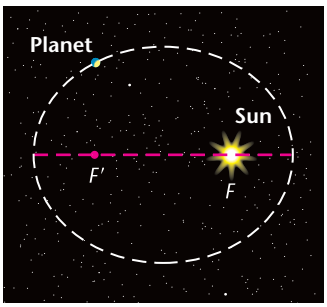
(A) Is a circle a special case of an ellipse? Before you answer, review the coordinate-free definition of an ellipse in this section and the coordinate-free definition of a circle in Appendix A, Section A-3.

(B) Why did we require $a>b$ in Theorem 1?

(C) State a theorem similar to Theorem 1 for circles.

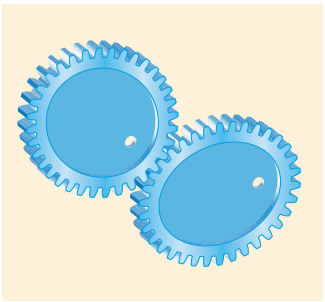
› Applications

Elliptical forms have many application: orbits of satellites, planets, and comets; shapes of galaxies; gears and cams; some airplane wings, boat keels, and rudders; tabletops; public fountains; and domes in buildings are a few examples (Fig. 9).



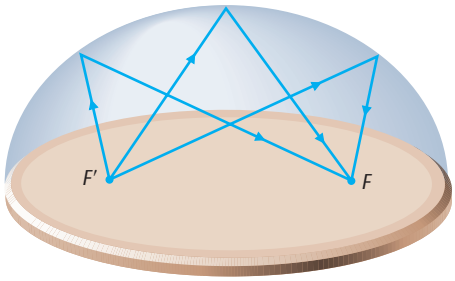
Planetary motion

(a)



Elliptical gears

(b)



Elliptical dome

(c)

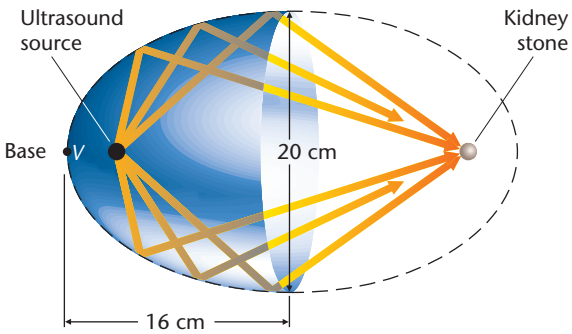
› **Figure 9** Uses of elliptical forms.

Johannes Kepler (1571–1630), a German astronomer, discovered that planets move in elliptical orbits, with the sun at a focus, and not in circular orbits as had been thought before [Fig. 9(a)]. Figure 9(b) shows a pair of elliptical gears with pivot points at foci. Such gears transfer constant rotational speed to variable rotational speed, and vice versa. Figure 9(c) shows an elliptical dome. An interesting property of such a dome is that a sound or light source at one focus will reflect off the dome and pass through the other focus. One of the chambers in the Capitol Building in Washington, D.C., has such a dome, and is referred to as a whispering room because a whispered sound at one focus can be easily heard at the other focus.

A fairly recent application in medicine is the use of elliptical reflectors and ultrasound to break up kidney stones. A device called a lithotripter is used to generate intense sound waves that break up the stone from outside the body, thus avoiding surgery. To be certain that the waves do not damage other parts of the body, the reflecting property of the ellipse is used to design and correctly position the lithotripter.

EXAMPLE 4 **Medicinal Lithotripsy**

A lithotripter is formed by rotating the portion of an ellipse below the minor axis around the major axis (Fig. 10). The lithotripter is 20 centimeters wide and 16 centimeters deep. If the ultrasound source is positioned at one focus of the ellipse and the kidney stone at the other, then all the sound waves will pass through the kidney stone. How far from the kidney stone should the point V on the base of the lithotripter be positioned to focus the sound waves on the kidney stone? Round the answer to one decimal place.



› **Figure 10** Lithotripter.

SOLUTION

From Figure 10 we see that $a = 16$ and $b = 10$ for the ellipse used to form the lithotripter. Thus, the distance c from the center to either the kidney stone or the ultrasound source is given by

$$c = \sqrt{a^2 - b^2} = \sqrt{16^2 - 10^2} = \sqrt{156} \approx 12.5$$

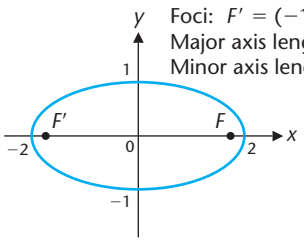
and the distance from the base of the lithotripter to the kidney stone is $16 + 12.5 = 28.5$ centimeters.

MATCHED PROBLEM 4

Because lithotripsy is an external procedure, the lithotripter described in Example 4 can be used only on stones within 12.5 centimeters of the surface of the body. Suppose a kidney stone is located 14 centimeters from the surface. If the diameter is kept fixed at 20 centimeters, how deep must a lithotripter be to focus on this kidney stone? Round answer to one decimal place.

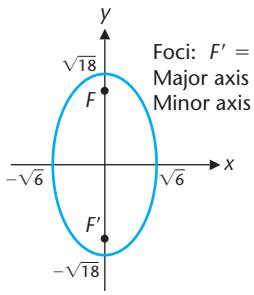
ANSWERS TO MATCHED PROBLEMS

1.



Foci: $F' = (-\sqrt{3}, 0)$, $F = (\sqrt{3}, 0)$
Major axis length = 4
Minor axis length = 2

2.



Foci: $F' = (0, -\sqrt{12})$, $F = (0, \sqrt{12})$
Major axis length $= 2\sqrt{18} \approx 8.49$
Minor axis length $= 2\sqrt{6} \approx 4.90$

3. (A) $\frac{x^2}{625} + \frac{y^2}{225} = 1$ (B) $\frac{x^2}{100} + \frac{y^2}{64} = 1$ **4.** 17.2 centimeters

11-2 Exercises

1. Does the graph of an ellipse pass the vertical line test (Section 1-2)? Explain.

2. Why are two equations required to graph an ellipse on a graphing calculator?

3. Does the graph of either equation in Problem 2 pass the horizontal line test (Section 1-6)? Explain.

4. Given the x and y intercepts of an ellipse centered at the origin, describe a procedure for sketching the graph of the ellipse.

5. Repeat Problem 4 for a circle.

6. Some say that the distinction between an ellipse and a circle is a distinction without a difference. Do you agree or disagree? Why?

In Problems 7–12, sketch a graph of each equation, find the coordinates of the foci, and find the lengths of the major and minor axes.

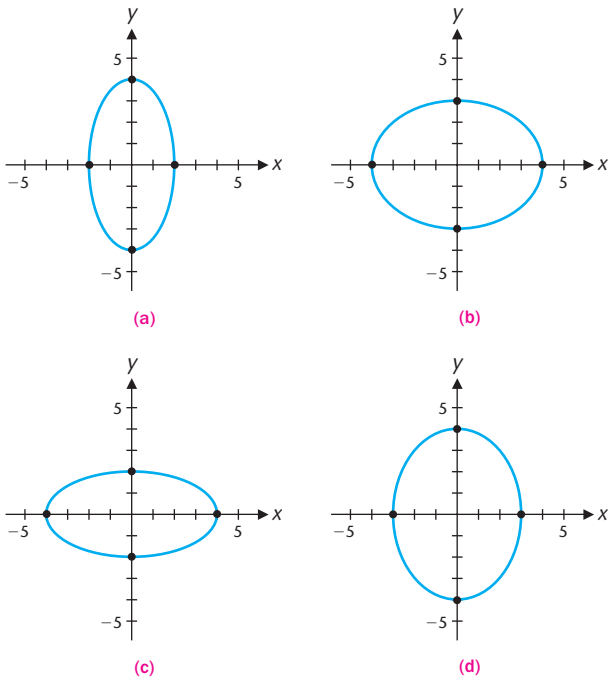
7. $\frac{x^2}{25} + \frac{y^2}{4} = 1$ **8.** $\frac{x^2}{9} + \frac{y^2}{4} = 1$ **9.** $\frac{x^2}{4} + \frac{y^2}{25} = 1$

10. $\frac{x^2}{4} + \frac{y^2}{9} = 1$ **11.** $x^2 + 9y^2 = 9$ **12.** $4x^2 + y^2 = 4$

In Problems 13–16, match each equation with one of graphs (a)–(d).

13. $9x^2 + 16y^2 = 144$ **14.** $16x^2 + 9y^2 = 144$

15. $4x^2 + y^2 = 16$ **16.** $x^2 + 4y^2 = 16$



In Problems 17–22, sketch a graph of each equation, find the coordinates of the foci, and find the lengths of the major and minor axes.

17. $25x^2 + 9y^2 = 225$ **18.** $16x^2 + 25y^2 = 400$

19. $2x^2 + y^2 = 12$ **20.** $4x^2 + 3y^2 = 24$

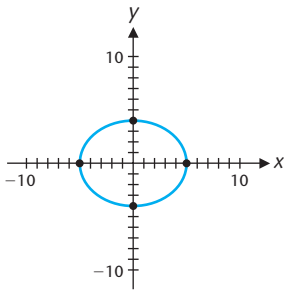
21. $4x^2 + 7y^2 = 28$ **22.** $3x^2 + 2y^2 = 24$

In Problems 23–34, find an equation of an ellipse in the form

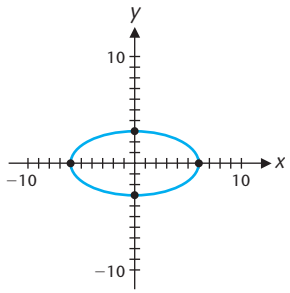
$$\frac{x^2}{M} + \frac{y^2}{N} = 1 \qquad M, N > 0$$

if the center is at the origin, and

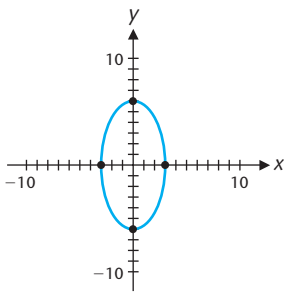
23. The graph is



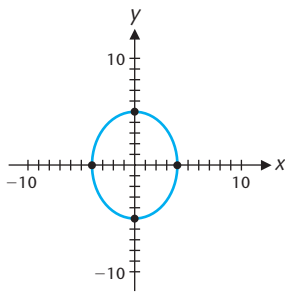
24. The graph is



25. The graph is



26. The graph is



27. Major axis on x axis
Major axis length $= 10$
Minor axis length $= 6$

28. Major axis on x axis
Major axis length $= 14$
Minor axis length $= 10$

29. Major axis on y axis
Major axis length $= 22$
Minor axis length $= 16$

30. Major axis on y axis
Major axis length $= 24$
Minor axis length $= 18$

31. Major axis on x axis
Major axis length $= 16$
Distance of foci from center $= 6$

32. Major axis on y axis
Major axis length $= 24$
Distance of foci from center $= 10$

33. Major axis on y axis
Minor axis length $= 20$
Distance of foci from center $= \sqrt{70}$

34. Major axis on x axis
Minor axis length $= 14$
Distance of foci from center $= \sqrt{200}$

35. Explain why an equation whose graph is an ellipse does not define a function.

36. Consider all ellipses having $(0, \pm 1)$ as the ends of the minor axis. Describe the connection between the elongation of the ellipse and the distance from a focus to the origin.

37. Find an equation of the set of points in a plane, each of whose distance from $(2, 0)$ is one-half its distance from the line $x = 8$. Identify the geometric figure.

38. Find an equation of the set of points in a plane, each of whose distance from $(0, 9)$ is three-fourths its distance from the line $y = 16$. Identify the geometric figure.

39. Let F and F' be two points in the plane and let c denote the constant $d(F, F')$. Describe the set of all points P in the plane such that the sum of the distances from P to F and F' is equal to the constant c.

40. Let F and F' be two points in the plane and let c be a constant such that $0 < c < d(F, F')$. Describe the set of all points P in the plane such that the sum of the distances from P to F and F' is equal to the constant c.

41. Study the following derivation of the standard equation of an ellipse with foci $(\pm c, 0)$, x intercepts $(\pm a, 0)$, and y intercepts $(0, \pm b)$. Explain why each equation follows from the equation that precedes it. [*Hint:* Recall from Figure 2 that $a^2 = b^2 + c^2$.]

$$d_1 + d_2 = 2a$$

$$\sqrt{(x + c)^2 + y^2} = 2a - \sqrt{(x - c)^2 + y^2}$$

$$(x + c)^2 + y^2 = 4a^2 - 4a\sqrt{(x - c)^2 + y^2} + (x - c)^2 + y^2$$

$$\sqrt{(x - c)^2 + y^2} = a - \frac{cx}{a}$$

$$(x - c)^2 + y^2 = a^2 - 2cx + \frac{c^2x^2}{a^2}$$

$$\left(1 - \frac{c^2}{a^2}\right)x^2 + y^2 = a^2 - c^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

42. Study the following derivation of the standard equation of an ellipse with foci $(0, \pm c)$, y intercepts $(0, \pm a)$, and x intercepts $(\pm b, 0)$. Explain why each equation follows from the equation that precedes it. [*Hint:* Recall from Figure 2 that $a^2 = b^2 + c^2$.]

$$d_1 + d_2 = 2a$$

$$\sqrt{x^2 + (y + c)^2} = 2a - \sqrt{x^2 + (y - c)^2}$$

$$x^2 + (y + c)^2 = 4a^2 - 4a\sqrt{x^2 + (y - c)^2} + x^2 + (y - c)^2$$

$$\sqrt{x^2 + (y - c)^2} = a - \frac{cy}{a}$$

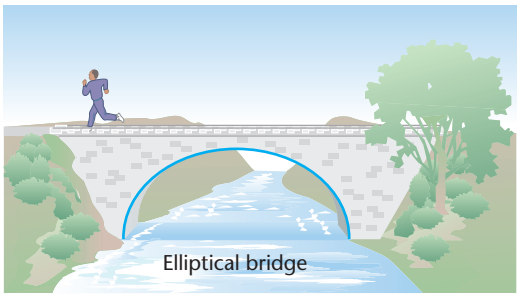
$$x^2 + (y - c)^2 = a^2 - 2cy + \frac{c^2y^2}{a^2}$$

$$x^2 + \left(1 - \frac{c^2}{a^2}\right)y^2 = a^2 - c^2$$

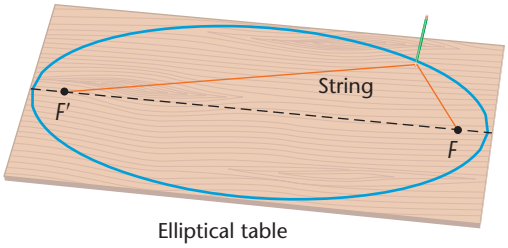
$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

APPLICATIONS

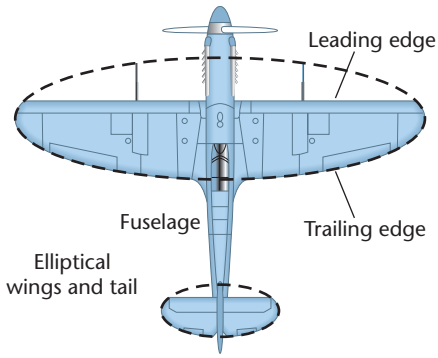
43. ENGINEERING The semielliptical arch in the concrete bridge in the figure must have a clearance of 12 feet above the water and span a distance of 40 feet. Find the equation of the ellipse after inserting a coordinate system with the center of the ellipse at the origin and the major axis on the x axis. The y axis points up, and the x axis points to the right. How much clearance above the water is there 5 feet from the bank?



44. DESIGN A 4×8 foot elliptical tabletop is to be cut out of a 4×8 foot rectangular sheet of teak plywood (see the figure). To draw the ellipse on the plywood, how far should the foci be located from each edge and how long a piece of string must be fastened to each focus to produce the ellipse (see Fig. 1 in the text)? Compute the answer to two decimal places.



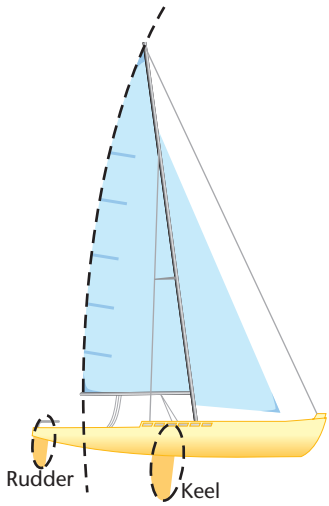
45. AERONAUTICAL ENGINEERING Of all possible wing shapes, it has been determined that the one with the least drag along the trailing edge is an ellipse. The leading edge may be a straight line, as shown in the figure. One of the most famous planes with this design was the World War II British Spitfire. The plane in the figure has a wingspan of 48.0 feet.



(A) If the straight-line leading edge is parallel to the major axis of the ellipse and is 1.14 feet in front of it, and if the leading edge is 46.0 feet long (including the width of the fuselage), find the equation of the ellipse. Let the x axis lie along the major axis (positive right), and let the y axis lie along the minor axis (positive forward).
(B) How wide is the wing in the center of the fuselage (assuming the wing passes through the fuselage)?

Compute quantities to three significant digits.

46. NAVAL ARCHITECTURE Currently, many high-performance racing sailboats use elliptical keels, rudders, and main sails for the reasons stated in Problem 45—less drag along the trailing edge. In the accompanying figure, the ellipse containing the keel has a 12.0-foot major axis. The straight-line leading edge is parallel to the major axis of the ellipse and 1.00 foot in front of it. The chord is 1.00 foot shorter than the major axis.



(A) Find the equation of the ellipse. Let the y axis lie along the minor axis of the ellipse, and let the x axis lie along the major axis, both with positive direction upward.
(B) What is the width of the keel, measured perpendicular to the major axis, 1 foot up the major axis from the bottom end of the keel?

Compute quantities to three significant digits.

11-3 Hyperbola

- Defining a Hyperbola
- Drawing a Hyperbola
- Standard Equations of Hyperbolas and Their Graphs
- Applications

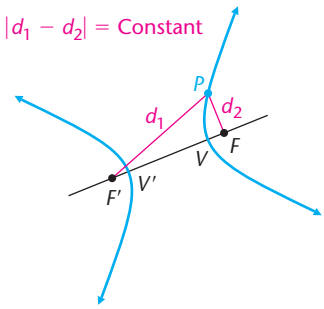
As before, we start with a coordinate-free definition of a hyperbola. Using this definition, we show how a hyperbola can be drawn and we derive standard equations for hyperbolas specially located in a rectangular coordinate system.

Defining a Hyperbola

The following is a coordinate-free definition of a hyperbola:

DEFINITION 1 Hyperbola

A **hyperbola** is the set of all points P in a plane such that the absolute value of the difference of the distances of P to two distinct fixed points in the plane is a positive constant (the constant is required to be less than the distance between the two fixed points). Each of the fixed points, F' and F, is called a **focus.** The intersection points V' and V of the line through the foci and the two branches of the hyperbola are called **vertices,** and each is called a **vertex.** The line segment $V'V$ is called the **transverse axis.** The midpoint of the transverse axis is the **center** of the hyperbola.

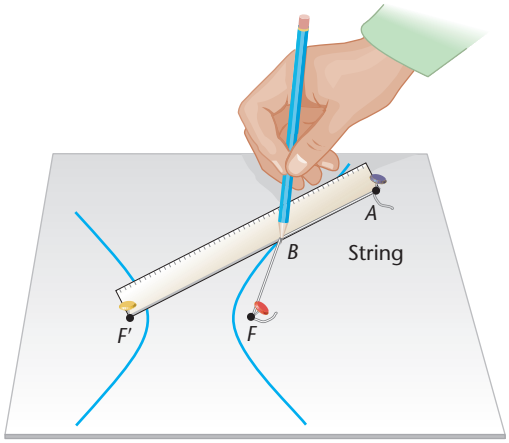


Drawing a Hyperbola

Thumbtacks, a straightedge, string, and a pencil are all that are needed to draw a hyperbola (Fig. 1). Place two thumbtacks in a piece of cardboard—these form the foci of the hyperbola. Rest one corner of the straightedge at the focus F' so that it is free to rotate about this point. Cut a piece of string shorter than the length of the straightedge, and fasten one end to the straightedge corner A and the other end to the thumbtack at F. Now push the string with a pencil up against the straightedge at B. Keeping the string taut, rotate the straightedge about F', keeping the corner at F'. The resulting curve will be part of a hyperbola. Other parts of the hyperbola can be drawn by changing the

position of the straightedge and string. To see that the resulting curve meets the conditions of the definition, note that the difference of the distances BF' and BF is

$$\begin{aligned} BF' - BF &= BF' + BA - BF - BA \\ &= AF' - (BF + BA) \\ &= \begin{pmatrix}\text{Straightedge}\\ \text{length}\end{pmatrix} - \begin{pmatrix}\text{String}\\ \text{length}\end{pmatrix} \\ &= \text{Constant} \end{aligned}$$

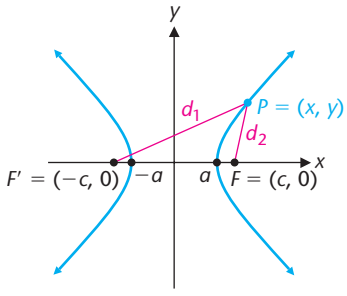


› **Figure 1** Drawing a hyperbola.

› Standard Equations of Hyperbolas and Their Graphs

Using the definition of a hyperbola and the distance-between-two-points formula, we can derive standard equations for a hyperbola located in a rectangular coordinate system. We start by placing a hyperbola in the coordinate system with the foci on the x axis at $F' = (-c, 0)$ and $F = (c, 0)$ with $c > 0$ (Fig. 2). By Definition 1, the constant difference $|d_1 - d_2|$ is required to be less than $2c$ (the distance between F and F'). Therefore, the hyperbola intersects the x axis at points $V' = (-a, 0)$ and $V = (a, 0)$ with $c > a > 0$. The hyperbola does not intersect the y axis, because the constant difference $|d_1 - d_2|$ is required to be positive by Definition 1.

› **Figure 2** Hyperbola with foci on the x axis.



$c > 0$

$|d_1 - d_2|$ = Positive constant $< d(F, F')$

Study Figure 2: Note that if $P = (a, 0)$, then $|d_1 - d_2| = 2a$. (Why?) Therefore the constant $|d_1 - d_2|$ is equal to the distance between the vertices. It is convenient to let $b = \sqrt{c^2 - a^2}$, so that $c^2 = a^2 + b^2$. (Unlike the situation for ellipses, b may be greater than or equal to a.)

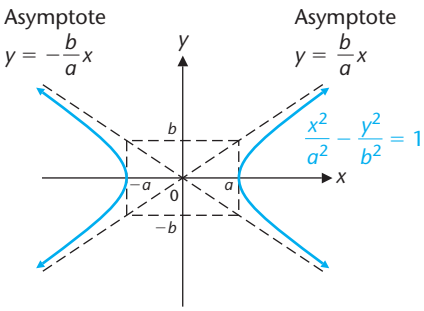
Referring again to Figure 2, the point $P = (x, y)$ is on the hyperbola if and only if

$$|d_1 - d_2| = 2a$$

Using the distance formula for d_1 and d_2, eliminating radicals, and simplifying (see Problem 53 in the exercises), we obtain the equation of the hyperbola pictured in Figure 2:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Although the hyperbola does not intersect the y axis, the points $(0, b)$ and $(0, -b)$ are significant; the line segment joining them is called the **conjugate axis** of the hyperbola. Note that the conjugate axis is perpendicular to the transverse axis, that is, the line segment joining the vertices $(a, 0)$ and $(-a, 0)$. The rectangle with corners (a, b), $(a, -b)$, $(-a, -b)$, and $(-a, b)$ is called the **asymptote rectangle** because its extended diagonals are asymptotes for the hyperbola (Fig. 3). In other words, the hyperbola approaches the lines $y = \pm\frac{b}{a}x$ as $|x|$ becomes larger (see Problems 49 and 50 in the exercises). As a result, it is helpful to include the asymptote rectangle and its extended diagonals when sketching the graph of a hyperbola.



› **Figure 3** Asymptotes.

Note that the four corners of the asymptote rectangle (Fig. 3) are equidistant from the origin, at distance $\sqrt{a^2 + b^2} = c$. Therefore,

A circle, with center at the origin, that passes through all four corners of the asymptote rectangle of a hyperbola also passes through its foci.

By similar reasoning (see Problem 54 in the exercises) we obtain the equation of a hyperbola centered at the origin with foci on the y axis. Both cases are summarized in Theorem 1.

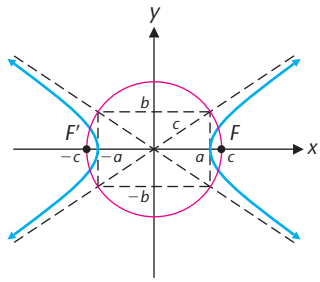
› THEOREM 1 Standard Equations of a Hyperbola with Center at (0, 0)

1. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

x intercepts: $\pm a$ (vertices)
y intercepts: none
Foci: $F' = (-c, 0)$, $F = (c, 0)$

$$c^2 = a^2 + b^2$$

Transverse axis length $= 2a$
Conjugate axis length $= 2b$
Asymptotes: $y = \pm\dfrac{b}{a}x$

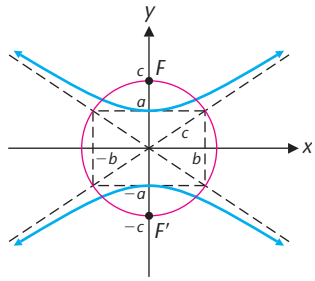


2. $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$

x intercepts: none
y intercepts: $\pm a$ (vertices)
Foci: $F' = (0, -c)$, $F = (0, c)$

$$c^2 = a^2 + b^2$$

Transverse axis length $= 2a$
Conjugate axis length $= 2b$
Asymptotes: $y = \pm\dfrac{a}{b}x$



[*Note:* Both graphs are symmetric with respect to the x axis, y axis, and origin.]

››› EXPLORE-DISCUSS 1

The line through a focus F of a hyperbola that is perpendicular to the transverse axis intersects the hyperbola in two points G and H. For each of the two standard equations of a hyperbola with center (0, 0), find an expression in terms of a and b for the distance from G to H.

EXAMPLE 1 Graphing Hyperbolas

Find the coordinates of the foci, find the lengths of the transverse and conjugate axes, find the equations of the asymptotes, and graph the following equation:

$$9x^2 - 16y^2 = 144$$

SOLUTIONS

First, write the equation in standard form by dividing both sides by 144 and determine a and b:

$$9x^2 - 16y^2 = 144 \qquad \text{Divide both sides by 144.}$$

$$\frac{9x^2}{144} - \frac{16y^2}{144} = \frac{144}{144} \qquad \text{Simplify.}$$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

$$a = 4 \quad \text{and} \quad b = 3$$

x intercepts: ± 4 Transverse axis length $= 2(4) = 8$

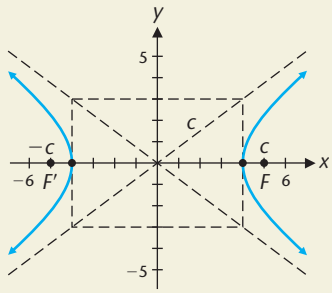
y intercepts: none Conjugate axis length $= 2(3) = 6$

$$\text{Foci: } c^2 = a^2 + b^2 \qquad \text{Substitute } a = 4 \text{ and } b = 3.$$
$$= 16 + 9$$
$$= 25$$
$$c = 5$$

Thus, the foci are $F' = (-5, 0)$ and $F = (5, 0)$.

Hand-Drawn Solution

Plot the foci and x intercepts, sketch the asymptote rectangle and the asymptotes, then sketch the hyperbola (Fig. 4). The equations of the asymptotes are $y = \pm\frac{3}{4}x$ (note that the diagonals of the asymptote rectangle have slope $\pm\frac{3}{4}$).



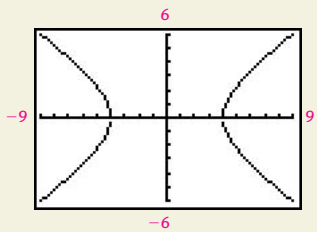
› Figure 4

Graphical Solution

Solve for y:

$$9x^2 - 16y^2 = 144 \qquad \text{Subtract } 9x^2 \text{ from both sides.}$$
$$-16y^2 = 144 - 9x^2 \qquad \text{Divide both sides by } -16.$$
$$y^2 = (9x^2 - 144)/16 \qquad \text{Take the square root of both sides.}$$
$$y = \pm\sqrt{(9x^2 - 144)/16}$$

This produces the functions $y_1 = \sqrt{(9x^2 - 144)/16}$ and $y_2 = -\sqrt{(9x^2 - 144)/16}$ whose graphs are shown in Figure 5.



› Figure 5

MATCHED PROBLEM 1

Find the coordinates of the foci, find the lengths of the transverse and conjugate axes, and graph the following equation:

$$16x^2 - 25y^2 = 400$$

EXAMPLE 2 Graphing Hyperbolas

Find the coordinates of the foci, find the lengths of the transverse and conjugate axes, find the equations of the asymptotes, and graph the following equation:

$$16y^2 - 9x^2 = 144$$

SOLUTIONS

Write the equation in standard form:

$$16y^2 - 9x^2 = 144 \qquad \text{Divide both sides by 144.}$$
$$\frac{y^2}{9} - \frac{x^2}{16} = 1$$
$$a = 3 \quad \text{and} \quad b = 4$$

y intercepts: ± 3 Transverse axis length $= 2(3) = 6$

x intercepts: none Conjugate axis length $= 2(4) = 8$

$$\text{Foci: } c^2 = a^2 + b^2 \qquad \text{Substitute } a = 3 \text{ and } b = 4.$$
$$= 9 + 16$$
$$= 25$$
$$c = 5$$

Thus, the foci are $F' = (0, -5)$ and $F = (0, 5)$.

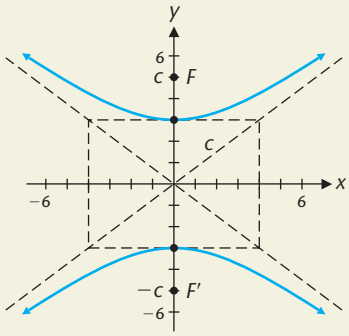
Hand-Drawn Solution

Plot the foci and y intercepts, sketch the asymptote rectangle and the asymptotes, then sketch the hyperbola (Fig. 6). The equations of the asymptotes are $y = \pm\frac{3}{4}x$ (note that the diagonals of the asymptote rectangle have slope $\pm\frac{3}{4}$).

Graphical Solution

Solve for y:

$$16y^2 - 9x^2 = 144 \qquad \text{Add } 9x^2 \text{ to both sides.}$$
$$16y^2 = 144 + 9x^2 \qquad \text{Divide both sides by 16.}$$
$$y^2 = (144 + 9x^2)/16 \qquad \text{Take the square root of both sides.}$$
$$y = \pm\sqrt{(144 + 9x^2)/16}$$

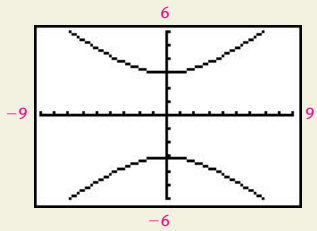


› Figure 6

This gives us the functions:

$$y_1 = \sqrt{(144 + 9x^2)/16} \quad \text{and}$$
$$y_2 = -\sqrt{(144 + 9x^2)/16}$$

whose graphs are shown in Figure 7.



› Figure 7

MATCHED PROBLEM 2

Find the coordinates of the foci, find the lengths of the transverse and conjugate axes, and graph the following equation:

$$25y^2 - 16x^2 = 400$$

Two hyperbolas of the form

$$\frac{x^2}{M} - \frac{y^2}{N} = 1 \quad \text{and} \quad \frac{y^2}{N} - \frac{x^2}{M} = 1 \qquad M, N > 0$$

are called **conjugate hyperbolas.** In Examples 1 and 2 and in Matched Problems 1 and 2, the hyperbolas are conjugate hyperbolas—they share the same asymptotes.

››› CAUTION ›››

When making a quick sketch of a hyperbola, it is a common error to have the hyperbola opening up and down when it should open left and right, or vice versa. The mistake can be avoided if you first locate the intercepts accurately.

EXAMPLE 3 Graphing Hyperbolas

Find the coordinates of the foci, find the lengths of the transverse and conjugate axes, and graph the following equation:

$$2x^2 - y^2 = 10$$

SOLUTIONS

$$2x^2 - y^2 = 10 \qquad \text{Divide both sides by 10.}$$
$$\frac{x^2}{5} - \frac{y^2}{10} = 1$$
$$a = \sqrt{5} \quad \text{and} \quad b = \sqrt{10}$$

x intercepts: $\pm\sqrt{5}$ Transverse axis length $= 2\sqrt{5} \approx 4.47$

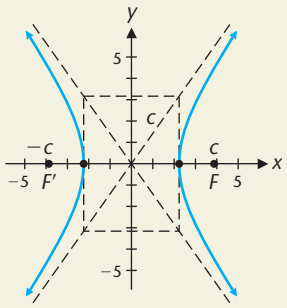
y intercepts: none Conjugate axis length $= 2\sqrt{10} \approx 6.32$

$$\text{Foci: } c^2 = a^2 + b^2 \qquad \text{Substitute } a = \sqrt{5} \text{ and } b = \sqrt{10}.$$
$$= 5 + 10$$
$$= 15$$
$$c = \sqrt{15}$$

Thus, the foci are $F' = (-\sqrt{15}, 0)$ and $F = (\sqrt{15}, 0)$.

Hand-Drawn Solution

Plot the foci and x intercepts, sketch the asymptote rectangle and the asymptotes, then sketch the hyperbola (Fig. 8).



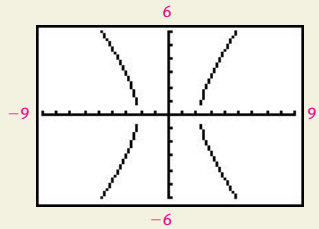
› Figure 8

Graphical Solution

Solve for y:

$$2x^2 - y^2 = 10 \qquad \text{Add } y^2 \text{ and } -10 \text{ to both sides.}$$
$$y^2 = 2x^2 - 10 \qquad \text{Take the square root of both sides.}$$
$$y = \pm\sqrt{2x^2 - 10}$$

This gives us two functions, $y_1 = \sqrt{2x^2 - 10}$ and $y_2 = -\sqrt{2x^2 - 10}$, which are graphed in Figure 9.



› Figure 9

MATCHED PROBLEM 3

Find the coordinates of the foci, find the lengths of the transverse and conjugate axes, and graph the following equation:

$$y^2 - 3x^2 = 12$$

EXAMPLE 4 **Finding the Equation of a Hyperbola**

Find an equation of a hyperbola in the form

$$\frac{y^2}{M} - \frac{x^2}{N} = 1 \qquad M, N > 0$$

if the center is at the origin, and:

(A) Length of transverse axis is 12
Length of conjugate axis is 20

(B) Length of transverse axis is 6
Distance of foci from center is 5

SOLUTIONS

(A) Start with

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

and find a and b:

$$a = \frac{12}{2} = 6 \qquad \text{and} \qquad b = \frac{20}{2} = 10$$

Thus, the equation is

$$\frac{y^2}{36} - \frac{x^2}{100} = 1$$

(B) Start with

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

and find a and b:

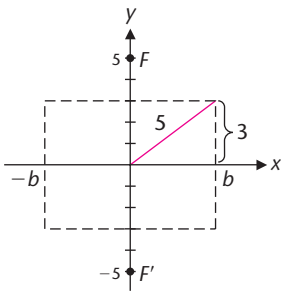
$$a = \frac{6}{2} = 3$$

To find b, sketch the asymptote rectangle (Fig. 10), label known parts, and use the Pythagorean theorem:

$$\begin{aligned} b^2 &= 5^2 - 3^2 \\ &= 16 \\ b &= 4 \end{aligned}$$

Thus, the equation is

$$\frac{y^2}{9} - \frac{x^2}{16} = 1$$



› **Figure 10** Asymptote rectangle.

MATCHED PROBLEM 4

Find an equation of a hyperbola in the form

$$\frac{x^2}{M} - \frac{y^2}{N} = 1 \qquad M, N > 0$$

if the center is at the origin, and:

(A) Length of transverse axis is 50
Length of conjugate axis is 30

(B) Length of conjugate axis is 12
Distance of foci from center is 9

››› EXPLORE-DISCUSS 2

(A) Does the line with equation $y = x$ intersect the hyperbola with equation $x^2 - (y^2/4) = 1$? If so, find the coordinates of all intersection points.

(B) Does the line with equation $y = 3x$ intersect the hyperbola with equation $x^2 - (y^2/4) = 1$? If so, find the coordinates of all intersection points.

(C) For which values of m does the line with equation $y = mx$ intersect the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$? Find the coordinates of all intersection points.

› Applications

You may not be aware of the many important uses of hyperbolic forms. They are encountered in the study of comets; the loran system of navigation for pleasure boats, ships, and aircraft; sundials; capillary action; nuclear reactor cooling towers; optical and radio telescopes; and contemporary architectural structures. The TWA building at Kennedy Airport is a *hyperbolic paraboloid,* and the St. Louis Science Center Planetarium is a *hyperboloid.* With such structures, thin concrete shells can span large spaces [Fig. 11(a)]. Some comets from outer space occasionally enter the sun's gravitational field, follow a hyperbolic path around the sun (with the sun as a focus), and then leave, never to be seen again [Fig. 11(b)]. Example 5 illustrates the use of hyperbolas in navigation.

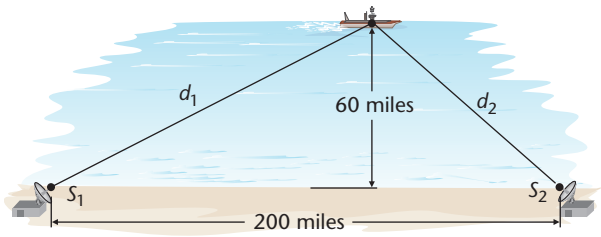
› **Figure 11** Uses of hyperbolic forms.

St. Louis Planetarium
(a)

Comet around sun
(b)

EXAMPLE 5 Navigation

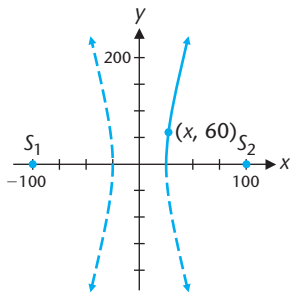
A ship is traveling on a course parallel to and 60 miles from a straight shoreline. Two transmitting stations, S_1 and S_2, are located 200 miles apart on the shoreline (Fig. 12). By timing radio signals from the stations, the ship's navigator determines that the ship is between the two stations and 50 miles closer to S_2 than to S_1. Find the distance from the ship to each station. Round answers to one decimal place.



› **Figure 12** $d_1 - d_2 = 50$.

SOLUTION

If d_1 and d_2 are the distances from the ship to S_1 and S_2, respectively, then $d_1 - d_2 = 50$ and the ship must be on the hyperbola with foci at S_1 and S_2 and fixed difference 50, as illustrated in Figure 13. In the derivation of the equation of a hyperbola, we represented the fixed difference as $2a$. Thus, for the hyperbola in Figure 13 we have



› **Figure 13**

$$c = 100$$
$$a = \tfrac{1}{2}(50) = 25$$
$$b = \sqrt{100^2 - 25^2} = \sqrt{9{,}375}$$

The equation for this hyperbola is

$$\frac{x^2}{625} - \frac{y^2}{9{,}375} = 1$$

Substitute $y = 60$ and solve for x (see Fig. 13):

$$\frac{x^2}{625} - \frac{60^2}{9{,}375} = 1 \qquad \text{Add } \frac{60^2}{9{,}375} \text{ to both sides.}$$

$$\frac{x^2}{625} = \frac{3{,}600}{9{,}375} + 1 \qquad \text{Multiply both sides by 625.}$$

$$x^2 = 625\,\frac{3{,}600 + 9{,}375}{9{,}375} \qquad \text{Simplify.}$$

$$= 865$$

Thus, $x = \sqrt{865} \approx 29.41$ (The negative square root is discarded, because the ship is closer to S_2 than to S_1.)

Distance from ship to S_1

$$d_1 = \sqrt{(29.41 + 100)^2 + 60^2}$$
$$= \sqrt{20{,}346.9841}$$
$$\approx 142.6 \text{ miles}$$

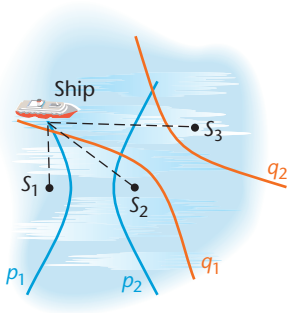
Distance from ship to S_2

$$d_2 = \sqrt{(29.41 - 100)^2 + 60^2}$$
$$= \sqrt{8{,}582.9841}$$
$$\approx 92.6 \text{ miles}$$

Notice that the difference between these two distances is 50, as it should be.

MATCHED PROBLEM 5

Repeat Example 5 if the ship is 80 miles closer to S_2 than to S_1.

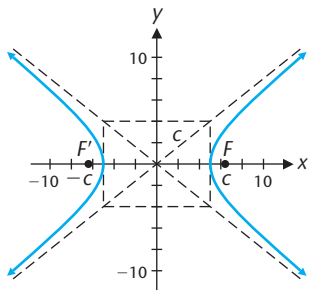


› **Figure 14** Loran navigation.

Example 5 illustrates a simplified form of the loran (LOng RAnge Navigation) system. In practice, three transmitting stations are used to send out signals simultaneously (Fig. 14), instead of the two used in Example 5. A computer onboard a ship will record these signals and use them to determine the differences of the distances that the ship is to S_1 and S_2, and to S_2 and S_3. Plotting all points so that these distances remain constant produces two branches, p_1 and p_2, of a hyperbola with foci S_1 and S_2, and two branches, q_1 and q_2, of a hyperbola with foci S_2 and S_3. It is easy to tell which branches the ship is on by comparing the signals from each station. The intersection of a branch of each hyperbola locates the ship and the computer expresses this in terms of longitude and latitude.

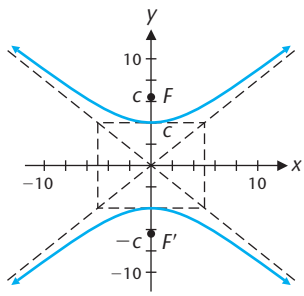
ANSWERS TO MATCHED PROBLEMS

1.



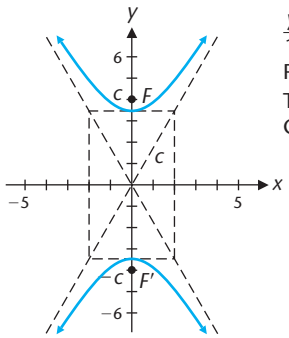
$\frac{x^2}{25} - \frac{y^2}{16} = 1$
Foci: $F' = (-\sqrt{41}, 0)$, $F = (\sqrt{41}, 0)$
Transverse axis length = 10
Conjugate axis length = 8

2.



$\frac{y^2}{16} - \frac{x^2}{25} = 1$
Foci: $F' = (0, -\sqrt{41})$, $F = (0, \sqrt{41})$
Transverse axis length = 8
Conjugate axis length = 10

3.



$\frac{y^2}{12} - \frac{x^2}{4} = 1$
Foci: $F' = (0, -4)$, $F = (0, 4)$
Transverse axis length $= 2\sqrt{12} \approx 6.93$
Conjugate axis length $= 4$

4. (A) $\frac{x^2}{625} - \frac{y^2}{225} = 1$ (B) $\frac{x^2}{45} - \frac{y^2}{36} = 1$ **5.** $d_1 = 159.5$ miles, $d_2 = 79.5$ miles

11-3 Exercises

1. What is the transverse axis of a hyperbola and how do you find it?

2. What is the conjugate axis of a hyperbola and how do you find it?

3. How do you find the foci of a hyperbola?

4. What is the asymptote rectangle and how is it used to graph a hyperbola?

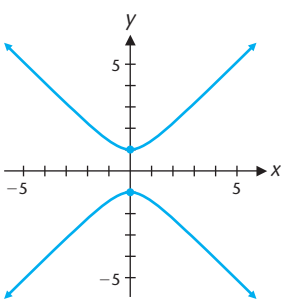
5. Given the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, replace 1 with 0 and then solve for y. Discuss how the results can serve as a memory aid when graphing a hyperbola.

6. Given the equation $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, replace 1 with 0 and then solve for y. Discuss how the results can serve as a memory aid when graphing a hyperbola.

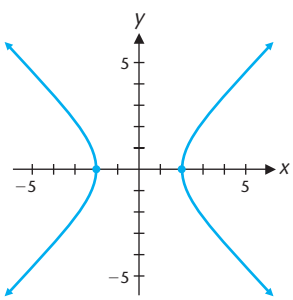
In Problems 7–10, match each equation with one of graphs (a)–(d).

7. $x^2 - y^2 = 1$ **8.** $y^2 - x^2 = 1$

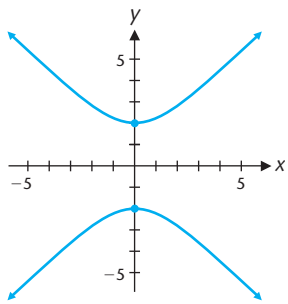
9. $y^2 - x^2 = 4$ **10.** $x^2 - y^2 = 4$



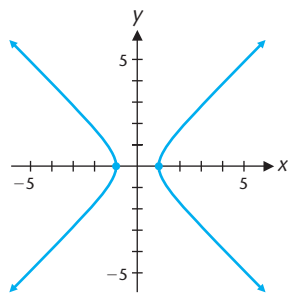
(a)



(b)



(c)



(d)

Sketch a graph of each equation in Problems 11–18, find the coordinates of the foci, and find the lengths of the transverse and conjugate axes.

11. $\frac{x^2}{9} - \frac{y^2}{4} = 1$ **12.** $\frac{x^2}{9} - \frac{y^2}{25} = 1$ **13.** $\frac{y^2}{4} - \frac{x^2}{9} = 1$

14. $\frac{y^2}{25} - \frac{x^2}{9} = 1$ **15.** $4x^2 - y^2 = 16$ **16.** $x^2 - 9y^2 = 9$

17. $9y^2 - 16x^2 = 144$ **18.** $4y^2 - 25x^2 = 100$

Sketch a graph of each equation in Problems 19–22, find the coordinates of the foci, and find the lengths of the transverse and conjugate axes.

19. $3x^2 - 2y^2 = 12$ **20.** $3x^2 - 4y^2 = 24$

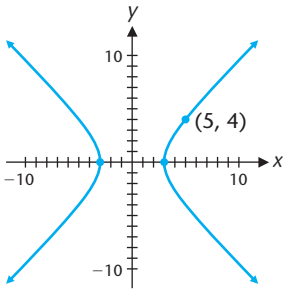
21. $7y^2 - 4x^2 = 28$ **22.** $3y^2 - 2x^2 = 24$

In Problems 23–34, find an equation of a hyperbola in the form

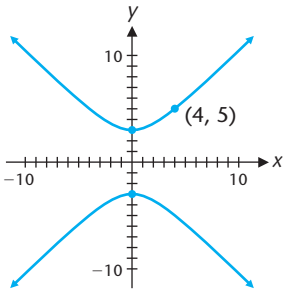
$$\frac{x^2}{M} - \frac{y^2}{N} = 1 \quad \text{or} \quad \frac{y^2}{N} - \frac{x^2}{M} = 1 \qquad M, N > 0$$

if the center is at the origin, and:

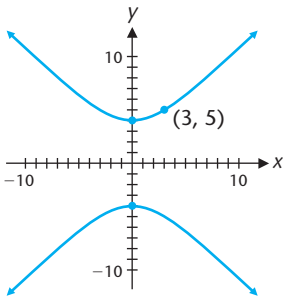
23. The graph is



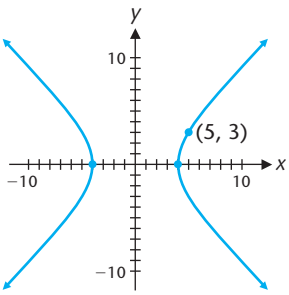
24. The graph is



25. The graph is



26. The graph is



27. Transverse axis on x axis
Transverse axis length $= 14$
Conjugate axis length $= 10$

28. Transverse axis on x axis
Transverse axis length $= 8$
Conjugate axis length $= 6$

29. Transverse axis on y axis
Transverse axis length $= 24$
Conjugate axis length $= 18$

30. Transverse axis on y axis
Transverse axis length $= 16$
Conjugate axis length $= 22$

31. Transverse axis on x axis
Transverse axis length $= 18$
Distance of foci from center $= 11$

32. Transverse axis on x axis
Transverse axis length $= 16$
Distance of foci from center $= 10$

33. Conjugate axis on x axis
Conjugate axis length $= 14$
Distance of foci from center $= \sqrt{200}$

34. Conjugate axis on x axis
Conjugate axis length $= 10$
Distance of foci from center $= \sqrt{70}$

In Problems 35–42, find the equations of the asymptotes of each hyperbola.

35. $\dfrac{x^2}{25} - \dfrac{y^2}{4} = 1$

36. $\dfrac{x^2}{16} - \dfrac{y^2}{36} = 1$

37. $\dfrac{y^2}{4} - \dfrac{x^2}{16} = 1$

38. $\dfrac{y^2}{9} - \dfrac{x^2}{25} = 1$

39. $9x^2 - y^2 = 9$

40. $x^2 - 4y^2 = 4$

41. $2y^2 - 3x^2 = 1$

42. $5y^2 - 6x^2 = 1$

43. (A) How many hyperbolas have center at (0, 0) and a focus at (1, 0)? Find their equations.
(B) How many ellipses have center at (0, 0) and a focus at (1, 0)? Find their equations.
(C) How many parabolas have center at (0, 0) and focus at (1, 0)? Find their equations.

44. How many hyperbolas have the lines $y = \pm 2x$ as asymptotes? Find their equations.

45. Find all intersection points of the graph of the hyperbola $x^2 - y^2 = 1$ with the graph of each of the following lines:
(A) $y = 0.5x$ (B) $y = 2x$

For what values of m will the graph of the hyperbola and the graph of the line $y = mx$ intersect? Find the coordinates of these intersection points.

46. Find all intersection points of the graph of the hyperbola $y^2 - x^2 = 1$ with the graph of each of the following lines:
(A) $y = 0.5x$ (B) $y = 2x$

For what values of m will the graph of the hyperbola and the graph of the line $y = mx$ intersect? Find the coordinates of these intersection points.

47. Find all intersection points of the graph of the hyperbola $y^2 - 4x^2 = 1$ with the graph of each of the following lines:
(A) $y = x$ (B) $y = 3x$

For what values of m will the graph of the hyperbola and the graph of the line $y = mx$ intersect? Find the coordinates of these intersection points.

48. Find all intersection points of the graph of the hyperbola $4x^2 - y^2 = 1$ with the graph of each of the following lines:
(A) $y = x$ (B) $y = 3x$

For what values of m will the graph of the hyperbola and the graph of the line $y = mx$ intersect? Find the coordinates of these intersection points.

49. Consider the hyperbola with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(A) Show that $y = \pm\frac{b}{a}x\sqrt{1 - \frac{a^2}{x^2}}$.
(B) Explain why the hyperbola approaches the lines $y = \pm\frac{b}{a}x$ as $|x|$ becomes larger.
(C) Does the hyperbola approach its asymptotes from above or below? Explain.

50. Consider the hyperbola with equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

(A) Show that $y = \pm\frac{a}{b}x\sqrt{1 + \frac{b^2}{x^2}}$.
(B) Explain why the hyperbola approaches the lines $y = \pm\frac{a}{b}x$ as $|x|$ becomes larger.
(C) Does the hyperbola approach its asymptotes from above or below? Explain.

51. Let F and F' be two points in the plane and let c be a constant such that $c > d(F, F')$. Describe the set of all points P in the plane such that the absolute value of the difference of the distances from P to F and F' is equal to the constant c.

52. Let F and F' be two points in the plane and let c denote the constant $d(F, F')$. Describe the set of all points P in the plane such that the absolute value of the difference of the distances from P to F and F' is equal to the constant c.

53. Study the following derivation of the standard equation of a hyperbola with foci $(\pm c, 0)$, x intercepts $(\pm a, 0)$, and endpoints of the conjugate axis $(0, \pm b)$. Explain why each equation follows from the equation that precedes it. [*Hint:* Recall that $c^2 = a^2 + b^2$.]

$$|d_1 - d_2| = 2a$$

$$\sqrt{(x + c)^2 + y^2} = \pm 2a + \sqrt{(x - c)^2 + y^2}$$

$$(x + c)^2 + y^2 = 4a^2 \pm 4a\sqrt{(x - c)^2 + y^2} + (x - c)^2 + y^2$$

$$\pm\sqrt{(x - c)^2 + y^2} = a - \frac{cx}{a}$$

$$(x - c)^2 + y^2 = a^2 - 2cx + \frac{c^2x^2}{a^2}$$

$$\left(1 - \frac{c^2}{a^2}\right)x^2 + y^2 = a^2 - c^2$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

54. Study the following derivation of the standard equation of a hyperbola with foci $(0, \pm c)$, y intercepts $(0, \pm a)$, and endpoints of the conjugate axis $(\pm b, 0)$. Explain why each equation follows from the equation that precedes it. [*Hint:* Recall that $c^2 = a^2 + b^2$.]

$$|d_1 - d_2| = 2a$$

$$\sqrt{x^2 + (y + c)^2} = \pm 2a + \sqrt{x^2 + (y - c)^2}$$

$$x^2 + (y + c)^2 = 4a^2 \pm 4a\sqrt{x^2 + (y - c)^2} + x^2 + (y - c)^2$$

$$\pm\sqrt{x^2 + (y - c)^2} = a - \frac{cy}{a}$$

$$x^2 + (y - c)^2 = a^2 - 2cy + \frac{c^2y^2}{a^2}$$

$$x^2 + \left(1 - \frac{c^2}{a^2}\right)y^2 = a^2 - c^2$$

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

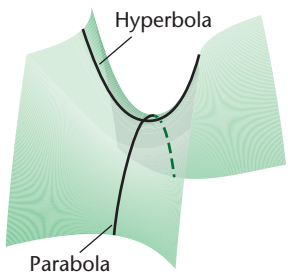
ECCENTRICITY *Problems 55 and 56 and Problems 37 and 38 in Exercise 11-2 are related to a property of conics called* ***eccentricity,*** *which is denoted by a positive real number E. Parabolas, ellipses, and hyperbolas all can be defined in terms of E, a fixed point called a focus, and a fixed line not containing the focus called a directrix as follows: The set of points in a plane each of whose distance from a fixed point is E times its distance from a fixed line is an ellipse if $0 < E < 1$, a parabola if $E = 1$, and a hyperbola if $E > 1$.*

55. Find an equation of the set of points in a plane each of whose distance from (3, 0) is three-halves its distance from the line $x = \frac{4}{3}$. Identify the geometric figure.

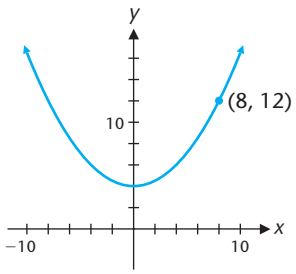
56. Find an equation of the set of points in a plane each of whose distance from (0, 4) is four-thirds its distance from the line $y = \frac{9}{4}$. Identify the geometric figure.

APPLICATIONS

57. **ARCHITECTURE** An architect is interested in designing a thin-shelled dome in the shape of a hyperbolic paraboloid, as shown in Figure (a). Find the equation of the hyperbola located in a coordinate system [Fig. (b)] satisfying the indicated conditions. How far is the hyperbola above the vertex 6 feet to the right of the vertex? Compute the answer to two decimal places.



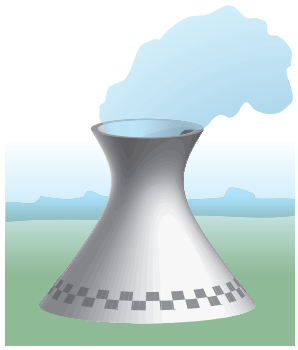
Hyperbolic paraboloid
(a)



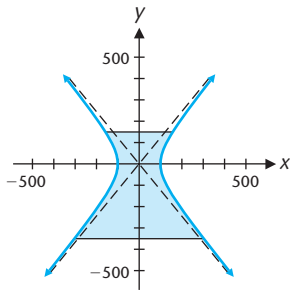
Hyperbola part of dome
(b)

58. NUCLEAR POWER A nuclear reactor cooling tower is a **hyperboloid,** that is, a hyperbola rotated around its conjugate axis, as shown in Figure (a). The equation of the hyperbola in Figure (b) used to generate the hyperboloid is

$$\frac{x^2}{100^2} - \frac{y^2}{150^2} = 1$$



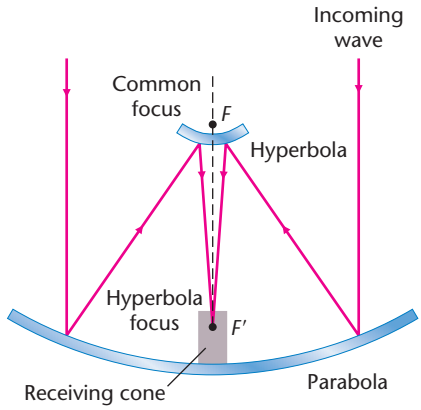
Nuclear reactor cooling tower
(a)



Hyperbola part of dome
(b)

If the tower is 500 feet tall, the top is 150 feet above the center of the hyperbola, and the base is 350 feet below the center, what is the radius of the top and the base? What is the radius of the smallest circular cross section in the tower? Compute answers to three significant digits.

59. SPACE SCIENCE In tracking space probes to the outer planets, NASA uses large parabolic reflectors with diameters equal to two-thirds the length of a football field. Needless to say, many design problems are created by the weight of these reflectors. One weight problem is solved by using a hyperbolic reflector sharing the parabola's focus to reflect the incoming electromagnetic waves to the other focus of the hyperbola where receiving equipment is installed (see the figure).



(a)

(b)

For the receiving antenna shown in the figure, the common focus F is located 120 feet above the vertex of the parabola, and focus F' (for the hyperbola) is 20 feet above the vertex. The vertex of the reflecting hyperbola is 110 feet above the vertex for the parabola. Introduce a coordinate system by using the axis of the parabola as the y axis (up positive), and let the x axis pass through the center of the hyperbola (right positive). What is the equation of the reflecting hyperbola? Write y in terms of x.

11-4 Translation and Rotation of Axes

- Translation of Axes
- Translation Used in Graphing
- Rotation of Axes
- Rotation Used in Graphing
- Identifying Conics

In Sections 11-1, 11-2, and 11-3 we found standard equations for parabolas, ellipses, and hyperbolas with axes on the coordinate axes and centered relative to the origin. Each of those standard equations was a special case of the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{1}$$

for appropriate constants A, B, C, D, E, and F. In this section we show that *every* equation of the form (1) has a graph that is either a conic, a degenerate conic (that is, a point, a line, or a pair of lines), or the empty set. The difficulty is that a conic with an equation of form (1) might not be centered at the origin, and might have axes that are skewed with respect to the coordinate axes. To overcome the difficulty we use two basic mathematical tools: *translation of axes* and *rotation of axes.* With these tools we will be able to choose a new coordinate system (that depends on the constants A, B, C, D, E, and F) in which the equation has an especially transparent and useful form.

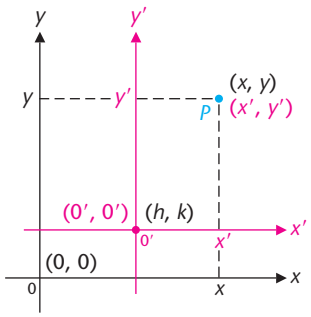


Figure 1 Translation of coordinates.

Translation of Axes

If you move a sheet of paper on a desk top, without rotating the paper and without flipping it over, you *translate* the paper to its new position. Similarly, a **translation of coordinate axes** occurs when the new coordinate axes have the same direction as, and are parallel to, the original coordinate axes. To see how coordinates in the original system are changed when moving to the translated system, and vice versa, refer to Figure 1.

A point P in the plane has two sets of coordinates: (x, y) in the original system and (x', y') in the translated system. If the coordinates of the origin of the translated system are (h, k) relative to the original system, then the old and new coordinates are related as given in Theorem 1.

THEOREM 1 Translation Formulas

1. $x = x' + h$	**2.** $x' = x - h$
$y = y' + k$	$y' = y - k$

It can be shown that these formulas hold for (h, k) located anywhere in the original coordinate system.

EXAMPLE 1 Equation of a Curve in a Translated System

A curve has the equation

$$(x - 4)^2 + (y + 1)^2 = 36$$

If the origin is translated to $(4, -1)$, find the equation of the curve in the translated system and identify the curve.

SOLUTION

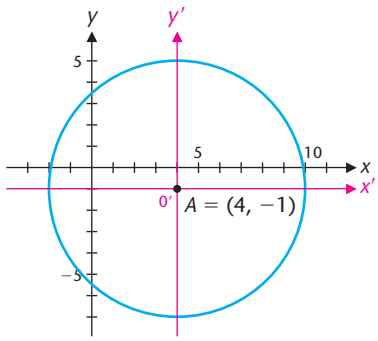
Because $(h, k) = (4, -1)$, use translation formulas

$$x' = x - h = x - 4$$
$$y' = y - k = y + 1$$

to obtain, after substitution,

$$x'^2 + y'^2 = 36$$

This is the equation of a circle of radius 6 with center at the new origin. The coordinates of the new origin in the original coordinate system are $(4, -1)$ (Fig. 2). Note that this result agrees with our general treatment of the circle in Section B-3.



› **Figure 2** $(x - 4)^2 + (y + 1)^2 = 36$.

MATCHED PROBLEM 1

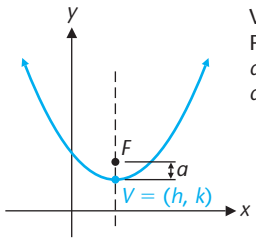
A curve has the equation $(y + 2)^2 = 8(x - 3)$. If the origin is translated to $(3, -2)$, find an equation of the curve in the translated system and identify the curve.

Suppose the coordinate axes in the xy system have been translated to (h, k), as in Figure 1 on page 1001. Then, as illustrated by Example 1, the circle $x'^2 + y'^2 = r^2$ has the equation $(x - h)^2 + (y - k)^2 = r^2$ in the original xy system. In a similar manner we use the standard equations for the parabola, ellipse, and hyperbola centered at the origin to obtain more general standard equations for conics centered at the point (h, k) (see Table 1). Note that when $h = 0$ and $k = 0$ the standard equations of Table 1 are exactly the standard equations obtained in Sections 11-1, 11-2, and 11-3.

Table 1 Standard Equations for Conics

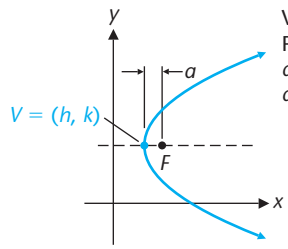
Parabolas

$(x - h)^2 = 4a(y - k)$



Vertex (h, k)
Focus $(h, k + a)$
$a > 0$ opens up
$a < 0$ opens down

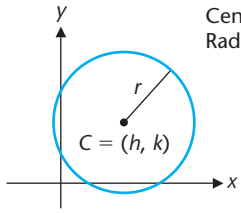
$(y - k)^2 = 4a(x - h)$



Vertex (h, k)
Focus $(h + a, k)$
$a < 0$ opens left
$a > 0$ opens right

Circles

$(x - h)^2 + (y - k)^2 = r^2$

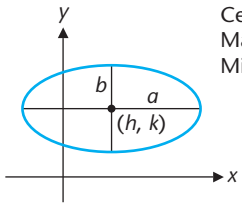


Center (h, k)
Radius r

Ellipses

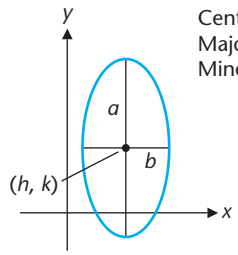
$\dfrac{(x - h)^2}{b^2} + \dfrac{(y - k)^2}{a^2} = 1$

$a > b > 0$



Center (h, k)
Major axis $2a$
Minor axis $2b$

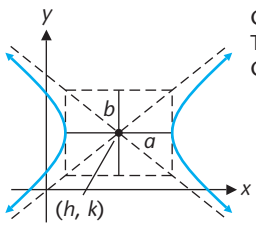
$\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$



Center (h, k)
Major axis $2a$
Minor axis $2b$

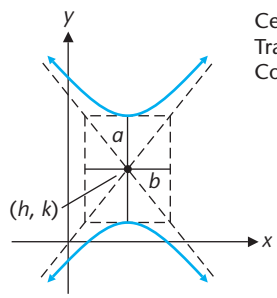
Hyperbolas

$\dfrac{(x - h)^2}{a^2} - \dfrac{(y - k)^2}{b^2} = 1$



Center (h, k)
Transverse axis $2a$
Conjugate axis $2b$

$\dfrac{(y - h)^2}{a^2} - \dfrac{(x - h)^2}{b^2} = 1$



Center (h, k)
Transverse axis $2a$
Conjugate axis $2b$

Translation Used in Graphing

Any equation of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \tag{2}$$

has a graph that is a conic, a degenerate conic, or the empty set [note that equation (2) is the same as equation (1) on page 1001 with $B = 0$]. To see this, we use the technique of completing the square discussed in Section 2-3. If we can transform equation (2) into one of the standard forms of Table 1, then we will be able to identity its graph and sketch it rather quickly. Some examples should help make the process clear.

EXAMPLE 2 Graphing a Conic

Given the equation

$$y^2 - 6y - 4x + 1 = 0 \tag{3}$$

(A) Transform the equation into one of the standard forms in Table 1 and identify the conic.

(B) Find the equation in the translated system.

(C) Graph the conic.

SOLUTIONS

(A) Complete the square in equation (3) relative to each variable that is squared—in this case y:

$$\begin{aligned} y^2 - 6y - 4x + 1 &= 0 && \text{Add } 4x - 1 \text{ to both sides.} \\ y^2 - 6y &= 4x - 1 && \text{Add 9 to both sides to complete the square on the left side.} \\ y^2 - 6y + 9 &= 4x + 8 && \text{Factor.} \\ (y - 3)^2 &= 4(x + 2) && (4) \end{aligned}$$

From Table 1 we recognize equation (4) as an equation of a parabola opening to the right with vertex at $(h, k) = (-2, 3)$.

(B) Find the equation of the parabola in the translated system with origin $0'$ at $(h, k) = (-2, 3)$. The equations of translation are read directly from equation (4):

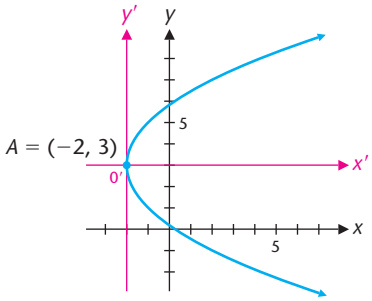
$$\begin{aligned} x' &= x + 2 \\ y' &= y - 3 \end{aligned}$$

Making these substitutions in equation (4) we obtain

$$y'^2 = 4x' \tag{5}$$

the equation of the parabola in the $x'y'$ system.

(C) Graph equation (5) in the $x'y'$ system following the process discussed in Section 11-1. The resulting graph is the graph of the original equation relative to the original xy coordinate system (Fig. 3).



› Figure 3

MATCHED PROBLEM 2

Repeat Example 2 for the equation $x^2 + 4x + 4y - 12 = 0$.

EXAMPLE 3 **Graphing a Conic**

Given the equation

$$9x^2 - 4y^2 - 36x - 24y - 36 = 0$$

(A) Transform the equation into one of the standard forms in Table 1 and identify the conic.

(B) Find the equation in the translated system.

(C) Graph the conic.

(D) Find the coordinates of any foci relative to the original system.

SOLUTIONS

(A) Complete the square relative to both x and y.

$$9x^2 - 4y^2 - 36x - 24y - 36 = 0 \qquad \text{Add 36 to both sides.}$$

$$9x^2 - 36x - 4y^2 - 24y = 36 \qquad \text{Factor out coefficients of } x^2 \text{ and } y^2.$$

$$9(x^2 - 4x\ \ ) - 4(y^2 + 6y\ \ ) = 36 \qquad \text{Complete squares.}$$

$$9(x^2 - 4x + 4) - 4(y^2 + 6y + 9) = 36 + 36 - 36 \qquad \text{Factor.}$$

$$9(x - 2)^2 - 4(y + 3)^2 = 36 \qquad \text{Divide both sides by 36.}$$

$$\frac{(x - 2)^2}{4} - \frac{(y + 3)^2}{9} = 1$$

From Table 1 we recognize the last equation as an equation of a hyperbola opening left and right with center at $(h, k) = (2, -3)$.

(B) Find the equation of the hyperbola in the translated system with origin $0'$ at $(h, k) = (2, -3)$. The equations of translation are read directly from the last equation in part A:

$$x' = x - 2$$
$$y' = y + 3$$

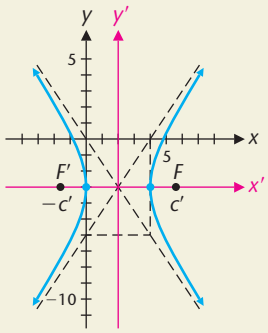
Making these substitutions, we obtain

$$\frac{x'^2}{4} - \frac{y'^2}{9} = 1$$

the equation of the hyperbola in the $x'y'$ system.

(C) Hand-Drawn Solution

Graph the equation obtained in part B in the $x'y'$ system following the process discussed in the last section. The resulting graph is the graph of the original equation relative to the original xy coordinate system (Fig. 4).



› Figure 4

(C) Graphing Calculator Solution

To graph the equation of this example on a graphing calculator, write it as a quadratic equation in the variable y, and use the quadratic formula to solve for y.

$$9x^2 - 4y^2 - 36x - 24y - 36 = 0$$

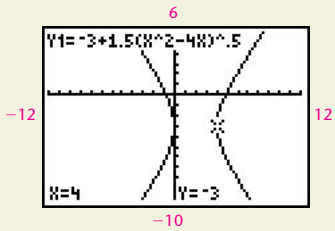
Write in the form $ay^2 + by + c = 0$.

$$4y^2 + 24y + (-9x^2 + 36x + 36) = 0$$

Use the quadratic formula with $a = 4$, $b = 24$, and $c = -9x^2 + 36x + 36$.

$$y = \frac{-24 \pm \sqrt{24^2 - 4(4)(-9x^2 + 36x + 36)}}{8}$$
$$= -3 \pm 1.5\sqrt{x^2 - 4x} \qquad (6)$$

The two functions determined by equation (6) are graphed in Figure 5.



› Figure 5

(D) Find the coordinates of the foci. To find the coordinates of the foci in the original system, first find the coordinates in the translated system:

$$c'^2 = 2^2 + 3^2 = 13$$
$$c' = \sqrt{13}$$
$$-c' = -\sqrt{13}$$

Thus, the coordinates in the translated system are

$$F' = (-\sqrt{13}, 0) \quad \text{and} \quad F = (\sqrt{13}, 0)$$

Now, use

$$x = x' + h = x' + 2$$
$$y = y' + k = y' - 3$$

to obtain

$$F' = (-\sqrt{13} + 2, -3) \quad \text{and} \quad F = (\sqrt{13} + 2, -3)$$

as the coordinates of the foci in the original system.

MATCHED PROBLEM 3

Repeat Example 3 for the equation

$$9x^2 + 16y^2 + 36x - 32y - 92 = 0$$

››› EXPLORE-DISCUSS 1

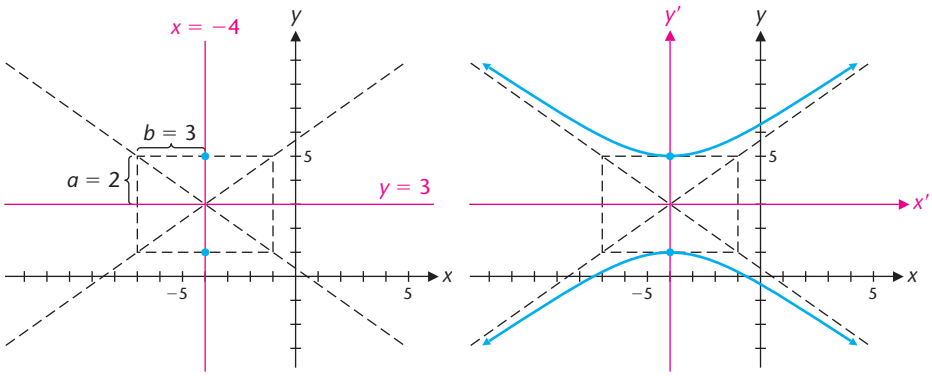
If $A \neq 0$ and $C \neq 0$, show that the translation of axes $x' = x + \frac{D}{2A}$ and $y' = y + \frac{E}{2C}$ transforms the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ into an equation of the form $Ax'^2 + Cy'^2 = K$.

EXAMPLE 4 Finding the Equation of a Conic

Find the equation of a hyperbola with vertices on the line $x = -4$, conjugate axis on the line $y = 3$, length of the transverse axis $= 4$, and length of the conjugate axis $= 6$.

SOLUTION

Locate the vertices, asymptote rectangle, and asymptotes in the original coordinate system [Fig. 6(a)], then sketch the hyperbola and translate the origin to the center of the hyperbola [Fig. 6(b)].



(a) Asymptote rectangle (b) Hyperbola

› Figure 6

Next write the equation of the hyperbola in the translated system:

$$\frac{y'^2}{4} - \frac{x'^2}{9} = 1$$

The origin in the translated system is at $(h, k) = (-4, 3)$, and the translation formulas are

$$x' = x - h = x - (-4) = x + 4$$
$$y' = y - k = y - 3$$

Thus, the equation of the hyperbola in the original system is

$$\frac{(y-3)^2}{4} - \frac{(x+4)^2}{9} = 1$$

or, after simplifying and writing in the form of equation (1) on page 1001,

$$4x^2 - 9y^2 + 32x + 54y + 19 = 0$$

MATCHED PROBLEM 4

Find the equation of an ellipse with foci on the line $x = 4$, minor axis on the line $y = -3$, length of the major axis = 8, and length of the minor axis = 4.

››› EXPLORE-DISCUSS 2

Use the strategy of completing the square to transform each equation into an equation in an $x'y'$ coordinate system. Note that the equation you obtain is not one of the standard forms in Table 1; instead, it is either the equation of a degenerate conic or the equation has no solution. If the solution set of the equation is not empty, graph it and identify the graph (a point, a line, two parallel lines, or two intersecting lines).

(A) $x^2 + 2y^2 - 2x + 16y + 33 = 0$

(B) $4x^2 - y^2 - 24x - 2y + 35 = 0$

(C) $y^2 - 2y - 15 = 0$

(D) $5x^2 + y^2 + 12y + 40 = 0$

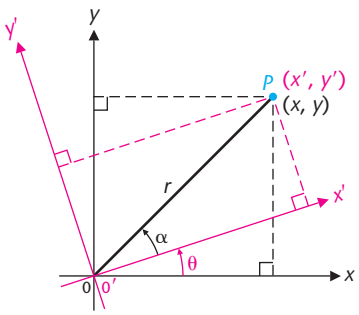
(E) $x^2 - 18x + 81 = 0$

› Rotation of Axes

To handle the general equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{1}$$

when $B \neq 0$, we need to be able to *rotate,* not just translate, coordinate axes. If you hold a sheet of paper to a desk top with a pencil point, and move the paper without moving the pencil point, you *rotate* the paper. Similarly, a **rotation of coordinate axes** occurs when the origin is kept fixed and the x' and y' axes are obtained by rotating the x and y axes counterclockwise through an angle θ, as shown in Figure 7.



› Figure 7

Referring to Figure 7 and using trigonometry, we have

$$x' = r\cos\alpha \qquad y' = r\sin\alpha \tag{7}$$

and

$$x = r\cos(\theta + \alpha) \qquad y = r\sin(\theta + \alpha) \tag{8}$$

Using sum identities from trigonometry for the equations in (8), we obtain

$$\begin{aligned} x &= r\cos(\theta + \alpha) && \text{Use sum identity for cosine.}\\ &= r(\cos\theta\cos\alpha - \sin\theta\sin\alpha) && \text{Distribute } r.\\ &= r\cos\theta\cos\alpha - r\sin\theta\sin\alpha && \text{Use associative property.}\\ &= (r\cos\alpha)\cos\theta - (r\sin\alpha)\sin\theta && \text{Substitute } x' = r\cos\alpha \text{ and } y' = r\sin\alpha.\\ &= x'\cos\theta - y'\sin\theta \end{aligned} \tag{9}$$

$$\begin{aligned} y &= r\sin(\theta + \alpha) && \text{Use sum identity for sine.}\\ &= r(\sin\theta\cos\alpha + \cos\theta\sin\alpha) && \text{Distribute } r.\\ &= r\sin\theta\cos\alpha + r\cos\theta\sin\alpha && \text{Use associative property.}\\ &= (r\cos\alpha)\sin\theta + (r\sin\alpha)\cos\theta && \text{Substitute } x' = r\cos\alpha \text{ and } y' = r\sin\alpha.\\ &= x'\sin\theta + y'\cos\theta \end{aligned} \tag{10}$$

Thus, equations (9) and (10) together transform the xy coordinate system into the $x'y'$ coordinate system.

Equations (9) and (10) can be solved for x' and y' in terms of x and y to produce formulas that transform the $x'y'$ coordinate system back into the xy coordinate system. Omitting the details, the formulas for the transformation in the reverse direction are

$$x' = x\cos\theta + y\sin\theta \qquad y' = -x\sin\theta + y\cos\theta \tag{11}$$

These results are summarized in Theorem 2.

› THEOREM 2 Rotation Formulas

If the xy coordinate axes are rotated counterclockwise through an angle of θ, then the xy and $x'y'$ coordinates of a point P are related by

1. $x = x'\cos\theta - y'\sin\theta$
$y = x'\sin\theta + y'\cos\theta$

2. $x' = x\cos\theta + y\sin\theta$
$y' = -x\sin\theta + y\cos\theta$

These formulas hold for P any point in the original coordinate system and θ any counterclockwise rotation.

››› EXPLORE-DISCUSS 3

Let θ be the first quadrant angle satisfying $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$ and let an xy coordinate system be transformed into an $x'y'$ coordinate system by a counterclockwise rotation through the angle θ.

(A) Sketch the $x'y'$ coordinate system in the xy coordinate system.

(B) Express x' and y' in terms of x and y.

(C) Solve $x' = 0$ to find the equation of the y' axis in the xy coordinate system.

(D) Solve $y' = 0$ to find the equation of the x' axis in the xy coordinate system.

(E) Use the results found in parts C and D to graph the $x'y'$ coordinate system in the xy coordinate system on a graphing calculator, using a squared viewing window.

› Rotation Used in Graphing

We now investigate how rotation formulas are used in graphing.

EXAMPLE 5 Using the Rotation of Axes Formulas

Transform the equation $xy = -2$ using a rotation of axes through 45°. Graph the new equation and identify the curve.

SOLUTION

Use the rotation formulas:

$$x = x'\cos 45° - y'\sin 45° = \frac{\sqrt{2}}{2}(x' - y')$$

$$y = x'\sin 45° + y'\cos 45° = \frac{\sqrt{2}}{2}(x' + y')$$

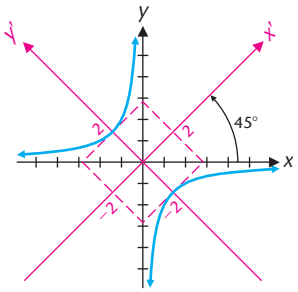
$$xy = -2 \qquad \text{Substitute for } x \text{ and } y.$$

$$\frac{\sqrt{2}}{2}(x' - y')\frac{\sqrt{2}}{2}(x' + y') = -2 \qquad \text{Simplify.}$$

$$\frac{1}{2}(x'^2 - y'^2) = -2 \qquad \text{Distribute } \tfrac{1}{2}.$$

$$\frac{x'^2}{2} - \frac{y'^2}{2} = -2 \qquad \text{Divide both sides by } -2.$$

$$\frac{y'^2}{4} - \frac{x'^2}{4} = 1$$



› Figure 8

This is a standard equation for a hyperbola. Summarizing, the graph of $xy = -2$ in the $x'y'$ coordinate system is a hyperbola with equation

$$\frac{y'^2}{4} - \frac{x'^2}{4} = 1$$

as shown in Figure 8.

Notice that the asymptotes in the rotated system are the x and y axes in the original system.

MATCHED PROBLEM 5

Transform the equation $2xy = 1$ using a rotation of axes through 45°. Graph the new equation and identify the curve. Check by graphing on a graphing calculator.

In Example 5, a 45° rotation transformed the original equation into one with no $x'y'$ term. This made it easy to recognize that the graph of the transformed equation was a hyperbola. In general, how do we determine the angle of rotation that will transform an equation with an xy term into one with no $x'y'$ term? To find out, we substitute

$$x = x'\cos\theta - y'\sin\theta \qquad \text{and} \qquad y = x'\sin\theta + y'\cos\theta$$

into equation (1) to obtain

$$\begin{aligned} &A(x'\cos\theta - y'\sin\theta)^2 + B(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) \\ &\quad + C(x'\sin\theta + y'\cos\theta)^2 + D(x'\cos\theta - y'\sin\theta) \\ &\quad + E(x'\sin\theta + y'\cos\theta) + F = 0 \end{aligned}$$

After multiplying and collecting terms, we have

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F = 0 \tag{12}$$

where

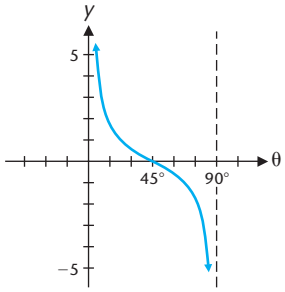
$$B' = 2(C - A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta) \tag{13}$$

For the $x'y'$ term in equation (12) to drop out, B' must be 0. We won't worry about A', C', D', and E' at this point; they will automatically be determined once we find θ so that $B' = 0$. We set the right side of equation (13) equal to 0 and solve for θ:

$$2(C - A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta) = 0$$

Using the double-angle identities from trigonometry, $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$, we obtain

$$\begin{aligned} (C - A)\sin 2\theta + B\cos 2\theta &= 0 && \text{Add } (A - C)\sin 2\theta \text{ to both sides.} \\ B\cos 2\theta &= (A - C)\sin 2\theta && \text{Divide both sides by } B\sin 2\theta. \\ \frac{\cos 2\theta}{\sin 2\theta} &= \frac{A - C}{B} && \text{Use quotient identity.} \\ \cot 2\theta &= \frac{A - C}{B} && (14) \end{aligned}$$



› Figure 9

Therefore, if we choose θ so that $\cot 2\theta = (A - C)/B$, then $B' = 0$ and the $x'y'$ term in equation (12) will drop out. There is always an angle θ between 0° and 90° that solves equation (14), because the range of $y = \cot 2\theta$ for $0° < \theta < 90°$ is the set of all real numbers (Fig. 9).

› **THEOREM 3 Angle of Rotation to Eliminate the $x'y'$ Term**

To transform the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

into an equation in x' and y' with no $x'y'$ term, find θ so that

$$\cot 2\theta = \frac{A - C}{B} \quad \text{and} \quad 0° < \theta < 90°$$

and use the rotation formulas in Theorem 2.

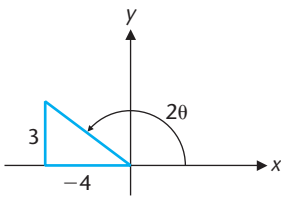
EXAMPLE 6 **Identifying and Graphing an Equation with an xy Term**

Given the equation $17x^2 - 6xy + 9y^2 = 72$, find the angle of rotation so that the transformed equation will have no $x'y'$ term. Sketch and identify the graph.

SOLUTION

$$17x^2 - 6xy + 9y^2 = 72 \tag{15}$$

$$\cot 2\theta = \frac{A - C}{B} = \frac{17 - 9}{-6} = -\frac{4}{3}$$



Therefore, 2θ is a Quadrant II angle, and using the reference triangle in the figure, we can see that $\cos 2\theta = -\frac{4}{5}$. We can find the rotation formulas exactly by the use of the half-angle identities

$$\sin\theta = \sqrt{\frac{1 - \cos 2\theta}{2}}$$

and

$$\cos\theta = \sqrt{\frac{1 + \cos 2\theta}{2}}$$

Using these identities and substituting $\cos 2\theta = -\frac{4}{5}$, we obtain

$$\sin\theta = \sqrt{\frac{1 - (-\frac{4}{5})}{2}} = \frac{3}{\sqrt{10}}$$

and

$$\cos\theta = \sqrt{\frac{1 + (-\frac{4}{5})}{2}} = \frac{1}{\sqrt{10}}$$

Hence, the rotation formulas (Theorem 2) are

$$x = \frac{1}{\sqrt{10}}x' - \frac{3}{\sqrt{10}}y'$$

and (16)

$$y = \frac{3}{\sqrt{10}}x' + \frac{1}{\sqrt{10}}y'$$

Substituting equations (16) into equation (15), we have

$$17\left(\frac{1}{\sqrt{10}}x' - \frac{3}{\sqrt{10}}y'\right)^2 - 6\left(\frac{1}{\sqrt{10}}x' - \frac{3}{\sqrt{10}}y'\right)\left(\frac{3}{\sqrt{10}}x' + \frac{1}{\sqrt{10}}y'\right) + 9\left(\frac{3}{\sqrt{10}}x' + \frac{1}{\sqrt{10}}y'\right)^2 = 72$$

$$\frac{17}{10}(x' - 3y')^2 - \frac{6}{10}(x' - 3y')(3x' + y') + \frac{9}{10}(3x' + y')^2 = 72$$

Further simplification leads to

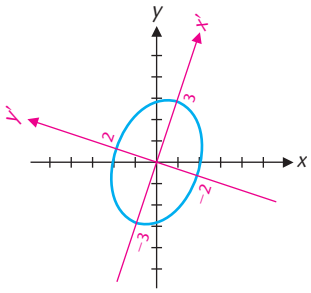
$$\frac{x'^2}{9} + \frac{y'^2}{4} = 1$$

which is a standard equation for an ellipse. To graph, we rotate the original axes through an angle θ determined as follows:

$$\cot 2\theta = -\frac{4}{3}$$
$$2\theta \approx 143.1301^\circ$$
$$\theta \approx 71.57^\circ$$

We could also use either

$$\sin\theta = \frac{3}{\sqrt{10}} \quad \text{or} \quad \cos\theta = \frac{1}{\sqrt{10}}$$



› Figure 10

to determine the angle of rotation. Summarizing these results, the graph of $17x^2 - 6xy + 9y^2 = 72$ in the $x'y'$ coordinate system formed by a rotation of 71.57° is an ellipse with equation

$$\frac{x'^2}{9} + \frac{y'^2}{4} = 1$$

as shown in Figure 10.

MATCHED PROBLEM 6

Given the equation $3x^2 + 26\sqrt{3}xy - 23y^2 = 144$, find the angle of rotation so that the transformed equation will have no $x'y'$ term. Sketch and identify the graph. Check by graphing on a graphing calculator.

› Identifying Conics

The **discriminant** of the general second-degree equation in two variables [equation (1)] is $B^2 - 4AC$. It can be shown that the value of this expression does not change when the axes are rotated. This forms the basis for Theorem 4.

› THEOREM 4 Identifying Conics

The graph of the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad (1)$$

is, excluding degenerate cases,

1. A hyperbola if $B^2 - 4AC > 0$
2. A parabola if $B^2 - 4AC = 0$
3. An ellipse if $B^2 - 4AC < 0$

The proof of Theorem 4 is beyond the scope of this book. Its use is best illustrated by example.

EXAMPLE 7 Identifying Conics

Identify the following conics.

(A) $x^2 - xy + y^2 = 5$
(B) $x^2 - xy - y^2 = 5$
(C) $x^2 - 4xy + 4y^2 + x = 5$

SOLUTIONS

(A) The discriminant is

$$B^2 - 4AC = (-1)^2 - 4(1)(1) = -3 < 0$$

so by Theorem 4 the conic is an ellipse.

(B) The discriminant is

$$B^2 - 4AC = (-1)^2 - 4(1)(-1) = 5 > 0$$

so by Theorem 4 the conic is a hyperbola.

(C) The discriminant is

$$B^2 - 4AC = (-4)^2 - 4(1)(4) = 0$$

so by Theorem 4 the conic is a parabola.

MATCHED PROBLEM 7

Identify the following conics.

(A) $x^2 + xy + 2y^2 = 10$
(B) $x^2 + xy - 2y^2 = 10$
(C) $x^2 - 2xy + y^2 - x = 10$

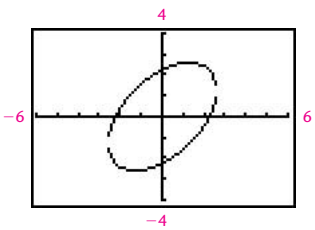
Each of the equations in Example 7 can be graphed by the method illustrated in Example 6, or, as an alternative, by a graphing calculator. For example, to graph the equation $x^2 - xy + y^2 = 5$ using a graphing calculator, first write the equation as a quadratic in the variable y, then use the quadratic formula to solve for y:

$$x^2 - xy + y^2 = 5 \quad \text{Write as a quadratic in } y.$$
$$y^2 - xy + x^2 - 5 = 0 \quad \text{Use the quadratic formula with } a = 1,\ b = -x, \text{ and } c = x^2 - 5.$$
$$y = \frac{x \pm \sqrt{(-x)^2 - 4(1)(x^2 - 5)}}{2} \quad \text{Simplify.}$$
$$= \frac{x \pm \sqrt{20 - 3x^2}}{2}$$

Graphing

$$y_1 = \frac{x + \sqrt{20 - 3x^2}}{2} \quad \text{and} \quad y_2 = \frac{x - \sqrt{20 - 3x^2}}{2}$$

produces the ellipse of Example 7A (Fig. 11).

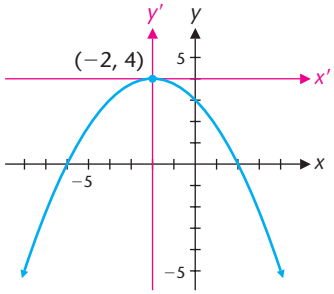


› Figure 11

ANSWERS TO MATCHED PROBLEMS

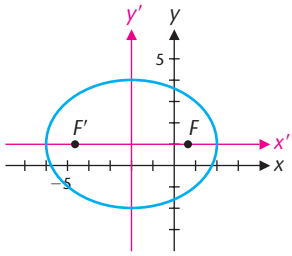
1. $y'^2 = 8x'$; a parabola **2.** (A) $(x + 2)^2 = -4(y - 4)$; a parabola (B) $x'^2 = -4y'$

(C)



3. (A) $\dfrac{(x + 2)^2}{16} + \dfrac{(y - 1)^2}{9} = 1$; ellipse (B) $\dfrac{x'^2}{16} + \dfrac{y'^2}{9} = 1$

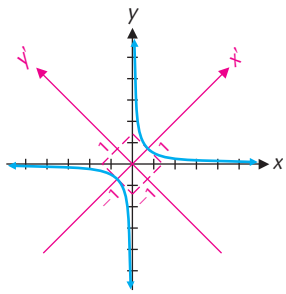
(C)

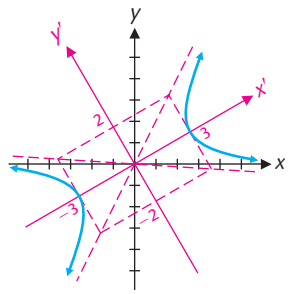


(D) Foci: $F' = (-\sqrt{7} - 2, 1)$, $F = (\sqrt{7} - 2, 1)$

4. $\dfrac{(x - 4)^2}{4} + \dfrac{(y + 3)^2}{16} = 1$, or $4x^2 + y^2 - 32x + 6y + 57 = 0$

5. $x'^2 - y'^2 = 1$; hyperbola **6.** $\dfrac{x'^2}{9} - \dfrac{y'^2}{4} = 1$; $\theta = 30°$; hyperbola





7. (A) Ellipse (B) Hyperbola (C) Parabola

11-4 Exercises

In Problems 1–8:
(A) Find translation formulas that translate the origin to the indicated point (h, k).
(B) Write the equation of the curve for the translated system.
(C) Identify the curve.

1. $(x-3)^2+(y-5)^2=81;\ (3,5)$

2. $(x-3)^2=8(y+2);\ (3,-2)$

3. $\dfrac{(x+7)^2}{9}+\dfrac{(y-4)^2}{16}=1;\ (-7,4)$

4. $(x+2)^2+(y+6)^2=36;\ (-2,-6)$

5. $(y+9)^2=16(x-4);\ (4,-9)$

6. $\dfrac{(y-9)^2}{10}-\dfrac{(x+5)^2}{6}=1;\ (-5,9)$

7. $\dfrac{(x+8)^2}{12}+\dfrac{(y+3)^2}{8}=1;\ (-8,-3)$

8. $\dfrac{(x+7)^2}{25}-\dfrac{(y-8)^2}{50}=1;\ (-7,8)$

In Problems 9–14:
(A) Write each equation in one of the standard forms listed in Table 1.
(B) Identify the curve.

9. $16(x-3)^2-9(y+2)^2=144$

10. $(y+2)^2-12(x-3)=0$

11. $6(x+5)^2+5(y+7)^2=30$

12. $12(y-5)^2-8(x-3)^2=24$

13. $(x+6)^2+24(y-4)=0$

14. $4(x-7)^2+7(y-3)^2=28$

In Problems 15–18, find the $x'y'$ coordinates of the given points if the coordinate axes are rotated through the indicated angle.

15. $(1,0),(0,1),(1,-1),(-3,4),\ \theta=30°$

16. $(1,0),(0,1),(-1,2),(-2,5),\ \theta=60°$

17. $(1,0),(0,1),(-1,-2),(1,-3),\ \theta=45°$

18. $(1,1),(-1,-1),(1,-1),(-1,1),\ \theta=90°$

In Problems 19–22, find the equations of the x' and y' axes in terms of x and y if the xy coordinate axes are rotated through the indicated angle.

19. $\theta=30°$ **20.** $\theta=60°$

21. $\theta=45°$ **22.** $\theta=90°$

In Problems 23–30, transform each equation into one of the standard forms in Table 1. Identify the curve and graph it.

23. $4x^2+9y^2-16x-36y+16=0$

24. $16x^2+9y^2+64x+54y+1=0$

25. $x^2+8x+8y=0$

26. $y^2+12x+4y-32=0$

27. $x^2+y^2+12x+10y+45=0$

28. $x^2+y^2-8x-6y=0$

29. $-9x^2+16y^2-72x-96y-144=0$

30. $16x^2-25y^2-160x=0$

In Problems 31–36, find the coordinates of any foci relative to the original coordinate system.

31. Problem 23 **32.** Problem 24 **33.** Problem 25

34. Problem 26 **35.** Problem 29 **36.** Problem 30

In Problems 37–40, complete the square in each equation, identify the transformed equation, and graph.

37. $x^2-2x+y^2+4y+5=0$

38. $x^2-6x+2y^2+4y+11=0$

39. $x^2+8x-4y^2+8y+12=0$

40. $x^2+4x-y^2+6y-5=0$

41. If $A\neq 0$, $C=0$, and $E\neq 0$, find h and k so that the translation of axes $x=x'+h, y=y'+k$ transforms the equation $Ax^2+Cy^2+Dx+Ey+F=0$ into one of the standard forms of Table 1.

42. If $A=0$, $C\neq 0$, and $D\neq 0$, find h and k so that the translation of axes $x=x'+h, y=y'+k$ transforms the equation $Ax^2+Cy^2+Dx+Ey+F=0$ into one of the standard forms of Table 1.

In Problems 43–46, find the transformed equation when the axes are rotated through the indicated angle. Sketch and identify the graph.

43. $x^2 + y^2 = 49, \theta = 45°$

44. $x^2 + y^2 = 25, \theta = 60°$

45. $2x^2 + \sqrt{3}xy + y^2 - 10 = 0, \theta = 30°$

46. $x^2 + 8xy + y^2 - 75 = 0, \theta = 45°$

In Problems 47–52, find the angle of rotation so that the transformed equation will have no $x'y'$ term. Sketch and identify the graph.

47. $x^2 - 4xy + y^2 = 12$

48. $x^2 + xy + y^2 = 6$

49. $8x^2 - 4xy + 5y^2 = 36$

50. $5x^2 - 4xy + 8y^2 = 36$

51. $x^2 - 2\sqrt{3}xy + 3y^2 - 16\sqrt{3}x - 16y = 0$

52. $x^2 + 2\sqrt{3}xy + 3y^2 + 8\sqrt{3}x - 8y = 0$

In Problems 53–62, find the equations (in the original xy coordinate system) of the asymptotes of each hyperbola.

53. $(x - 3)^2 - (y + 2)^2 = 1$

54. $(x + 1)^2 - (y - 4)^2 = 1$

55. $\dfrac{x^4}{4} - \dfrac{(y + 1)^2}{25} = 1$

56. $\dfrac{(x - 5)^2}{36} - \dfrac{y^2}{4} = 1$

57. $9(y - 5)^2 - 16(x + 2)^2 = 144$

58. $25(y + 3)^2 - 9(x - 1)^2 = 225$

59. $3(y + 4)^2 - x^2 = 1$

60. $y^2 - 5(x + 2)^2 = 1$

61. $xy - 9 = 0$

62. $4xy + 1 = 0$

In Problems 63–74, use the given information to find the equation of each conic. Express the answer in the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$ with integer coefficients and $A > 0$.

63. A parabola with vertex at (2, 5), axis the line $x = 2$, and passing through the point $(-2, 1)$.

64. A parabola with vertex at $(4, -1)$, axis the line $y = -1$, and passing through the point (2, 3).

65. An ellipse with major axis on the line $y = -3$, minor axis on the line $x = -2$, length of major axis = 8, and length of minor axis = 4.

66. An ellipse with major axis on the line $x = -4$, minor axis on the line $y = 1$, length of major axis = 4, and length of minor axis = 2.

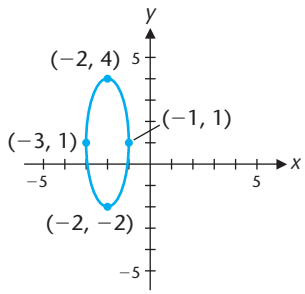
67. An ellipse with vertices $(4, -7)$ and (4, 3) and foci $(4, -6)$ and (4, 2).

68. An ellipse with vertices $(-3, 1)$ and (7, 1) and foci $(-1, 1)$ and (5, 1).

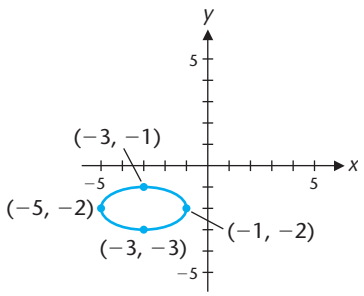
69. A hyperbola with transverse axis on the line $x = 2$, length of transverse axis = 4, conjugate axis on the line $y = 3$, and length of conjugate axis = 2.

70. A hyperbola with transverse axis on the line $y = -5$, length of transverse axis = 6, conjugate axis on the line $x = 2$, and length of conjugate axis = 6.

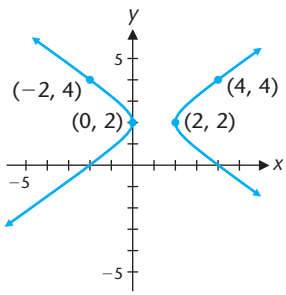
71. An ellipse with the following graph:



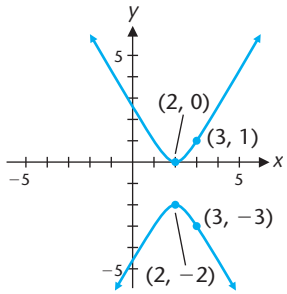
72. An ellipse with the following graph:



73. A hyperbola with the following graph:



74. A hyperbola with the following graph:



In Problems 75–80, use the discriminant to identify each graph. Graph on a graphing calculator.

75. $13x^2 + 10xy + 13y^2 - 72 = 0$

76. $3x^2 - 10xy + 3y^2 + 8 = 0$

77. $x^2 - 6\sqrt{3}xy - 5y^2 - 8 = 0$

78. $16x^2 + 24xy + 9y^2 + 15x - 20y = 0$

79. $16x^2 - 24xy + 9y^2 + 60x + 80y = 0$

80. $7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$

In Problems 81 and 82, use a rotation followed by a translation to transform each equation into a standard form. Sketch and identify the curve.

81. $x^2 + 2\sqrt{3}xy + 3y^2 - 8\sqrt{3}x - 8y - 4 = 0$

82. $73x^2 + 72xy + 52y^2 - 260x - 320y + 400 = 0$

11-5 Systems of Nonlinear Equations

- Solving by Substitution
- Other Solution Methods

If a system of equations contains any equations that are not linear, then the system is called a **nonlinear system.** In this section, we will investigate a special type of nonlinear system involving two first- or second-degree equations of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = F \tag{1}$$

Notice that the standard equations for a parabola, an ellipse, and a hyperbola are second-degree equations. It can be shown that a system of two equations of form (1) will have at most four solutions, some of which may be imaginary. Since we are interested in finding both real and imaginary solutions to the systems we consider, we now assume that the replacement set for each variable is the set of complex numbers, rather than the set of real numbers.

Solving by Substitution

The substitution method used to solve linear systems of two equations in two variables is also an effective method for solving nonlinear systems. This process is best illustrated by examples.

EXAMPLE 1 Solving a Nonlinear System by Substitution

Solve the system:
$$\begin{aligned} x^2 + y^2 &= 5 \\ 3x + y &= 1 \end{aligned}$$

SOLUTIONS

Algebraic Solution

We can start with either equation. But since the y term in the second equation is a first-degree term with coefficient 1, our calculations will be simplified if we start by solving for y in terms of x in the second equation. Next we substitute for y in the first equation to obtain an equation that involves x alone:

$$3x + y = 1 \qquad \text{Add } -3x \text{ to both sides.}$$
$$y = 1 - 3x \qquad \text{Substitute for } y \text{ in the second equation.}$$
$$x^2 + y^2 = 5$$
$$x^2 + (1 - 3x)^2 = 5 \qquad \text{Square the binomial and collect like terms on the left side.}$$
$$10x^2 - 6x - 4 = 0 \qquad \text{Divide both sides by 2.}$$
$$5x^2 - 3x - 2 = 0 \qquad \text{Factor.}$$
$$(x - 1)(5x + 2) = 0 \qquad \text{Use the zero product property.}$$
$$x = 1, -\tfrac{2}{5}$$

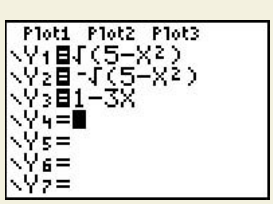
If we substitute these values back into the equation $y = 1 - 3x$, we obtain two solutions to the system:

$$x = 1 \qquad x = -\tfrac{2}{5}$$
$$y = 1 - 3(1) = -2 \qquad y = 1 - 3(-\tfrac{2}{5}) = \tfrac{11}{5}$$

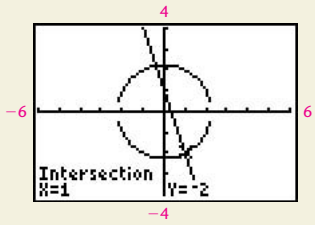
A check, which you should provide, verifies that $(1, -2)$ and $(-\frac{2}{5}, \frac{11}{5})$ are both solutions to the system.

Graphical Solution

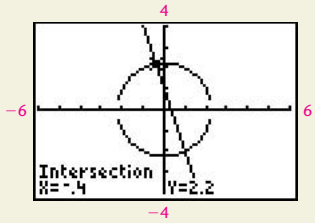
We enter two equations to graph the circle and one to graph the line (Fig. 1). Using the INTERSECT command, we find two solutions, $(1, -2)$ (Fig. 2) and $(-0.4, 2.2) = (-\frac{2}{5}, \frac{11}{5})$ (Fig. 3).



› Figure 1



› Figure 2



› Figure 3

MATCHED PROBLEM 1

Solve the system:
$$\begin{aligned} x^2 + y^2 &= 10 \\ 2x + y &= 1 \end{aligned}$$

Refer to the algebraic solution of Example 1. If we substitute the values of x back into the equation $x^2 + y^2 = 5$, we obtain

$$\begin{aligned} x &= 1 \\ 1^2 + y^2 &= 5 \\ y^2 &= 4 \\ y &= \pm 2 \end{aligned} \qquad \begin{aligned} x &= -\tfrac{2}{5} \\ (-\tfrac{2}{5})^2 + y^2 &= 5 \\ y^2 &= \tfrac{121}{25} \\ y &= \pm\tfrac{11}{5} \end{aligned}$$

It appears that we have found two additional solutions, (1, 2) and $(-\frac{2}{5}, -\frac{11}{5})$. But neither of these solutions satisfies the equation $3x + y = 1$, which you should verify. So, neither is a solution of the original system. We have produced two **extraneous roots,** apparent solutions that do not actually satisfy both equations in the system. This is a common occurrence when solving nonlinear systems.

It is always very important to check the solutions of any nonlinear system to ensure that extraneous roots have not been introduced.

››› EXPLORE-DISCUSS 1

In Example 1, we saw that the line $3x + y = 1$ intersected the circle $x^2 + y^2 = 5$ in two points.

(A) Consider the system

$$\begin{aligned} x^2 + y^2 &= 5 \\ 3x + y &= 10 \end{aligned}$$

Graph both equations in the same coordinate system. Are there any real solutions to this system? Are there any complex solutions? Find any real or complex solutions.

(B) Consider the family of lines given by

$$3x + y = b \qquad b \text{ any real number}$$

What do all these lines have in common? Illustrate graphically the lines in this family that intersect the circle $x^2 + y^2 = 5$ in exactly one point. How many such lines are there? What are the corresponding value(s) of b? What are the intersection points? How are these lines related to the circle?

EXAMPLE 2 Solving a Nonlinear System by Substitution

Solve: $\begin{aligned} x^2 - 2y^2 &= 2 \\ xy &= 2 \end{aligned}$

SOLUTIONS

Algebraic Solution

Solve the second equation for y, substitute into the first equation, and proceed as before.

$$xy = 2 \qquad \text{Divide both sides by } x.$$

$$y = \frac{2}{x} \qquad \text{Substitute for } y \text{ in the first equation.}$$

$$x^2 - 2\left(\frac{2}{x}\right)^2 = 2 \qquad \text{Simplify.}$$

$$x^2 - \frac{8}{x^2} = 2 \qquad \text{Multiply both sides by } x^2 \text{ and simplify.}$$

$$x^4 - 2x^2 - 8 = 0 \qquad \text{Substitute } u = x^2 \text{ (see Section 2-6).}$$

$$u^2 - 2u - 8 = 0 \qquad \text{Factor.}$$

$$(u - 4)(u + 2) = 0 \qquad \text{Use the zero product property.}$$

$$u = 4, -2$$

Thus,

$$x^2 = 4 \quad \text{or} \quad x^2 = -2$$
$$x = \pm 2 \qquad x = \pm\sqrt{-2} = \pm i\sqrt{2}$$

For $x = 2, y = \dfrac{2}{2} = 1.$ For $x = i\sqrt{2}, y = \dfrac{2}{i\sqrt{2}} = -i\sqrt{2}.$

For $x = -2, y = \dfrac{2}{-2} = -1.$ For $x = -i\sqrt{2}, y = \dfrac{2}{-i\sqrt{2}} = i\sqrt{2}.$

Thus, the four solutions to this system are $(2, 1), (-2, -1), (i\sqrt{2}, -i\sqrt{2})$, and $(-i\sqrt{2}, i\sqrt{2})$. You should verify that each of these satisfies both equations in the system.

Graphical Solution

Solving the first equation for y, we have

$$x^2 - 2y^2 = 2 \qquad \text{Subtract } x^2 \text{ from both sides.}$$

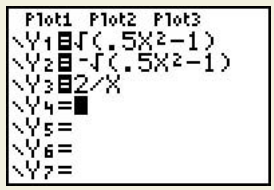
$$-2y^2 = -x^2 + 2 \qquad \text{Multiply both sides by } \tfrac{1}{-2} = -0.5.$$

$$y^2 = 0.5x^2 - 1 \qquad \text{Take the square root of both sides.}$$

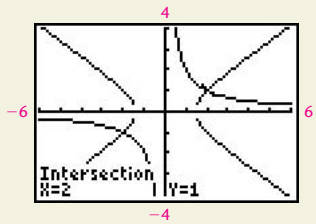
$$y = \pm\sqrt{0.5x^2 - 1}$$

We enter these two equations and $y = 2/x$ (Fig. 4) in a graphing calculator. Using the INTERSECT command, we find the two real solutions, (2, 1) (Fig. 5) and $(-2, -1)$ (Fig. 6). But we cannot find the two complex solutions.

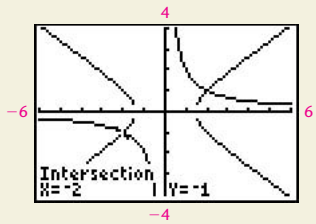
› Figure 4



› Figure 5



› Figure 6



MATCHED PROBLEM 2

Solve: $3x^2 - y^2 = 6$
$xy = 3$

››› EXPLORE-DISCUSS 2

Study the graphing calculator technique used in the graphical solution of Example 2. Explain why this technique does not produce the imaginary solutions of a system of equations.

EXAMPLE 3 Design

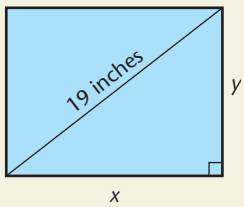
An engineer is to design a rectangular computer screen with a 19-inch diagonal and a 175-square-inch area. Find the dimensions of the screen to the nearest tenth of an inch.

SOLUTIONS

Algebraic Solution

Sketch a rectangle letting x be the width and y the height (Fig. 7). We obtain the following system using the Pythagorean theorem and the formula for the area of a rectangle:

$$x^2 + y^2 = 19^2$$
$$xy = 175$$



› Figure 7

This system is solved using the procedures outlined in Example 2. However, in this case, we are only interested in real solutions. We start by solving the second equation for y in terms of x and substituting the result into the first equation.

$$y = \frac{175}{x}$$

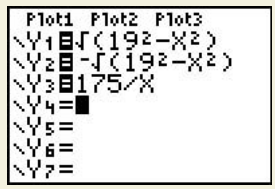
$$x^2 + \frac{175^2}{x^2} = 19^2 \qquad \text{Multiply both sides by } x^2 \text{ and simplify.}$$

$$x^4 + 30{,}625 = 361x^2 \qquad \text{Subtract } 361x^2 \text{ from each side.}$$

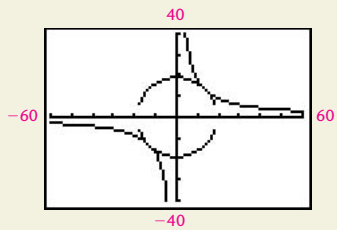
$$x^4 - 361x^2 + 30{,}625 = 0 \qquad \text{Quadratic in } x^2.$$

Graphical Solution

Figure 8 shows the three functions required to graph this system. The graph is shown in Figure 9. We are only interested in the solutions in the first quadrant. Zooming in and using INTERSECT produces the results in Figures 10 and 11. Assuming that the screen is wider than it is high, its dimensions are 15.0 inches by 11.7 inches.



› Figure 8



› Figure 9

Solve the last equation for x^2 using the quadratic formula, then solve for x:

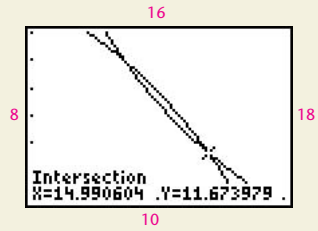
$$x = \sqrt{\frac{361 \pm \sqrt{361^2 - 4(1)(30{,}625)}}{2}}$$
$$\approx 15.0 \text{ inches or } 11.7 \text{ inches}$$

Substitute each choice of x into $y = 175/x$ to find the corresponding y values:

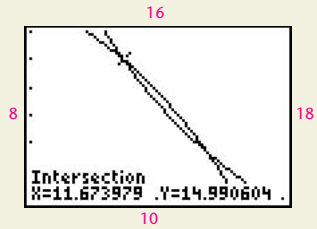
For $x = 15.0$ inches, $y = \frac{175}{15} = 11.7$ inches

For $x = 11.7$ inches, $y = \frac{175}{11.7} = 15.0$ inches

Assuming the screen is wider than it is high, the dimensions are 15.0 inches by 11.7 inches.



› Figure 10



› Figure 11

MATCHED PROBLEM 3

An engineer is to design a rectangular television screen with a 21-inch diagonal and a 209-square-inch area. Find the dimensions of the screen to the nearest tenth of an inch.

› Other Solution Methods

We now look at some other techniques for solving nonlinear systems of equations.

EXAMPLE 4 Solving a Nonlinear System by Elimination

Solve:
$$x^2 - y^2 = 5$$
$$x^2 + 2y^2 = 17$$

SOLUTIONS

Algebraic Solution

This type of system can be solved using elimination by addition.* Multiply the second equation by -1 and add:

$$\begin{aligned} x^2 - y^2 &= 5 \\ -x^2 - 2y^2 &= -17 \\ \hline -3y^2 &= -12 \\ y^2 &= 4 \\ y &= \pm 2 \end{aligned}$$

Graphical Solution

Solving each equation for y gives us the four functions shown in Figure 12. Examining the graph in Figure 13, we see that there are four intersection points. Using the INTERSECT command repeatedly (details omitted), we find that the solutions are $(3, -2)$, $(3, 2)$, $(-3, -2)$, and $(-3, 2)$.

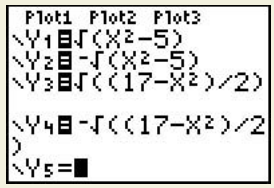
*This system can also be solved by substitution.

Now substitute $y = 2$ and $y = -2$ back into either original equation to find x.

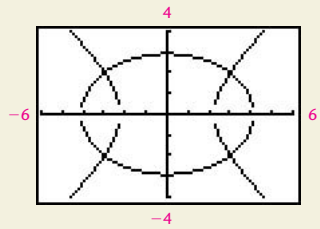
For $y = 2$,
$$x^2 - (2)^2 = 5$$
$$x = \pm 3$$

For $y = -2$,
$$x^2 - (-2)^2 = 5$$
$$x = \pm 3$$

Thus, $(3, -2)$, $(3, 2)$, $(-3, -2)$, and $(-3, 2)$, are the four solutions to the system. The check of the solutions is left to you.



› Figure 12



› Figure 13

MATCHED PROBLEM 4

Solve:
$$2x^2 - 3y^2 = 5$$
$$3x^2 + 4y^2 = 16$$

EXAMPLE 5 Solving a Nonlinear System Using Factoring and Substitution

Solve:
$$x^2 + 3xy + y^2 = 20$$
$$xy - y^2 = 0$$

SOLUTION

$$xy - y^2 = 0$$ Factor the left side of the equation that has a 0 constant term.
$$y(x - y) = 0$$ Use the zero product property.
$$y = 0 \quad \text{or} \quad y = x$$

Thus, the original system is equivalent to the two systems:

$$\begin{aligned} y &= 0 \\ x^2 + 3xy + y^2 &= 20 \end{aligned} \quad \text{or} \quad \begin{aligned} y &= x \\ x^2 + 3xy + y^2 &= 20 \end{aligned}$$

These systems are solved by substitution.

FIRST SYSTEM

$$\begin{aligned} y &= 0 \\ x^2 + 3xy + y^2 &= 20 \end{aligned}$$ Substitute $y = 0$ in the second equation, and solve for x.
$$x^2 + 3x(0) + (0)^2 = 20$$ Simplify.
$$x^2 = 20$$ Take the square root of both sides.
$$x = \pm\sqrt{20} = \pm 2\sqrt{5}$$
$$(2\sqrt{5}, 0) \qquad (-2\sqrt{5}, 0)$$ Solutions to the first system.

SECOND SYSTEM

$$
\begin{aligned}
y &= x && \\
x^2 + 3xy + y^2 &= 20 && \text{Substitute } y = x \text{ in the second equation and solve for } x.\\
x^2 + 3xx + x^2 &= 20 && \text{Simplify.}\\
5x^2 &= 20 && \text{Divide both sides by 5.}\\
x^2 &= 4 && \text{Take the square root of both sides.}\\
x &= \pm 2 && \text{Substitute these values back into } y = x \text{ to find } y.\\
(2,\ 2) \qquad & (-2,\ -2) && \text{Solutions to the second system.}
\end{aligned}
$$

Combining the solutions for the first system with the solutions for the second system, the solutions for the original system are $(2\sqrt{5}, 0)$, $(-2\sqrt{5}, 0)$, $(2, 2)$, and $(-2, -2)$. The check of the solutions is left to you.

MATCHED PROBLEM 5

Solve:
$$
\begin{aligned}
x^2 - xy + y^2 &= 9\\
2x^2 - xy &= 0
\end{aligned}
$$

Example 5 is somewhat specialized. However, it suggests a procedure that is effective for some problems.

EXAMPLE 6 Graphical Approximations of Real Solutions

Use a graphing calculator to approximate real solutions to two decimal places:

$$
\begin{aligned}
x^2 - 4xy + y^2 &= 12\\
2x^2 + 2xy + y^2 &= 6
\end{aligned}
$$

SOLUTION

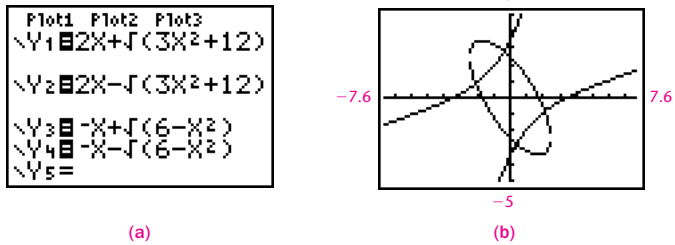
Before we can enter these equations in our calculator, we must solve for y:

$$x^2 - 4xy + y^2 = 12 \qquad 2x^2 + 2xy + y^2 = 6$$
$$y^2 - 4xy + (x^2 - 12) = 0 \qquad y^2 + 2xy + (2x^2 - 6) = 0$$
$a = 1,\ b = -4x,\ c = x^2 - 12$ $\qquad$ $a = 1,\ b = 2x,\ c = 2x^2 - 6$

Applying the quadratic formula to each equation, we have

$$y = \frac{4x \pm \sqrt{16x^2 - 4(x^2 - 12)}}{2} \qquad y = \frac{-2x \pm \sqrt{4x^2 - 4(2x^2 - 6)}}{2}$$
$$= \frac{4x \pm \sqrt{12x^2 + 48}}{2} \qquad = \frac{-2x \pm \sqrt{24 - 4x^2}}{2}$$
$$= 2x \pm \sqrt{3x^2 + 12} \qquad = -x \pm \sqrt{6 - x^2}$$

Since each equation has two solutions, we must enter four functions in the graphing calculator, as shown in Figure 14(a). Examining the graph in Figure 14(b), we see that there are four intersection points. Using the INTERSECT command repeatedly (details omitted), we find that the solutions to two decimal places are $(-2.10, 0.83)$, $(-0.37, 2.79)$, $(0.37, -2.79)$, and $(2.10, -0.83)$.



(a) (b)

› Figure 14

MATCHED PROBLEM 6

Use a graphing calculator to approximate real solutions to two decimal places:

$$x^2 + 8xy + y^2 = 70$$
$$2x^2 - 2xy + y^2 = 20$$

ANSWERS TO MATCHED PROBLEMS

1. $(-1, 3)$, $(\frac{9}{5}, -\frac{13}{5})$
2. $(\sqrt{3}, \sqrt{3})$, $(-\sqrt{3}, -\sqrt{3})$, $(i, -3i)$, $(-i, 3i)$
3. 17.1 by 12.2 in.
4. $(2, 1)$, $(2, -1)$, $(-2, 1)$, $(-2, -1)$
5. $(0, 3)$, $(0, -3)$, $(\sqrt{3}, 2\sqrt{3})$, $(-\sqrt{3}, -2\sqrt{3})$
6. $(-3.89, -1.68)$, $(-0.96, -5.32)$, $(0.96, 5.32)$, $(3.89, 1.68)$

11-5 Exercises

1. Would you choose substitution or elimination to solve the following nonlinear system? Assume a, b, c, d, e, and $f \neq 0$.

$$\begin{aligned} ax + by &= c \\ dx^2 + ey^2 &= f \end{aligned}$$

Justify your answer by describing the steps you would take to solve this system.

2. Repeat Problem 1 for the following nonlinear system.

$$\begin{aligned} ax^2 + by^2 &= c \\ dx^2 + ey^2 &= f \end{aligned}$$

Solve each system in Problems 3–14.

3. $\begin{aligned} x^2 + y^2 &= 169 \\ x &= -12 \end{aligned}$

4. $\begin{aligned} x^2 + y^2 &= 25 \\ y &= -4 \end{aligned}$

5. $\begin{aligned} 8x^2 - y^2 &= 16 \\ y &= 2x \end{aligned}$

6. $\begin{aligned} y^2 &= 2x \\ x &= y - \tfrac{1}{2} \end{aligned}$

7. $\begin{aligned} 3x^2 - 2y^2 &= 25 \\ x + y &= 0 \end{aligned}$

8. $\begin{aligned} x^2 + 4y^2 &= 32 \\ x + 2y &= 0 \end{aligned}$

9. $\begin{aligned} y^2 &= x \\ x - 2y &= 2 \end{aligned}$

10. $\begin{aligned} x^2 &= 2y \\ 3x &= y + 2 \end{aligned}$

11. $\begin{aligned} 2x^2 + y^2 &= 24 \\ x^2 - y^2 &= -12 \end{aligned}$

12. $\begin{aligned} x^2 - y^2 &= 3 \\ x^2 + y^2 &= 5 \end{aligned}$

13. $\begin{aligned} x^2 + y^2 &= 10 \\ 16x^2 + y^2 &= 25 \end{aligned}$

14. $\begin{aligned} x^2 - 2y^2 &= 1 \\ x^2 + 4y^2 &= 25 \end{aligned}$

Solve each system in Problems 15–26.

15. $\begin{aligned} xy - 4 &= 0 \\ x - y &= 2 \end{aligned}$

16. $\begin{aligned} xy - 6 &= 0 \\ x - y &= 4 \end{aligned}$

17. $\begin{aligned} x^2 + 2y^2 &= 6 \\ xy &= 2 \end{aligned}$

18. $\begin{aligned} 2x^2 + y^2 &= 18 \\ xy &= 4 \end{aligned}$

19. $\begin{aligned} 2x^2 + 3y^2 &= -4 \\ 4x^2 + 2y^2 &= 8 \end{aligned}$

20. $\begin{aligned} 2x^2 - 3y^2 &= 10 \\ x^2 + 4y^2 &= -17 \end{aligned}$

21. $\begin{aligned} x^2 - y^2 &= 2 \\ y^2 &= x \end{aligned}$

22. $\begin{aligned} x^2 + y^2 &= 20 \\ x^2 &= y \end{aligned}$

23. $\begin{aligned} x^2 + y^2 &= 9 \\ x^2 &= 9 - 2y \end{aligned}$

24. $\begin{aligned} x^2 + y^2 &= 16 \\ y^2 &= 4 - x \end{aligned}$

25. $\begin{aligned} x^2 - y^2 &= 3 \\ xy &= 2 \end{aligned}$

26. $\begin{aligned} y^2 &= 5x^2 + 1 \\ xy &= 2 \end{aligned}$

An important type of calculus problem is to find the area between the graphs of two functions. To solve some of these problems it is necessary to find the coordinates of the points of intersections of the two graphs. In Problems 27–34, find the coordinates of the points of intersections of the two given equations.

27. $y = 5 - x^2, y = 2 - 2x$

28. $y = 5x - x^2, y = x + 3$

29. $y = x^2 - x, y = 2x$

30. $y = x^2 + 2x, y = 3x$

31. $y = x^2 - 6x + 9, y = 5 - x$

32. $y = x^2 + 2x + 3, y = 2x + 4$

33. $y = 8 + 4x - x^2, y = x^2 - 2x$

34. $y = x^2 - 4x - 10, y = 14 - 2x - x^2$

35. Consider the circle with equation $x^2 + y^2 = 5$ and the family of lines given by $2x - y = b$, where b is any real number.
(A) Illustrate graphically the lines in this family that intersect the circle in exactly one point, and describe the relationship between the circle and these lines.
(B) Find the values of b corresponding to the lines in part A, and find the intersection points of the lines and the circle.
(C) How is the line with equation $x + 2y = 0$ related to this family of lines? How could this line be used to find the intersection points in part B?

36. Consider the circle with equation $x^2 + y^2 = 25$ and the family of lines given by $3x + 4y = b$, where b is any real number.
(A) Illustrate graphically the lines in this family that intersect the circle in exactly one point, and describe the relationship between the circle and these lines.
(B) Find the values of b corresponding to the lines in part A, and find the intersection points of the lines and the circle.
(C) How is the line with equation $4x - 3y = 0$ related to this family of lines? How could this line be used to find the intersection points and the values of b in part B?

Solve each system in Problems 37–44.

37. $\begin{aligned} 2x + 5y + 7xy &= 8 \\ xy - 3 &= 0 \end{aligned}$

38. $\begin{aligned} 2x + 3y + xy &= 16 \\ xy - 5 &= 0 \end{aligned}$

39. $\begin{aligned} x^2 - 2xy + y^2 &= 1 \\ x - 2y &= 2 \end{aligned}$

40. $\begin{aligned} x^2 + xy - y^2 &= -5 \\ y - x &= 3 \end{aligned}$

41. $\begin{aligned} 2x^2 - xy + y^2 &= 8 \\ x^2 - y^2 &= 0 \end{aligned}$

42. $\begin{aligned} x^2 + 2xy + y^2 &= 36 \\ x^2 - xy &= 0 \end{aligned}$

43. $\begin{aligned} x^2 + xy - 3y^2 &= 3 \\ x^2 + 4xy + 3y^2 &= 0 \end{aligned}$

44. $\begin{aligned} x^2 - 2xy + 2y^2 &= 16 \\ x^2 - y^2 &= 0 \end{aligned}$

In Problems 45–50, use a graphing calculator to approximate the real solutions of each system to two decimal places.

45. $-x^2 + 2xy + y^2 = 1$
$3x^2 - 4xy + y^2 = 2$

46. $-x^2 + 4xy + y^2 = 2$
$8x^2 - 2xy + y^2 = 9$

47. $3x^2 - 4xy - y^2 = 2$
$2x^2 + 2xy + y^2 = 9$

48. $5x^2 + 4xy + y^2 = 4$
$4x^2 - 2xy + y^2 = 16$

49. $2x^2 - 2xy + y^2 = 9$
$4x^2 - 4xy + y^2 + x = 3$

50. $2x^2 + 2xy + y^2 = 12$
$4x^2 - 4xy + y^2 + x + 2y = 9$

APPLICATIONS

51. **NUMBERS** Find two numbers such that their sum is 3 and their product is 1.

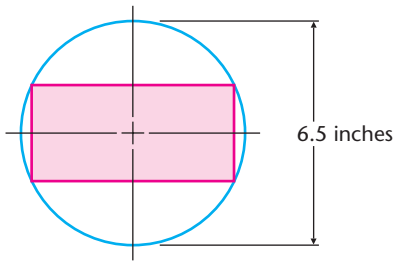
52. **NUMBERS** Find two numbers such that their difference is 1 and their product is 1. (Let x be the larger number and y the smaller number.)

53. **GEOMETRY** Find the lengths of the legs of a right triangle with an area of 30 square inches if its hypotenuse is 13 inches long.

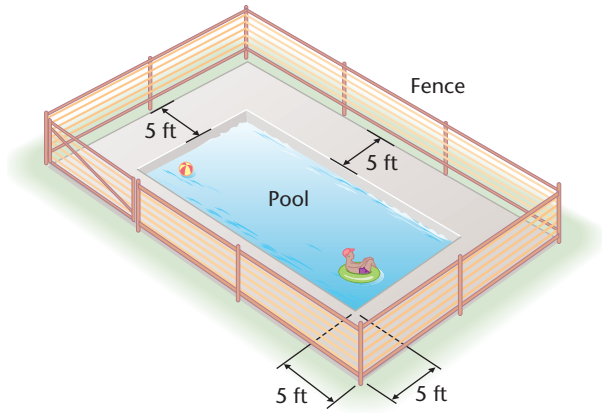
54. **GEOMETRY** Find the dimensions of a rectangle with an area of 32 square meters if its perimeter is 36 meters long.

55. **DESIGN** An engineer is designing a small portable television set. According to the design specifications, the set must have a rectangular screen with a 7.5-inch diagonal and an area of 27 square inches. Find the dimensions of the screen.

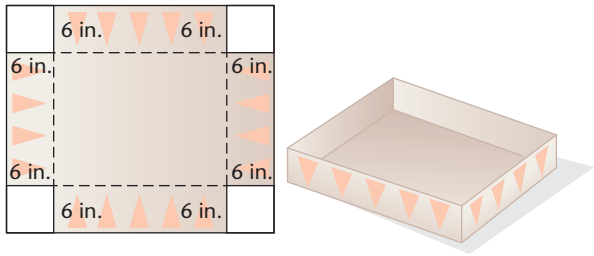
56. **DESIGN** An artist is designing a logo for a business in the shape of a circle with an inscribed rectangle. The diameter of the circle is 6.5 inches, and the area of the rectangle is 15 square inches. Find the dimensions of the rectangle.



57. **CONSTRUCTION** A rectangular swimming pool with a deck 5 feet wide is enclosed by a fence as shown in the figure. The surface area of the pool is 572 square feet, and the total area enclosed by the fence (including the pool and the deck) is 1,152 square feet. Find the dimensions of the pool.



58. **CONSTRUCTION** An open-topped rectangular box is formed by cutting a 6-inch square from each corner of a rectangular piece of cardboard and bending up the ends and sides. The area of the cardboard before the corners are removed is 768 square inches, and the volume of the box is 1,440 cubic inches. Find the dimensions of the original piece of cardboard.



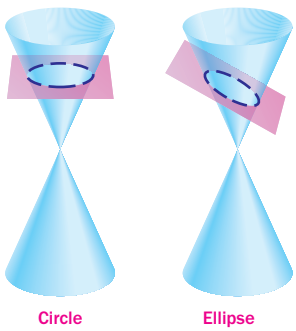
59. **TRANSPORTATION** Two boats leave Bournemouth, England, at the same time and follow the same route on the 75-mile trip across the English Channel to Cherbourg, France. The average speed of boat A is 5 miles per hour greater than the average speed of boat B. Consequently, boat A arrives at Cherbourg 30 minutes before boat B. Find the average speed of each boat.

60. **TRANSPORTATION** Bus A leaves Milwaukee at noon and travels west on Interstate 94. Bus B leaves Milwaukee 30 minutes later, travels the same route, and overtakes bus A at a point 210 miles west of Milwaukee. If the average speed of bus B is 10 miles per hour greater than the average speed of bus A, at what time did bus B overtake bus A?

CHAPTER 11 Review

11-1 Conic Sections; Parabola

The plane curves obtained by intersecting a right circular cone with a plane are called **conic sections.** If the plane cuts clear through one nappe, then the intersection curve is called a **circle** if the plane is perpendicular to the axis and an **ellipse** if the plane is not perpendicular to the axis. If a plane cuts only one nappe, but does not cut clear through, then the intersection curve is called a **parabola.** If a plane cuts through both nappes, but not through the vertex, the resulting intersection curve is called a **hyperbola.** A plane passing through the vertex of the cone produces a **degenerate conic**—a point, a line, or a pair of lines. The figure illustrates the four nondegenerate conics.



The graph of

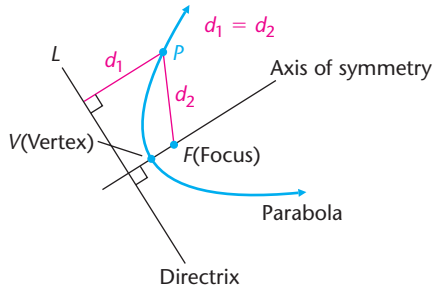
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is a conic, a degenerate conic, or the empty set.

The following is a coordinate-free definition of a parabola:

Parabola

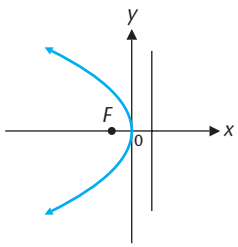
A **parabola** is the set of all points in a plane equidistant from a fixed point F and a fixed line L (not containing F) in the plane. The fixed point F is called the **focus,** and the fixed line L is called the **directrix.** A line through the focus perpendicular to the directrix is called the **axis of symmetry,** and the point on the axis halfway between the directrix and focus is called the **vertex.**



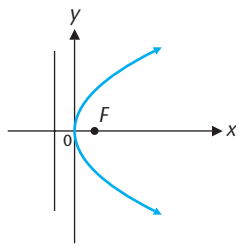
From the definition of a parabola, we can obtain the following standard equations:

Standard Equations of a Parabola with Vertex at (0, 0)

1. $y^2 = 4ax$
Vertex (0, 0)
Focus: $(a, 0)$
Directrix: $x = -a$
Symmetric with respect to the x axis
Axis of symmetry the x axis

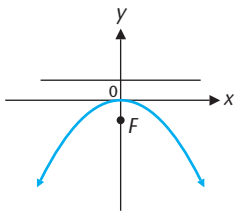


$a < 0$ (opens left)

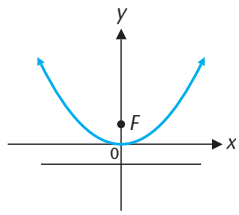


$a > 0$ (opens right)

2. $x^2 = 4ay$
Vertex: (0, 0)
Focus: $(0, a)$
Directrix: $y = -a$
Symmetric with respect to the y axis
Axis of symmetry the y axis



$a < 0$ (opens down)



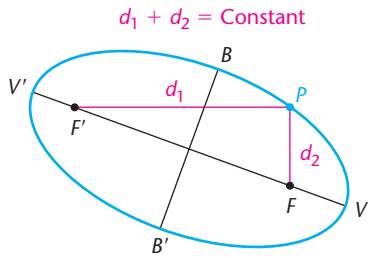
$a > 0$ (opens up)

11-2 Ellipse

The following is a coordinate-free definition of an ellipse:

Ellipse

An **ellipse** is the set of all points P in a plane such that the sum of the distances from P to two fixed points in the plane is a constant. (The constant is required to be greater than the distance between the two fixed points.) Each of the fixed points, F' and F, is called a **focus,** and together they are called **foci.** Referring to the figure, the line segment $V'V$ through the foci is the **major axis.** The perpendicular bisector $B'B$ of the major axis is the **minor axis.** Each end of the major axis, V' and V, is called a **vertex.** The midpoint of the line segment $F'F$ is called the **center** of the ellipse.



From the definition of an ellipse, we can obtain the following standard equations:

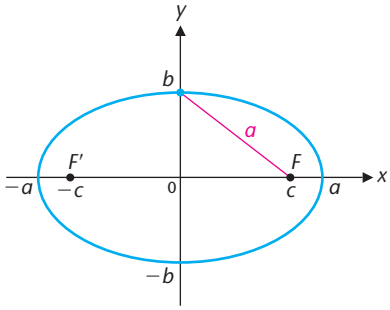
Standard Equations of an Ellipse with Center at (0, 0)

1. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \qquad a > b > 0$

x intercepts: $\pm a$ (vertices)
y intercepts: $\pm b$
Foci: $F' = (-c, 0)$, $F = (c, 0)$

$$c^2 = a^2 - b^2$$

Major axis length $= 2a$
Minor axis length $= 2b$

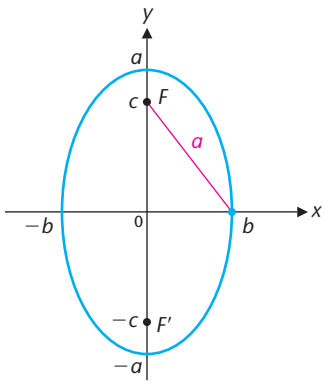


2. $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1 \qquad a > b > 0$

x intercepts: $\pm b$
y intercepts: $\pm a$ (vertices)
Foci: $F' = (0, -c)$, $F = (0, c)$

$$c^2 = a^2 - b^2$$

Major axis length $= 2a$
Minor axis length $= 2b$



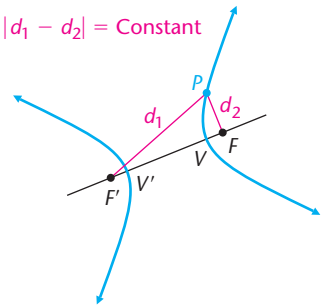
[*Note:* Both graphs are symmetric with respect to the x axis, y axis, and origin. Also, the major axis is always longer than the minor axis.]

11-3 Hyperbola

The following is a coordinate-free definition of a hyperbola:

Hyperbola

A **hyperbola** is the set of all points P in a plane such that the absolute value of the difference of the distances from P to two fixed points in the plane is a positive constant. (The constant is required to be less than the distance between the two fixed points.) Each of the fixed points, F' and F, is called a **focus.** The intersection points V' and V of the line through the foci and the two branches of the hyperbola are called **vertices,** and each is called a **vertex.** The line segment $V'V$ is called the **transverse axis.** The midpoint of the transverse axis is the **center** of the hyperbola.



From the definition of a hyperbola, we can obtain the following standard equations:

Standard Equations of a Hyperbola with Center at (0, 0)

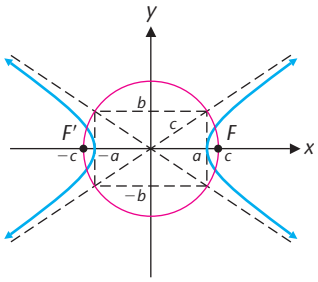
1. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

x intercepts: $\pm a$ (vertices)
y intercepts: none
Foci: $F' = (-c, 0)$, $F = (c, 0)$

$$c^2 = a^2 + b^2$$

Transverse axis length $= 2a$
Conjugate axis length $= 2b$

Asymptotes: $y = \pm\dfrac{b}{a}x$



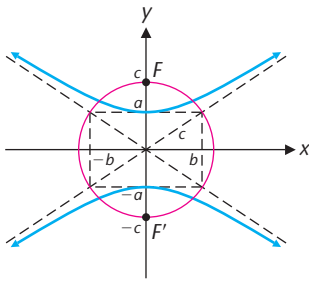
2. $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$

x intercepts: none
y intercepts: $\pm a$ (vertices)
Foci: $F' = (0, -c)$, $F = (0, c)$

$$c^2 = a^2 + b^2$$

Transverse axis length $= 2a$
Conjugate axis length $= 2b$

Asymptotes: $y = \pm\dfrac{a}{b}x$



[*Note:* Both graphs are symmetric with respect to the x axis, y axis, and origin.]

11-4 Translation and Rotation of Axes

In Sections 11-1, 11-2, and 11-3 we found standard equations for parabolas, ellipses, and hyperbolas located with their axes on the coordinate axes and centered relative to the origin. We now move the conics away from the origin while keeping their axes parallel to the coordinate axes. In this process we obtain new standard equations that are special cases of the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$, where A and C are not both zero. The basic mathematical tool used is *translation of axes.*

A **translation of coordinate axes** occurs when the new coordinate axes have the same direction as, and are parallel to, the original coordinate axes. **Translation formulas** are as follows:

1. $x = x' + h$ **2.** $x' = x - h$
$y = y' + k$ $y' = y - k$

where (h, k) are the coordinates of the origin $0'$ relative to the original system.

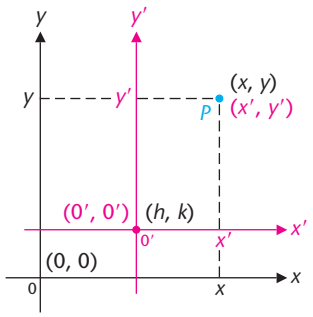


Table 1 on page 1034 lists the standard equations for conics.

If the xy coordinate axes are rotated counterclockwise through an angle θ into the $x'y'$ coordinate axes, then the xy and $x'y'$ coordinate systems are related by the **rotation formulas:**

1. $x = x' \cos\theta - y' \sin\theta$ **2.** $x' = x \cos\theta + y \sin\theta$
$y = x' \sin\theta + y' \cos\theta$ $y' = -x \sin\theta + y \cos\theta$

To transform the general quadratic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

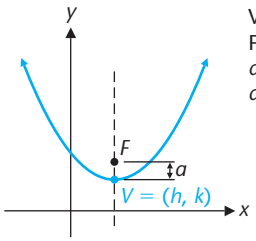
into an equation in x' and y' with no $x'y'$ term, choose the *angle of rotation* θ to satisfy $\cot 2\theta = (A - C)/B$ and $0° < \theta < 90°$. The **discriminant** of the general second-degree equation in two variables is $B^2 - 4AC$ and the graph is

1. A hyperbola if $B^2 - 4AC > 0$
2. A parabola if $B^2 - 4AC = 0$
3. An ellipse if $B^2 - 4AC < 0$

Table 1 Standard Equations for Conics

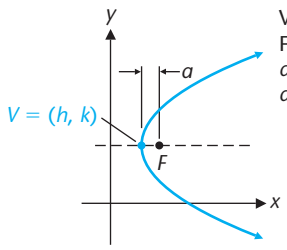
Parabolas

$(x - h)^2 = 4a(y - k)$



Vertex (h, k)
Focus $(h, k + a)$
$a > 0$ opens up
$a < 0$ opens down

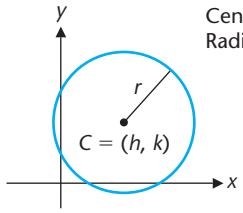
$(y - k)^2 = 4a(x - h)$



Vertex (h, k)
Focus $(h + a, k)$
$a < 0$ opens left
$a > 0$ opens right

Circles

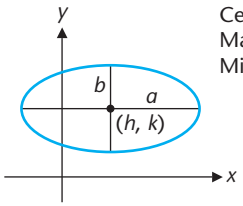
$(x - h)^2 + (y - k)^2 = r^2$



Center (h, k)
Radius r

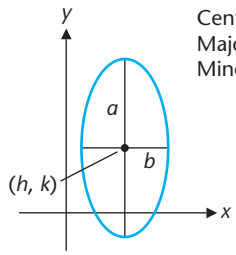
Ellipses

$\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$ $\quad a > b > 0$



Center (h, k)
Major axis $2a$
Minor axis $2b$

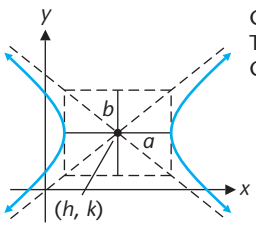
$\dfrac{(x - h)^2}{b^2} + \dfrac{(y - k)^2}{a^2} = 1$



Center (h, k)
Major axis $2a$
Minor axis $2b$

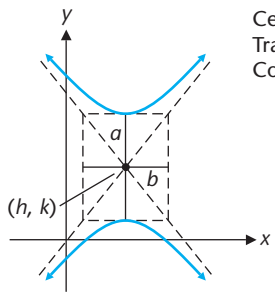
Hyperbolas

$\dfrac{(x - h)^2}{a^2} - \dfrac{(y - k)^2}{b^2} = 1$



Center (h, k)
Transverse axis $2a$
Conjugate axis $2b$

$\dfrac{(y - k)^2}{a^2} - \dfrac{(x - h)^2}{b^2} = 1$



Center (h, k)
Transverse axis $2a$
Conjugate axis $2b$

11-5 Systems of Nonlinear Equations

If a system of equations contains any equations that are not linear, then the system is called a **nonlinear system.** In this section we investigated nonlinear systems involving second-degree terms such as

$$\begin{array}{lll} x^2 + y^2 = 5 & x^2 - 2y^2 = 2 & x^2 + 3xy + y^2 = 20 \\ 3x + y = 1 & xy = 2 & xy - y^2 = 0 \end{array}$$

It can be shown that such systems have at most four solutions, some of which may be imaginary.

Several methods were used to solve nonlinear systems of the indicated form: *solution by substitution, solution using elimination by addition,* and *solution using factoring and substitution.* It is always important to check the solutions of any nonlinear system to ensure that **extraneous roots** have not been introduced.

CHAPTER 11 Review Exercises

Work through all the problems in this chapter review and check answers in the back of the book. Answers to all review problems are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

In Problems 1–6, graph each equation and locate foci. Locate the directrix for any parabolas. Find the lengths of major, minor, transverse, and conjugate axes where applicable.

1. $9x^2 + 25y^2 = 225$

2. $x^2 = -12y$

3. $25y^2 - 9x^2 = 225$

4. $x^2 - y^2 = 16$

5. $y^2 = 8x$

6. $2x^2 + y^2 = 8$

In Problems 7–9:

(A) Write each equation in one of the standard forms listed in Table 1 of the review.

(B) Identify the curve.

7. $4(y + 2)^2 - 25(x - 4)^2 = 100$

8. $(x + 5)^2 + 12(y + 4) = 0$

9. $16(x - 6)^2 + 9(y - 4)^2 = 144$

10. Find the $x'y'$ coordinates of the point (3, 4) when the axes are rotated through

(A) 30° (B) 45° (C) 60°

11. Find the equations of the x' and y' axes in terms of x and y if the axes are rotated through an angle of 75°.

In Problems 12–14, solve the system.

12. $\begin{array}{l} y = x^2 - 5x - 3 \\ y = -x + 2 \end{array}$

13. $\begin{array}{l} x^2 + y^2 = 2 \\ 2x - y = 3 \end{array}$

14. $\begin{array}{l} 3x^2 - y^2 = -6 \\ 2x^2 + 3y^2 = 29 \end{array}$

15. Find the equation of the parabola having its vertex at the origin, its axis of symmetry the x axis, and $(-4, -2)$ on its graph.

In Problems 16 and 17, find the equation of the ellipse in the form

$$\frac{x^2}{M} + \frac{y^2}{N} = 1 \qquad M, N > 0$$

if the center is at the origin, and:

16. Major axis on x axis
Major axis length = 12
Minor axis length = 10

17. Major axis on y axis
Minor axis length = 12
Distance between foci = 16

In Problems 18 and 19, find the equation of the hyperbola in the form

$$\frac{x^2}{M} - \frac{y^2}{N} = 1 \text{ or } \frac{y^2}{M} - \frac{x^2}{N} = 1 \qquad M, N > 0$$

if the center is at the origin, and:

18. Transverse axis on y axis
Conjugate axis length = 6
Distance between foci = 8

19. Transverse axis on x axis
Transverse axis length = 14
Conjugate axis length = 16

In Problems 20–25, solve the system.

20. $\begin{array}{l} x^2 + 4y^2 = 32 \\ x + 2y = 0 \end{array}$

21. $\begin{array}{l} 16x^2 + 25y^2 = 400 \\ 16x^2 - 45y = 0 \end{array}$

22. $\begin{array}{l} x^2 + y^2 = 10 \\ 16x^2 + y^2 = 25 \end{array}$

23. $\begin{array}{l} x^2 - y^2 = 2 \\ y^2 = x \end{array}$

24. $\begin{array}{l} x^2 + 2xy + y^2 = 1 \\ xy = -2 \end{array}$

25. $\begin{array}{l} 2x^2 + xy + y^2 = 8 \\ x^2 - y^2 = 0 \end{array}$

26. Find the equation of the parabola having directrix $y = 5$ and focus $(0, -5)$.

27. Find the foci of the ellipse through the point $(-6, 0)$ if the center is at the origin, the major axis is on the x axis, and the major axis has twice the length of the minor axis.

28. Find the y intercepts of a hyperbola if the center is at the origin, the conjugate axis is on the x axis and has length 4, and $(0, -3)$ is a focus.

29. Find the directrix of a parabola having its vertex at the origin and focus $(-4, 0)$.

30. Find the points of intersection of the parabolas $x^2 = 8y$ and $y^2 = -x$.

31. Find the x intercepts of an ellipse if the center is at the origin, the major axis is on the y axis and has length 14, and $(0, -1)$ is a focus.

32. Find the foci of the hyperbola through the point $(0, -4)$ if the center is at the origin, the transverse axis is on the y axis, and the conjugate axis has twice the length of the transverse axis.

In Problems 33–35, transform each equation into one of the standard forms in Table 1 in the review. Identify the curve and graph it.

33. $16x^2 + 4y^2 + 96x - 16y + 96 = 0$

34. $x^2 - 4x - 8y - 20 = 0$

35. $4x^2 - 9y^2 + 24x - 36y - 36 = 0$

36. Given the equation $x^2 - \sqrt{3}xy + 2y^2 - 10 = 0$, find the transformed equation when the axes are rotated through 30°. Sketch and identify the graph.

37. Given the equation $5x^2 + 26xy + 5y^2 + 72 = 0$, find the angle of rotation so that the transformed equation will have no $x'y'$ term. Sketch and identify the graph.

38. Given the equation $3x^2 + 4xy + 2y^2 - 20 = 0$, identify the curve.

39. Use the definition of a parabola and the distance formula to find the equation of a parabola with directrix $x = 6$ and focus at $(2, 4)$.

40. Find an equation of the set of points in a plane each of whose distance from $(4, 0)$ is twice its distance from the line $x = 1$. Identify the geometric figure.

41. Find an equation of the set of points in a plane each of whose distance from $(4, 0)$ is two-thirds its distance from the line $x = 9$. Identify the geometric figure.

In Problems 42–44, find the coordinates of any foci relative to the original coordinate system.

42. Problem 33 **43.** Problem 34 **44.** Problem 35

In Problems 45–47, find the equations of the asymptotes of each hyperbola.

45. $\dfrac{x^2}{49} - \dfrac{y^2}{25} = 1$ **46.** $\dfrac{y^2}{64} - \dfrac{x^2}{4} = 1$ **47.** $4x^2 - y^2 = 1$

APPLICATIONS

48. **COMMUNICATIONS** A parabolic satellite television antenna has a diameter of 8 feet and is 1 foot deep. How far is the focus from the vertex?

49. **ENGINEERING** An elliptical gear is to have foci 8 centimeters apart and a major axis 10 centimeters long. Letting the x axis lie along the major axis (right positive) and the y axis lie along the minor axis (up positive), write the equation of the ellipse in the standard form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

50. **SPACE SCIENCE** A hyperbolic reflector for a radio telescope (such as that illustrated in Problem 59, Exercises 11-3) has the equation

$$\frac{y^2}{40^2} - \frac{x^2}{30^2} = 1$$

If the reflector has a diameter of 30 feet, how deep is it? Compute the answer to three significant digits.

CHAPTER 11

››› GROUP ACTIVITY Focal Chords

Many of the applications of the conic sections are based on their reflective or focal properties. One of the interesting algebraic properties of the conic sections concerns their focal chords.

If a line through a focus F contains two points G and H of a conic section, then the line segment GH is called a **focal chord.** Let $G = (x_1, y_1)$ and $H = (x_2, y_2)$ be points on the graph of $x^2 = 4ay$ such that GH is a focal chord. Let u denote the length of GF and v the length of FH (Fig. 1).

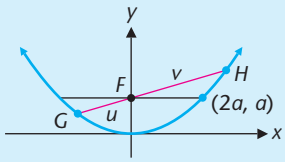


Figure 1 Focal chord GH of the parabola $x^2 = 4ay$.

(A) Use the distance formula to show that $u = y_1 + a$.
(B) Show that G and H lie on the line $y - a = mx$, where $m = (y_2 - y_1)/(x_2 - x_1)$.
(C) Solve $y - a = mx$ for x and substitute in $x^2 = 4ay$, obtaining a quadratic equation in y. Explain why $y_1y_2 = a^2$.
(D) Show that $\frac{1}{u} + \frac{1}{v} = \frac{1}{a}$.
(E) Show that $u + v - 4a = \frac{(u - 2a)^2}{u - a}$. Explain why this implies that $u + v \geq 4a$, with equality if and only if $u = v = 2a$.
(F) Which focal chord is the shortest? Is there a longest focal chord?
(G) Is $\frac{1}{u} + \frac{1}{v}$ a constant for focal chords of the ellipse? For focal chords of the hyperbola? Obtain evidence for your answers by considering specific examples.

CHAPTERS 10–11 Cumulative Review

Work through all the problems in this cumulative review and check answers in the back of the book. Answers to all review problems are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

1. Determine whether each of the following can be the first three terms of an arithmetic sequence, a geometric sequence, or neither.

(A) 20, 15, 10, . . . (B) 5, 25, 125, . . . (C) 5, 25, 50, . . .
(D) 27, −9, 3, . . . (E) −9, −6, −3, . . .

In Problems 2–4:
(A) Write the first four terms of each sequence.
(B) Find a_8. (C) Find S_8.

2. $a_n = 2 \cdot 5^n$ **3.** $a_n = 3n - 1$

4. $a_1 = 100$; $a_n = a_{n-1} - 6$, $n \geq 2$

5. Evaluate each of the following:

(A) 8! (B) $\frac{32!}{30!}$ (C) $\frac{9!}{3!(9 - 3)!}$

6. Evaluate each of the following:

(A) $\binom{7}{2}$ (B) $C_{7,2}$ (C) $P_{7,2}$

In Problems 7–9, graph each equation and locate foci. Locate the directrix for any parabolas. Find the lengths of major, minor, transverse, and conjugate axes where applicable.

7. $25x^2 - 36y^2 = 900$ **8.** $25x^2 + 36y^2 = 900$

9. $25x^2 - 36y = 0$

10. Solve
$$x^2 + y^2 = 2$$
$$2x - y = 1$$

11. What type of curve is the graph of
$$3x^2 - 4xy + 2y^2 - 7 = 0$$

12. A coin is flipped three times. How many combined outcomes are possible? Solve

(A) By using a tree diagram

(B) By using the multiplication principle

13. How many ways can four distinct books be arranged on a shelf? Solve

(A) By using the multiplication principle

(B) By using permutations or combinations, whichever is applicable

14. In a single deal of 3 cards from a standard 52-card deck, what is the probability of being dealt three diamonds?

15. Each of the 10 digits 0 through 9 is printed on 1 of 10 different cards. Four of these cards are drawn in succession without replacement. What is the probability of drawing the digits 4, 5, 6, and 7 by drawing 4 on the first draw, 5 on the second draw, 6 on the third draw, and 7 on the fourth draw? What is the probability of drawing the digits 4, 5, 6, and 7 in any order?

16. A thumbtack lands point down in 38 out of 100 tosses. What is the approximate empirical probability of the tack landing point up?

Verify Problems 17 and 18 for $n = 1$, 2, and 3.

17. P_n: $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$

18. P_n: $n^2 + n + 2$ is divisible by 2

In Problems 19 and 20, write P_k and P_{k+1}.

19. For P_n in Problem 17 **20.** For P_n in Problem 18

21. Find the equation of the parabola having its vertex at the origin, its axis the y axis, and $(2, -8)$ on its graph.

22. Find an equation of an ellipse in the form
$$\frac{x^2}{M} + \frac{y^2}{N} = 1 \qquad M, N > 0$$
if the center is at the origin, the major axis is the x axis, the major axis length is 10, and the distance of the foci from the center is 3.

23. Find an equation of a hyperbola in the form
$$\frac{x^2}{M} - \frac{y^2}{N} = 1 \qquad M, N > 0$$
if the center is at the origin, the transverse axis length is 16, and the distance of the foci from the center is $\sqrt{89}$.

In Problems 24 and 25, find the angle of rotation so that the transformed equation will have no $x'y'$ term. Identify the curve and graph it.

24. $2\sqrt{3}xy + 2y^2 + 3 = 0$

25. $x^2 + 2xy + y^2 + 4\sqrt{2}x - 4\sqrt{2}y = 0$

In Problems 26 and 27, solve the system.

26.
$$x^2 - 3xy + 3y^2 = 1$$
$$xy = 1$$

27.
$$x^2 - 3xy + y^2 = -1$$
$$x^2 - xy = 0$$

28. Find all real solutions to two decimal places
$$x^2 + 2xy - y^2 = 1$$
$$9x^2 + 4xy + y^2 = 15$$

29. Write $\sum_{k=1}^{5} k^k$ without summation notation and find the sum.

30. Write the series $\frac{2}{2!} - \frac{2^2}{3!} + \frac{2^3}{4!} - \frac{2^4}{5!} + \frac{2^5}{6!} - \frac{2^6}{7!}$ using summation notation with the summation index k starting at $k = 1$.

31. Find S_∞ for the geometric series $108 - 36 + 12 - 4 + \cdots$

32. How many four-letter code words are possible using the first six letters of the alphabet if no letter can be repeated? If letters can be repeated? If adjacent letters cannot be alike?

33. A basketball team with 12 members has two centers. If 5 players are selected at random, what is the probability that both centers are selected? Express the answer in terms of $C_{n,r}$ or $P_{n,r}$, as appropriate, and evaluate.

34. A single die is rolled 1,000 times with the frequencies of outcomes shown in the table.

(A) What is the approximate empirical probability that the number of dots showing is divisible by 3?

(B) What is the theoretical probability that the number of dots showing is divisible by 3?

Number of dots facing up	1	2	3	4	5	6
Frequency	160	155	195	180	140	170

35. Let $a_n = 100(0.9)^n$ and $b_n = 10 + 0.03n$. Find the least positive integer n such that $a_n < b_n$ by graphing the sequences $\{a_n\}$ and $\{b_n\}$ with a graphing calculator. Check your answer by using a graphing calculator to display both sequences in table form.

36. Evaluate each of the following:

(A) $P_{25,5}$ (B) $C(25, 5)$ (C) $\binom{25}{20}$

37. Expand $(a + \frac{1}{2}b)^6$ using the binomial formula.

38. Find the fifth and the eighth terms in the expansion of $(3x - y)^{10}$.

Establish each statement in Problems 39 and 40 for all positive integers using mathematical induction.

39. P_n in Problem 15 **40.** P_n in Problem 16

41. Find the sum of all the odd integers between 50 and 500.

42. Use the formula for the sum of an infinite geometric series to write $2.\overline{45} = 2.454\,545\cdots$ as the quotient of two integers.

43. Let $a_k = \binom{30}{k}(0.1)^{30-k}(0.9)^k$ for $k = 0, 1, \ldots, 30$. Use a graphing calculator to find the largest term of the sequence $\{a_k\}$ and the number of terms that are greater than 0.01.

In Problems 44–46, use a translation of coordinates to transform each equation into a standard equation for a nondegenerate conic. Identify the curve and graph it.

44. $4x + 4y - y^2 + 8 = 0$

45. $x^2 + 2x - 4y^2 - 16y + 1 = 0$

46. $4x^2 - 16x + 9y^2 + 54y + 61 = 0$

47. How many nine-digit zip codes are possible? How many of these have no repeated digits?

48. Use mathematical induction to prove that the following statement holds for all positive integers:

$$P_n: \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

49. Three-digit numbers are randomly formed from the digits 1, 2, 3, 4, and 5. What is the probability of forming an even number if digits cannot be repeated? If digits can be repeated?

50. Use the binomial formula to expand $(x - 2i)^6$, where i is the imaginary unit.

51. Use the definition of a parabola and the distance formula to find the equation of a parabola with directrix $y = 3$ and focus $(6, 1)$.

52. An ellipse has vertices $(\pm 4, 0)$ and foci $(\pm 2, 0)$. Find the y intercepts.

53. A hyperbola has vertices $(2, \pm 3)$ and foci $(2, \pm 5)$. Find the length of the conjugate axis.

54. Seven distinct points are selected on the circumference of a circle. How many triangles can be formed using these seven points as vertices?

55. Use mathematical induction to prove that $2^n < n!$ for all integers $n > 3$.

56. Use mathematical induction to show that $\{a_n\} = \{b_n\}$, where $a_1 = 3$, $a_n = 2a_{n-1} - 1$ for $n > 1$, and $b_n = 2^n + 1$, $n \geq 1$.

57. Find an equation of the set of points in the plane each of whose distance from $(1, 4)$ is three times its distance from the x axis. Write the equation in the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$, and identify the curve.

58. A box of 12 lightbulbs contains 4 defective bulbs. If three bulbs are selected at random, what is the probability of selecting at least one defective bulb?

APPLICATIONS

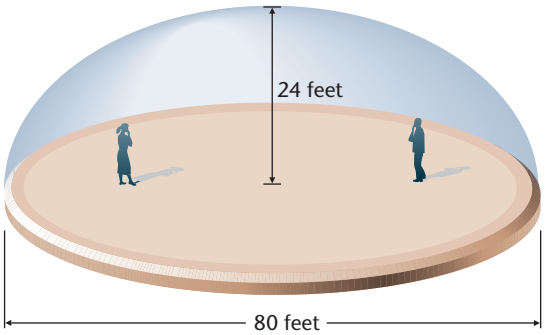
59. **ECONOMICS** The government, through a subsidy program, distributes $2,000,000. If we assume that each individual or agency spends 75% of what it receives, and 75% of this is spent, and so on, how much total increase in spending results from this government action?

60. **GEOMETRY** Find the dimensions of a rectangle with perimeter 24 meters and area 32 square meters.

61. **ENGINEERING** An automobile headlight contains a parabolic reflector with a diameter of 8 inches. If the light source is located at the focus, which is 1 inch from the vertex, how deep is the reflector?

62. **ARCHITECTURE** A sound whispered at one focus of a whispering chamber can be easily heard at the other focus. Suppose that a cross section of this chamber is a semielliptical arch

which is 80 feet wide and 24 feet high (see the figure). How far is each focus from the center of the arch? How high is the arch above each focus?



63. POLITICAL SCIENCE A random survey of 1,000 residents in a state produced the following results:

	Party affiliation			
Age	Democrat	Republican	Independent	Totals
Under 30	130	80	40	250
30–39	120	90	20	230
40–49	70	80	20	170
50–59	50	60	10	120
Over 59	90	110	30	230
Totals	460	420	120	1,000

Find the empirical probability that a person selected at random:

(A) Is under 30 *and* a Democrat

(B) Is under 40 *and* a Republican

(C) Is over 59 *or* is an Independent

Basic Algebra Review

APPENDIX

A

OUTLINE

A-1 Algebra and Real Numbers
A-2 Exponents
A-3 Radicals
A-4 Polynomials: Basic Operations
A-5 Polynomials: Factoring
A-6 Rational Expressions: Basic Operations

A-1 Algebra and Real Numbers

- Sets
- The Set of Real Numbers
- The Real Number Line
- Addition and Multiplication of Real Numbers
- Further Operations and Properties

The numbers 14, -3, 0, $\frac{7}{5}$, $\sqrt{2}$, and $\sqrt[3]{6}$ are all examples of *real numbers*. Because the symbols we use in algebra often stand for real numbers, we will discuss important properties of the real number system. We first introduce some useful notions about sets.

Sets

Georg Cantor (1845–1918) developed a theory of sets as an outgrowth of his studies on infinity. His work has become a milestone in the development of mathematics.

Our use of the word "set" will not differ appreciably from the way it is used in everyday language. Words such as "set," "collection," "bunch," and "flock" all convey the same idea. Thus, we think of a **set** as a collection of objects with the important property that we can tell whether any given object is or is not in the set.

Each object in a set is called an **element,** or **member,** of the set. Symbolically,

$a \in A$	means	"a is an element of set A"	$3 \in \{1, 3, 5\}$
$a \notin A$	means	"a is not an element of set A"	$2 \notin \{1, 3, 5\}$

Capital letters are often used to represent sets and lowercase letters to represent elements of a set.

A set is **finite** if the number of elements in the set can be counted and **infinite** if there is no end in counting its elements. A set is **empty** if it contains no elements. The empty set is also called the **null** set and is denoted by $\varnothing$. It is important to observe that the empty set is *not* written as $\{\varnothing\}$.

A set is usually described in one of two ways—by **listing** the elements between braces, $\{\ \}$, or by enclosing within braces a **rule** that determines its elements. For example, if D is the set of all numbers x such that $x^2 = 4$, then using the listing method we write

$$D = \{-2, 2\} \quad \text{Listing method}$$

or, using the rule method we write

$$D = \{x \mid x^2 = 4\} \quad \text{Rule method}$$

The notation used in the rule method is sometimes called **set-builder notation;** the vertical bar $\mid$ represents "such that," and $\{x \mid x^2 = 4\}$ is read, "The set of all x such that $x^2 = 4$."

The letter x introduced in the rule method is a *variable.* In general, a **variable** is a symbol that is used as a placeholder for the elements of a set with two or more elements. This set is called the **replacement set** for the variable. A **constant,** on the other hand, is a symbol that names exactly one object. The symbol "8" is a constant, since it always names the number eight.

If each element of set A is also an element of set B, we say that A is a **subset** of set B, and we write

$$A \subset B \qquad \{1, 5\} \subset \{1, 3, 5\}$$

Note that the definition of a subset allows a set to be a subset of itself.

Since the empty set $\varnothing$ has no elements, every element of $\varnothing$ is also an element of any given set. Therefore, the empty set is a subset of every set. For example,

$$\varnothing \subset \{1, 3, 5\} \qquad \text{and} \qquad \varnothing \subset \{2, 4, 6\}$$

If two sets A and B have exactly the same elements, the sets are said to be **equal,** and we write

$$A = B \qquad \{4, 2, 6\} = \{6, 4, 2\}$$

Notice that the order of listing elements in a set does not matter.

We can now begin our discussion of the real number system. Additional set concepts will be introduced as needed.

The Set of Real Numbers

The real number system is the number system you have used most of your life. Informally, a **real number** is any number that has a decimal representation. Table 1 describes the set of real numbers and some of its important subsets. Figure 1 on page A-4 illustrates how these sets of numbers are related to each other.

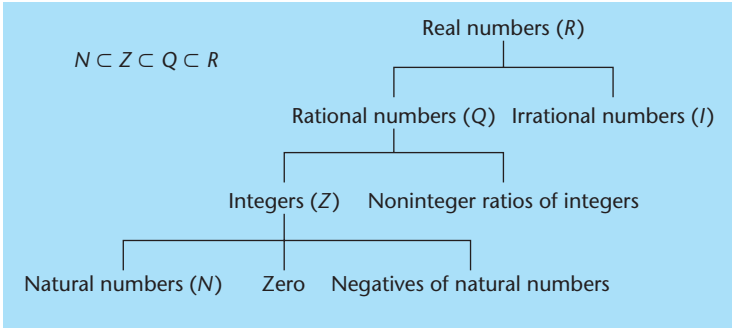
Table 1 The Set of Real Numbers

Symbol	Name	Description	Examples
N	Natural numbers	Counting numbers (also called positive integers)	1, 2, 3, . . .
Z	Integers	Natural numbers, their negatives, and 0 (also called whole numbers)	. . . , −2, −1, 0, 1, 2, . . .
Q	Rational numbers	Numbers that can be represented as a/b, where a and b are integers and $b \neq 0$; decimal representations are repeating or terminating	$-4, 0, 1, 25, \frac{-3}{5}, \frac{2}{3}, 3.67, -0.33\overline{3}$,* $5.272\overline{727}$
I	Irrational numbers	Numbers that can be represented as nonrepeating and nonterminating decimal numbers	$\sqrt{2}, \pi, \sqrt[3]{7}$, 1.414213 . . . ,† 2.71828182 . . .†
R	Real numbers	Rational numbers and irrational numbers	

*The overbar indicates that the number (or block of numbers) repeats indefinitely.

†Note that the ellipsis does *not* indicate that a number (or block of numbers) repeats indefinitely.

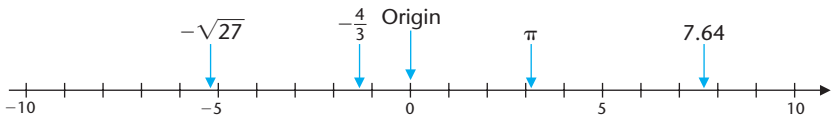
› **Figure 1** Real numbers and important subsets.



› The Real Number Line

A one-to-one correspondence exists between the set of real numbers and the set of points on a line. That is, each real number corresponds to exactly one point, and each point to exactly one real number. A line with a real number associated with each point, and vice versa, as in Figure 2, is called a **real number line,** or simply a **real line.** Each number associated with a point is called the **coordinate** of the point. The point with coordinate 0 is called the **origin.** The arrow on the right end of the line indicates a positive direction. The coordinates of all points to the right of the origin are called **positive real numbers,** and those to the left of the origin are called **negative real numbers.** The real number 0 is neither positive nor negative.

› **Figure 2** A real number line.



› Addition and Multiplication of Real Numbers

How do you add or multiply two real numbers that have nonrepeating and nonterminating decimal expansions? The answer to this difficult question relies on a solid understanding of the arithmetic of rational numbers. The **rational numbers** are numbers that can be written in the form a/b, where a and b are integers and $b \neq 0$ (see Table 1 on page A-3). The numbers $7/5$ and $-2/3$ are rational, and any integer a is rational because it can be written in the form $a/1$. Two rational numbers a/b and c/d are **equal** if $ad = bc$; for example, $35/10 = 7/2$. Recall how the sum and product of rational numbers are defined.

› DEFINITION 1 Addition and Multiplication of Rationals

For rational numbers a/b and c/d, where a, b, c, and d are integers and $b \neq 0$, $d \neq 0$:

Addition: $$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Multiplication: $$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Addition and multiplication of rational numbers are **commutative;** changing the order in which two numbers are added or multiplied does not change the result.

$$\frac{3}{2}+\frac{5}{7}=\frac{5}{7}+\frac{3}{2} \qquad \text{Addition is commutative.}$$

$$\frac{3}{2}\cdot\frac{5}{7}=\frac{5}{7}\cdot\frac{3}{2} \qquad \text{Multiplication is commutative.}$$

Addition and multiplication of rational numbers is also **associative;** changing the grouping of three numbers that are added or multiplied does not change the result:

$$\frac{3}{2}+\left(\frac{5}{7}+\frac{9}{4}\right)=\left(\frac{3}{2}+\frac{5}{7}\right)+\frac{9}{4} \qquad \text{Addition is associative.}$$

$$\frac{3}{2}\cdot\left(\frac{5}{7}\cdot\frac{9}{4}\right)=\left(\frac{3}{2}\cdot\frac{5}{7}\right)\cdot\frac{9}{4} \qquad \text{Multiplication is associative.}$$

Furthermore, the operations of addition and multiplication are related in that multiplication **distributes** over addition:

$$\frac{3}{2}\cdot\left(\frac{5}{7}+\frac{9}{4}\right)=\frac{3}{2}\cdot\frac{5}{7}+\frac{3}{2}\cdot\frac{9}{4} \qquad \text{Left distributive law}$$

$$\left(\frac{5}{7}+\frac{9}{4}\right)\cdot\frac{3}{2}=\frac{5}{7}\cdot\frac{3}{2}+\frac{9}{4}\cdot\frac{3}{2} \qquad \text{Right distributive law}$$

The rational number 0 is an **additive identity;** adding 0 to a number does not change it. The rational number 1 is a **multiplicative identity;** multiplying a number by 1 does not change it. Every rational number r has an **additive inverse,** denoted $-r$; the additive inverse of $4/5$ is $-4/5$, and the additive inverse of $-3/2$ is $3/2$. The sum of a number and its additive inverse is 0. Every *nonzero* rational number r has a **multiplicative inverse,** denoted r^{-1}; the multiplicative inverse of $4/5$ is $5/4$, and the multiplicative inverse of $-3/2$ is $-2/3$. The product of a number and its multiplicative inverse is 1. The rational number 0 has no multiplicative inverse.

EXAMPLE 1 Arithmetic of Rational Numbers

Perform the indicated operations.

(A) $\frac{1}{3}+\frac{6}{5}$ (B) $\frac{8}{3}\cdot\frac{5}{4}$

(C) $(-17/9)^{-1}$ (D) $(-6+9/2)^{-1}$

SOLUTION

(A) $\frac{1}{3}+\frac{6}{5}=\frac{5+18}{15}=\frac{23}{15}$

(B) $\frac{8}{3}\cdot\frac{5}{4}=\frac{40}{12}=\frac{10}{3}$ $\frac{40}{12}=\frac{10}{3}$ because $40\cdot 3=12\cdot 10$

(C) $(-17/9)^{-1} = -9/17$

(D) $(-6 + 9/2)^{-1} = \left(\frac{-6}{1} + \frac{9}{2}\right)^{-1} = \left(\frac{-12 + 9}{2}\right)^{-1} = \left(\frac{-3}{2}\right)^{-1} = \frac{-2}{3}$

MATCHED PROBLEM 1

Perform the indicated operations.

(A) $-(5/2 + 7/3)$ (B) $-(8/17)^{-1}$

(C) $\frac{21}{20} \cdot \frac{15}{14}$ (D) $5 \cdot (1/2 + 1/3)$

Rational numbers have decimal expansions that are repeating or terminating. For example, by long division,

$$\frac{2}{3} = 0.666\overline{6} \quad \text{The number 6 repeats indefinitely.}$$

$$\frac{22}{7} = 3.\overline{142857} \quad \text{The block 142857 repeats indefinitely.}$$

$$\frac{13}{8} = 1.625 \quad \text{Terminating expansion}$$

Conversely, any decimal expansion that is repeating or terminating represents a rational number (see Problems 69 and 70 in the exercises).

The number $\sqrt{2}$ is *irrational* because it cannot be written in the form a/b, where a and b are integers, $b \neq 0$ (for an explanation, see Problem 83 in Section A-5 or the Appendix A Group Activity). Similarly, $\sqrt{3}$ is irrational. But $\sqrt{4}$, which is equal to 2, is a rational number. In fact, if n is a positive integer, then $\sqrt{n}$ is irrational unless n belongs to the sequence of perfect squares 1, 4, 9, 16, 25, . . . (see Problem 84 in Section A-5).

We now return to our original question: how do you add or multiply two real numbers that have nonrepeating and nonterminating decimal expansions? Although we will not give a detailed answer to this question, the key idea is that every real number can be approximated to any desired precision by rational numbers. For example, the irrational number

$$\sqrt{2} \approx 1.414\,213\,562\ldots$$

is approximated by the rational numbers

$$\frac{14}{10} = 1.4$$

$$\frac{141}{100} = 1.41$$

$$\frac{1{,}414}{1{,}000} = 1.414$$
$$\frac{14{,}142}{10{,}000} = 1.4142$$
$$\frac{141{,}421}{100{,}000} = 1.41421$$
$$\vdots$$

Using the idea of approximation by rational numbers, we can extend the definitions of rational number operations to include real number operations. The following box summarizes the basic properties of real number operations.

BASIC PROPERTIES OF THE SET OF REAL NUMBERS

Let R be the set of real numbers, and let x, y, and z be arbitrary elements of R.

Addition Properties

Closure:	$x + y$ is a unique element in R.
Associative:	$(x + y) + z = x + (y + z)$
Commutative:	$x + y = y + x$
Identity:	0 is the additive identity; that is, $0 + x = x + 0 = x$ for all x in R, and 0 is the only element in R with this property.
Inverse:	For each x in R, $-x$ is its unique additive inverse; that is, $x + (-x) = (-x) + x = 0$, and $-x$ is the only element in R relative to x with this property.

Multiplication Properties

Closure:	xy is a unique element in R.
Associative:	$(xy)z = x(yz)$
Commutative:	$xy = yx$
Identity:	1 is the multiplicative identity; that is, for all x in R, $(1)x = x(1) = x$, and 1 is the only element in R with this property.
Inverse:	For each x in R, $x \neq 0$, x^{-1} is its unique multiplicative inverse; that is, $xx^{-1} = x^{-1}x = 1$, and x^{-1} is the only element in R relative to x with this property.

Combined Property

Distributive:	$x(y + z) = xy + xz \qquad (x + y)z = xz + yz$

EXAMPLE 2 Using Real Number Properties

Which real number property justifies the indicated statement?

Statement	Property Illustrated
(A) $(7x)y = 7(xy)$	Associative $(\cdot)$
(B) $a(b + c) = (b + c)a$	Commutative $(\cdot)$
(C) $(2x + 3y) + 5y = 2x + (3y + 5y)$	Associative $(+)$
(D) $(x + y)(a + b) = (x + y)a + (x + y)b$	Distributive
(E) If $a + b = 0$, then $b = -a$.	Inverse $(+)$

MATCHED PROBLEM 2

Which real number property justifies the indicated statement?

(A) $4 + (2 + x) = (4 + 2) + x$ (B) $(a + b) + c = c + (a + b)$

(C) $3x + 7x = (3 + 7)x$ (D) $(2x + 3y) + 0 = 2x + 3y$

(E) If $ab = 1$, then $b = 1/a$.

Further Operations and Properties

Subtraction of real numbers can be defined in terms of addition and the additive inverse. If a and b are real numbers, then $a - b$ is defined to be $a + (-b)$. Similarly, division can be defined in terms of multiplication and the multiplicative inverse. If a and b are real numbers and $b \neq 0$, then $a \div b$ (also denoted a/b) is defined to be $a \cdot b^{-1}$.

DEFINITION 2 Subtraction and Division of Real Numbers

For all real numbers a and b:

Subtraction: $a - b = a + (-b)$ $5 - 3 = 5 + (-3) = 2$

Division: $a \div b = a \cdot b^{-1}$ $b \neq 0$ $3 \div 2 = 3 \cdot 2^{-1} = 3 \cdot \frac{1}{2} = 1.5$

It is important to remember that

Division by 0 is never allowed.

››› EXPLORE-DISCUSS 1

(A) Give an example that shows that subtraction of real numbers is not commutative.

(B) Give an example that shows that division of real numbers is not commutative.

The basic properties of the set of real numbers, together with the definitions of subtraction and division, imply the following properties of negatives and zero.

› THEOREM 1 Properties of Negatives

For all real numbers a and b:

1. $-(-a) = a$
2. $(-a)b = -(ab) = a(-b) = -ab$
3. $(-a)(-b) = ab$
4. $(-1)a = -a$
5. $\dfrac{-a}{b} = -\dfrac{a}{b} = \dfrac{a}{-b} \qquad b \neq 0$
6. $\dfrac{-a}{-b} = -\dfrac{-a}{b} = -\dfrac{a}{-b} = \dfrac{a}{b} \qquad b \neq 0$

› THEOREM 2 Zero Properties

For all real numbers a and b:

1. $a \cdot 0 = 0 \cdot a = 0$
2. $ab = 0$ if and only if* $a = 0$ or $b = 0$ or both

Note that if $b \neq 0$, then $\frac{0}{b} = 0 \cdot b^{-1} = 0$ by Theorem 2. In particular, $\frac{0}{3} = 0$; but the expressions $\frac{3}{0}$ and $\frac{0}{0}$ are undefined.

*Given statements P and Q, "P if and only if Q" stands for both "if P, then Q" and "if Q, then P."

EXAMPLE 3 Using Negative and Zero Properties

Which real number property or definition justifies each statement?

Statement	Property or Definition Illustrated
(A) $3 - (-2) = 3 + [-(-2)] = 5$	Subtraction (Definition 1 and Theorem 1, part 1)
(B) $-(-2) = 2$	Negatives (Theorem 1, part 1)
(C) $-\frac{-3}{2} = \frac{3}{2}$	Negatives (Theorem 1, part 6)
(D) $\frac{5}{-2} = -\frac{5}{2}$	Negatives (Theorem 1, part 5)
(E) If $(x - 3)(x + 5) = 0$, then either $x - 3 = 0$ or $x + 5 = 0$.	Zero (Theorem 2, part 2)

MATCHED PROBLEM 3

Which real number property or definition justifies each statement?

(A) $\frac{3}{5} = 3\left(\frac{1}{5}\right)$ (B) $(-5)(2) = -(5 \cdot 2)$ (C) $(-1)3 = -3$

(D) $\frac{-7}{9} = -\frac{7}{9}$ (E) If $x + 5 = 0$, then $(x - 3)(x + 5) = 0$.

››› EXPLORE-DISCUSS 2

A set of numbers is **closed** under an operation if performing the operation on numbers of the set always produces another number in the set. For example, the set of odd integers is closed under multiplication, but is not closed under addition.

(A) Give an example that shows that the set of irrational numbers is not closed under addition.

(B) Explain why the set of irrational numbers is closed under taking multiplicative inverses.

If a and b are real numbers, $b \neq 0$, the quotient $a \div b$, when written in the form a/b, is called a **fraction.** The number a is the **numerator,** and b is the **denominator.** It can be shown that fractions satisfy the following properties. (Note that some of these properties, under the restriction that numerators and denominators be integers, were used earlier to define arithmetic operations on the rationals.)

THEOREM 3 Fraction Properties

For all real numbers a, b, c, d, and k (division by 0 excluded):

1. $\dfrac{a}{b} = \dfrac{c}{d}$ if and only if $ad = bc$

   $\dfrac{4}{6} = \dfrac{6}{9}$ since $4 \cdot 9 = 6 \cdot 6$

2. $\dfrac{ka}{kb} = \dfrac{a}{b}$

   $\dfrac{7 \cdot 3}{7 \cdot 5} = \dfrac{3}{5}$

3. $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$

   $\dfrac{3}{5} \cdot \dfrac{7}{8} = \dfrac{3 \cdot 7}{5 \cdot 8} = \dfrac{21}{40}$

4. $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \cdot \dfrac{d}{c}$

   $\dfrac{2}{3} \div \dfrac{5}{7} = \dfrac{2}{3} \cdot \dfrac{7}{5} = \dfrac{14}{15}$

5. $\dfrac{a}{b} + \dfrac{c}{b} = \dfrac{a + c}{b}$

   $\dfrac{3}{6} + \dfrac{4}{6} = \dfrac{3 + 4}{6} = \dfrac{7}{6}$

6. $\dfrac{a}{b} - \dfrac{c}{b} = \dfrac{a - c}{b}$

   $\dfrac{7}{8} - \dfrac{2}{8} = \dfrac{7 - 2}{8} = \dfrac{5}{8}$

7. $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$

   $\dfrac{2}{3} + \dfrac{1}{5} = \dfrac{2 \cdot 5 + 3 \cdot 1}{3 \cdot 5} = \dfrac{13}{15}$

ANSWERS TO MATCHED PROBLEMS

1. (A) $-29/6$ (B) $-17/8$ (C) $9/8$ (D) $25/6$
2. (A) Associative (+) (B) Commutative (+) (C) Distributive (D) Identity (+) (E) Inverse (·)
3. (A) Division (Definition 1) (B) Negatives (Theorem 1, part 2) (C) Negatives (Theorem 1, part 4) (D) Negatives (Theorem 1, part 5) (E) Zero (Theorem 2, part 1)

A-1 Exercises

*Additional answers can be found in the Instructor Answer Appendix.

A

In Problems 1–12, indicate true (T) or false (F).

1. $4 \in \{3, 4, 5\}$ T

2. $6 \in \{2, 4, 6\}$ T

3. $3 \notin \{3, 4, 5\}$ F

4. $7 \notin \{2, 4, 6\}$ T

5. $\{1, 2\} \subset \{1, 3, 5\}$ F

6. $\{2, 6\} \subset \{2, 4, 6\}$ T

7. $\{7, 3, 5\} \subset \{3, 5, 7\}$ T

8. $\{7, 3, 5\} = \{3, 5, 7\}$ T

9. $\varnothing \in \{A, B, C\}$ F

10. $\varnothing \subset \{A, B, C, D\}$ T

11. $\{A, L, G, E, B, R, A\} = \{G, A, R, B, L, E\}$ T

12. $\{A, B, C, D, E\} \subset \{A, C, E\}$ F

In Problems 13–20, perform the indicated operations, if defined, and express the result as an integer.

13. $\dfrac{4}{9} \cdot \dfrac{9}{4}$ 1

14. $\left(-\dfrac{3}{5}\right)\left(-\dfrac{5}{3}\right)$ 1

15. $100 \div 0$ Undefined

16. $0 \div 0$ Undefined

17. $0 \div (9 \cdot 8)$ 0

18. $\dfrac{4}{7} \div \left(3 - \dfrac{6}{2}\right)$ Undefined

19. $0 \cdot \left(100 + \dfrac{1}{100}\right)$ 0

20. $[(0 + 5) \div 20]^{-1}$ 4

In Problems 21–26, perform the indicated operations and express the result in the form a/b, where a and b are integers.

21. $\frac{4}{9} + \frac{12}{5}$ $\frac{128}{45}$

22. $\frac{17}{8} \cdot \frac{2}{7}$ $\frac{17}{28}$

23. $-\left(\frac{1}{100} + \frac{4}{25}\right)$ $\frac{-17}{100}$

24. $\left(\frac{5}{2} \cdot \frac{14}{25}\right)^{-1}$ $\frac{5}{7}$

25. $\left(\frac{3}{8}\right)^{-1} + 2^{-1}$ $\frac{19}{6}$

26. $-(1 + 3^{-1})$ $\frac{-4}{3}$

In Problems 27–38, each statement illustrates the use of one of the following properties or definitions. Indicate which one.

Commutative (+)
Commutative (·)
Associative (+)
Associative (·)
Distributive
Identity (+)
Identity (·)
Inverse (+)
Inverse (·)
Subtraction
Division
Negatives (Theorem 1)
Zero (Theorem 2)

27. $x + ym = x + my$ Commutative (·)

28. $7(3m) = (7 \cdot 3)m$ Associative (·)

29. $7u + 9u = (7 + 9)u$ Distributive

30. $-\frac{u}{-v} = \frac{u}{v}$ Negatives (Theorem 1)

31. $(-2)(\frac{1}{-2}) = 1$ Inverse (·)

32. $8 - 12 = 8 + (-12)$ Subtraction

33. $w + (-w) = 0$ Inverse (+)

34. $5 \div (-6) = 5(\frac{1}{-6})$ Division

35. $3(xy + z) + 0 = 3(xy + z)$ Identity (+)

36. $ab(c + d) = abc + abd$ Distributive

37. $\frac{-x}{-y} = \frac{x}{y}$ Negatives (Theorem 1)

38. $(x + y) \cdot 0 = 0$ Zero (Theorem 2)

B

Write each set in Problems 39–44 using the listing method; that is, list the elements between braces. If the set is empty, write ∅.

39. $\{x \mid x$ is an even integer between -3 and $5\}$ $\{-2, 0, 2, 4\}$

40. $\{x \mid x$ is an odd integer between -4 and $6\}$ $\{-3, -1, 1, 3, 5\}$

41. $\{x \mid x$ is a letter in "status"$\}$ {a, s, t, u}

42. $\{x \mid x$ is a letter in "consensus"$\}$ {c, o, n, s, e, u}

43. $\{x \mid x$ is a month starting with B$\}$ ∅

44. $\{x \mid x$ is a month with 32 days$\}$ ∅

45. The set $S_1 = \{a\}$ has only two subsets, S_1 and ∅. How many subsets does each of the following sets have?
(A) $S_2 = \{a, b\}$ 4
(B) $S_3 = \{a, b, c\}$ 8
(C) $S_4 = \{a, b, c, d\}$ 16

46. Based on the results in Problem 45, how many subsets do you think a set with n elements will have? 2^n

47. If $ab = 0$, does either a or b have to be 0? Yes

48. If $ab = 1$, does either a or b have to be 1? No

49. Indicate which of the following are true:
(A) All natural numbers are integers. T
(B) All real numbers are irrational. F
(C) All rational numbers are real numbers. T

50. Indicate which of the following are true:
(A) All integers are natural numbers. F
(B) All rational numbers are real numbers. T
(C) All natural numbers are rational numbers. T

51. Give an example of a rational number that is not an integer.
$\frac{3}{5}$ and -1.43 are two examples of infinitely many.

52. Give an example of a real number that is not a rational number. $\sqrt{2}$ and π are two examples of infinitely many.

In Problems 53 and 54, list the subset of S consisting of (A) natural numbers, (B) integers, (C) rational numbers, and (D) irrational numbers.

53. $S = \{-3, -\frac{2}{3}, 0, 1, \sqrt{3}, \frac{9}{5}, \sqrt{144}\}$

54. $S = \{-\sqrt{5}, -1, -\frac{1}{2}, 2, \sqrt{7}, 6, \sqrt{625/9}, \pi\}$

In Problems 55 and 56, use a calculator to express each number in decimal form. Classify each decimal number as terminating, repeating, or nonrepeating and nonterminating. Identify the pattern of repeated digits in any repeating decimal numbers.*

55. (A) $\frac{8}{9}$ (B) $\frac{3}{11}$ (C) $\sqrt{5}$ (D) $\frac{11}{8}$

56. (A) $\frac{13}{6}$ (B) $\sqrt{21}$ (C) $\frac{7}{16}$ (D) $\frac{29}{111}$

57. Indicate true (T) or false (F), and for each false statement find real number replacements for a and b that will provide a counterexample. For all real numbers a and b:
(A) $a + b = b + a$ T
(B) $a - b = b - a$ F; since, for example, $5 - 3 \neq 3 - 5$
(C) $ab = ba$ T
(D) $a \div b = b \div a$ F; since, for example, $9 \div 3 \neq 3 \div 9$

58. Indicate true (T) or false (F), and for each false statement find real number replacements for a, b, and c that will provide a counterexample. For all real numbers a, b, and c:
(A) $(a + b) + c = a + (b + c)$ T
(B) $(a - b) - c = a - (b - c)$ F
(C) $a(bc) = (ab)c$ T
(D) $(a \div b) \div c = a \div (b \div c)$ F

*Problem numbers that appear in blue indicate that require students to apply their reasoning and writing skills in the solution of the problem.

C

In Problems 59–66, indicate true (T) or false (F), and for each false statement give a specific counterexample.

59. The difference of any two natural numbers is a natural number. F

60. The quotient of any two nonzero integers is an integer. F

61. The sum of any two rational numbers is a rational number. T

62. The sum of any two irrational numbers is an irrational number. F

63. The product of any two irrational numbers is an irrational number. F

64. The product of any two rational numbers is a rational number. T

65. The multiplicative inverse of any irrational number is an irrational number. T

66. The multiplicative inverse of any nonzero rational number is a rational number. T

67. If $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$, find:
(A) $\{x \mid x \in A \text{ or } x \in B\}$ $\{1, 2, 3, 4, 6\}$
(B) $\{x \mid x \in A \text{ and } x \in B\}$ $\{2, 4\}$

68. If $F = \{-2, 0, 2\}$ and $G = \{-1, 0, 1, 2\}$, find:
(A) $\{x \mid x \in F \text{ or } x \in G\}$ $\{-2, -1, 0, 1, 2\}$
(B) $\{x \mid x \in F \text{ and } x \in G\}$ $\{0, 2\}$

69. If $c = 0.151515\ldots$, then $100c = 15.1515\ldots$ and

$$100c - c = 15.1515\ldots - 0.151515\ldots$$
$$99c = 15$$
$$c = \tfrac{15}{99} = \tfrac{5}{33}$$

Proceeding similarly, convert the repeating decimal $0.090909\ldots$ into a fraction. (All repeating decimals are rational numbers, and all rational numbers have repeating decimal representations.) $\frac{1}{11}$

70. Repeat Problem 69 for $0.181818\ldots$. $\frac{2}{11}$

71. To see how the distributive property is behind the mechanics of long multiplication, compute each of the following and compare:

Long Multiplication	Use of the Distributive Property
23	$23 \cdot 12$
$\times\ 12$	$= 23(2 + 10)$
	$= 23 \cdot 2 + 23 \cdot 10 =$

72. For a and b real numbers, justify each step using a property in this section.

Statement		Reason
1. $(a + b) + (-a)$	$= (-a) + (a + b)$	**1.** Commutative (+)
2.	$= [(-a) + a] + b$	**2.** Associative (+)
3.	$= 0 + b$	**3.** Inverse (+)
4.	$= b$	**4.** Identity (+)

A-2 Exponents

- Integer Exponents
- Roots of Real Numbers
- Rational Exponents
- Scientific Notation

The French philosopher/mathematician René Descartes (1596–1650) is generally credited with the introduction of the very useful exponent notation "x^n." This notation as well as other improvements in algebra may be found in his *Geometry,* published in 1637.

If n is a natural number, x^n denotes the product of n factors, each equal to x. In this section, the meaning of x^n will be expanded to allow the exponent n to be any rational number. Each of the following expressions will then represent a unique real number:

$$7^5 \qquad 5^{-4} \qquad 3.14^0 \qquad 6^{1/2} \qquad 14^{-5/3}$$

Integer Exponents

Definition 1 generalizes exponent notation to include 0 and negative integer exponents.

DEFINITION 1 a^n, n an integer and a a real number

1. For n a positive integer:

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors of } a} \qquad 3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$$

2. For $n = 0$:

$$a^0 = 1 \qquad a \neq 0 \qquad 132^0 = 1$$

0^0 is not defined

3. For n a negative integer:

$$a^n = \frac{1}{a^{-n}} \qquad a \neq 0 \qquad 7^{-3} = \frac{1}{7^{-(-3)}} = \frac{1}{7^3}$$

Note: It can be shown that for any $a \neq 0$ and for *all* integers n

$$a^{-n} = \frac{1}{a^n} \qquad a^{-5} = \frac{1}{a^5} \qquad a^{-(-3)} = \frac{1}{a^{-3}}$$

EXAMPLE 1 Using the Definition of Integer Exponents

Write each part as a decimal fraction or using positive exponents. Assume all variables represent nonzero real numbers.

(A) $(u^3v^2)^0 = 1$ (B) $10^{-3} = \dfrac{1}{10^3} = \dfrac{1}{1{,}000} = 0.001$

(C) $x^{-8} = \dfrac{1}{x^8}$ (D) $\dfrac{x^{-3}}{y^{-5}} = \dfrac{x^{-3}}{1} \cdot \dfrac{1}{y^{-5}} = \dfrac{1}{x^3} \cdot \dfrac{y^5}{1} = \dfrac{y^5}{x^3}$

MATCHED PROBLEM 1

Write parts A–D as decimal fractions and parts E and F with positive exponents. Assume all variables represent nonzero real numbers.

(A) 636^0 (B) $(x^2)^0$ (C) 10^{-5}

(D) $\dfrac{1}{10^{-3}}$ (E) $\dfrac{1}{x^{-4}}$ (F) $\dfrac{u^{-7}}{v^{-3}}$

*Throughout the book, dashed boxes—called think boxes—are used to represent steps that may be performed mentally.

The basic properties of integer exponents are summarized in Theorem 1. The proof of this theorem involves *mathematical induction,* which is discussed in Chapter 7.

› THEOREM 1 Properties of Integer Exponents

For n and m integers and a and b real numbers:

1. $a^m a^n = a^{m+n}$ $\quad a^5 a^{-7} = a^{5+(-7)} = a^{-2}$
2. $(a^n)^m = a^{mn}$ $\quad (a^3)^{-2} = a^{(-2)3} = a^{-6}$
3. $(ab)^m = a^m b^m$ $\quad (ab)^3 = a^3 b^3$
4. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ $\quad b \neq 0$ $\quad \left(\frac{a}{b}\right)^4 = \frac{a^4}{b^4}$
5. $\frac{a^m}{a^n} = \begin{cases} a^{m-n} \\ \frac{1}{a^{n-m}} \end{cases}$ $\quad a \neq 0$ $\quad \frac{a^3}{a^{-2}} = a^{3-(-2)} = a^5$; $\frac{a^3}{a^{-2}} = \frac{1}{a^{-2-3}} = \frac{1}{a^{-5}}$

››› EXPLORE-DISCUSS 1

Property 1 in Theorem 1 can be expressed verbally as follows:

To find the product of two exponential forms with the same base, add the exponents and use the same base.

Express the other properties in Theorem 1 verbally. Decide which you find easier to remember, a formula or a verbal description.

EXAMPLE 2 Using Exponent Properties

Simplify using exponent properties, and express answers using positive exponents only.*

(A) $(3a^5)(2a^{-3}) = (3 \cdot 2)(a^5 a^{-3}) = 6a^2$

(B) $\frac{6x^{-2}}{8x^{-5}} = \frac{3x^{-2-(-5)}}{4} = \frac{3x^3}{4}$

*By "simplify" we mean eliminate common factors from numerators and denominators and reduce to a minimum the number of times a given constant or variable appears in an expression. We ask that answers be expressed using positive exponents only in order to have a definite form for an answer. Later (in this section and elsewhere) we will encounter situations where we will want negative exponents in a final answer.

(C) $-4y^3 - (-4y)^3 = -4y^3 - (-4)^3y^3 = -4y^3 - (-64)y^3$

$= -4y^3 + 64y^3 = 60y^3$

MATCHED PROBLEM 2

Simplify using exponent properties, and express answers using positive exponents only.

(A) $(5x^{-3})(3x^4)$ (B) $\dfrac{9y^{-7}}{6y^{-4}}$ (C) $2x^4 - (-2x)^4$

››› CAUTION ›››

Be careful when using the relationship $a^{-n} = \dfrac{1}{a^n}$:

$ab^{-1} \neq \dfrac{1}{ab}$ $ab^{-1} = \dfrac{a}{b}$ and $(ab)^{-1} = \dfrac{1}{ab}$

$\dfrac{1}{a+b} \neq a^{-1} + b^{-1}$ $\dfrac{1}{a+b} = (a+b)^{-1}$ and $\dfrac{1}{a} + \dfrac{1}{b} = a^{-1} + b^{-1}$

Do not confuse properties 1 and 2 in Theorem 1:

$a^3a^4 \neq a^{3\cdot 4}$ $a^3a^4 = a^{3+4} = a^7$ property 1, Theorem 1

$(a^3)^4 \neq a^{3+4}$ $(a^3)^4 = a^{3\cdot 4} = a^{12}$ property 2, Theorem 1

From the definition of negative exponents and the five properties of exponents, we can easily establish the following properties, which are used very frequently when dealing with exponent forms.

› THEOREM 2 Further Exponent Properties

For a and b any real numbers and m, n, and p any integers (division by 0 excluded):

1. $(a^m b^n)^p = a^{pm}b^{pn}$
2. $\left(\dfrac{a^m}{b^n}\right)^p = \dfrac{a^{pm}}{b^{pn}}$
3. $\dfrac{a^{-n}}{b^{-m}} = \dfrac{b^m}{a^n}$
4. $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$

PROOF We prove properties 1 and 4 in Theorem 2 and leave the proofs of 2 and 3 to you.

1. $(a^m b^n)^p = (a^m)^p (b^n)^p$ property 3, Theorem 1

$= a^{pm} b^{pn}$ property 2, Theorem 1

4. $\left(\frac{a}{b}\right)^{-n} = \frac{a^{-n}}{b^{-n}}$ property 4, Theorem 1

$= \frac{b^n}{a^n}$ property 3, Theorem 2

$= \left(\frac{b}{a}\right)^n$ property 4, Theorem 1

EXAMPLE 3 Using Exponent Properties

Simplify using exponent properties, and express answers using positive exponents only.

(A) $(2a^{-3}b^2)^{-2} = 2^{-2}a^6 b^{-4} = \frac{a^6}{4b^4}$

(B) $\left(\frac{a^3}{b^5}\right)^{-2} = \frac{a^{-6}}{b^{-10}} = \frac{b^{10}}{a^6}$ or $\left(\frac{a^3}{b^5}\right)^{-2} = \left(\frac{b^5}{a^3}\right)^2 = \frac{b^{10}}{a^6}$

(C) $\frac{4x^{-3}y^{-5}}{6x^{-4}y^3} = \frac{2x^{-3-(-4)}}{3y^{3-(-5)}} = \frac{2x}{3y^8}$

(D) $\left(\frac{m^{-3}m^3}{n^{-2}}\right)^{-2} = \left(\frac{m^{-3+3}}{n^{-2}}\right)^{-2} = \left(\frac{m^0}{n^{-2}}\right)^{-2} = \left(\frac{1}{n^{-2}}\right)^{-2} = \frac{1}{n^4}$

(E) $(x + y)^{-3} = \frac{1}{(x + y)^3}$

MATCHED PROBLEM 3

Simplify using exponent properties, and express answers using positive exponents only.

(A) $(3x^4y^{-3})^{-2}$ (B) $\left(\frac{x^2}{y^4}\right)^{-3}$ (C) $\frac{6m^{-2}n^3}{15m^{-1}n^{-2}}$

(D) $\left(\frac{x^{-3}}{y^4y^{-4}}\right)^{-3}$ (E) $\frac{1}{(a - b)^{-2}}$

In simplifying exponent forms there is often more than one sequence of steps that will lead to the same result (see Example 3B). Use whichever sequence of steps makes sense to you.

Roots of Real Numbers

Perhaps you recall that a **square root** of a number b is a number c such that $c^2 = b$, and a **cube root** of a number b is a number d such that $d^3 = b$.

What are the square roots of 9?

$$3 \text{ is a square root of } 9, \text{ since } 3^2 = 9.$$
$$-3 \text{ is a square root of } 9, \text{ since } (-3)^2 = 9.$$

Therefore, 9 has two real square roots, one the negative of the other.

What are the cube roots of 8?

$$2 \text{ is a cube root of } 8, \text{ since } 2^3 = 8.$$

And 2 is the only real number with this property. In general,

DEFINITION 2 Definition of an nth Root

For a natural number n and a and b real numbers:

a is an ***n*th root** of b if $a^n = b$ 3 is a fourth root of 81, since $3^4 = 81$.

How many real square roots of 4 exist? Of 5? Of -9? How many real fourth roots of 5 exist? Of -5? How many real cube roots of 27 are there? Of -27? The following important theorem (which we state without proof) answers these questions.

THEOREM 3 Number of Real nth Roots of a Real Number b*

	n even	n odd
b positive	Two real nth roots -3 and 3 are both fourth roots of 81.	One real nth root 2 is the only real cube root of 8.
b negative	No real nth root -9 has no real square roots.	One real nth root -2 is the only real cube root of -8.

*In this section, we limit our discussion to real roots of real numbers. After the real numbers are extended to the complex numbers (see Section 2-4), additional roots may be considered. For example, it turns out that 1 has three cube roots: in addition to the real number 1, there are two other cube roots of 1 in the complex number system.

Thus, 4 and 5 have two real square roots each, and -9 has none. There are two real fourth roots of 5 and none for -5. And 27 and -27 have one real cube root each. What symbols do we use to represent these roots? We turn to this question now.

Rational Exponents

If all exponent properties are to continue to hold even if some of the exponents are rational numbers, then

$$(5^{1/3})^3 = 5^{3/3} = 5 \qquad \text{and} \qquad (7^{1/2})^2 = 7^{2/2} = 7$$

Since Theorem 3 states that the number 5 has one real cube root, it seems reasonable to use the symbol $5^{1/3}$ to represent this root. On the other hand, Theorem 3 states that 7 has two real square roots. Which real square root of 7 does $7^{1/2}$ represent? We answer this question in the following definition.

DEFINITION 3 $b^{1/n}$, Principal nth Root

For n a natural number and b a real number,

$$b^{1/n} \text{ is the } \textbf{principal } \boldsymbol{n}\textbf{th root of } \boldsymbol{b}$$

defined as follows:

1. If n is even and b is positive, then $b^{1/n}$ represents the positive nth root of b.

   $16^{1/2} = 4$ not -4 and 4.

   $-16^{1/2} = -4$ $-16^{1/2}$ and $(-16)^{1/2}$ are not the same.

2. If n is even and b is negative, then $b^{1/n}$ does not represent a real number. (More will be said about this case later.)

   $(-16)^{1/2}$ is not real.

3. If n is odd, then $b^{1/n}$ represents the real nth root of b (there is only one).

   $32^{1/5} = 2$ $(-32)^{1/5} = -2$

4. $0^{1/n} = 0$ $0^{1/9} = 0$ $0^{1/6} = 0$

EXAMPLE 4 Principal nth Roots

(A) $9^{1/2} = 3$

(B) $-9^{1/2} = -3$ Compare parts (B) and (C).

(C) $(-9)^{1/2}$ is not a real number.

(D) $27^{1/3} = 3$

(E) $(-27)^{1/3} = -3$

(F) $0^{1/7} = 0$

MATCHED PROBLEM 4

Find each of the following:

(A) $4^{1/2}$ (B) $-4^{1/2}$ (C) $(-4)^{1/2}$

(D) $8^{1/3}$ (E) $(-8)^{1/3}$ (F) $0^{1/8}$

How should a symbol such as $7^{2/3}$ be defined? If the properties of exponents are to hold for rational exponents, then $7^{2/3} = (7^{1/3})^2$; that is, $7^{2/3}$ must represent the square of the cube root of 7. This leads to the following general definition:

DEFINITION 4 $b^{m/n}$ and $b^{-m/n}$, Rational Number Exponent

For m and n natural numbers and b any real number (except b cannot be negative when n is even):

$$b^{m/n} = (b^{1/n})^m \quad \text{and} \quad b^{-m/n} = \frac{1}{b^{m/n}}$$

$4^{3/2} = (4^{1/2})^3 = 2^3 = 8$ $\quad 4^{-3/2} = \frac{1}{4^{3/2}} = \frac{1}{8}$ $\quad (-4)^{3/2}$ is not real

$(-32)^{3/5} = [(-32)^{1/5}]^3 = (-2)^3 = -8$

We have now discussed $b^{m/n}$ for all rational numbers m/n and real numbers b. It can be shown, though we will not do so, that all five properties of exponents listed in Theorem 1 continue to hold for rational exponents as long as we avoid even roots of negative numbers. With the latter restriction in effect, the following useful relationship is an immediate consequence of the exponent properties:

THEOREM 4 Rational Exponent Property

For m and n natural numbers and b any real number (except b cannot be negative when n is even):

$$b^{m/n} = \begin{cases} (b^{1/n})^m \\ (b^m)^{1/n} \end{cases} \qquad 8^{2/3} = \begin{cases} (8^{1/3})^2 \\ (8^2)^{1/3} \end{cases}$$

EXPLORE-DISCUSS 2

Find the contradiction in the following chain of equations:

$$-1 = (-1)^{2/2} = [(-1)^2]^{1/2} = 1^{1/2} = 1 \qquad (1)$$

Where did we try to use Theorem 4? Why was this not correct?

The three exponential forms in Theorem 4 are equal as long as only real numbers are involved. But if b is negative and n is even, then $b^{1/n}$ is not a real number and Theorem 4 does not necessarily hold, as illustrated in Explore-Discuss 2. One way to avoid this difficulty is to assume that m and n have no common factors.

EXAMPLE 5 Using Rational Exponents

Simplify, and express answers using positive exponents only. All letters represent positive real numbers.

(A) $8^{2/3} = (8^{1/3})^2 = 2^2 = 4$ or $8^{2/3} = (8^2)^{1/3} = 64^{1/3} = 4$

(B) $(-8)^{5/3} = [(-8)^{1/3}]^5 = (-2)^5 = -32$

(C) $(3x^{1/3})(2x^{1/2}) = 6x^{1/3+1/2} = 6x^{5/6}$

(D) $\left(\dfrac{4x^{1/3}}{x^{1/2}}\right)^{1/2} = \dfrac{4^{1/2}x^{1/6}}{x^{1/4}} = \dfrac{2}{x^{1/4-1/6}} = \dfrac{2}{x^{1/12}}$

(E) $(u^{1/2} - 2v^{1/2})(3u^{1/2} + v^{1/2}) = 3u - 5u^{1/2}v^{1/2} - 2v$

MATCHED PROBLEM 5

Simplify, and express answers using positive exponents only. All letters represent positive real numbers.

(A) $9^{3/2}$ (B) $(-27)^{4/3}$ (C) $(5y^{3/4})(2y^{1/3})$ (D) $(2x^{-3/4}y^{1/4})^4$

(E) $\left(\dfrac{8x^{1/2}}{x^{2/3}}\right)^{1/3}$ (F) $(2x^{1/2} + y^{1/2})(x^{1/2} - 3y^{1/2})$

Scientific Notation

Scientific work often involves the use of very large numbers or very small numbers. For example, the average cell contains about 200,000,000,000,000 molecules, and the diameter of an electron is about 0.000 000 000 0004 centimeter. It is generally troublesome to write and work with numbers of this type in standard decimal form. The

two numbers written here cannot even be entered into most calculators as they are written. However, each can be expressed as the product of a number between 1 and 10 and an integer power of 10:

$$200{,}000{,}000{,}000{,}000 = 2 \times 10^{14}$$
$$0.000\ 000\ 000\ 0004 = 4 \times 10^{-13}$$

In fact any positive number written in decimal form can be expressed in **scientific notation,** that is, in the form

$$a \times 10^n \qquad 1 \le a < 10,\ n \text{ an integer, } a \text{ in decimal form}$$

EXAMPLE 6 Scientific Notation

Each number is written in scientific notation:

$$7 = 7 \times 10^0 \qquad 0.5 = 5 \times 10^{-1}$$
$$720 = 7.2 \times 10^2 \qquad 0.08 = 8 \times 10^{-2}$$
$$6{,}430 = 6.43 \times 10^3 \qquad 0.000\ 32 = 3.2 \times 10^{-4}$$
$$5{,}350{,}000 = 5.35 \times 10^6 \qquad 0.000\ 000\ 0738 = 7.38 \times 10^{-8}$$

Can you discover a rule relating the number of decimal places the decimal point is moved to the power of 10 that is used?

$$7{,}320{,}000 = 7.320\ 000. \times 10^6 = 7.32 \times 10^6$$

6 places left
Positive exponent

$$0.000\ 000\ 54 = 0.000\ 000\ 5.4 \times 10^{-7} = 5.4 \times 10^{-7}$$

7 places right
Negative exponent

MATCHED PROBLEM 6

(A) Write each number in scientific notation: 430; 23,000; 345,000,000; 0.3; 0.0031; 0.000 000 683

(B) Write in standard decimal form: 4×10^3; 5.3×10^5; 2.53×10^{-2}; 7.42×10^{-6}

Most calculators express very large and very small numbers in scientific notation. Consult the manual for your calculator to see how numbers in scientific notation are

entered in your calculator. Some common methods for displaying scientific notation on a calculator are shown here.

Number Represented	Typical Scientific Calculator Display	Typical Graphing Calculator Display
$5.427\ 493 \times 10^{-17}$	5.427493 – 17	5.427493E–17
$2.359\ 779 \times 10^{12}$	2.359779 12	2.359779E12

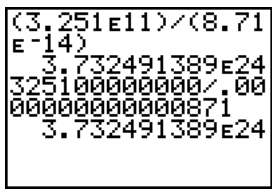
EXAMPLE 7 Using Scientific Notation on a Calculator

Write each number in scientific notation; then carry out the computations using your calculator. (Refer to the user's manual accompanying your calculator for the procedure.) Express the answer to three significant digits* in scientific notation.

$$\frac{325{,}100{,}000{,}000}{0.000\ 000\ 000\ 000\ 0871} = \frac{3.251 \times 10^{11}}{8.71 \times 10^{-14}}$$

$$= \boxed{3.732491389\text{E}24} \quad \text{Calculator display}$$

$$= 3.73 \times 10^{24} \quad \text{To three significant digits}$$



› Figure 1

Figure 1 shows two solutions to this problem on a graphing calculator. In the first solution we entered the numbers in scientific notation, and in the second we used standard decimal notation. Although the multiple-line screen display on a graphing calculator allows us to enter very long standard decimals, scientific notation is usually more efficient and less prone to errors in data entry. Furthermore, as Figure 1 shows, the calculator uses scientific notation to display the answer, regardless of the manner in which the numbers are entered.

MATCHED PROBLEM 7

Repeat Example 7 for:

$$\frac{0.000\ 000\ 006\ 932}{62{,}600{,}000{,}000}$$

EXAMPLE 8 Measuring Time with an Atomic Clock

An atomic clock that counts the radioactive emissions of cesium is used to provide a precise definition of a second. One second is defined to be the time it takes cesium to emit 9,192,631,770 cycles of radiation. How many of these cycles will occur in 1 hour? Express the answer to five significant digits in scientific notation.

SOLUTION

$$(9{,}192{,}631{,}770)(60^2) = \boxed{3.309347437\text{E}13}$$

$$= 3.3093 \times 10^{13}$$

*For those not familiar with the meaning of *significant digits,* see Appendix C, Section C-1, for a brief discussion of this concept.

MATCHED PROBLEM 8

Refer to Example 8. How many of these cycles will occur in 1 year? Express the answer to five significant digits in scientific notation.

EXAMPLE 9 Evaluating Rational Exponential Forms with a Calculator

Evaluate to four significant digits using a calculator. (Refer to the instruction book for your particular calculator to see how exponential forms are evaluated.)

(A) $11^{3/4}$ (B) $3.1046^{-2/3}$ (C) $(0.000\ 000\ 008\ 437)^{3/11}$

SOLUTIONS

(A) First change $\frac{3}{4}$ to the standard decimal form 0.75; then evaluate $11^{0.75}$ using a calculator.

$$11^{3/4} = 6.040$$

(B) $3.1046^{-2/3} = 0.4699$

(C) $(0.000\ 000\ 008\ 437)^{3/11} = (8.437 \times 10^{-9})^{3/11}$
$= 0.006\ 281$

MATCHED PROBLEM 9

Evaluate to four significant digits using a calculator.

(A) $2^{3/8}$ (B) $57.28^{-5/6}$ (C) $(83{,}240{,}000{,}000)^{5/3}$

ANSWERS TO MATCHED PROBLEMS

1. (A) 1 (B) 1 (C) 0.000 01 (D) 1,000 (E) x^4 (F) v^3/u^7
2. (A) $15x$ (B) $3/(2y^3)$ (C) $-14x^4$
3. (A) $y^6/(9x^8)$ (B) y^{12}/x^6 (C) $2n^5/(5m)$ (D) x^9 (E) $(a-b)^2$
4. (A) 2 (B) -2 (C) Not real (D) 2 (E) -2 (F) 0
5. (A) 27 (B) 81 (C) $10y^{13/12}$ (D) $16y/x^3$ (E) $2/x^{1/18}$
(F) $2x - 5x^{1/2}y^{1/2} - 3y$
6. (A) 4.3×10^2; 2.3×10^4; 3.45×10^8; 3×10^{-1}; 3.1×10^{-3}; 6.83×10^{-7}
(B) 4,000; 530,000; 0.0253; 0.000 007 42
7. 1.11×10^{-19} **8.** 2.8990×10^{17}
9. (A) 1.297 (B) 0.034 28 (C) 1.587×10^{18}

A-2 Exercises

All variables are restricted to prevent division by 0.

In Problems 1–26, evaluate each expression that results in a rational number.

1. 2^8 **2.** 3^8 **3.** $\left(\frac{2}{3}\right)^4$

4. $\left(\frac{3}{5}\right)^3$ **5.** 4^{-4} **6.** 2^{-6}

7. $(-5)^4$ **8.** $(-2)^5$ **9.** $(-3)^{-1}$

10. $(-3)^{-2}$ **11.** -7^{-2} **12.** -7^0

13. $\left(\frac{1}{3}\right)^0$ **14.** $\left(\frac{1}{5}\right)^{-1}$ **15.** $100^{1/2}$

16. $361^{1/2}$ **17.** $125^{1/3}$ **18.** $27^{2/3}$

19. $9^{-3/2}$ **20.** $2^{-1/2}$ **21.** $\left(\frac{1}{2}\right)^{-1/2}$

22. $64^{-4/3}$ **23.** $1^{-3} + 3^{-1}$ **24.** $4^{-1} + 4^2$

25. $\frac{5}{5^4}$ **26.** $\frac{8}{8^{-2}}$

Simplify Problems 27–38 and express answers using positive exponents only.

27. x^5x^{-2} **28.** y^6y^{-8} **29.** $(2y)(3y^2)(5y^4)$

30. $(6x^3)(4x^7)(x^{-5})$ **31.** $(a^2b^3)^5$ **32.** $(2c^4d^{-2})^{-3}$

33. $u^{1/3}u^{5/3}$ **34.** $v^{-1/5}v^{6/5}$ **35.** $(x^{-3})^{1/6}$

36. $(y^{-2/3})^{9/4}$ **37.** $(49a^4b^{-2})^{1/2}$ **38.** $(125a^{-9}b^{12})^{1/3}$

Write the numbers in Problems 39–44 in scientific notation.

39. 45,320,000 **40.** 3,670

41. 0.066 **42.** 0.029

43. 0.000 000 084 **44.** 0.000 497

In Problems 45–50, write each number in standard decimal form.

45. 9×10^{-5} **46.** 3×10^{-3}

47. 3.48×10^6 **48.** 8.63×10^8

49. 4.2×10^{-9} **50.** 1.6×10^{-7}

Simplify Problems 51–62, and write the answers using positive exponents only.

51. $\left(\frac{x^4y^{-1}}{x^{-2}y^3}\right)^2$ **52.** $\left(\frac{m^{-2}n^3}{m^4n^{-1}}\right)^2$ **53.** $\left(\frac{2x^{-3}y^2}{4xy^{-1}}\right)^{-2}$

54. $\left(\frac{6mn^{-2}}{3m^{-1}n^2}\right)^{-3}$ **55.** $\left(\frac{a^{-3}}{b^4}\right)^{1/12}$ **56.** $\left(\frac{m^{-2/3}}{n^{-1/2}}\right)^{-6}$

57. $\left(\frac{4x^{-2}}{y^4}\right)^{-1/2}$ **58.** $\left(\frac{w^4}{9x^{-2}}\right)^{-1/2}$

59. $\left(\frac{8a^{-4}b^3}{27a^2b^{-3}}\right)^{1/3}$ **60.** $\left(\frac{25x^5y^{-1}}{16x^{-3}y^{-5}}\right)^{1/2}$

∫ *61. $-3(x^3 + 3)^{-4}(3x^2)$ ∫ **62.** $-2(x^2 + 3x)^{-3}(2x + 3)$

63. What is the result of entering 2^{3^2} on a calculator?

64. Refer to Problem 63. What is the difference between $2^{(3^2)}$ and $(2^3)^2$? Which agrees with the value of 2^{3^2} obtained with a calculator?

65. If $n = 0$, then property 1 in Theorem 1 implies that $a^m a^0 = a^{m+0} = a^m$. Explain how this helps motivate the definition of a^0.

66. If $m = -n$, then property 1 in Theorem 1 implies that $a^{-n}a^n = a^0 = 1$. Explain how this helps motivate the definition of a^{-n}.

Evaluate Problems 67–70, to three significant digits using scientific notation and a calculator.

67. $\frac{(32.7)(0.000\ 000\ 008\ 42)}{(0.0513)(80{,}700{,}000{,}000)}$

68. $\frac{(4{,}320)(0.000\ 000\ 000\ 704)}{(835)(635{,}000{,}000{,}000)}$

69. $\frac{(5{,}760{,}000{,}000)}{(527)(0.000\ 007\ 09)}$

70. $\frac{0.000\ 000\ 007\ 23}{(0.0933)(43{,}700{,}000{,}000)}$

*The symbol ∫ denotes problems that are related to calculus.

In Problems 71–78, evaluate to four significant digits using a calculator. (Refer to the instruction book for your calculator to see how exponential forms are evaluated.)

71. $15^{5/4}$ 29.52 **72.** $22^{3/2}$ 103.2 **73.** $103^{-3/4}$ 0.03093

74. $827^{-3/8}$ 0.08053 **75.** $2.876^{8/5}$ 5.421 **76.** $37.09^{7/3}$ 4,588

77. $(0.000\ 000\ 077\ 35)^{-2/7}$ 107.6 **78.** $(491{,}300{,}000{,}000)^{7/4}$ 2.883×10^{20}

C

Problems 79–82, illustrate common errors involving rational exponents. In each case, find numerical examples that show that the left side is not always equal to the right side.

79. $(x + y)^{1/2} \neq x^{1/2} + y^{1/2}$ $x = y = 1$ is one of many choices.

80. $(x^3 + y^3)^{1/3} \neq x + y$ $x = y = 1$ is one of many choices.

81. $(x + y)^{1/3} \neq \dfrac{1}{(x + y)^3}$ $x = y = 1$ is one of many choices.

82. $(x + y)^{-1/2} \neq \dfrac{1}{(x + y)^2}$ $x = y = 1$ is one of many choices.

In Problems 83–86, m and n represent positive integers. Simplify and express answers using positive exponents.

83. $(a^{3/n}b^{3/m})^{1/3}$ $a^{1/n}b^{1/m}$ **84.** $(a^{n/2}b^{n/3})^{1/n}$ $a^{1/2}b^{1/3}$

85. $(x^{m/4}y^{n/3})^{-12}$ $1/(x^{3m}y^{4n})$ **86.** $(a^{m/3}b^{n/2})^{-6}$ $\dfrac{1}{a^{2m}b^{3n}}$

87. If possible, find a real value of x such that:
(A) $(x^2)^{1/2} \neq x$ (B) $(x^2)^{1/2} = x$ (C) $(x^3)^{1/3} \neq x$

88. If possible, find a real value of x such that:
(A) $(x^2)^{1/2} \neq -x$ (B) $(x^2)^{1/2} = -x$ (C) $(x^3)^{1/3} = -x$

89. If n is even and b is negative, then $b^{1/n}$ is not real. If m is odd, n is even, and b is negative, is $(b^m)^{1/n}$ real? No

90. If we assume that m is odd and n is even, is it possible that one of $(b^{1/n})^m$ and $(b^m)^{1/n}$ is real and the other is not? No

APPLICATIONS

91. **EARTH SCIENCE** If the mass of Earth is approximately 6.1×10^{27} grams and each gram is 2.2×10^{-3} pound, what is the mass of Earth in pounds? Express the answer to two significant digits in scientific notation. 1.3×10^{25} lb

92. **BIOLOGY** In 1929 Vernadsky, a biologist, estimated that all the free oxygen of the earth weighs 1.5×10^{21} grams and that it is produced by life alone. If 1 gram is approximately 2.2×10^{-3} pound, what is the weight of the free oxygen in pounds? Express the answer to two significant digits in scientific notation. 3.3×10^{18} lb

93. **COMPUTER SCIENCE** If a computer can perform a single operation in 10^{-10} second, how many operations can it perform in 1 second? In 1 minute? 10^{10} or 10 billion; 6×10^{11} or 600 billion

***94.** **COMPUTER SCIENCE** If electricity travels in a computer circuit at the speed of light (1.86×10^5 miles per second), how far will electricity travel in the superconducting computer (see Problem 93) in the time it takes it to perform one operation? (Size of circuits is a critical problem in computer design.) Give the answer in miles, feet, and inches (1 mile = 5,280 feet). Compute answers to three significant digits. 1.86×10^{-5} mi, 9.82×10^{-2} ft, 1.18 in

95. **ECONOMICS** If in the United States in 2003 the national debt was about \$6,760,000,000,000 and the population was about 291,000,000, estimate to three significant digits each individual's share of the national debt. Write your answer in scientific notation and in standard decimal form. 2.32×10^4 or \$23,200

96. **ECONOMICS** If in the United States in 2003 the gross domestic product (GDP) was about \$10,990,000,000,000 and the population was about 291,000,000, estimate to three significant digits the GDP per person. Write your answer in scientific notation and in standard decimal form. 3.78×10^4 or \$37,800

97. **ECONOMICS** The number of units N of a finished product produced from the use of x units of labor and y units of capital for a particular Third World country is approximated by

$$N = 10x^{3/4}y^{1/4} \quad \text{Cobb-Douglas equation}$$

Estimate how many units of a finished product will be produced using 256 units of labor and 81 units of capital. 1,920 units

98. **ECONOMICS** The number of units N of a finished product produced by a particular automobile company where x units of labor and y units of capital are used is approximated by

$$N = 50x^{1/2}y^{1/2} \quad \text{Cobb-Douglas equation}$$

Estimate how many units will be produced using 256 units of labor and 144 units of capital. 9,600 units

99. **BRAKING DISTANCE** R. A. Moyer of Iowa State College found, in comprehensive tests carried out on 41 wet pavements, that the braking distance d (in feet) for a particular automobile traveling at v miles per hour was given approximately by

$$d = 0.0212v^{7/3}$$

Approximate the braking distance to the nearest foot for the car traveling on wet pavement at 70 miles per hour. 428 ft

100. **BRAKING DISTANCE** Approximately how many feet would it take the car in Problem 99 to stop on wet pavement if it were traveling at 50 miles per hour? (Compute answer to the nearest foot.) 195 feet

A-3 Radicals

- From Rational Exponents to Radicals, and Vice Versa
- Properties of Radicals
- Simplifying Radicals
- Rationalizing Denominators

What do the following algebraic expressions have in common?

$$2^{1/2} \qquad 2x^{2/3} \qquad \frac{1}{x^{1/2}+y^{1/2}}$$

$$\sqrt{2} \qquad 2\sqrt[3]{x^2} \qquad \frac{1}{\sqrt{x}+\sqrt{y}}$$

Each vertical pair represents the same quantity, one in rational exponent form and the other in *radical form.* There are occasions when it is more convenient to work with radicals than with rational exponents, or vice versa. In this section, we see how the two forms are related and investigate some basic operations on radicals.

From Rational Exponents to Radicals, and Vice Versa

We start this discussion by defining an ***n*th-root radical:**

DEFINITION 1 $\sqrt[n]{b}$, *n*th-Root Radical

For n a natural number greater than 1 and b a real number, we define $\sqrt[n]{b}$ to be the **principal *n*th root of *b*** (see Definition 3 in the last section); that is,

$$\sqrt[n]{b} = b^{1/n}$$

The symbol $\sqrt{\ }$ is called a **radical,** n is called the **index,** and b is called the **radicand.**

If $n = 2$, we write $\sqrt{b}$ in place of $\sqrt[2]{b}$.

$\sqrt{25} = 25^{1/2} = 5$ $\qquad$ $\sqrt[5]{32} = 32^{1/5} = 2$

$-\sqrt{25} = -25^{1/2} = -5$ $\qquad$ $\sqrt[5]{-32} = (-32)^{1/5} = -2$

$\sqrt{-25}$ is not real $\qquad$ $\sqrt[4]{0} = 0^{1/4} = 0$

As already stated, it is often an advantage to be able to shift back and forth between rational exponent forms and radical forms. The following relationships, which are direct consequences of Definition 4 and Theorem 4 in the last section, are useful in this regard:

› RATIONAL EXPONENT/RADICAL CONVERSIONS

For m and n positive integers ($n > 1$), and b not negative when n is even,

$$b^{m/n} = \begin{cases} (b^m)^{1/n} = \sqrt[n]{b^m} \\ (b^{1/n})^m = (\sqrt[n]{b})^m \end{cases} \qquad 2^{2/3} = \begin{cases} \sqrt[3]{2^2} \\ (\sqrt[3]{2})^2 \end{cases}$$

Note: Unless stated to the contrary, all variables in the rest of the discussion represent positive real numbers.

››› EXPLORE-DISCUSS 1

In each of the following, evaluate both radical forms.

$$16^{3/2} = \sqrt{16^3} = (\sqrt{16})^3$$
$$27^{2/3} = \sqrt[3]{27^2} = (\sqrt[3]{27})^2$$

Which radical conversion form is easier to use if you are performing the calculations by hand?

EXAMPLE 1 Rational Exponent/Radical Conversions

Change from rational exponent form to radical form.

(A) $x^{1/7} = \sqrt[7]{x}$

(B) $(3u^2v^3)^{3/5} = \sqrt[5]{(3u^2v^3)^3}$ or $(\sqrt[5]{3u^2v^3})^3$ The first is usually preferred.

(C) $y^{-2/3} = \dfrac{1}{y^{2/3}} = \dfrac{1}{\sqrt[3]{y^2}}$ or $\sqrt[3]{y^{-2}}$ or $\sqrt[3]{\dfrac{1}{y^2}}$

Change from radical form to rational exponent form.

(D) $\sqrt[5]{6} = 6^{1/5}$ (E) $-\sqrt[3]{x^2} = -x^{2/3}$ (F) $\sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2}$

MATCHED PROBLEM 1

Change from rational exponent form to radical form.

(A) $u^{1/5}$ (B) $(6x^2y^5)^{2/9}$ (C) $(3xy)^{-3/5}$

Change from radical form to rational exponent form.

(D) $\sqrt[4]{9u}$ (E) $-\sqrt[7]{(2x)^4}$ (F) $\sqrt[3]{x^3 + y^3}$

Properties of Radicals

The exponent properties considered earlier imply the following properties of radicals.

THEOREM 1 Properties of Radicals

For n a natural number greater than 1, and x and y positive real numbers:

1. $\sqrt[n]{x^n} = x$ $\quad \sqrt[3]{x^3} = x$
2. $\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$ $\quad \sqrt[5]{xy} = \sqrt[5]{x}\sqrt[5]{y}$
3. $\sqrt[n]{\dfrac{x}{y}} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$ $\quad \sqrt[4]{\dfrac{x}{y}} = \dfrac{\sqrt[4]{x}}{\sqrt[4]{y}}$

EXAMPLE 2 Simplifying Radicals

Simplify:

(A) $\sqrt[5]{(3x^2y)^5} = 3x^2y$

(B) $\sqrt{10}\sqrt{5} = \sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$

(C) $\sqrt[3]{\dfrac{x}{27}} = \dfrac{\sqrt[3]{x}}{\sqrt[3]{27}} = \dfrac{\sqrt[3]{x}}{3}$ or $\dfrac{1}{3}\sqrt[3]{x}$

MATCHED PROBLEM 2

Simplify:

(A) $\sqrt[7]{(u^2 + v^2)^7}$ (B) $\sqrt{6}\sqrt{2}$ (C) $\sqrt[3]{\dfrac{x^2}{8}}$

››› CAUTION ›››

In general, properties of radicals can be used to simplify terms raised to powers, not sums of terms raised to powers. Thus, for x and y positive real numbers,

$$\sqrt{x^2 + y^2} \neq \sqrt{x^2} + \sqrt{y^2} = x + y$$

but

$$\sqrt{x^2 + 2xy + y^2} = \sqrt{(x + y)^2} = x + y$$

› Simplifying Radicals

The properties of radicals provide us with the means of changing algebraic expressions containing radicals to a variety of equivalent forms. One form that is often useful is a *simplified form.* An algebraic expression that contains radicals is said to be in **simplified form** if all four of the conditions listed in the following definition are satisfied.

› DEFINITION 2 Simplified (Radical) Form

1. No radicand (the expression within the radical sign) contains a factor to a power greater than or equal to the index of the radical.
   For example, $\sqrt{x^5}$ violates this condition.
2. No power of the radicand and the index of the radical have a common factor other than 1.
   For example, $\sqrt[6]{x^4}$ violates this condition.
3. No radical appears in a denominator.
   For example, $y/\sqrt{x}$ violates this condition.
4. No fraction appears within a radical.
   For example, $\sqrt{\frac{3}{5}}$ violates this condition.

EXAMPLE 3 Finding Simplified Form

Express radicals in simplified form.

(A) $\sqrt{12x^3y^5z^2} = \sqrt{(4x^2y^4z^2)(3xy)}$ $\quad x^{pm}y^{pn} = (x^m y^n)^p$

$= \sqrt{(2xy^2z)^2(3xy)}$ $\quad \sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$

$= \sqrt{(2xy^2z)^2}\sqrt{3xy}$ $\quad \sqrt[n]{x^n} = x$

$= 2xy^2z\sqrt{3xy}$

(B) $\sqrt[3]{6x^2y}\sqrt[3]{4x^5y^2} = \sqrt[3]{(6x^2y)(4x^5y^2)}$ Simplify radicand.

$= \sqrt[3]{24x^7y^3}$ Condition 1 is not met.

$= \sqrt[3]{(8x^6y^3)(3x)}$ $x^{pm}y^{pn} = (x^m y^n)^p$

$= \sqrt[3]{(2x^2y)^3(3x)}$ $\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$

$= \sqrt[3]{(2x^2y)^3}\sqrt[3]{3x}$ $\sqrt[n]{x^n} = x$

$= 2x^2y\sqrt[3]{3x}$

(C) $\sqrt[6]{16x^4y^2} = [(4x^2y)^2]^{1/6}$ Condition 2 is not met.

$= (4x^2y)^{2/6}$ Note the convenience of using rational exponents.

$= (4x^2y)^{1/3}$ Write as a radical.

$= \sqrt[3]{4x^2y}$

(D) $\sqrt[3]{\sqrt{27}} = [(3^3)^{1/2}]^{1/3}$ $(x^m)^n = x^{mn}$

$= (3^3)^{1/6} = 3^{3/6}$ Write as a radical.

$= 3^{1/2} = \sqrt{3}$

MATCHED PROBLEM 3

Express radicals in simplified form.

(A) $\sqrt{18x^5y^2z^3}$ (B) $\sqrt[4]{27a^3b^3}\sqrt[4]{3a^5b^3}$ (C) $\sqrt[9]{8x^6y^3}$ (D) $\sqrt{\sqrt[3]{4}}$

Algebraic expressions involving radicals often can be simplified by adding and subtracting terms that contain exactly the same radical expressions. The distributive property of real numbers plays a central role in this process.

EXAMPLE 4 Combining Like Terms

Combine as many terms as possible:

(A) $5\sqrt{3} + 4\sqrt{3} = (5 + 4)\sqrt{3} = 9\sqrt{3}$

(B) $2\sqrt[3]{xy^2} - 7\sqrt[3]{xy^2} = (2 - 7)\sqrt[3]{xy^2} = -5\sqrt[3]{xy^2}$

(C) $3\sqrt{xy} - 2\sqrt[3]{xy} + 4\sqrt{xy} - 7\sqrt[3]{xy} = 3\sqrt{xy} + 4\sqrt{xy} - 2\sqrt[3]{xy} - 7\sqrt[3]{xy}$

$= 7\sqrt{xy} - 9\sqrt[3]{xy}$

MATCHED PROBLEM 4

Combine as many terms as possible:

(A) $6\sqrt{2} + 2\sqrt{2}$ (B) $3\sqrt[5]{2x^2y^3} - 8\sqrt[5]{2x^2y^3}$

(C) $5\sqrt[3]{mn^2} - 3\sqrt{mn} - 2\sqrt[3]{mn^2} + 7\sqrt{mn}$

EXAMPLE 5 Multiplication with Radical Forms

Multiply and simplify:

(A) $\sqrt{2}(\sqrt{10} - 3) = \sqrt{2}\sqrt{10} - \sqrt{2} \cdot 3 = \sqrt{20} - 3\sqrt{2} = 2\sqrt{5} - 3\sqrt{2}$

(B) $(\sqrt{2} - 3)(\sqrt{2} + 5) = \sqrt{2}\sqrt{2} - 3\sqrt{2} + 5\sqrt{2} - 15$

$= 2 + 2\sqrt{2} - 15$

$= 2\sqrt{2} - 13$

(C) $(\sqrt{x} - 3)(\sqrt{x} + 5) = \sqrt{x}\sqrt{x} - 3\sqrt{x} + 5\sqrt{x} - 15$

$= x + 2\sqrt{x} - 15$

(D) $(\sqrt[3]{m} + \sqrt[3]{n^2})(\sqrt[3]{m^2} - \sqrt[3]{n}) = \sqrt[3]{m^3} + \sqrt[3]{m^2n^2} - \sqrt[3]{mn} - \sqrt[3]{n^3}$

$= m - \sqrt[3]{mn} + \sqrt[3]{m^2n^2} - n$

MATCHED PROBLEM 5

Multiply and simplify:

(A) $\sqrt{3}(\sqrt{6} - 4)$ (B) $(\sqrt{3} - 2)(\sqrt{3} + 4)$

(C) $(\sqrt{y} - 2)(\sqrt{y} + 4)$ (D) $(\sqrt[3]{x^2} - \sqrt[3]{y^2})(\sqrt[3]{x} + \sqrt[3]{y})$

› Rationalizing Denominators

Eliminating a radical from a denominator is referred to as **rationalizing the denominator.** To rationalize the denominator, we multiply the numerator and denominator by a suitable factor that will rationalize the denominator—that is, will leave the denominator free of radicals. This factor is called a **rationalizing factor.** The following special products are of use in finding some rationalizing factors (see Examples 6C, D):

$$(a - b)(a + b) = a^2 - b^2 \quad (1)$$

$$(a - b)(a^2 + ab + b^2) = a^3 - b^3 \quad (2)$$

$$(a + b)(a^2 - ab + b^2) = a^3 + b^3 \quad (3)$$

››› EXPLORE-DISCUSS 2

Use the preceding special products (1) to (3) to find a rationalizing factor for each of the following:

(A) $\sqrt{a} - \sqrt{b}$ **(B)** $\sqrt{a} + \sqrt{b}$ **(C)** $\sqrt[3]{a} - \sqrt[3]{b}$ **(D)** $\sqrt[3]{a} + \sqrt[3]{b}$

EXAMPLE 6 Rationalizing Denominators

Rationalize denominators.

(A) $\dfrac{3}{\sqrt{5}}$ (B) $\sqrt[3]{\dfrac{2a^2}{3b^2}}$ (C) $\dfrac{\sqrt{x} + \sqrt{y}}{3\sqrt{x} - 2\sqrt{y}}$ (D) $\dfrac{1}{\sqrt[3]{m} + 2}$

SOLUTIONS

(A) $\sqrt{5}$ is a rationalizing factor for $\sqrt{5}$, since $\sqrt{5}\sqrt{5} = \sqrt{5^2} = 5$. Therefore, we multiply the numerator and denominator by $\sqrt{5}$ to rationalize the denominator:

$$\frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{\sqrt{5}\sqrt{5}} = \frac{3\sqrt{5}}{5}$$

(B) $$\sqrt[3]{\frac{2a^2}{3b^2}} = \frac{\sqrt[3]{2a^2}}{\sqrt[3]{3b^2}} = \frac{\sqrt[3]{2a^2}\sqrt[3]{3^2b}}{\sqrt[3]{3b^2}\sqrt[3]{3^2b}} = \frac{\sqrt[3]{2 \cdot 3^2a^2b}}{\sqrt[3]{3^3b^3}} = \frac{\sqrt[3]{18a^2b}}{3b}$$

(C) Special product (1) suggests that if we multiply the denominator $3\sqrt{x} - 2\sqrt{y}$ by $3\sqrt{x} + 2\sqrt{y}$, we will obtain the difference of two squares and the denominator will be rationalized.

$$\frac{\sqrt{x} + \sqrt{y}}{3\sqrt{x} - 2\sqrt{y}} = \frac{(\sqrt{x} + \sqrt{y})(3\sqrt{x} + 2\sqrt{y})}{(3\sqrt{x} - 2\sqrt{y})(3\sqrt{x} + 2\sqrt{y})} \quad \text{Expand.}$$

$$= \frac{3\sqrt{x^2} + 2\sqrt{xy} + 3\sqrt{xy} + 2\sqrt{y^2}}{(3\sqrt{x})^2 - (2\sqrt{y})^2} \quad \text{Combine like terms.}$$

$$= \frac{3x + 5\sqrt{xy} + 2y}{9x - 4y}$$

(D) Special product (3) suggests that if we multiply the denominator $\sqrt[3]{m} + 2$ by $(\sqrt[3]{m})^2 - 2\sqrt[3]{m} + 2^2$, we will obtain the sum of two cubes and the denominator will be rationalized.

$$\frac{1}{\sqrt[3]{m} + 2} = \frac{1[(\sqrt[3]{m})^2 - 2\sqrt[3]{m} + 2^2]}{(\sqrt[3]{m} + 2)[(\sqrt[3]{m})^2 - 2\sqrt[3]{m} + 2^2]} \quad \text{Expand.}$$

$$= \frac{\sqrt[3]{m^2} - 2\sqrt[3]{m} + 4}{(\sqrt[3]{m})^3 + 2^3} \quad (\sqrt[n]{x})^n = x$$

$$= \frac{\sqrt[3]{m^2} - 2\sqrt[3]{m} + 4}{m + 8}$$

MATCHED PROBLEM 6

Rationalize denominators.

(A) $\dfrac{6}{\sqrt{2x}}$ (B) $\dfrac{10x^3}{\sqrt[3]{4x}}$ (C) $\dfrac{\sqrt{x}+2}{2\sqrt{x}+3}$ (D) $\dfrac{1}{1-\sqrt[3]{y}}$

ANSWERS TO MATCHED PROBLEMS

1. (A) $\sqrt[5]{u}$ (B) $\sqrt[9]{(6x^2y^5)^2}$ or $(\sqrt[9]{6x^2y^5})^2$ (C) $1/\sqrt[5]{(3xy)^3}$ (D) $(9u)^{1/4}$ (E) $-(2x)^{4/7}$ (F) $(x^3+y^3)^{1/3}$
2. (A) u^2+v^2 (B) $2\sqrt{3}$ (C) $(\sqrt[3]{x^2})/2$ or $\frac{1}{2}\sqrt[3]{x^2}$
3. (A) $3x^2yz\sqrt{2xz}$ (B) $3a^2b\sqrt[4]{b^2} = 3a^2b\sqrt{b}$ (C) $\sqrt[3]{2x^2y}$ (D) $\sqrt[3]{2}$
4. (A) $8\sqrt{2}$ (B) $-5\sqrt[5]{2x^2y^3}$ (C) $3\sqrt[3]{mn^2}+4\sqrt{mn}$
5. (A) $3\sqrt{2}-4\sqrt{3}$ (B) $2\sqrt{3}-5$ (C) $y+2\sqrt{y}-8$ (D) $x+\sqrt[3]{x^2y}-\sqrt[3]{xy^2}-y$
6. (A) $\dfrac{3\sqrt{2x}}{x}$ (B) $5x^2\sqrt[3]{2x^2}$ (C) $\dfrac{2x+\sqrt{x}-6}{4x-9}$ (D) $\dfrac{1+\sqrt[3]{y}+\sqrt[3]{y^2}}{1-y}$

A-3 Exercises

*Additional answers can be found in the Instructor Answer Appendix.

A

Unless stated to the contrary, all variables represent positive real numbers

In Problems 1–8, change to radical form. Do not simplify.

1. $32^{1/5}$ $\sqrt[5]{32}$ **2.** $625^{3/4}$ $\sqrt[4]{625^3}$ **3.** $8x^{2/3}$ $8\sqrt[3]{x^2}$

4. $(3x^2y)^{3/7}$ **5.** $4x^{-1/2}$ $4/\sqrt{x}$ **6.** $32y^{-2/5}$ $32\sqrt[5]{y^{-2}}$

7. $x^{1/3}-y^{1/3}$ $\sqrt[3]{x}-\sqrt[3]{y}$ **8.** $(x-y)^{1/3}$ $\sqrt[3]{x-y}$

In Problems 9–14, change to rational exponent form. Do not simplify.

9. $\sqrt{361}$ $361^{1/2}$ **10.** $\sqrt[3]{17^2}$ $17^{2/3}$ **11.** $4x\sqrt[5]{y^3}$ $4xy^{3/5}$

12. $\sqrt[4]{7x^3y^2}$ $(7x^3y^2)^{1/4}$ **13.** $\sqrt[3]{x^2+y^2}$ $(x^2+y^2)^{1/3}$ **14.** $\sqrt[3]{x^2}+\sqrt[3]{y^2}$ $(x^2)^{1/3}+(y^2)^{1/3}$

In Problems 15–34, write in simplified form.

15. $\sqrt[3]{-8}$ -2 **16.** $\sqrt[3]{-27}$ -3 **17.** $-\sqrt{128}$ $-8\sqrt{2}$

18. $-\sqrt{125}$ $-5\sqrt{5}$ **19.** $\sqrt{27}-5\sqrt{3}$ **20.** $2\sqrt{8}+\sqrt{18}$ $7\sqrt{2}$

21. $\sqrt[3]{5}-\sqrt[3]{25}+\sqrt[3]{625}$ **22.** $\sqrt{20}+\sqrt[3]{40}-\sqrt[3]{5}$

23. $\sqrt[3]{25}\sqrt[3]{10}$ **24.** $\sqrt{6}\sqrt{14}$ $2\sqrt{21}$ **25.** $\sqrt{9x^8y^4}$ $3x^4y^2$

26. $\sqrt{16m^4y^8}$ $4m^2y^4$ **27.** $\sqrt[4]{16m^4n^8}$ $2mn^2$ **28.** $\sqrt[5]{32a^{15}b^{10}}$ $2a^3b^2$

29. $\sqrt[4]{m^2}$ $\sqrt{m}$ **30.** $\sqrt[10]{n^6}$ $\sqrt[5]{n^3}$ **31.** $\sqrt[5]{\sqrt[3]{xy}}$ $\sqrt[15]{xy}$

32. $\sqrt{\sqrt[4]{5x}}$ $\sqrt[8]{5x}$ **33.** $\sqrt[3]{9x^2}\sqrt[3]{9x}$ $3x\sqrt[3]{3}$ **34.** $\sqrt{2x}\sqrt{8xy}$ $4x\sqrt{y}$

In Problems 35–46, rationalize denominators, and write in simplified form.

35. $\dfrac{1}{2\sqrt{5}}$ $\dfrac{\sqrt{5}}{10}$ **36.** $\dfrac{1}{3\sqrt{7}}$ $\dfrac{\sqrt{7}}{21}$ **37.** $\dfrac{1}{\sqrt[3]{7}}$ $\dfrac{\sqrt[3]{49}}{7}$

38. $\dfrac{10}{\sqrt[4]{5}}$ $2\sqrt[4]{125}$ **39.** $\dfrac{2}{\sqrt{44}}$ $\dfrac{\sqrt{11}}{11}$ **40.** $\dfrac{3}{\sqrt[3]{54}}$ $\dfrac{\sqrt[3]{4}}{2}$

41. $\dfrac{6x}{\sqrt{3x}}$ $2\sqrt{3x}$ **42.** $\dfrac{12y^2}{\sqrt{6y}}$ $2y\sqrt{6y}$ **43.** $\dfrac{2}{\sqrt{2}-1}$ $2\sqrt{2}+2$

44. $\dfrac{4}{\sqrt{6}-2}$ $2\sqrt{6}+4$ **45.** $\dfrac{\sqrt{2}}{\sqrt{6}+2}$ $\sqrt{3}-\sqrt{2}$ **46.** $\dfrac{\sqrt{2}}{\sqrt{10}-2}$ $\dfrac{\sqrt{5}+\sqrt{2}}{3}$

B

In Problems 47–54, write in simplified form.

47. $x\sqrt[5]{3^6x^7y^{11}}$ $3x^2y^2\sqrt[5]{3x^2y}$ **48.** $2a\sqrt[3]{8a^8b^{13}}$ $4a^3b^4\sqrt[3]{a^2b}$ **49.** $\dfrac{\sqrt[4]{32m^7n^9}}{2mn}$ $n\sqrt[4]{2m^3n}$

50. $\dfrac{\sqrt[5]{32u^{12}v^8}}{uv}$ $2u\sqrt[5]{u^2v^3}$ **51.** $\sqrt[3]{\sqrt[4]{a^9b^3}}$ $\sqrt[4]{a^3b}$ **52.** $\sqrt{\sqrt[6]{x^8y^6}}$ $\sqrt[6]{x^4y^3}$

53. $\sqrt[3]{2x^2y^4}\sqrt[3]{3x^5y}$ $x^2y\sqrt[3]{6xy^2}$

54. $\sqrt[4]{4m^5n}\sqrt[4]{6m^3n^4}$ $m^2n\sqrt[4]{24n}$

In Problems 55–60, rationalize denominators and write in simplified form.

55. $\dfrac{\sqrt{2m}\sqrt{5}}{\sqrt{20m}}$

56. $\dfrac{\sqrt{6}\sqrt{8c}}{\sqrt{18c}}$ $2\sqrt{6}$ 3

57. $\dfrac{3\sqrt{y}}{2\sqrt{y}-3}$

58. $\dfrac{5\sqrt{x}}{3-2\sqrt{x}}$

59. $\dfrac{2\sqrt{5}+3\sqrt{2}}{5\sqrt{5}+2\sqrt{2}}$ $(38+11\sqrt{10})/117$

60. $\dfrac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{3}-2\sqrt{2}}$

Problems 61–64 are calculus-related. Rationalize the numerators; that is, perform operations on the fractions that eliminate radicals from the numerators. (This is a particularly useful operation in some problems in calculus.)

61. $\dfrac{\sqrt{t}-\sqrt{x}}{t-x}$ $1/(\sqrt{t}+\sqrt{x})$

62. $\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ $\dfrac{x-y}{x+2\sqrt{xy}+y}$

63. $\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$ $1/(\sqrt{x+h}+\sqrt{x})$

64. $\dfrac{\sqrt{2+h}+\sqrt{2}}{h}$ $\dfrac{1}{\sqrt{2+h}-\sqrt{2}}$

In Problems 65–72, evaluate to four significant digits using a calculator. (Read the instruction booklet accompanying your calculator for the process required to evaluate $\sqrt[n]{x}$.)

65. $\sqrt{0.032\ 965}$ 0.1816

66. $\sqrt{419.763}$ 20.49

67. $\sqrt[3]{45.0218}$ 3.557

68. $\sqrt[4]{0.098\ 553}$ 0.5603

69. $\sqrt[8]{5.477\times10^{-9}}$

70. $\sqrt[7]{4.892\times10^{16}}$ 242.2

71. $\sqrt[5]{9}+\sqrt[5]{9}$ 1.602

72. $\sqrt[3]{2}+\sqrt[3]{2}$ 1.483

C

For what real numbers are Problems 73–76 true?

73. $\sqrt{x^2}=-x$ $x\le0$

74. $\sqrt{x^2}=x$ $x\ge0$

75. $\sqrt[3]{x^3}=x$ All real numbers

76. $\sqrt[3]{x^3}=-x$ $x=0$

In Problems 77 and 78, evaluate each expression on a calculator and determine which pairs have the same value. Verify these results algebraically.

77. (A) $\sqrt{3}+\sqrt{5}$ (B) $\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}$
(C) $1+\sqrt{3}$ (D) $\sqrt[3]{10+6\sqrt{3}}$
(E) $\sqrt{8+\sqrt{60}}$ (F) $\sqrt{6}$ A and E, B and F, C and D

78. (A) $2\sqrt[3]{2+\sqrt{5}}$ (B) $\sqrt{8}$
(C) $\sqrt{3}+\sqrt{7}$ (D) $\sqrt{3+\sqrt{8}}+\sqrt{3-\sqrt{8}}$
(E) $\sqrt{10+\sqrt{84}}$ (F) $1+\sqrt{5}$
A and F, B and D, C and E

In Problems 79–82, rationalize denominators.

79. $\dfrac{1}{\sqrt[3]{a}-\sqrt[3]{b}}$ $\dfrac{\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}}{a-b}$

80. $\dfrac{1}{\sqrt[3]{m}+\sqrt[3]{n}}$ $\dfrac{\sqrt[3]{m^2}-\sqrt[3]{mn}+\sqrt[3]{n^2}}{m+n}$

81. $\dfrac{1}{\sqrt{x}-\sqrt{y}+\sqrt{z}}$

82. $\dfrac{1}{\sqrt{x}+\sqrt{y}-\sqrt{z}}$

[*Hint for Problem 81:* Start by multiplying numerator and denominator by $(\sqrt{x}-\sqrt{y})-\sqrt{z}$.]

Problems 83 and 84 are calculus-related. Rationalize numerators.

83. $\dfrac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}$

84. $\dfrac{\sqrt[3]{t}-\sqrt[3]{x}}{t-x}$ $\dfrac{1}{\sqrt[3]{t^2}+\sqrt[3]{t}\sqrt[3]{x}+\sqrt[3]{x^2}}$

85. Show that $\sqrt[kn]{x^{km}}=\sqrt[n]{x^m}$ for k, m, and n natural numbers greater than 1.

86. Show that $\sqrt[m]{\sqrt[n]{x}}=\sqrt[mn]{x}$ for m and n natural numbers greater than 1. $\sqrt[m]{\sqrt[n]{x}}=((x^{1/n})^{1/m}=x^{1/mn}=\sqrt[mn]{x}$

APPLICATIONS

87. **PHYSICS—RELATIVISTIC MASS** The mass M of an object moving at a velocity v is given by

$$M=\frac{M_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$$

where M_0 = mass at rest and c = velocity of light. The mass of an object increases with velocity and tends to infinity as the velocity approaches the speed of light. Show that M can be written in the form

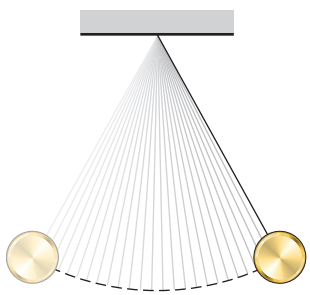
$$M=\frac{M_0c\sqrt{c^2-v^2}}{c^2-v^2}$$

88. **PHYSICS—PENDULUM** A simple pendulum is formed by hanging a bob of mass M on a string of length L from a fixed support (see the figure). The time it takes the bob to swing from right to left and back again is called the **period** T and is given by

$$T=2\pi\sqrt{\frac{L}{g}}$$

where g is the gravitational constant. Show that T can be written in the form

$$T=\frac{2\pi\sqrt{gL}}{g}$$



A-4 Polynomials: Basic Operations

- Polynomials
- Combining Like Terms
- Addition and Subtraction
- Multiplication
- Application

In this section, we review the basic operations on *polynomials*. Polynomials are expressions such as $x^4 - 5x^2 + 1$ or $3xy - 2x + 5y + 6$ that are built from constants and variables using only addition, subtraction, and multiplication (the power x^4 is the product $x \cdot x \cdot x \cdot x$). Polynomials are used throughout mathematics to describe and approximate mathematical relationships.

Polynomials

Algebraic expressions are formed by using constants and variables and the algebraic operations of addition, subtraction, multiplication, division, raising to powers, and taking roots. Some examples are

$$\sqrt[3]{x^3 + 5} \qquad 5x^4 + 2x^2 - 7$$
$$x + y - 7 \qquad (2x - y)^2$$
$$\frac{x - 5}{x^2 + 2x - 5} \qquad 1 + \frac{1}{1 + \frac{1}{x}}$$

An algebraic expression involving only the operations of addition, subtraction, multiplication, and raising to natural number powers is called a **polynomial.** (Note that raising to a natural number power is repeated multiplication.) Some examples are

$$2x - 3 \qquad 4x^2 - 3x + 7$$
$$x - 2y \qquad 5x^3 - 2x^2 - 7x + 9$$
$$5 \qquad x^2 - 3xy + 4y^2$$
$$0 \qquad x^3 - 3x^2y + xy^2 + 2y^7$$

In a polynomial, a variable cannot appear in a denominator, as an exponent, or within a radical. Accordingly, a **polynomial in one variable** x is constructed by adding or subtracting constants and terms of the form ax^n, where a is a real number and n is a natural number. A **polynomial in two variables** x and y is constructed by adding and subtracting constants and terms of the form ax^my^n, where a is a real number and m and n are natural numbers. Polynomials in three or more variables are defined in a similar manner.

Polynomials can be classified according to their *degree*. If a term in a polynomial has only one variable as a factor, then the **degree of that term** is the power of the variable. If two or more variables are present in a term as factors, then the **degree of the term** is the sum of the powers of the variables. The **degree of a polynomial** is the degree of the nonzero term with the highest degree in the polynomial. Any nonzero constant is defined to be a **polynomial of degree 0.** The number 0 is also a polynomial but is not assigned a degree.

EXAMPLE 1 Polynomials and Nonpolynomials

(A) Polynomials in one variable:

$$x^2 - 3x + 2 \qquad 6x^3 - \sqrt{2}x - \tfrac{1}{3}$$

(B) Polynomials in several variables:

$$3x^2 - 2xy + y^2 \qquad 4x^3y^2 - \sqrt{3}xy^2z^5$$

(C) Nonpolynomials:

$$\sqrt{2x} - \frac{3}{x} + 5 \qquad \frac{x^2 - 3x + 2}{x - 3} \qquad \sqrt{x^2 - 3x + 1}$$

(D) The degree of the first term in $6x^3 - \sqrt{2}x - \frac{1}{3}$ is 3, the degree of the second term is 1, the degree of the third term is 0, and the degree of the whole polynomial is 3.

(E) The degree of the first term in $4x^3y^2 - \sqrt{3}xy^2$ is 5, the degree of the second term is 3, and the degree of the whole polynomial is 5.

MATCHED PROBLEM 1

(A) Which of the following are polynomials?

$$3x^2 - 2x + 1 \qquad \sqrt{x - 3} \qquad x^2 - 2xy + y^2 \qquad \frac{x - 1}{x^2 + 2}$$

(B) Given the polynomial $3x^5 - 6x^3 + 5$, what is the degree of the first term? The second term? The whole polynomial?

(C) Given the polynomial $6x^4y^2 - 3xy^3$, what is the degree of the first term? The second term? The whole polynomial?

In addition to classifying polynomials by degree, we also call a single-term polynomial a **monomial,** a two-term polynomial a **binomial,** and a three-term polynomial a **trinomial.**

$\frac{5}{2}x^2y^3$ Monomial

$x^3 + 4.7$ Binomial

$x^4 - \sqrt{2}x^2 + 9$ Trinomial

Combining Like Terms

We start with a word about *coefficients.* A constant in a term of a polynomial, including the sign that precedes it, is called the **numerical coefficient,** or simply, the **coefficient,** of the term. If a constant doesn't appear, or only a + sign appears, the coefficient is understood to be 1. If only a − sign appears, the coefficient is understood to be −1. Thus, given the polynomial

$$2x^4 - 4x^3 + x^2 - x + 5 \qquad 2x^4 + (-4)x^3 + 1x^2 + (-1)x + 5$$

the coefficient of the first term is 2, the coefficient of the second term is −4, the coefficient of the third term is 1, the coefficient of the fourth term is −1, and the coefficient of the last term is 5.

At this point, it is useful to state two additional distributive properties of real numbers that follow from the distributive properties stated in Section A-1.

ADDITIONAL DISTRIBUTIVE PROPERTIES

1. $a(b - c) = (b - c)a = ab - ac$

2. $a(b + c + \cdots + f) = ab + ac + \cdots + af$

Two terms in a polynomial are called **like terms** if they have exactly the same variable factors to the same powers. The numerical coefficients may or may not be the same. Since constant terms involve no variables, all constant terms are like terms. If a polynomial contains two or more like terms, these terms can be combined into a single term by making use of distributive properties. Consider the following example:

$$\begin{aligned} 5x^3y - 2xy - x^3y - 2x^3y &= 5x^3y - x^3y - 2x^3y - 2xy && \text{Group like terms.} \\ &= (5x^3y - x^3y - 2x^3y) - 2xy && \text{Factor out } x^3y. \\ &= (5 - 1 - 2)x^3y - 2xy && \text{Simplify.} \\ &= 2x^3y - 2xy \end{aligned}$$

It should be clear that free use has been made of the real number properties discussed earlier. The steps done in the dashed box are usually done mentally, and the process is quickly mechanized as follows:

Like terms in a polynomial are combined by adding their numerical coefficients.

EXAMPLE 2 Simplifying Polynomials

Remove parentheses and combine like terms:

(A) $2(3x^2 - 2x + 5) + (x^2 + 3x - 7)$

$= 2(3x^2 - 2x + 5) + 1(x^2 + 3x - 7)$ Distribute, remove parentheses.

$= 6x^2 - 4x + 10 + x^2 + 3x - 7$ Combine like terms.

$= 7x^2 - x + 3$

(B) $(x^3 - 2x - 6) - (2x^3 - x^2 + 2x - 3)$

$= 1(x^3 - 2x - 6) + (-1)(2x^3 - x^2 + 2x - 3)$ Distribute, remove parentheses.

$= x^3 - 2x - 6 - 2x^3 + x^2 - 2x + 3$ Combine like terms.

$= -x^3 + x^2 - 4x - 3$

(C) $[3x^2 - (2x + 1)] - (x^2 - 1) = [3x^2 - 2x - 1] - (x^2 - 1)$ Remove parentheses.

$= 3x^2 - 2x - 1 - x^2 + 1$ Combine like terms.

$= 2x^2 - 2x$

MATCHED PROBLEM 2

Remove parentheses and combine like terms:

(A) $3(u^2 - 2v^2) + (u^2 + 5v^2)$

(B) $(m^3 - 3m^2 + m - 1) - (2m^3 - m + 3)$

(C) $(x^3 - 2) - [2x^3 - (3x + 4)]$

› Addition and Subtraction

Addition and subtraction of polynomials can be thought of in terms of removing parentheses and combining like terms, as illustrated in Example 2. Horizontal and vertical arrangements are illustrated in the next two examples. You should be able to work either way, letting the situation dictate the choice.

EXAMPLE 3 Adding Polynomials

Add: $x^4 - 3x^3 + x^2$, $-x^3 - 2x^2 + 3x$, and $3x^2 - 4x - 5$

SOLUTION

Add horizontally:

$(x^4 - 3x^3 + x^2) + (-x^3 - 2x^2 + 3x) + (3x^2 - 4x - 5)$ Remove parentheses.

$= x^4 - 3x^3 + x^2 - x^3 - 2x^2 + 3x + 3x^2 - 4x - 5$ Combine like terms.

$= x^4 - 4x^3 + 2x^2 - x - 5$

Or vertically, by lining up like terms and adding their coefficients:

$$\begin{array}{r} x^4 - 3x^3 + x^2 \\ -\ x^3 - 2x^2 + 3x \\ \hline 3x^2 - 4x - 5 \\ \hline x^4 - 4x^3 + 2x^2 - x - 5 \end{array}$$

MATCHED PROBLEM 3

Add horizontally and vertically:

$$3x^4 - 2x^3 - 4x^2, \quad x^3 - 2x^2 - 5x, \quad \text{and} \quad x^2 + 7x - 2$$

EXAMPLE 4 Subtracting Polynomials

Subtract: $4x^2 - 3x + 5$ from $x^2 - 8$

SOLUTION

$$(x^2 - 8) - (4x^2 - 3x + 5)$$
$$= x^2 - 8 - 4x^2 + 3x - 5$$
$$= -3x^2 + 3x - 13$$

or

$$\begin{array}{r} x^2 - 8 \\ -4x^2 + 3x - 5 \\ \hline -3x^2 + 3x - 13 \end{array}$$

← Change signs and add.

MATCHED PROBLEM 4

Subtract: $2x^2 - 5x + 4$ from $5x^2 - 6$

›››CAUTION›››

When you use a horizontal arrangement to subtract a polynomial with more than one term, you must enclose the polynomial in parentheses. Therefore, to subtract $2x + 5$ from $4x - 11$, you must write

$$4x - 11 - (2x + 5) \quad \text{and not} \quad 4x - 11 - 2x + 5$$

› Multiplication

Multiplication of algebraic expressions involves the extensive use of distributive properties for real numbers, as well as other real number properties.

EXAMPLE 5 Multiplying Polynomials

Multiply: $(2x - 3)(3x^2 - 2x + 3)$

SOLUTION

$$(2x - 3)(3x^2 - 2x + 3)$$

$$= 2x(3x^2 - 2x + 3) - 3(3x^2 - 2x + 3)$$ Distribute, remove parentheses.

$$= 6x^3 - 4x^2 + 6x - 9x^2 + 6x - 9$$ Combine like terms.

$$= 6x^3 - 13x^2 + 12x - 9$$

Or, using a vertical arrangement,

$$\begin{array}{r} 3x^2 - 2x + 3 \\ 2x - 3 \\ \hline 6x^3 - 4x^2 + 6x \\ \quad - 9x^2 + 6x - 9 \\ \hline 6x^3 - 13x^2 + 12x - 9 \end{array}$$

MATCHED PROBLEM 5

Multiply:

$$(2x - 3)(2x^2 + 3x - 2)$$

To multiply two polynomials, multiply each term of one by each term of the other, and combine like terms.

EXAMPLE 6 Multiplying Binomials

Multiply:

(A) $(2x - 3y)(5x + 2y) = 10x^2 + 4xy - 15xy - 6y^2 = 10x^2 - 11xy - 6y^2$

(B) $(3a - 2b)(3a + 2b) = (3a)^2 - (2b)^2 = 9a^2 - 4b^2$

(C) $(5x - 3)^2 = (5x)^2 - 2(5x)(3) + 3^2 = 25x^2 - 30x + 9$

(D) $(m + 2n)^2 = m^2 + 4mn + 4n^2$

MATCHED PROBLEM 6

Multiply:

(A) $(4u - 3v)(2u + v)$ (B) $(2xy + 3)(2xy - 3)$

(C) $(m + 4n)(m - 4n)$ (D) $(2u - 3v)^2$ (E) $(6x + y)^2$

Products of certain binomial factors occur so frequently that it is useful to remember formulas for their products. The following formulas are easily verified by multiplying the factors on the left:

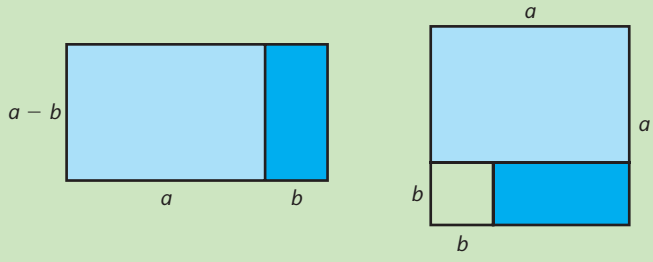
SPECIAL PRODUCTS

1. $(a - b)(a + b) = a^2 - b^2$
2. $(a + b)^2 = a^2 + 2ab + b^2$
3. $(a - b)^2 = a^2 - 2ab + b^2$

EXPLORE-DISCUSS 1

(A) Explain the relationship between special product formula 1 and the areas of the rectangles in the figures.

$$(a - b)(a + b) \quad = \quad a^2 - b^2$$



(B) Construct similar figures to provide geometric interpretations for special product formulas 2 and 3.

The following conventions govern the order in which algebraic operations are performed.

› ORDER OF OPERATIONS

1. Simplify inside the innermost grouping first, then the next innermost, and so on.

$$\begin{aligned} 2[3 - (x - 4)] &= 2[3 - x + 4] \\ &= 2(7 - x) = 14 - 2x \end{aligned}$$

2. Unless grouping symbols indicate otherwise, apply exponents before multiplication or division is performed.

$$2(x - 2)^2 = 2(x^2 - 4x + 4) = 2x^2 - 8x + 8$$

3. Unless grouping symbols indicate otherwise, perform multiplication and division before addition and subtraction. In either case, proceed from left to right.

$$5 - 2(x - 3) = 5 - 2x + 6 = 11 - 2x$$

EXAMPLE 7 Combined Operations

Perform the indicated operations and simplify:

(A) $3x - \{5 - 3[x - x(3 - x)]\}$

$= 3x - \{5 - 3[x - 3x + x^2]\}$ Combine like terms.

$= 3x - \{5 - 3[-2x + x^2]\}$ Distribute, remove square brackets.

$= 3x - \{5 + 6x - 3x^2\}$ Distribute, remove braces.

$= 3x - 5 - 6x + 3x^2$ Combine like terms.

$= 3x^2 - 3x - 5$

(B) $(x - 2y)(2x + 3y) - (2x + y)^2$

$= 2x^2 + 3xy - 4xy - 6y^2 - (4x^2 + 4xy + y^2)$ Distribute, remove parentheses.

$= 2x^2 - xy - 6y^2 - 4x^2 - 4xy - y^2$ Combine like terms.

$= -2x^2 - 5xy - 7y^2$

(C) $(2m + 3n)^3 = (2m + 3n)(2m + 3n)^2$ Expand the square.

$= (2m + 3n)(4m^2 + 12mn + 9n^2)$ Expand.

$= 8m^3 + 24m^2n + 18mn^2 + 12m^2n + 36mn^2 + 27n^3$ Combine like terms.

$= 8m^3 + 36m^2n + 54mn^2 + 27n^3$

MATCHED PROBLEM 7

Perform the indicated operations and simplify:

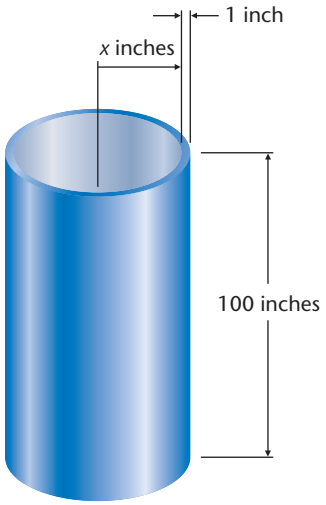
(A) $2t - \{7 - 2[t - t(4 + t)]\}$ (B) $(u - 3v)^2 - (2u - v)(2u + v)$

(C) $(4x - y)^3$

› Application

EXAMPLE 8 Volume of a Cylindrical Shell

A plastic water pipe with a hollow center is 100 inches long, 1 inch thick, and has an inner radius of x inches (see the figure). Write an algebraic expression in terms of x that represents the volume of the plastic used to construct the pipe. Simplify the expression. [*Recall:* The volume V of a right circular cylinder of radius r and height h is given by $V = \pi r^2 h$.]



SOLUTION

A right circular cylinder with a hollow center is called a *cylindrical shell.* The volume of the shell is equal to the volume of the cylinder minus the volume of the hole. Since the radius of the hole is x inches and the pipe is 1 inch thick, the radius of the cylinder is $x + 1$ inches. Therefore, we have

$$\begin{pmatrix}\text{Volume of}\\ \text{shell}\end{pmatrix} = \begin{pmatrix}\text{Volume of}\\ \text{cylinder}\end{pmatrix} - \begin{pmatrix}\text{Volume of}\\ \text{hole}\end{pmatrix}$$

$$\begin{aligned}\text{Volume} &= \pi(x + 1)^2 100 - \pi x^2 100 && \text{Expand the square.}\\ &= 100\pi(x^2 + 2x + 1) - 100\pi x^2 && \text{Distribute, remove parentheses.}\\ &= 100\pi x^2 + 200\pi x + 100\pi - 100\pi x^2 && \text{Combine like terms.}\\ &= 200\pi x + 100\pi\end{aligned}$$

MATCHED PROBLEM 8

A plastic water pipe is 200 inches long, 2 inches thick, and has an outer radius of x inches. Write an algebraic expression in terms of x that represents the volume of the plastic used to construct the pipe. Simplify the expression.

ANSWERS TO MATCHED PROBLEMS

1. (A) $3x^2 - 2x + 1, x^2 - 2xy + y^2$ (B) 5, 3, 5 (C) 6, 4, 6
2. (A) $4u^2 - v^2$ (B) $-m^3 - 3m^2 + 2m - 4$ (C) $-x^3 + 3x + 2$
3. $3x^4 - x^3 - 5x^2 + 2x - 2$ **4.** $3x^2 + 5x - 10$ **5.** $4x^3 - 13x + 6$
6. (A) $8u^2 - 2uv - 3v^2$ (B) $4x^2y^2 - 9$ (C) $m^2 - 16n^2$
(D) $4u^2 - 12uv + 9v^2$ (E) $36x^2 + 12xy + y^2$
7. (A) $-2t^2 - 4t - 7$ (B) $-3u^2 - 6uv + 10v^2$ (C) $64x^3 - 48x^2y + 12xy^2 - y^3$
8. Volume $= 200\pi x^2 - 200\pi(x - 2)^2 = 800\pi x - 800\pi$

A-4 Exercises

*Additional answers can be found in the Instructor Answer Appendix.

A

Problems 1–8 refer to the following polynomials: (a) $x^4 - 2x^2 + 3$ and (b) $x^3 - 1$.

1. What is the degree of (a)? 4

2. What is the degree of (b)? 3

3. What is the degree of the sum of (a) and (b)? 4

4. What is the degree of the product of (a) and (b)? 7

5. Multiply (a) and (b). $x^7 - 2x^3 - x^4 + 3x^3 + 2x^2 - 3$

6. Add (a) and (b). $x^4 + x^3 - 2x^2 + 2$

7. Subtract (b) from (a). $x^4 - x^3 - 2x^2 + 4$

8. Subtract (a) from (b). $-x^4 + x^3 + 2x^2 - 4$

In Problems 9–14, is the algebraic expression a polynomial? If so, give its degree.

9. $6 - 5x + x^2$ Yes; 2

10. $3x^2 + 5x^4 - 8$ Yes; 4

11. $x^3 - 5x + 4\sqrt{x}$ No

12. $x^5 + 7x^3 + \sqrt{2}$ Yes; 5

13. $x^3 + 9x^2 + 5^{-2}$ Yes; 3

14. $2x^4 + 3x^{-1} + 10$ No

In Problems 15–36, perform the indicated operations and simplify.

15. $2(x - 1) + 3(2x - 3) - (4x - 5)$ $4x - 6$

16. $2(u - 1) - (3u + 2) - 2(2u - 3)$ $-5u + 2$

17. $2y - 3y[4 - 2(y - 1)]$ $6y^2 - 16y$

18. $4a - 2a[5 - 3(a + 2)]$ $6a^2 + 6a$

19. $(m - n)(m + n)$ $m^2 - n^2$

20. $(a + b)(a - b)$ $a^2 - b^2$

21. $(4t - 3)(t - 2)$ $4t^2 - 11t + 6$

22. $(3x - 5)(2x + 1)$ $6x^2 - 7x - 5$

23. $(5y - 1)(3 - 2y)$ $-10y^2 + 17y - 3$

24. $(-2 + m)(6m + 5)$ $6m^2 - 7m - 10$

25. $(3x + 2y)(x - 3y)$ $3x^2 - 7xy - 6y^2$

26. $(2x - 3y)(x + 2y)$ $2x^2 + xy - 6y^2$

27. $(2m - 7)(2m + 7)$ $4m^2 - 49$

28. $(3y + 2)(3y - 2)$ $9y^2 - 4$

29. $(6x - 4y)(5x + 3y)$ $30x^2 - 2xy - 12y^2$

30. $(3m + 7n)(2m - 5n)$ $6m^2 - mn - 35n^2$

31. $(3x - 2y)(3x + 2y)$ $9x^2 - 4y^2$

32. $(4m + 3n)(4m - 3n)$ $16m^2 - 9n^2$

33. $(4x - y)^2$ $16x^2 - 8xy + y^2$

34. $(3u + 4v)^2$ $9u^2 + 24uv + 16v^2$

35. $(a + b)(a^2 - ab + b^2)$ $a^3 + b^3$

36. $(a - b)(a^2 + ab + b^2)$ $a^3 - b^3$

B

In Problems 37–50, perform the indicated operations and simplify.

37. $2x - 3\{x + 2[x - (x + 5)] + 1\}$ $-x + 27$

38. $m - \{m - [m - (m - 1)]\}$ 1

39. $2\{3[a - 4(1 - a)] - (5 - a)\}$ $32a - 34$

40. $5b - 3\{-[2 - 4(2b - 1)] + 2(2 - 3b)\}$ $-b + 6$

41. $(2x^2 - 3x + 1)(x^2 + x - 2)$ $2x^4 - x^3 - 6x^2 + 7x - 2$

42. $(x^2 - 3xy + y^2)(x^2 + 3xy + y^2)$ $x^4 - 7x^2y^2 + y^4$

43. $(x - 2y)^2(x + 2y)^2$ $x^4 - 8x^2y^2 + 16y^4$

44. $(n^2 + 4nm + m^2)(n^2 - 4nm + m^2)$ $m^4 - 14n^2m^2 + n^4$

45. $(3u - 2v)^2 - (2u - 3v)(2u + 3v)$ $5u^2 - 12uv + 13v^2$

46. $(2a - b)^2 - (a + 2b)^2$ $3a^2 - 8ab - 3b^2$

47. $(z + 2)(z^2 - 2z + 3) + z - 7$ $z^3 - 1$

48. $(y + 3)(y^2 - 3y + 1) + 8y - 1$ $y^3 - 2$

49. $(2m - n)^3$ $8m^3 - 12m^2n + 6mn^2 - n^2$

50. $(3a + 2b)^3$

Problems 51–58 are calculus-related. Perform the indicated operations and simplify.

51. $3(x + h) - 7 - (3x - 7)$ $3h$

52. $(x + h)^2 - x^2$ $2xh + h^2$

53. $2(x + h)^2 - 3(x + h) - (2x^2 - 3x)$ $4hx - 3h + 2h^2$

54. $-4(x + h)^2 + 6(x + h) - (-4x^2 + 6x)$ $-8xh + 6h - 4h^2$

55. $2(x + h)^2 - 4(x + h) - 9 - (2x^2 - 4x - 9)$ $4hx - 4h + 2h^2$

56. $3(x + h)^2 + 5(x + h) + 7 - (3x^2 + 5x + 7)$ $6hx + 5h + 3h^2$

57. $(x + h)^3 - 2(x + h)^2 - (x^3 - 2x^2)$ $3hx^2 - 4hx + 3h^2x - 2h^2 + h^3$

58. $(x + h)^3 + 3(x + h) - (x^3 + 3x)$ $3hx^2 + 3h^2x + 3h + h^3$

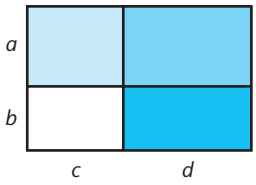
59. Subtract the sum of the first two polynomials from the sum of the last two: $3m^2 - 2m + 5$, $4m^2 - m$, $3m^2 - 3m - 2$, $m^3 + m^2 + 2$. $m^3 - 3m^2 - 5$

60. Subtract the sum of the last two polynomials from the sum of the first two: $2x^2 - 4xy + y^2$, $3xy - y^2$, $x^2 - 2xy - y^2$, $-x^2 + 3xy - 2y^2$. $2x^2 - 2xy + 3y^2$

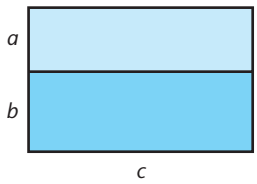
61. Explain the relationship between the equation

$$(a + b)(c + d) = ac + ad + bc + bd$$

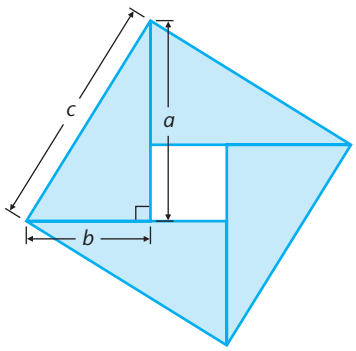
and the rectangles in the figure.



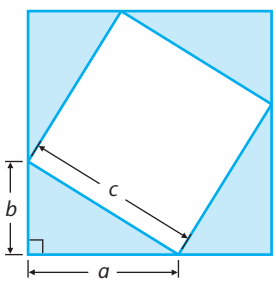
62. Explain the relationship between the distributive property of real numbers and the areas of the rectangles in the figure.



63. In the figure, the area of the large square is equal to the sum of the area of the small square and the areas of the four congruent triangles. Translate this verbal statement into a mathematical equation and simplify. What famous theorem have you proven? The Pythagorean theorem



64. In the figure, the area of the large square is equal to the sum of the area of the small square and the areas of the four congruent triangles. Translate this verbal statement into a mathematical equation and simplify. What famous theorem have you proven? The Pythagorean theorem



C

In Problems 65–68, perform the indicated operations and simplify.

65. $2(x - 2)^3 - (x - 2)^2 - 3(x - 2) - 4$ $2x^3 - 13x^2 + 25x - 18$

66. $(2x - 1)^3 - 2(2x - 1)^2 + 3(2x - 1) + 7$
$8x^3 - 20x^2 + 20x + 1$

67. $-3x\{x[x - x(2 - x)] - (x + 2)(x^2 - 3)\}$ $9x^3 - 9x^2 - 18x$

68. $2\{(x - 3)(x^2 - 2x + 1) - x[3 - x(x - 2)]\}$
$4x^3 - 14x^2 + 8x - 6$

69. Show by example that, in general, $(a + b)^2 \neq a^2 + b^2$. Discuss possible conditions on a and b that would make this a valid equation. $(1 + 1)^2 \neq 1^2 + 1^2$; either a or b must be zero.

70. Show by example that, in general, $(a - b)^2 \neq a^2 - b^2$. Discuss possible conditions on a and b that would make this a valid equation. $(2 - 1)^2 \neq 2^2 - 1^2$; b must be zero

71. If you are given two polynomials, one of degree m and the other of degree n, $m > n$, what is the degree of the sum?
m

72. What is the degree of the product of the two polynomials in Problem 71? $m + n$

73. How does the answer to Problem 71 change if the two polynomials can have the same degree?
Now the degree is less than or equal to m.

74. How does the answer to Problem 72 change if the two polynomials can have the same degree? It doesn't change.

APPLICATIONS

75. **GEOMETRY** The width of a rectangle is 5 centimeters less than its length. If x represents the length, write an algebraic expression in terms of x that represents the perimeter of the rectangle. Simplify the expression.
Perimeter $= 2x + 2(x - 5) = 4x - 10$

76. **GEOMETRY** The length of a rectangle is 8 meters more than its width. If x represents the width of the rectangle, write an algebraic expression in terms of x that represents its area. Change the expression to a form without parentheses. Area $= x^2 + 8x$

★**77.** **COIN PROBLEM** A parking meter contains nickels, dimes, and quarters. There are 5 fewer dimes than nickels, and 2 more quarters than dimes. If x represents the number of nickels, write an algebraic expression in terms of x that represents the value of all the coins in the meter in cents. Simplify the expression.
Value $= 5x + 10(x - 5) + 25(x - 3) = 40x - 125$

★**78.** **COIN PROBLEM** A vending machine contains dimes and quarters only. There are 4 more dimes than quarters. If x represents the number of quarters, write an algebraic expression in terms of x that represents the value of all the coins in the vending machine in cents. Simplify the expression.
Value $= 35x + 40$

79. PACKAGING A spherical plastic container for designer wristwatches has an inner radius of x centimeters (see the figure). If the plastic shell is 0.3 centimeters thick, write an algebraic expression in terms of x that represents the volume of the plastic used to construct the container. Simplify the expression. [*Recall:* The volume V of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.] Volume $= \frac{4}{3}\pi(x + 0.3)^3 - \frac{4}{3}\pi x^3 = 1.2\pi x^2 + 0.36\pi x + 0.036\pi$

80. PACKAGING A cubical container for shipping computer components is formed by coating a metal mold with polystyrene. If the metal mold is a cube with sides x centimeters long and the polystyrene coating is 2 centimeters thick, write an algebraic expression in terms of x that represents the volume of the polystyrene used to construct the container. Simplify the expression. [*Recall:* The volume V of a cube with sides of length t is given by $V = t^3$.] Volume $= 12x^2 + 48x + 64$

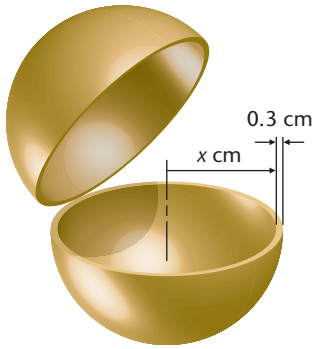


Figure for 79

A-5 Polynomials: Factoring

- Factoring—What Does It Mean?
- Common Factors and Factoring by Grouping
- Factoring Second-Degree Polynomials
- More Factoring

Factoring—What Does It Mean?

A **factor of a number** is one of two or more numbers whose product is the given number. Similarly, a **factor of an algebraic expression** is one of two or more algebraic expressions whose product is the given algebraic expression. For example,

$30 = 2 \cdot 3 \cdot 5$ 2, 3, and 5 are each factors of 30.

$x^2 - 4 = (x - 2)(x + 2)$ $(x - 2)$ and $(x + 2)$ are each factors of $x^2 - 4$.

The process of writing a number or algebraic expression as the product of other numbers or algebraic expressions is called **factoring.** We start our discussion of factoring with the positive integers.

An integer such as 30 can be represented in a factored form in many ways. The products

$$6 \cdot 5 \qquad (\tfrac{1}{2})(10)(6) \qquad 15 \cdot 2 \qquad 2 \cdot 3 \cdot 5$$

all yield 30. A particularly useful way of factoring positive integers greater than 1 is in terms of *prime* numbers.

› DEFINITION 1 Prime and Composite Numbers

An integer greater than 1 is **prime** if its only positive integer factors are itself and 1. An integer greater than 1 that is not prime is called a **composite number.** The integer 1 is neither prime nor composite.

Examples of prime numbers: 2, 3, 5, 7, 11, 13

Examples of composite numbers: 4, 6, 8, 9, 10, 12

››› EXPLORE-DISCUSS 1

In the array below, cross out all multiples of 2, except 2 itself. Then cross out all multiples of 3, except 3 itself. Repeat this for each integer in the array that has not yet been crossed out. Describe the set of numbers that remains when this process is completed.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100

This process is referred to as the **sieve of Eratosthenes.** (Eratosthenes was a Greek mathematician and astronomer who was a contemporary of Archimedes, circa 200 B.C.)

A composite number is said to be **factored completely** if it is represented as a product of prime factors. The only factoring of 30 given on page A-47 that meets this condition is $30 = 2 \cdot 3 \cdot 5$.

EXAMPLE 1 Factoring a Composite Number

Write 60 in completely factored form.

SOLUTION

$$60 = 6 \cdot 10 = 2 \cdot 3 \cdot 2 \cdot 5 = 2^2 \cdot 3 \cdot 5$$

or

$$60 = 5 \cdot 12 = 5 \cdot 4 \cdot 3 = 2^2 \cdot 3 \cdot 5$$

or

$$60 = 2 \cdot 30 = 2 \cdot 2 \cdot 15 = 2^2 \cdot 3 \cdot 5$$

MATCHED PROBLEM 1

Write 180 in completely factored form.

Notice in Example 1 that we end up with the same prime factors for 60 irrespective of how we progress through the factoring process. This illustrates an important property of integers:

THEOREM 1 The Fundamental Theorem of Arithmetic

Each integer greater than 1 is either prime or can be expressed uniquely, except for the order of factors, as a product of prime factors.

We can also write polynomials in completely factored form. A polynomial such as $2x^2 - x - 6$ can be written in factored form in many ways. The products

$$(2x + 3)(x - 2) \qquad 2(x^2 - \tfrac{1}{2}x - 3) \qquad 2(x + \tfrac{3}{2})(x - 2)$$

all yield $2x^2 - x - 6$. A particularly useful way of factoring polynomials is in terms of prime polynomials.

DEFINITION 2 Prime Polynomials

A polynomial of degree greater than 0 is said to be **prime** relative to a given set of numbers if: (1) all of its coefficients are from that set of numbers; and (2) it cannot be written as a product of two polynomials (excluding constant polynomials that are factors of 1) having coefficients from that set of numbers.

Relative to the set of integers:

$x^2 - 2$ is prime
$x^2 - 9$ is not prime, since $x^2 - 9 = (x - 3)(x + 3)$

[*Note:* The set of numbers most frequently used in factoring polynomials is the set of integers.]

A nonprime polynomial is said to be **factored completely relative to a given set of numbers** if it is written as a product of prime polynomials relative to that set of numbers.

Our objective in this section is to review some of the standard factoring techniques for polynomials with integer coefficients. In Chapter 3 we treated in detail the topic of factoring polynomials of higher degree with arbitrary coefficients.

Common Factors and Factoring by Grouping

Example 2 illustrates the use of the distributive properties in factoring.

EXAMPLE 2 **Factoring Out Common Factors**

Factor out, relative to the integers, all factors common to all terms:

(A) $2x^3y - 8x^2y^2 - 6xy^3$ (B) $2x(3x - 2) - 7(3x - 2)$

SOLUTIONS

(A) $2x^3y - 8x^2y^2 - 6xy^3 = (2xy)x^2 - (2xy)4xy - (2xy)3y^2$ Factor out $2xy$.

$= 2xy(x^2 - 4xy - 3y^2)$

(B) $2x(3x - 2) - 7(3x - 2) = 2x(3x - 2) - 7(3x - 2)$ Factor out $3x - 2$.

$= (2x - 7)(3x - 2)$

MATCHED PROBLEM 2

Factor out, relative to the integers, all factors common to all terms:

(A) $3x^3y - 6x^2y^2 - 3xy^3$ (B) $3y(2y + 5) + 2(2y + 5)$

EXAMPLE 3 **Factoring Out Common Factors**

Factor completely, relative to the integers:

$$4(2x + 7)(x - 3)^2 + 2(2x + 7)^2(x - 3)$$

SOLUTION

$4(2x + 7)(x - 3)^2 + 2(2x + 7)^2(x - 3)$ Factor out $2(2x + 7)(x - 3)$.

$= 2(2x + 7)(x - 3)[2(x - 3) + (2x + 7)]$ Distribute, remove parentheses.

$= 2(2x + 7)(x - 3)(2x - 6 + 2x + 7)$ Combine like terms.

$= 2(2x + 7)(x - 3)(4x + 1)$

MATCHED PROBLEM 3

Factor completely, relative to the integers:

$$4(2x + 5)(3x + 1)^2 + 6(2x + 5)^2(3x + 1)$$

Some polynomials can be factored by first grouping terms to obtain an algebraic expression that looks something like Example 2B. We can then complete the factoring by the method used in that example.

EXAMPLE 4 Factoring by Grouping

Factor completely, relative to the integers, by grouping:

(A) $3x^2 - 6x + 4x - 8$ (B) $wy + wz - 2xy - 2xz$

(C) $3ac + bd - 3ad - bc$

SOLUTIONS

(A) $3x^2 - 6x + 4x - 8$ Group the first two and last two terms.

$= (3x^2 - 6x) + (4x - 8)$ Remove common factors from each group.

$= 3x(x - 2) + 4(x - 2)$ Factor out the common factor $(x - 2)$.

$= (3x + 4)(x - 2)$

(B) $wy + wz - 2xy - 2xz$ Group the first two and last two terms—be careful of signs.

$= (wy + wz) - (2xy + 2xz)$ Remove common factors from each group.

$= w(y + z) - 2x(y + z)$ Factor out the common factor $(y + z)$.

$= (w - 2x)(y + z)$

(C) $3ac + bd - 3ad - bc$

In parts A and B the polynomials are arranged in such a way that grouping the first two terms and the last two terms leads to common factors. In this problem, neither the first two terms nor the last two terms have a common factor. Sometimes rearranging terms will lead to a factoring by grouping. In this case, we interchange the second and fourth terms to obtain a problem comparable to part B, which can be factored as follows:

$3ac - bc - 3ad + bd = (3ac - bc) - (3ad - bd)$ Factor out c, d.

$= c(3a - b) - d(3a - b)$ Factor out $3a - b$.

$= (c - d)(3a - b)$

MATCHED PROBLEM 4

Factor completely, relative to the integers, by grouping:

(A) $2x^2 + 6x + 5x + 15$ (B) $2pr + ps - 6qr - 3qs$

(C) $6wy - xz - 2xy + 3wz$

Factoring Second-Degree Polynomials

We now turn our attention to factoring second-degree polynomials of the form

$$2x^2 - 5x - 3 \qquad \text{and} \qquad 2x^2 + 3xy - 2y^2$$

into the product of two first-degree polynomials with integer coefficients. Example 5 will illustrate an approach to the problem.

EXAMPLE 5 Factoring Second-Degree Polynomials

Factor each polynomial, if possible, using integer coefficients:

(A) $2x^2 + 3xy - 2y^2$ (B) $x^2 - 3x + 4$ (C) $6x^2 + 5xy - 4y^2$

SOLUTIONS

(A) $2x^2 + 3xy - 2y^2 = (2x + y)(x - y)$

↑ ↑
? ?

Put in what we know. Signs must be opposite. (We can reverse this choice if we get $-3xy$ instead of $+3xy$ for the middle term.)

Now, what are the factors of 2 (the coefficient of y^2)?

$$\begin{array}{r} 2 \\ \hline 1 \cdot 2 \\ 2 \cdot 1 \end{array}$$

$(2x + y)(x - 2y) = 2x^2 - 3xy - 2y^2$

$(2x + 2y)(x - y) = 2x^2 - 2y^2$

The first choice gives us $-3xy$ for the middle term—close, but not there—so we reverse our choice of signs to obtain

$$2x^2 + 3xy - 2y^2 = (2x - y)(x + 2y)$$

(B) $x^2 - 3x + 4 = (x -)(x -)$

Signs must be the same because the third term is positive and must be negative because the middle term is negative.

$$\begin{array}{r} 4 \\ \hline 2 \cdot 2 \\ 1 \cdot 4 \\ 4 \cdot 1 \end{array}$$

$(x - 2)(x - 2) = x^2 - 4x + 4$

$(x - 1)(x - 4) = x^2 - 5x + 4$

$(x - 4)(x - 1) = x^2 - 5x + 4$

No choice produces the middle term; hence $x^2 - 3x + 4$ is not factorable using integer coefficients.

(C) $6x^2 + 5xy - 4y^2 = (\ x + \ y)(\ x - \ y)$

↑ ↑ ↑ ↑
? ? ? ?

The signs must be opposite in the factors, because the third term is negative. We can reverse our choice of signs later if necessary. We now write all factors of 6 and of 4:

6	4
$2 \cdot 3$	$2 \cdot 2$
$3 \cdot 2$	$1 \cdot 4$
$1 \cdot 6$	$4 \cdot 1$
$6 \cdot 1$	

and try each choice on the left with each on the right—a total of 12 combinations that give us the first and last terms in the polynomial $6x^2 + 5xy - 4y^2$. The question is: Does any combination also give us the middle term, $5xy$? After trial and error and, perhaps, some educated guessing among the choices, we find that $3 \cdot 2$ matched with $4 \cdot 1$ gives us the correct middle term. Thus,

$$6x^2 + 5xy - 4y^2 = (3x + 4y)(2x - y)$$

If none of the 24 combinations (including reversing our sign choice) had produced the middle term, then we would conclude that the polynomial is not factorable using integer coefficients.

MATCHED PROBLEM 5

Factor each polynomial, if possible, using integer coefficients:

(A) $x^2 - 8x + 12$ (B) $x^2 + 2x + 5$

(C) $2x^2 + 7xy - 4y^2$ (D) $4x^2 - 15xy - 4y^2$

More Factoring

The special factoring formulas listed here will enable us to factor certain polynomial forms that occur frequently.

SPECIAL FACTORING FORMULAS

1. $u^2 + 2uv + v^2 = (u + v)^2$	**Perfect Square**
2. $u^2 - 2uv + v^2 = (u - v)^2$	**Perfect Square**
3. $u^2 - v^2 = (u - v)(u + v)$	**Difference of Squares**
4. $u^3 - v^3 = (u - v)(u^2 + uv + v^2)$	**Difference of Cubes**
5. $u^3 + v^3 = (u + v)(u^2 - uv + v^2)$	**Sum of Cubes**

The formulas in the box can be established by multiplying the factors on the right.

››› CAUTION ›››

Note that we did not list a special factoring formula for the sum of two squares. In general,

$$u^2 + v^2 \neq (au + bv)(cu + dv)$$

for any choice of real number coefficients a, b, c, and d. In Chapter 1 we will see that $u^2 + v^2$ can be factored using complex numbers.

EXAMPLE 6 Using Special Factoring Formulas

Factor completely relative to the integers:

(A) $x^2 + 6xy + 9y^2$ (B) $9x^2 - 4y^2$ (C) $8m^3 - 1$ (D) $x^3 + y^3z^3$

SOLUTIONS

(A) $x^2 + 6xy + 9y^2 = x^2 + 2(x)(3y) + (3y)^2 = (x + 3y)^2$ Perfect square

(B) $9x^2 - 4y^2 = (3x)^2 - (2y)^2 = (3x - 2y)(3x + 2y)$ Difference of squares

(C) $8m^3 - 1 = (2m)^3 - 1^3$ Difference of cubes
$= (2m - 1)[(2m)^2 + (2m)(1) + 1^2]$ Simplify.
$= (2m - 1)(4m^2 + 2m + 1)$

(D) $x^3 + y^3z^3 = x^3 + (yz)^3$ Sum of cubes
$= (x + yz)(x^2 - xyz + y^2z^2)$

MATCHED PROBLEM 6

Factor completely relative to the integers:

(A) $4m^2 - 12mn + 9n^2$ (B) $x^2 - 16y^2$ (C) $z^3 - 1$ (D) $m^3 + n^3$

››› EXPLORE-DISCUSS 2

(A) Verify the following factor formulas for $u^4 - v^4$:

$$\begin{aligned} u^4 - v^4 &= (u - v)(u + v)(u^2 + v^2) \\ &= (u - v)(u^3 + u^2v + uv^2 + v^3) \end{aligned}$$

(B) Discuss the pattern in the following formulas:

$$\begin{aligned} u^2 - v^2 &= (u - v)(u + v) \\ u^3 - v^3 &= (u - v)(u^2 + uv + v^2) \\ u^4 - v^4 &= (u - v)(u^3 + u^2v + uv^2 + v^3) \end{aligned}$$

(C) Use the pattern you discovered in part (B) to write similar formulas for $u^5 - v^5$ and $u^6 - v^6$. Verify your formulas by multiplication.

We complete this section by considering factoring that involves combinations of the preceding techniques as well as a few additional ones. Generally speaking;

When asked to factor a polynomial, we first take out all factors common to all terms, if they are present, and then proceed to factor until all factors are prime.

EXAMPLE 7 Combining Factoring Techniques

Factor completely relative to the integers:

(A) $18x^3 - 8x$ (B) $x^2 - 6x + 9 - y^2$ (C) $4m^3n - 2m^2n^2 + 2mn^3$

(D) $2t^4 - 16t$ (E) $2y^4 - 5y^2 - 12$

SOLUTIONS

(A) $18x^3 - 8x = 2x(9x^2 - 4)$
$= 2x(3x - 2)(3x + 2)$

(B) $x^2 - 6x + 9 - y^2$ Group the first three terms.
$= (x^2 - 6x + 9) - y^2$ Factor $x^2 - 6x + 9$.
$= (x - 3)^2 - y^2$ Difference of squares
$= [(x - 3) - y][(x - 3) + y]$ Remove parentheses.
$= (x - 3 - y)(x - 3 + y)$

(C) $4m^3n - 2m^2n^2 + 2mn^3 = 2mn(2m^2 - mn + n^2)$

(D) $2t^4 - 16t = 2t(t^3 - 8)$ Difference of cubes
$= 2t(t - 2)(t^2 + 2t + 4)$

(E) $2y^4 - 5y^2 - 12 = (2y^2 + 3)(y^2 - 4)$ Difference of squares
$= (2y^2 + 3)(y - 2)(y + 2)$

MATCHED PROBLEM 7

Factor completely relative to the integers:

(A) $3x^3 - 48x$ (B) $x^2 - y^2 - 4y - 4$

(C) $3u^4 - 3u^3v + 9u^2v^2$ (D) $3m^4 - 24mn^3$

(E) $3x^4 - 5x^2 + 2$

ANSWERS TO MATCHED PROBLEMS

1. $2^2 \cdot 3^2 \cdot 5$ **2.** (A) $3xy(x^2 - 2xy - y^2)$ (B) $(3y + 2)(2y + 5)$
3. $2(2x + 5)(3x + 1)(12x + 17)$
4. (A) $(2x + 5)(x + 3)$ (B) $(p - 3q)(2r + s)$ (C) $(3w - x)(2y + z)$
5. (A) $(x - 2)(x - 6)$ (B) Not factorable using integers (C) $(2x - y)(x + 4y)$
(D) $(4x + y)(x - 4y)$
6. (A) $(2m - 3n)^2$ (B) $(x - 4y)(x + 4y)$ (C) $(z - 1)(z^2 + z + 1)$
(D) $(m + n)(m^2 - mn + n^2)$
7. (A) $3x(x - 4)(x + 4)$ (B) $(x - y - 2)(x + y + 2)$ (C) $3u^2(u^2 - uv - 3v^2)$
(D) $3m(m - 2n)(m^2 + 2mn + 4n^2)$ (E) $(3x^2 - 2)(x - 1)(x + 1)$

A-5 Exercises

*Additional answers can be found in the Instructor Answer Appendix.

A

In Problems 1–8, factor out, relative to the integers, all factors common to all terms.

1. $6x^4 - 8x^3 - 2x^2$
$2x^2(3x^2 - 4x - 1)$

2. $6m^4 - 9m^3 - 3m^2$
$3m^2(2m^2 - 3m - 1)$

3. $10x^3y + 20x^2y^2 - 15xy^3$
$5xy(2x^2 + 4xy - 3y^2)$

4. $8u^3v - 6u^2v^2 + 4uv^3$
$2uv(4u^2 - 3uv + 2v^2)$

5. $5x(x + 1) - 3(x + 1)$
$(5x - 3)(x + 1)$

6. $7m(2m - 3) + 5(2m - 3)$
$(2m - 3)(7m + 5)$

7. $2w(y - 2z) - x(y - 2z)$
$(2w - x)(y - 2z)$

8. $a(3c + d) - 4b(3c + d)$
$(3c + d)(a - 4b)$

In Problems 9–16, factor completely, relative to the integers.

9. $x^2 - 2x + 3x - 6$
$(x + 3)(x - 2)$

10. $2y^2 - 6y + 5y - 15$
$(y - 3)(2y + 5)$

11. $6m^2 + 10m - 3m - 5$
$(2m - 1)(3m + 5)$

12. $5x^2 - 40x - x + 8$
$(x - 8)(5x - 1)$

13. $2x^2 - 4xy - 3xy + 6y^2$ $(2x - 3y)(x - 2y)$

14. $3a^2 - 12ab - 2ab + 8b^2$ $(a - 4b)(3a - 2b)$

15. $8ac + 3bd - 6bc - 4ad$ $(4a - 3b)(2c - d)$

16. $3pr - 2qs - qr + 6ps$ $(r + 2s)(3p - q)$

In Problems 17–30, factor completely, relative to the integers. If a polynomial is prime relative to the integers, say so.

17. $2x^2 + x - 3$
$(2x + 3)(x - 1)$

18. $3y^2 - 8y - 3$ $(3y + 1)(y - 3)$

19. $x^2 + 3x - 8$ Prime

20. $u^2 + 4uv - 12v^2$
$(u - 2v)(u + 6v)$

21. $x^2 + 5xy - 14y^2$
$(x + 7y)(x - 2y)$

22. $m^2 + m - 20$ $(m + 5)(m - 4)$

23. $4a^2 - 9b^2$ $(2a + 3b)(2a - 3b)$

24. $x^2 + 4y^2$ Prime

25. $4x^2 - 20x + 25$ $(2x - 5)^2$

26. $a^2b^2 - c^2$ $(ab + c)(ab - c)$

27. $a^2b^2 + c^2$ Prime

28. $9x^2 - 4$ $(3x - 2)(3x + 2)$

29. $4x^2 + 9$ Prime

30. $16x^2 - 25$ $(4x - 5)(4x + 5)$

B

In Problems 31–46, factor completely, relative to the integers. If a polynomial is prime relative to the integers, say so.

31. $6x^2 + 48x + 72$ $6(x + 2)(x + 6)$

32. $3z^2 - 28z + 48$ Prime

33. $2y^3 - 22y^2 + 48y$ $2y(y - 3)(y - 8)$

34. $2x^4 - 24x^3 + 40x^2$ $2x^2(x - 10)(x - 2)$

35. $16x^2y - 8xy + y$ $y(4x - 1)^2$

36. $4xy^2 - 12xy + 9x$ $x(2y - 3)^2$

37. $6m^2 - mn - 12n^2$ $(3m + 4n)(2m - 3n)$

38. $6s^2 + 7st + 2t^2$ $(3s + 2t)(2s + t)$

39. $x^3y - 9xy^3$ $xy(x - 3y)(x + 3y)$

40. $4u^3v - uv^3$ $uv(2u - v)(2u + v)$

41. $3m^3 - 6m^2 + 15m$ $3m(m^2 - 2m + 5)$

42. $2x^3 - 2x^2 + 8x$ $2x(x^2 - x + 4)$

43. $m^3 + n^3$ $(m + n)(m^2 - mn + n^2)$

44. $r^3 - t^3$ $(r - t)(r^2 + rt + t^2)$

45. $8x^3 - 125$ $(2x - 5)(4x^2 + 10x + 25)$

46. $27y^3 + 1$ $(3y + 1)(9y^2 - 3y + 1)$

Problems 47–54 are calculus-related. Factor completely, relative to the integers.

47. $2x(x + 1)^4 + 4x^2(x + 1)^3$ $2x(3x + 1)(x + 1)^3$

48. $(x - 1)^3 + 3x(x - 1)^2$ $(x - 1)^2(4x - 1)$

49. $6(3x - 5)(2x - 3)^2 + 4(3x - 5)^2(2x - 3)$ $2(3x - 5)(2x - 3)(12x - 19)$

50. $2(x - 3)(4x + 7)^2 + 8(x - 3)^2(4x + 7)$ $2(x - 3)(4x + 7)(8x - 5)$

51. $5x^4(9 - x)^4 - 4x^5(9 - x)^3$ $9x^4(9 - x)^3(5 - x)$

52. $3x^4(x - 7)^2 + 4x^3(x - 7)^3$ $7x^3(x - 7)^2(x - 4)$

53. $2(x + 1)(x^2 - 5)^2 + 4x(x + 1)^2(x^2 - 5)$ $2(x + 1)(x^2 - 5)(3x + 5)(x - 1)$

54. $4(x - 3)^3(x^2 + 2)^3 + 6x(x - 3)^4(x^2 + 2)^2$ $2(x - 3)^3(x^2 + 2)^2(5x - 4)(x - 1)$

In Problems 55–60, factor completely, relative to the integers. In polynomials involving more than three terms, try grouping the terms in various combinations as a first step. If a polynomial is prime relative to the integers, say so.

55. $(a - b)^2 - 4(c - d)^2$ $[(a - b) - 2(c - d)][(a - b) + 2(c - d)]$

56. $(x + 2)^2 + 9$ Prime

57. $2am - 3an + 2bm - 3bn$ $(2m - 3n)(a + b)$

58. $15ac - 20ad + 3bc - 4bd$ $(5a + b)(3c - 4d)$

59. $3x^2 - 2xy - 4y^2$ Prime

60. $5u^2 + 4uv - v^2$ $(u + v)(5u - v)$

C

In Problems 61–78, factor completely, relative to the integers. In polynomials involving more than three terms, try grouping the terms as a first step. If a polynomial is prime relative to the integers, say so.

61. $x^3 - 3x^2 - 9x + 27$ $(x + 3)(x - 3)^2$

62. $x^3 - x^2 - x + 1$ $(x + 1)(x - 1)^2$

63. $a^3 - 2a^2 - a + 2$ $(a - 2)(a + 1)(a - 1)$

64. $t^3 - 2t^2 + t - 2$ $(t^2 + 1)(t - 2)$

65. $4(A + B)^2 - 5(A + B) - 5$ Prime

66. $6(x - y)^2 + 23(x - y) - 4$ $(x - y + 4)(6x - 6y - 1)$

67. $x^4 + 6x^2 + 8$ $(x^2 + 2)(x^2 + 4)$

68. $x^4 - x^2 - 6$ $(x^2 + 2)(x^2 - 3)$

69. $m^4 - n^4$ $(m - n)(m + n)(m^2 + n^2)$

70. $y^4 - 3y^2 - 4$ $(y - 2)(y + 2)(y^2 + 1)$

71. $y^4 - 11y^2 + 18$ $(y - 3)(y + 3)(y^2 - 2)$

72. $27a^2 + a^5b^3$ $a^2(3 + ab)(9 - 3ab + a^2b^2)$

73. $m^2 + 2mn + n^2 - m - n$ $(m + n)(m + n - 1)$

74. $y^2 - 2xy + x^2 - y + x$ $(y - x)(y - x - 1)$

75. $18a^3 - 8a(x^2 + 8x + 16)$ $2a[3a - 2(x + 4)][3a + 2(x + 4)]$

76. $25(4x^2 - 12xy + 9y^2) - 9a^2b^2$ $[10x - 15y + 3ab][10x - 15y - 3ab]$

77. $x^4 + 2x^2 + 1 - x^2$ $(x^2 - x + 1)(x^2 + x + 1)$

78. $a^4 + 2a^2b^2 + b^4 - a^2b^2$ $(a^2 + b^2 - ab)(a^2 + b^2 + ab)$

79. Find a factor formula for $u^7 - v^7$, where the first factor is $u - v$ and the second factor is a sum of terms of the form au^nv^m. Verify your formula by multiplication.

80. Show that $u^8 - v^8 = (u - v)(u + v)(u^2 + v^2)(u^4 + v^4)$.

81. Use the Sieve of Eratosthenes to find all prime numbers less than 200.

82. If p and $q = p + 2$ are both prime numbers, then p and q are called **twin primes.** Use the results of Problem 81 to find all twin primes less than 200.

83. To show that $\sqrt{2}$ is an irrational number, explain how the assumption that $\sqrt{2}$ is rational leads to a contradiction of Theorem 1, the fundamental theorem of arithmetic, by the following steps:

(A) Suppose that $\sqrt{2} = a/b$, where a and b are positive integers, $b \neq 0$. Explain why $a^2 = 2b^2$.

(B) Explain why the prime number 2 appears an even number of times (possibly 0 times) as a factor in the prime factorization of a^2.

(C) Explain why the prime number 2 appears an odd number of times as a factor in the prime factorization of $2b^2$.

(D) Explain why parts (B) and (C) contradict the fundamental theorem of arithmetic.

84. To show that $\sqrt{n}$ is an irrational number unless n is a perfect square, explain how the assumption that $\sqrt{n}$ is rational leads to a contradiction of the fundamental theorem of arithmetic by the following steps:

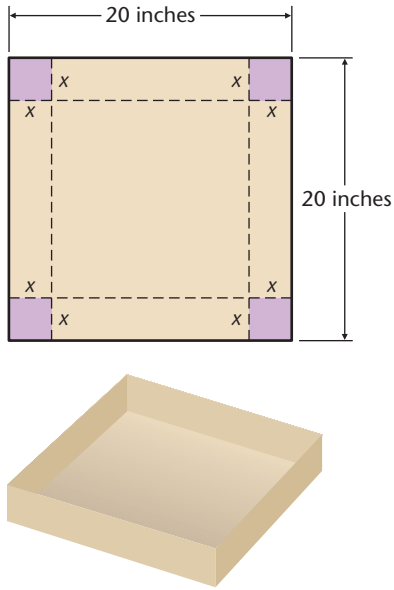
(A) Assume that n is not a perfect square, that is, does not belong to the sequence $1, 4, 9, 16, 25, \ldots$. Explain why some prime number p appears an odd number of times as a factor in the prime factorization of n.

(B) Suppose that $\sqrt{n} = a/b$, where a and b are positive integers, $b \neq 0$. Explain why $a^2 = nb^2$.

(C) Explain why the prime number p appears an even number of times (possibly 0 times) as a factor in the prime factorization of a^2.

(D) Explain why the prime number p appears an odd number of times as a factor in the prime factorization of nb^2.

(E) Explain why parts (C) and (D) contradict the fundamental theorem of arithmetic.



APPLICATIONS

85. CONSTRUCTION A rectangular open-topped box is to be constructed out of 20-inch-square sheets of thin cardboard by cutting x-inch squares out of each corner and bending the sides up as indicated in the figure. Express each of the following quantities as a polynomial in both factored and expanded form.

(A) The area of cardboard after the corners have been removed.

(B) The volume of the box. (A) $4(10 - x)(10 + x) = 400 - 4x^2$ (B) $4x(10 - x)^2 = 400x - 80x^2 + 4x^3$

86. CONSTRUCTION A rectangular open-topped box is to be constructed out of 9- by 16-inch sheets of thin cardboard by cutting x-inch squares out of each corner and bending the sides up. Express each of the following quantities as a polynomial in both factored and expanded form.

(A) The area of cardboard after the corners have been removed.

(B) The volume of the box. (A) $4(6 - x)(6 + x) = 144 - 4x^2$ (B) $2x(8 - x)(9 - 2x) = 144x - 50x^2 + 4x^3$

A-6 Rational Expressions: Basic Operations

- Reducing to Lowest Terms
- Multiplication and Division
- Addition and Subtraction
- Compound Fractions

A quotient of two algebraic expressions, division by 0 excluded, is called a **fractional expression.** If both the numerator and denominator of a fractional expression are polynomials, the fractional expression is called a **rational expression.** Some examples of rational expressions are the following (recall, a nonzero constant is a polynomial of degree 0):

$$\frac{x - 2}{2x^2 - 3x + 5} \qquad \frac{1}{x^4 - 1} \qquad \frac{3}{x} \qquad \frac{x^2 + 3x - 5}{1}$$

In this section, we discuss basic operations on rational expressions, including multiplication, division, addition, and subtraction.

Since variables represent real numbers in the rational expressions we are going to consider, the properties of real number fractions summarized in Section A-1 play a central role in much of the work that we will do.

Even though not always explicitly stated, we always assume that variables are restricted so that division by 0 is excluded.

Reducing to Lowest Terms

We start this discussion by restating the **fundamental property of fractions** (from Theorem 3 in Section A-1):

FUNDAMENTAL PROPERTY OF FRACTIONS

If a, b, and k are real numbers with $b, k \neq 0$, then

$$\frac{ka}{kb} = \frac{a}{b} \qquad \frac{2 \cdot 3}{2 \cdot 4} = \frac{3}{4} \qquad \frac{(x-3)2}{(x-3)x} = \frac{2}{x}, \; x \neq 0, x \neq 3$$

Using this property from left to right to eliminate all common factors from the numerator and the denominator of a given fraction is referred to as **reducing a fraction to lowest terms.** We are actually dividing the numerator and denominator by the same nonzero common factor.

Using the property from right to left—that is, multiplying the numerator and the denominator by the same nonzero factor—is referred to as **raising a fraction to higher terms.** We will use the property in both directions in the material that follows.

We say that a rational expression is **reduced to lowest terms** if the numerator and denominator do not have any factors in common. Unless stated to the contrary, factors will be relative to the integers.

EXAMPLE 1 Reducing Rational Expressions

Reduce each rational expression to lowest terms.

(A) $$\frac{x^2 - 6x + 9}{x^2 - 9} = \frac{(x-3)^2}{(x-3)(x+3)} = \frac{x-3}{x+3}$$

Factor numerator and denominator completely. Divide numerator and denominator by $(x - 3)$; this is a valid operation as long as $x \neq 3$.

(B) $$\frac{x^3 - 1}{x^2 - 1} = \frac{(x - 1)(x^2 + x + 1)}{(x - 1)(x + 1)} = \frac{x^2 + x + 1}{x + 1}$$

Dividing numerator and denominator by $(x - 1)$ can be indicated by drawing lines through both $(x - 1)$'s and writing the resulting quotients, 1's.

$x \neq -1$ and $x \neq 1$

MATCHED PROBLEM 1

Reduce each rational expression to lowest terms.

(A) $\dfrac{6x^2 + x - 2}{2x^2 + x - 1}$ (B) $\dfrac{x^4 - 8x}{3x^3 - 2x^2 - 8x}$

››› CAUTION ›››

Remember to always factor the numerator and denominator first, then divide out any *common factors*. Do not indiscriminately eliminate *terms* that appear in both the numerator and the denominator. For example,

$$\frac{2x^3 + y^2}{y^2} \neq \frac{2x^3 + y^2}{y^2}$$

$$\frac{2x^3 + y^2}{y^2} \neq 2x^3 + 1$$

Since the term y^2 is not a factor of the numerator, it cannot be eliminated. In fact, $(2x^3 + y^2)/y^2$ is already reduced to lowest terms.

› Multiplication and Division

Since we are restricting variable replacements to real numbers, multiplication and division of rational expressions follow the rules for multiplying and dividing real number fractions (Theorem 3 in Section A-1).

› MULTIPLICATION AND DIVISION

If a, b, c, and d are real numbers with $b, d \neq 0$, then:

1. $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$ $\qquad \dfrac{2}{3} \cdot \dfrac{x}{x - 1} = \dfrac{2x}{3(x - 1)}$

2. $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \cdot \dfrac{d}{c}$ $\quad c \neq 0$ $\qquad \dfrac{2}{3} \div \dfrac{x}{x - 1} = \dfrac{2}{3} \cdot \dfrac{x - 1}{x}$

››› EXPLORE-DISCUSS 1

Write a verbal description of the process of multiplying two fractions. Do the same for the quotient of two fractions.

EXAMPLE 2 Multiplying and Dividing Rational Expressions

Perform the indicated operations and reduce to lowest terms.

(A) $$\frac{10x^3y}{3xy+9y}\cdot\frac{x^2-9}{4x^2-12x}=\frac{\cancel{10}^{\,5x^2}\cancel{x^3}\,\cancel{y}}{\cancel{3}\,\cancel{y}(x+3)}\cdot\frac{(x-3)(x+3)}{\cancel{4}\,\cancel{x}(x-3)}$$

Factor numerators and denominators; then divide any numerator and any denominator with a like common factor.

$$=\frac{5x^2}{6}$$

(B) $$\frac{4-2x}{4}\div(x-2)=\frac{\cancel{2}^{\,1}(2-x)}{\cancel{4}_{\,2}}\cdot\frac{1}{x-2}$$

$x-2$ is the same as $\frac{x-2}{1}$.

$$=\frac{2-x}{2(x-2)}=\frac{-\cancel{(x-2)}^{\,-1}}{2\cancel{(x-2)}_{\,1}}$$

$b-a=-(a-b)$, a useful change in some problems.

$$=-\frac{1}{2}$$

(C) $$\frac{2x^3-2x^2y+2xy^2}{x^3y-xy^3}\div\frac{x^3+y^3}{x^2+2xy+y^2}$$

$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}$

$$=\frac{\cancel{2x}^{\,2}(x^2-xy+y^2)^{\,1}}{\cancel{xy}_{\,y}(x+y)_{\,1}(x-y)}\cdot\frac{(x+y)^{\,1}}{(x+y)_{\,1}\cancel{(x^2-xy+y^2)}_{\,1}}$$

Divide out common factors.

$$=\frac{2}{y(x-y)}$$

MATCHED PROBLEM 2

Perform the indicated operations and reduce to lowest terms.

(A) $$\frac{12x^2y^3}{2xy^2+6xy}\cdot\frac{y^2+6y+9}{3y^3+9y^2}$$

(B) $$(4-x)\div\frac{x^2-16}{5}$$

(C) $$\frac{m^3+n^3}{2m^2+mn-n^2}\div\frac{m^3n-m^2n^2+mn^3}{2m^3n^2-m^2n^3}$$

Addition and Subtraction

Again, because we are restricting variable replacements to real numbers, addition and subtraction of rational expressions follow the rules for adding and subtracting real number fractions (Theorem 3 in Section A-1).

ADDITION AND SUBTRACTION

For a, b, and c real numbers with $b \neq 0$:

1. $\dfrac{a}{b} + \dfrac{c}{b} = \dfrac{a + c}{b}$ $\quad \dfrac{x}{x-3} + \dfrac{2}{x-3} = \dfrac{x+2}{x-3}$

2. $\dfrac{a}{b} - \dfrac{c}{b} = \dfrac{a - c}{b}$ $\quad \dfrac{x}{2xy^2} - \dfrac{x-4}{2xy^2} = \dfrac{x-(x-4)}{2xy^2}$

Thus, we add rational expressions with the same denominators by adding or subtracting their numerators and placing the result over the common denominator. If the denominators are not the same, we raise the fractions to higher terms, using the fundamental property of fractions to obtain common denominators, and then proceed as described.

Even though any common denominator will do, our work will be simplified if the **least common denominator** (LCD) is used. Often, the LCD is obvious, but if it is not, the steps in the box describe how to find it.

THE LEAST COMMON DENOMINATOR (LCD)

The LCD of two or more rational expressions is found as follows:

1. Factor each denominator completely.

2. Identify each different prime factor from all the denominators.

3. Form a product using each different factor to the highest power that occurs in any one denominator. This product is the LCD.

EXAMPLE 3 Adding and Subtracting Rational Expressions

Combine into a single fraction and reduce to lowest terms.

(A) $\dfrac{3}{10} + \dfrac{5}{6} - \dfrac{11}{45}$ $\quad$ (B) $\dfrac{4}{9x} - \dfrac{5x}{6y^2} + 1$

(C) $\dfrac{x+3}{x^2 - 6x + 9} - \dfrac{x+2}{x^2 - 9} - \dfrac{5}{3 - x}$

SOLUTIONS

(A) To find the LCD, factor each denominator completely:

$$\left.\begin{aligned} 10 &= 2 \cdot 5 \\ 6 &= 2 \cdot 3 \\ 45 &= 3^2 \cdot 5 \end{aligned}\right\} \text{LCD} = 2 \cdot 3^2 \cdot 5 = 90$$

Now use the fundamental property of fractions to make each denominator 90:

$$\frac{3}{10} + \frac{5}{6} - \frac{11}{45} = \frac{\mathbf{9} \cdot 3}{\mathbf{9} \cdot 10} + \frac{\mathbf{15} \cdot 5}{\mathbf{15} \cdot 6} - \frac{\mathbf{2} \cdot 11}{\mathbf{2} \cdot 45} \qquad \text{Multiply.}$$

$$= \frac{27}{90} + \frac{75}{90} - \frac{22}{90} \qquad \text{Combine into a single fraction.}$$

$$= \frac{27 + 75 - 22}{90} = \frac{80}{90} = \frac{8}{9}$$

(B) $\left.\begin{aligned} 9x &= 3^2x \\ 6y^2 &= 2 \cdot 3y^2 \end{aligned}\right\} \text{LCD} = 2 \cdot 3^2xy^2 = 18xy^2$

$$\frac{4}{9x} - \frac{5x}{6y^2} + 1 = \frac{\mathbf{2y^2} \cdot 4}{\mathbf{2y^2} \cdot 9x} - \frac{\mathbf{3x} \cdot 5x}{\mathbf{3x} \cdot 6y^2} + \frac{\mathbf{18xy^2}}{\mathbf{18xy^2}} \qquad \text{Multiply, combine.}$$

$$= \frac{8y^2 - 15x^2 + 18xy^2}{18xy^2}$$

(C) $$\frac{x+3}{x^2 - 6x + 9} - \frac{x+2}{x^2 - 9} - \frac{5}{3 - x} = \frac{x+3}{(x-3)^2} - \frac{x+2}{(x-3)(x+3)} + \frac{5}{x-3}$$

Note: $-\dfrac{5}{3-x} = -\dfrac{5}{-(x-3)} = \dfrac{5}{x-3}$ We have again used the fact that $a - b = -(b - a)$.

The LCD $= (x-3)^2(x+3)$. Therefore,

$$\frac{(x+3)^2}{(x-3)^2(x+3)} - \frac{(x-3)(x+2)}{(x-3)^2(x+3)} + \frac{5(x-3)(x+3)}{(x-3)^2(x+3)} \qquad \text{Expand numerators.}$$

$$= \frac{(x^2 + 6x + 9) - (x^2 - x - 6) + 5(x^2 - 9)}{(x-3)^2(x+3)} \qquad \text{Be careful of sign errors here.}$$

$$= \frac{x^2 + 6x + 9 - x^2 + x + 6 + 5x^2 - 45}{(x-3)^2(x+3)} \qquad \text{Combine like terms.}$$

$$= \frac{5x^2 + 7x - 30}{(x-3)^2(x+3)}$$

MATCHED PROBLEM 3

Combine into a single fraction and reduce to lowest terms.

(A) $\frac{5}{28} - \frac{1}{10} + \frac{6}{35}$

(B) $\frac{1}{4x^2} - \frac{2x+1}{3x^3} + \frac{3}{12x}$

(C) $\frac{y-3}{y^2-4} - \frac{y+2}{y^2-4y+4} - \frac{2}{2-y}$

»» EXPLORE-DISCUSS 2

What is the value of $\frac{\frac{16}{4}}{2}$?

What is the result of entering $16 \div 4 \div 2$ on a calculator?

What is the difference between $16 \div (4 \div 2)$ and $(16 \div 4) \div 2$?

How could you use fraction bars to distinguish between these two cases when writing $\frac{\frac{16}{4}}{2}$?

› Compound Fractions

A fractional expression with fractions in its numerator, denominator, or both is called a **compound fraction.** It is often necessary to represent a compound fraction as a **simple fraction**—that is (in all cases we will consider), as the quotient of two polynomials. The process does not involve any new concepts. It is a matter of applying old concepts and processes in the right sequence. We will illustrate two approaches to the problem, each with its own merits, depending on the particular problem under consideration.

EXAMPLE 4 Simplifying Compound Fractions

Express as a simple fraction reduced to lowest terms:

$$\frac{\frac{2}{x} - 1}{\frac{4}{x^2} - 1}$$

SOLUTION

Method 1. Multiply the numerator and denominator by the LCD of all fractions in the numerator and denominator—in this case, x^2. (We are multiplying by $1 = x^2/x^2$.)

$$\frac{x^2\left(\frac{2}{x} - 1\right)}{x^2\left(\frac{4}{x^2} - 1\right)} = \frac{x^2\frac{2}{x} - x^2}{x^2\frac{4}{x^2} - x^2} = \frac{2x - x^2}{4 - x^2} = \frac{x(2 - x)}{(2 + x)(2 - x)}$$

$$= \frac{x}{2 + x}$$

Method 2. Write the numerator and denominator as single fractions. Then treat as a quotient.

$$\frac{\frac{2}{x} - 1}{\frac{4}{x^2} - 1} = \frac{\frac{2 - x}{x}}{\frac{4 - x^2}{x^2}} = \frac{2 - x}{x} \div \frac{4 - x^2}{x^2} = \frac{2 - x}{x} \cdot \frac{x^2}{(2 - x)(2 + x)}$$

$$= \frac{x}{2 + x}$$

MATCHED PROBLEM 4

Express as a simple fraction reduced to lowest terms. Use the two methods described in Example 4.

$$\frac{1 + \frac{1}{x}}{x - \frac{1}{x}}$$

ANSWERS TO MATCHED PROBLEMS

1. (A) $\frac{3x + 2}{x + 1}$ (B) $\frac{x^2 + 2x + 4}{3x + 4}$
2. (A) $2x$ (B) $\frac{-5}{x + 4}$ (C) mn
3. (A) $\frac{1}{4}$ (B) $\frac{3x^2 - 5x - 4}{12x^3}$ (C) $\frac{2y^2 - 9y - 6}{(y - 2)^2(y + 2)}$
4. $\frac{1}{x - 1}$

A-6 Exercises

*Additional answers can be found in the Instructor Answer Appendix.

A

In Problems 1–8, reduce each rational expression to lowest terms.

1. $\frac{42}{105}$ $\frac{2}{5}$

2. $\frac{360}{216}$ $\frac{5}{3}$

3. $\frac{x+1}{x^2+3x+2}$ $\frac{1}{x+2}$

4. $\frac{x^2-2x-24}{x-6}$ $x+4$

5. $\frac{x^2-9}{x^2+3x-18}$ $\frac{x+3}{x+6}$

6. $\frac{x^2+9x+20}{x^2-16}$ $\frac{x+5}{x-4}$

7. $\frac{3x^2y^3}{x^4y}$ $\frac{3y^2}{x^2}$

8. $\frac{2a^2b^4c^6}{6a^5b^3c}$ $\frac{bc^5}{3a^3}$

In Problems 9–28, perform the indicated operations and reduce answers to lowest terms. Represent any compound fractions as simple fractions reduced to lowest terms.

9. $\frac{5}{6}-\frac{11}{15}$ $\frac{1}{10}$

10. $\frac{7}{10}-\frac{19}{25}$ $\frac{-3}{50}$

11. $\left(\frac{b^2}{2a}\div\frac{b}{a^2}\right)\cdot\frac{a}{3b}$ $\frac{a^2}{6}$

12. $\frac{b^2}{2a}\div\left(\frac{b}{a^2}\cdot\frac{a}{3b}\right)$ $\frac{3b^2}{2}$

13. $\frac{1}{n}-\frac{1}{m}$ $\frac{m-n}{mn}$

14. $\frac{m}{n}-\frac{n}{m}$ $\frac{m^2-n^2}{mn}$

15. $\frac{x^2-1}{x+2}\div\frac{x+1}{x^2-4}$ $(x-1)(x-2)$

16. $\frac{x^2-9}{x^2-1}\div\frac{x-3}{x-1}$ $\frac{x+3}{x+1}$

17. $\frac{1}{c}+\frac{1}{b}+\frac{1}{a}$ $\frac{ab+ac+bc}{abc}$

18. $\frac{1}{bc}+\frac{1}{ac}+\frac{1}{ab}$ $\frac{a+b+c}{abc}$

19. $\frac{2a-b}{a^2-b^2}-\frac{2a+3b}{a^2+2ab+b^2}$ $\frac{2b^2}{(a+b)^2(a-b)}$

20. $\frac{x+2}{x^2-1}-\frac{x-2}{(x-1)^2}$ $\frac{2x}{(x+1)(x-1)^2}$

21. $m+2-\frac{m-2}{m-1}$ $\frac{m^2}{m-1}$

22. $\frac{x+1}{x-1}+x$ $\frac{x^2+1}{x-1}$

23. $\frac{3}{x-2}-\frac{2}{2-x}$ $\frac{5}{x-2}$

24. $\frac{1}{a-3}-\frac{2}{3-a}$ $\frac{3}{a-3}$

25. $\frac{3}{y+2}+\frac{2}{y-2}-\frac{4y}{y^2-4}$ $\frac{1}{y+2}$

26. $\frac{4x}{x^2-y^2}+\frac{3}{x+y}-\frac{2}{x-y}$ $\frac{5}{x+y}$

27. $\dfrac{\frac{x^2}{y^2}-1}{\frac{x}{y}+1}$ $\frac{x-y}{y}$

28. $\dfrac{\frac{4}{x}-x}{\frac{2}{x}-1}$ $x+2$

B

Problems 29–34 are calculus-related. Reduce each fraction to lowest terms.

29. $\frac{6x^3(x^2+2)^2-2x(x^2+2)^3}{x^4}$ $\frac{4(x+1)(x-1)(x^2+2)^2}{x^3}$

30. $\frac{4x^4(x^2+3)-3x^2(x^2+3)^2}{x^6}$ $\frac{(x+3)(x-3)(x^2+3)}{x^4}$

31. $\frac{2x(1-3x)^3+9x^2(1-3x)^2}{(1-3x)^6}$ $\frac{x(2+3x)}{(1-3x)^4}$

32. $\frac{2x(2x+3)^4-8x^2(2x+3)^3}{(2x+3)^8}$ $\frac{2x(3-2x)}{(2x+3)^5}$

33. $\frac{-2x(x+4)^3-3(3-x^2)(x+4)^2}{(x+4)^6}$ $\frac{(x+1)(x-9)}{(x+4)^4}$

34. $\frac{3x^2(x+1)^3-3(x^3+4)(x+1)^2}{(x+1)^6}$ $\frac{3(x+2)(x-2)}{(x+1)^4}$

In Problems 35–48, perform the indicated operations and reduce answers to lowest terms. Represent any compound fractions as simple fractions reduced to lowest terms.

35. $\dfrac{y}{y^2-2y-8}-\dfrac{2}{y^2-5y+4}+\dfrac{1}{y^2+y-2}$ $\quad \dfrac{1}{y-1}$

36. $\dfrac{x}{x^2-9x+18}+\dfrac{x-8}{x-6}+\dfrac{x+4}{x-3}$ $\quad \dfrac{2x}{x-3}$

37. $\dfrac{16-m^2}{m^2+3m-4}\cdot\dfrac{m-1}{m-4}$ $\quad -1$

38. $\dfrac{x+1}{x(1-x)}\cdot\dfrac{x^2-2x+1}{x^2-1}$ $\quad -\dfrac{1}{x}$

39. $\dfrac{x+7}{ax-bx}+\dfrac{y+9}{by-ay}$ $\quad \dfrac{7y-9x}{xy(a-b)}$

40. $\dfrac{c+2}{5c-5}-\dfrac{c-2}{3c-3}+\dfrac{c}{1-c}$ $\quad \dfrac{-17c+16}{15(c-1)}$

41. $\dfrac{x^2-16}{2x^2+10x+8}\div\dfrac{x^2-13x+36}{x^3+1}$ $\quad \dfrac{x^2-x+1}{2(x-9)}$

42. $\left(\dfrac{x^3-y^3}{y^3}\cdot\dfrac{y}{x-y}\right)\div\dfrac{x^2+xy+y^2}{y^2}$ $\quad 1$

43. $\dfrac{x^2-xy}{xy+y^2}\div\left(\dfrac{x^2-y^2}{x^2+2xy+y^2}\div\dfrac{x^2-2xy+y^2}{x^2y+xy^2}\right)$ $\quad \dfrac{(x-y)^2}{y^2(x+y)}$

44. $\left(\dfrac{x^2-xy}{xy+y^2}\div\dfrac{x^2-y^2}{x^2+2xy+y^2}\right)\div\dfrac{x^2-2xy+y^2}{x^2y+xy^2}$ $\quad \dfrac{x^2(x+y)}{(x-y)^2}$

45. $\left(\dfrac{x}{x^2-16}-\dfrac{1}{x+4}\right)\div\dfrac{4}{x+4}$ $\quad \dfrac{1}{x-4}$

46. $\left(\dfrac{3}{x-2}-\dfrac{1}{x+1}\right)\div\dfrac{x+4}{x-2}$ $\quad \dfrac{2x+5}{(x+1)(x+4)}$

47. $\dfrac{1+\dfrac{2}{x}-\dfrac{15}{x^2}}{1+\dfrac{4}{x}-\dfrac{5}{x^2}}$ $\quad \dfrac{x-3}{x-1}$

48. $\dfrac{\dfrac{x}{y}-2+\dfrac{y}{x}}{\dfrac{x}{y}-\dfrac{y}{x}}$ $\quad \dfrac{x-y}{x+y}$

Problems 49–52 are calculus-related. Perform the indicated operations and reduce answers to lowest terms. Represent any compound fractions as simple fractions reduced to lowest terms.

49. $\dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}$ $\quad \dfrac{-1}{x(x+h)}$

50. $\dfrac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}$ $\quad \dfrac{-2x-h}{x^2(x+h)^2}$

51. $\dfrac{\dfrac{(x+h)^2}{x+h+2}-\dfrac{x^2}{x+2}}{h}$ $\quad \dfrac{x^2+xh+4x+2h}{(x+h+2)(x+2)}$

52. $\dfrac{\dfrac{2x+2h+3}{x+h}-\dfrac{2x+3}{x}}{h}$ $\quad \dfrac{-3}{x(x+h)}$

In Problems 53–60, imagine that the indicated "solutions" were given to you by a student whom you were tutoring in this class.

(A) Is the solution correct? If the solution is incorrect, explain what is wrong and how it can be corrected.

(B) Show a correct solution for each incorrect solution.

53. $\dfrac{x^2+5x+4}{x+4}=\dfrac{x^2+5x}{x}=x+5$ (A) Incorrect (B) $x+1$

54. $\dfrac{x^2-2x-3}{x-3}=\dfrac{x^2-2x}{x}=x-2$ (A) Incorrect (B) $x+1$

55. $\dfrac{(x+h)^2-x^2}{h}=(x+1)^2-x^2=2x+1$ (A) Incorrect (B) $2x+h$

56. $\dfrac{(x+h)^3-x^3}{h}=(x+1)^3-x^3=3x^2+3x+1$ (A) Incorrect (B) $3x^2+3xh+h^2$

57. $\dfrac{x^2-2x}{x^2-x-2}+x-2=\dfrac{x^2-2x+x-2}{x^2-x-2}=1$ (A) Incorrect (B) $\dfrac{x^2-2}{x+1}$

58. $\dfrac{2}{x-1}-\dfrac{x+3}{x^2-1}=\dfrac{2x+2-x-3}{x^2-1}=\dfrac{1}{x+1}$ (A) Correct

59. $\dfrac{2x^2}{x^2-4}-\dfrac{x}{x-2}=\dfrac{2x^2-x^2-2x}{x^2-4}=\dfrac{x}{x+2}$ (A) Correct

60. $x+\dfrac{x-2}{x^2-3x+2}=\dfrac{x+x-2}{x^2-3x+2}=\dfrac{2}{x-2}$

C (A) Incorrect (B) $\dfrac{x^2-x+1}{x-1}$

In Problems 61–64, perform the indicated operations and reduce answers to lowest terms. Represent any compound fractions as simple fractions reduced to lowest terms.

61. $\dfrac{y-\dfrac{y^2}{y-x}}{1+\dfrac{x^2}{y^2-x^2}}$ $\quad \dfrac{-x(y+x)}{y}$

62. $\dfrac{\dfrac{s^2}{s-t}-s}{\dfrac{t^2}{s-t}+t}$ $\quad 1$

63. $2-\dfrac{1}{1-\dfrac{2}{a+2}}$ $\quad \dfrac{a-2}{a}$

64. $1-\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}$ $\quad x$

In Problems 65 and 66, a, b, c, and d represent real numbers.

65. (A) Prove that d/c is the multiplicative inverse of c/d $(c, d \neq 0)$.

(B) Use part (A) to prove that

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c} \qquad b, c, d \neq 0$$

66. Prove that

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b} \qquad b \neq 0$$

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Review of Equations and Graphing

APPENDIX

B

IN this appendix we provide review and reference material especially important for a course in mathematics that emphasizes the use of graphing technology. Further review of basic algebra operations and concepts usually studied in earlier courses is available at the Online Learning Center for *College Algebra: Graphs and Models,* Third Edition. To access the Online Learning Center, visit www.mhhe.com/barnett and click on the image of this book's cover.

OUTLINE

B-1 Linear Equations and Inequalities

B-2 Cartesian Coordinate System

B-3 Basic Formulas in Analytic Geometry

B-1 Linear Equations and Inequalities

- Equations
- Solving Linear Equations
- Inequality Relations and Interval Notation
- Solving Linear Inequalities

Equations

An **algebraic equation** is a mathematical statement that relates two algebraic expressions involving at least one variable. Some examples of equations with x as a variable are

$$3x - 2 = 7 \qquad \frac{1}{1+x} = \frac{x}{x-2}$$
$$2x^2 - 3x + 5 = 0 \qquad \sqrt{x+4} = x - 1$$

The **replacement set,** or **domain,** for a variable is defined to be the set of numbers that are permitted to replace the variable.

ASSUMPTION On Domains of Variables

Unless stated to the contrary, we assume that the domain for a variable is the set of those real numbers for which the algebraic expressions involving the variable are real numbers.

For example, the domain for the variable x in the expression

$$2x - 4$$

is R, the set of all real numbers, because $2x - 4$ represents a real number for all replacements of x by real numbers. The domain of x in the equation

$$\frac{1}{x} = \frac{2}{x-3}$$

is the set of all real numbers except 0 and 3. These values are excluded because the left member is not defined for $x = 0$ and the right member is not defined for $x = 3$. The left and right members represent real numbers for all other replacements of x by real numbers.

The **solution set** for an equation is defined to be the set of elements in the domain of the variable that make the equation true. Each element of the solution set is called a **solution,** or **root,** of the equation. To **solve an equation** is to find the solution set for the equation.

Knowing what we mean by the solution set of an equation is one thing; finding it is another. To this end we introduce the idea of equivalent equations. Two equations are said to be **equivalent** if they both have the same solution set for a given replacement set. A basic technique for solving equations is to perform operations on equations that produce simpler equivalent equations, and to continue the process until an equation is reached whose solution is obvious.

Application of any of the properties of equality given in Theorem 1 will produce equivalent equations.

› THEOREM 1 Properties of Equality

For a, b, and c any real numbers,

1. If $a = b$, then $a + c = b + c$.	**Addition Property**
2. If $a = b$, then $a - c = b - c$.	**Subtraction Property**
3. If $a = b$, then $ca = cb, c \neq 0$.	**Multiplication Property**
4. If $a = b$, then $\frac{a}{c} = \frac{b}{c}, c \neq 0$.	**Division Property**
5. If $a = b$, then either may replace the other in any statement without changing the truth or falsity of the statement.	**Substitution Property**

› Solving Linear Equations

We now turn our attention to methods of solving *first-degree,* or *linear, equations* in one variable.

› DEFINITION 1 Linear Equation in One Variable

Any equation that can be written in the form

$$ax + b = 0 \qquad a \neq 0 \qquad \textbf{Standard Form}$$

where a and b are real constants and x is a variable, is called a **linear,** or **first-degree, equation** in one variable.

$5x - 1 = 2(x + 3)$ is a linear equation, because it can be written in the standard form $3x - 7 = 0$.

EXAMPLE 1 Solving a Linear Equation

Solve $5x - 9 = 3x + 7$ and check.

SOLUTION

We use the properties of equality to transform the given equation into an equivalent equation whose solution is obvious.

$$\begin{aligned} 5x - 9 &= 3x + 7 && \text{Add 9 to both sides.} \\ 5x - 9 + 9 &= 3x + 7 + 9 && \text{Combine like terms.} \\ 5x &= 3x + 16 && \text{Subtract } 3x \text{ from both sides.} \\ 5x - 3x &= 3x + 16 - 3x && \text{Combine like terms.} \\ 2x &= 16 && \text{Divide both sides by 2.} \\ \frac{2x}{2} &= \frac{16}{2} && \text{Simplify.} \\ x &= 8 \end{aligned}$$

The solution set for this last equation is obvious:

$$\text{Solution set: } \{8\}$$

And because the equation $x = 8$ is equivalent to all the preceding equations in our solution, $\{8\}$ is also the solution set for all these equations, including the original equation. [*Note:* If an equation has only one element in its solution set, we generally use the last equation (in this case, $x = 8$) rather than set notation to represent the solution.]

CHECK

$$\begin{aligned} 5x - 9 &= 3x + 7 && \text{Substitute } x = 8. \\ 5(8) - 9 &\stackrel{?}{=} 3(8) + 7 && \text{Simplify each side.} \\ 40 - 9 &\stackrel{?}{=} 24 + 7 \\ 31 &\stackrel{\checkmark}{=} 31 && \text{A true statement} \end{aligned}$$

MATCHED PROBLEM 1

Solve $7x - 10 = 4x + 5$ and check.

EXAMPLE 2 Solving a Linear Equation

Solve $3x - 2(2x - 5) = 2(x + 3) - 8$ and check.

SOLUTION

$$\begin{aligned} 3x - 2(2x - 5) &= 2(x + 3) - 8 && \text{Multiply to remove parentheses.} \\ 3x - 4x + 10 &= 2x + 6 - 8 && \text{Combine like terms.} \\ -x + 10 &= 2x - 2 && \text{Subtract } 2x \text{ and } 10 \text{ from both sides.} \\ -3x &= -12 && \text{Divide both sides by } -3. \\ x &= 4 \end{aligned}$$

CHECK

$$\begin{aligned} 3x - 2(2x - 5) &= 2(x + 3) - 8 \\ 3(4) - 2[2(4) - 5] &\stackrel{?}{=} 2[(4) + 3] - 8 \\ 6 &\stackrel{\checkmark}{=} 6 \end{aligned}$$

MATCHED PROBLEM 2

Solve $2(3 - x) - (3x + 1) = 8 - 2(x + 2)$ and check.

Inequality Relations and Interval Notation

Just as we use $=$ to replace the words *is equal to,* we use the **inequality symbols** $<$ and $>$ to represent *is less than* and *is greater than,* respectively.

Although it probably seems obvious to you that

$$2 < 4 \qquad 5 > 0 \qquad 25{,}000 > 1$$

are true, it may not seem as obvious that

$$-4 < -2 \qquad 0 > -5 \qquad -25{,}000 < -1$$

To make the inequality relation precise so that we can interpret it relative to all real numbers, we need a precise definition of the concept.

DEFINITION 2 $a < b$ and $b > a$

For a and b real numbers, we say that ***a* is less than *b*** or ***b* is greater than *a*** and write

$$a < b \qquad \text{or} \qquad b > a$$

if there exists a positive real number p such that $a + p = b$ (or equivalently, $b - a = p$).

We certainly expect that if a positive number is added to *any* real number, the sum is larger than the original. That is essentially what the definition states.

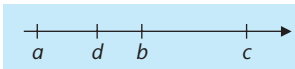
When we write

$$a \le b$$

we mean $a < b$ or $a = b$ and say ***a* is less than or equal to *b*.** When we write

$$a \ge b$$

we mean $a > b$ or $a = b$ and say ***a* is greater than or equal to *b*.**



› **Figure 1** $a < b$, $c > d$.

The inequality symbols $<$ and $>$ have a very clear geometric interpretation on the real number line. If $a < b$, then a is to the left of b; if $c > d$, then c is to the right of d (Fig. 1).

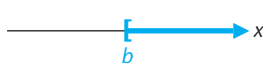
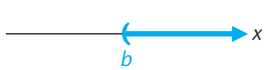
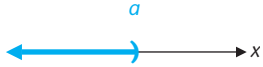
It is an interesting and useful fact that for any two real numbers a and b, either $a < b$, or $a > b$, or $a = b$. This is called the **trichotomy property** of real numbers.

The double inequality $a < x \le b$ means that $x > a$ and $x \le b$; that is, x is between a and b, including b but not including a. The set of all real numbers x satisfying the inequality $a < x \le b$ is called an **interval** and is represented by $(a, b]$. Therefore,

$$(a, b] = \{x | a < x \le b\}^*$$

The number a is called the **left endpoint** of the interval, and the symbol (indicates that a is not included in the interval. The number b is called the **right endpoint** of the interval, and the symbol] indicates that b is included in the interval. Other types of intervals of real numbers are shown in Table 1.

Table 1 Interval Notation

Interval Notation	Inequality Notation	Line Graph	Type
$[a, b]$	$a \le x \le b$		Closed
$[a, b)$	$a \le x < b$		Half-open
$(a, b]$	$a < x \le b$		Half-open
(a, b)	$a < x < b$		Open
$[b, \infty)$	$x \ge b$		Closed
(b, ∞)	$x > b$		Open
$(-\infty, a]$	$x \le a$		Closed
$(-\infty, a)$	$x < a$		Open

*In general, $\{x | P(x)\}$ represents the set of all x such that statement $P(x)$ is true. To express this set verbally, just read the vertical bar as "such that."

Note that the symbol ∞, read *infinity,* used in Table 1 is not a numeral. When we write $[b, \infty)$, we are simply referring to the interval starting at b and continuing indefinitely to the right. We would never write $[b, \infty]$ or $b \le x \le \infty$, because ∞ cannot be used as an endpoint of an interval. The interval $(-\infty, \infty)$ represents the set of real numbers R, because its graph is the entire real number line.

››› CAUTION ›››

It is important to note that

$$5 > x \ge -3 \qquad \text{is equivalent to } [-3, 5) \text{ and not to } (5, -3]$$

In interval notation, the smaller number is always written to the left. Thus, it may be useful to rewrite the inequality as $-3 \le x < 5$ before rewriting it in interval notation.

EXAMPLE 3 Graphing Intervals and Inequalities

Write each of the following in inequality notation and graph on a real number line:

(A) $[-2, 3)$ (B) $(-4, 2)$ (C) $[-2, \infty)$ (D) $(-\infty, 3)$

SOLUTIONS

(A) $-2 \le x < 3$

(B) $-4 < x < 2$

(C) $x \ge -2$

(D) $x < 3$

MATCHED PROBLEM 3

Write each of the following in interval notation and graph on a real number line:

(A) $-3 < x \le 3$ (B) $2 \ge x \ge -1$ (C) $x > 1$ (D) $x \le 2$

››› EXPLORE-DISCUSS 1

Example 3, part C, shows the graph of the inequality $x \geq -2$. What is the graph of $x < -2$? What is the corresponding interval? Describe the relationship between these sets.

Because intervals are sets of real numbers, the set operations of *union* and *intersection* are often useful when working with intervals. The **union** of sets A and B, denoted by $A \cup B$, is the set formed by combining all the elements of A and all the elements of B. The **intersection** of sets A and B, denoted by $A \cap B$, is the set of elements of A that are also in B. Symbolically:

› DEFINITION 3 Union and Intersection

Union: $A \cup B = \{x \mid x \text{ is in } A \text{ or } x \text{ is in } B\}$

$\{1, 2, 3\} \cup \{2, 3, 4, 5\} = \{1, 2, 3, 4, 5\}$

Intersection: $A \cap B = \{x \mid x \text{ is in } A \text{ and } x \text{ is in } B\}$

$\{1, 2, 3\} \cap \{2, 3, 4, 5\} = \{2, 3\}$

EXAMPLE 4 Graphing Unions and Intersections of Intervals

If $A = [-2, 3]$, $B = (1, 6)$, and $C = (4, \infty)$, graph the indicated sets and write as a single interval, if possible.

(A) $A \cup B$ and $A \cap B$ (B) $A \cup C$ and $A \cap C$

SOLUTIONS

(A)

$A = [-2, 3]$

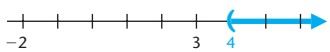
$B = (1, 6)$

$A \cup B = [-2, 6)$

$A \cap B = (1, 3]$

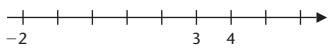
(B)

$A = [-2, 3]$



$C = (4, \infty)$

$A \cup C = [-2, 3] \cup (4, \infty)$



$A \cap C = \varnothing$

MATCHED PROBLEM 4

If $D = [-4, 1)$, $E = (-1, 3]$, and $F = [2, \infty)$, graph the indicated sets and write as a single interval, if possible.

(A) $D \cup E$ (B) $D \cap E$ (C) $E \cup F$ (D) $E \cap F$

››› EXPLORE-DISCUSS 2

Replace ? with $<$ or $>$ in each of the following:

(A) $-1 \; ? \; 3$ and $2(-1) \; ? \; 2(3)$

(B) $-1 \; ? \; 3$ and $-2(-1) \; ? \; -2(3)$

(C) $12 \; ? \; -8$ and $\frac{12}{4} \; ? \; \frac{-8}{4}$

(D) $12 \; ? \; -8$ and $\frac{12}{-4} \; ? \; \frac{-8}{-4}$

Based on these examples, describe verbally the effect of multiplying both sides of an inequality by a number.

› Solving Linear Inequalities

We now turn to the problem of solving linear inequalities in one variable, such as

$$2(2x + 3) < 6(x - 2) + 10 \quad \text{and} \quad -3 < 2x + 3 \leq 9$$

The **solution set** for an inequality is the set of all values of the variable that make the inequality a true statement. Each element of the solution set is called a **solution** of the inequality. To **solve an inequality** is to find its solution set. Two inequalities are **equivalent** if they have the same solution set for a given replacement set. Just as with equations, we perform operations on inequalities that produce simpler equivalent inequalities, and continue the process until an inequality is reached whose solution is obvious. The properties of inequalities given in Theorem 2 can be used to produce equivalent inequalities.

› THEOREM 2 Inequality Properties

For a, b, and c any real numbers,

1. If $a < b$ and $b < c$, then $a < c$. **Transitive Property**

2. If $a < b$, then $a + c < b + c$. **Addition Property**

$-2 < 4 \quad -2 + 3 < 4 + 3$

3. If $a < b$, then $a - c < b - c$. **Subtraction Property**

$-2 < 4 \quad -2 - 3 < 4 - 3$

4. If $a < b$ and c is positive, then $ca < cb$. **Multiplication Property (Note difference between 4 and 5.)**

$-2 < 4 \quad 3(-2) < 3(4)$

5. If $a < b$ and c is negative, then $ca > cb$.

$-2 < 4 \quad (-3)(-2) > (-3)4$

6. If $a < b$ and c is positive, then $\frac{a}{c} < \frac{b}{c}$. **Division Property (Note difference between 6 and 7.)**

$-2 < 4 \quad \frac{-2}{2} > \frac{4}{2}$

7. If $a < b$ and c is negative, then $\frac{a}{c} > \frac{b}{c}$.

$-2 < 4 \quad \frac{-2}{2} > \frac{4}{2}$

Similar properties hold if each inequality sign is reversed, or if $<$ is replaced with $\leq$ and $>$ is replaced with $\geq$. Therefore, we find that we can perform essentially the same operations on inequalities that we perform on equations. When working with inequalities, however, we have to be particularly careful of the use of the multiplication and division properties.

The order of the inequality reverses if we multiply or divide both sides of an inequality statement by a negative number.

››› EXPLORE-DISCUSS 3

Properties of equality are easily summarized. We can add, subtract, multiply, or divide both sides of an equation by any nonzero real number to produce an equivalent equation. Write a similar summary for the properties of inequalities.

Now let's see how the inequality properties are used to solve linear inequalities. Several examples will illustrate the process.

EXAMPLE 5 Solving a Linear Inequality

Solve $2(2x + 3) - 10 < 6(x - 2)$ and graph.

SOLUTION

$$2(2x + 3) - 10 < 6(x - 2)$$ Multiply to remove parentheses.

$$4x + 6 - 10 < 6x - 12$$ Combine like terms.

$$4x - 4 < 6x - 12$$

$$4x - 4 + 4 < 6x - 12 + 4$$ Add 4 to both sides.

$$4x < 6x - 8$$ Combine like terms.

$$4x - 6x < 6x - 8 - 6x$$ Subtract $-6x$ from both sides.

$$-2x < -8$$ Combine like terms.

$$\frac{-2x}{-2} > \frac{-8}{-2}$$ Divide both sides by -2 and reverse the inequality. Simplify.

$$x > 4 \quad \text{or} \quad (4, \infty)$$ Solution set

Graph of solution set

MATCHED PROBLEM 5

Solve $3(x - 1) \geq 5(x + 2) - 5$ and graph.

EXAMPLE 6 Solving a Double Inequality

Solve $-1 \leq 3 - 4x < 11$ and graph.

SOLUTION

We proceed as before, except we try to isolate x in the middle with a coefficient of 1.

$$-1 \leq 3 - 4x < 11$$ Subtract 3 from each member.

$$-1 - 3 \leq 3 - 4x - 3 < 11 - 3$$ Combine like terms.

$$-4 \leq -4x < 8$$ Divide each number by -4 and reverse each inequality.

$$\frac{-4}{-4} \geq \frac{-4x}{-4} > \frac{8}{-4}$$ Simplify.

$$1 \geq x > -2 \quad \text{or} \quad -2 < x \leq 1 \quad \text{or} \quad (-2, 1]$$

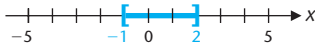
MATCHED PROBLEM 6

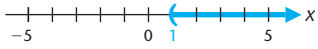
Solve $-5 < 10 - 3x \leq 10$ and graph.

ANSWERS TO MATCHED PROBLEMS

1. $x = 5$ **2.** $x = \frac{1}{3}$

3. (A) $(-3, 3]$

(B) $[-1, 2]$ 

(C) $(1, \infty)$ 

(D) $(-\infty, 2]$

4. (A)

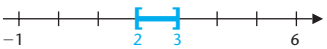
$D \cup E = [-4, 3]$

(B)

$D \cap E = (-1, 1)$

(C)

$E \cup F = (-1, \infty)$

(D) 

$E \cap F = [2, 3]$

5. $x \leq -4$ or $(-\infty, -4]$

6. $5 > x \geq 0$
or
$0 \leq x < 5$ or $[0, 5)$

B-1 Exercises

Solve Problems 1–6.

1. $x + 5 = 12$ **2.** $x - 9 = -2$

3. $2s - 7 = -2$ **4.** $7 - 3t = 1$

5. $2m + 8 = 5m - 7$ **6.** $3y + 5 = 6y - 10$

In Problems 7–12, rewrite in inequality notation and graph on a real number line.

7. $[-8, 7]$ **8.** $(-4, 8)$

9. $[-6, 6)$ **10.** $(-3, 3]$

11. $[-6, \infty)$ **12.** $(-\infty, 7)$

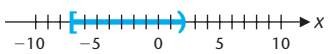
In Problems 13–18, rewrite in interval notation and graph on a real number line.

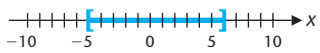
13. $-2 < x \leq 6$ **14.** $-5 \leq x \leq 5$

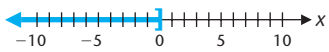
15. $-7 < x < 8$ **16.** $-4 \leq x < 5$

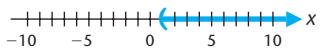
17. $x \leq -2$ **18.** $x > 3$

In Problems 19–22, write in interval and inequality notation.

19. 

20. 

21. 

22. 

In Problems 23–30, replace each ? with $>$ or $<$ to make the resulting statement true.

23. $12 \;?\; 6$ and $12 + 5 \;?\; 6 + 5$ $>, >$

24. $-4 \;?\; -2$ and $-4 - 7 \;?\; -2 - 7$ $<, <$

25. $-6 \;?\; -8$ and $-6 - 3 \;?\; -8 - 3$ $>, >$

26. $4 \;?\; 9$ and $4 + 2 \;?\; 9 + 2$ $<, <$

27. $2 \;?\; -1$ and $-2(2) \;?\; -2(-1)$ $>, <$

28. $-3 \;?\; 2$ and $4(-3) \;?\; 4(2)$ $<, <$

29. $2 \;?\; 6$ and $\frac{2}{2} \;?\; \frac{6}{2}$ $<, <$

30. $-10 \;?\; -15$ and $\frac{-10}{5} \;?\; \frac{-15}{5}$ $>, >$

In Problems 31–40, solve and graph.

31. $7x - 8 < 4x + 7$

32. $4x + 8 \geq x - 1$

33. $3 - x \geq 5(3 - x)$

34. $2(x - 3) + 5 < 5 - x$

35. $\frac{N}{-2} > 4$

36. $\frac{M}{-3} \leq -2$

37. $3 - m < 4(m - 3)$

38. $2(1 - u) \geq 5u$

39. $-2 - \frac{B}{4} \leq \frac{1 + B}{3}$

40. $\frac{y - 3}{4} - 1 < \frac{y}{2}$

B *Solve Problems 41–46.*

41. $3 - \frac{2x - 3}{3} = \frac{5 - x}{2}$ 9

42. $\frac{x - 2}{3} + 1 = \frac{x}{7}$ $-\frac{7}{4}$

43. $0.1(x - 7) + 0.05x = 0.8$ 10

44. $0.4(x + 5) - 0.3x = 17$ 150

45. $0.3x - 0.04(x + 1) = 2.04$ 8

46. $0.02x - 0.5(x - 2) = 5.32$ −9

In Problems 47–58, graph the indicated set and write as a single interval, if possible.

47. $(-5, 5) \cup [4, 7]$

48. $(-5, 5) \cap [4, 7]$

49. $[-1, 4) \cap (2, 6]$

50. $[-1, 4) \cup (2, 6]$

51. $(-\infty, 1) \cup (-2, \infty)$

52. $(-\infty, 1) \cap (2, \infty)$

53. $(-\infty, -1) \cup [3, 7)$

54. $(1, 6] \cup [9, \infty)$

55. $(2, 3] \cup [1, 5)$

56. $(2, 3] \cap [1, 5)$

57. $(-\infty, 4) \cup (-1, 6]$

58. $(-3, 2) \cup [0, \infty)$

In Problems 59–68, solve and graph.

59. $\frac{q}{7} - 3 > \frac{q - 4}{3} + 1$

60. $\frac{p}{3} - \frac{q - 2}{2} \leq \frac{p}{4} - 4$

61. $\frac{2x}{5} - \frac{1}{2}(x - 3) \leq \frac{2x}{3} - \frac{3}{10}(x + 2)$

62. $\frac{2}{3}(x + 7) - \frac{x}{4} > \frac{1}{2}(3 - x) + \frac{x}{6}$

63. $-4 \leq \frac{9}{5}x + 32 \leq 68$

64. $-1 \leq \frac{2}{3}A + 5 \leq 11$

65. $16 < 7 - 3x \leq 31$

66. $-1 \leq 9 - 3x < 5$

67. $-6 < -\frac{2}{5}(1 - x) \leq 4$

68. $15 \leq 7 - \frac{2}{5}x \leq 21$

C **69.** Indicate true (T) or false (F):
(A) If $p > q$ and $m > 0$, then $mp < mq$.
(B) If $p < q$ and $m < 0$, then $mp > mq$.
(C) If $p > 0$ and $p < 0$, then $p + q > q$.

70. Assume that $m > n > 0$; then

$$\begin{aligned} mn &> n^2 \\ mn - m^2 &> n^2 - m^2 \\ m(n - m) &> (n + m)(n - m) \\ m &> n + m \\ 0 &> n \end{aligned}$$

But it was assumed that $n > 0$. Find the error.

Prove each inequality property in Problems 71–74, given a, b, and c are arbitrary real numbers.

71. If $a < b$, then $a + c < b + c$.

72. If $a < b$, then $a - c < b - c$.

73. (A) If $a < b$ and c is positive, then $ca < cb$.
(B) If $a < b$ and c is negative, then $ca > cb$.

74. (A) If $a < b$ and c is positive, then $\frac{a}{c} > \frac{b}{c}$.

(B) If $a < b$ and c is negative, then $\frac{a}{c} > \frac{b}{c}$.

*Problem numbers that appear in blue indicate that require students to apply their reasoning and writing skills in the solution of the problem.

B-2 Cartesian Coordinate System

- Cartesian Coordinate System
- Graphing: Point by Point
- Data Analysis

Analytic geometry is the study of the relationship between geometric forms, such as circles and lines, and algebraic forms, such as equations and inequalities. The key to this relationship is the Cartesian coordinate system, named after the French mathematician and philosopher René Descartes (1596–1650) who was the first to combine the study of algebra and geometry into a single discipline. In this section, we develop some of the basic tools used to graph equations.

Cartesian Coordinate System

Just as a real number line establishes a one-to-one correspondence between the points on a line and the elements in the set of real numbers, we can form a **real plane** by establishing a one-to-one correspondence between the points in a plane and elements in the set of all ordered pairs of real numbers. This can be done by means of a Cartesian coordinate system.

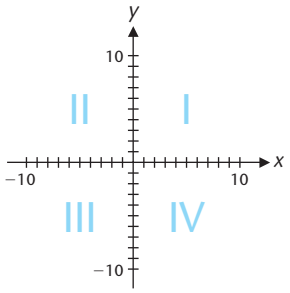


Figure 1 Cartesian coordinate system

Recall that to form a **Cartesian** or **rectangular coordinate system,** we select two real number lines, one horizontal and one vertical, and let them cross through their origins as indicated in Figure 1. Up and to the right are the usual choices for the positive directions. These two number lines are called the **horizontal axis** and the **vertical axis,** or together, the **coordinate axes.** The horizontal axis is usually referred to as the ***x* axis** and the vertical axis as the ***y* axis,** and each is labeled accordingly. Other labels may be used in certain situations. The coordinate axes divide the plane into four parts called **quadrants,** which are numbered counterclockwise from I to IV (see Fig. 1).

Now we want to assign *coordinates* to each point in the plane. Given an arbitrary point P in the plane, pass horizontal and vertical lines through the point (Fig. 2). The vertical line will intersect the horizontal axis at a point with coordinate a, and the horizontal line will intersect the vertical axis at a point with coordinate b. These two numbers written as the ordered pair (a, b) form the **coordinates** of the point P. The first coordinate a is called the **abscissa** of P; the second coordinate b is called the **ordinate** of P. The abscissa of Q in Figure 2 is -10, and the ordinate of Q is 5. The coordinates of a point can also be referenced in terms of the axis labels. The ***x* coordinate** of R in Figure 2 is 5, and the ***y* coordinate** of R is 10. The point with coordinates (0, 0) is called the **origin.**

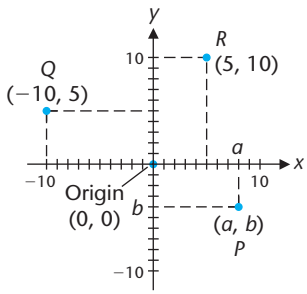


Figure 2 Coordinates in a plane.

The procedure we have just described assigns to each point P in the plane a unique pair of real numbers (a, b). Conversely, if we are given an ordered pair of real numbers (a, b), then, reversing this procedure, we can determine a unique point P in the plane. Therefore,

There is a one-to-one correspondence between the points in a plane and the elements in the set of all ordered pairs of real numbers.

This is often referred to as the **fundamental theorem of analytic geometry.** Because of this correspondence, we often speak of the point (a, b) when we are referring to the point with coordinates (a, b). We also write $P = (a, b)$ to identify the coordinates of the point P. Thus, in Figure 2, referring to Q as the point $(-10, 5)$ and writing $R = (5, 10)$ are both acceptable statements.

Given any set of ordered pairs S, the **graph** of S is the set of points in the plane corresponding to the ordered pairs in S.

Graphing: Point by Point

The fundamental theorem of analytic geometry enables us to look at algebraic forms geometrically and to look at geometric forms algebraically. We begin by considering an algebraic form, an equation in two variables:

$$y = x^2 - 4 \tag{1}$$

A **solution** to equation (1) is an ordered pair of real numbers (a, b) such that

$$b = a^2 - 4$$

The **solution set** of equation (1) is the set of all these ordered pairs. More formally,

$$\text{Solution set of equation (1): } \{(x, y) | y = x^2 - 4\} \tag{2}$$

To find a solution for equation (1) we simply replace x with a number and calculate the value of y. For example, if $x = 2$, then $y = 2^2 - 4 = 0$, and the ordered pair $(2, 0)$ is a solution. Similarly, if $x = -3$, then $y = (-3)^2 - 4 = 5$, and the ordered pair $(-3, 5)$ is also a solution of equation (1). In fact, any real number substituted for x in equation (1) will produce a solution to the equation. Therefore, the solution set shown in equation (2) must have an infinite number of elements. We now use a rectangular coordinate system to provide a geometric representation of this set.

The **graph of an equation** is the graph of its solution set. To *sketch the graph of an equation,* we plot enough points from its solution set so that the total graph is apparent and then connect these points with a smooth curve, proceeding from left to right. This process is called **point-by-point plotting.**

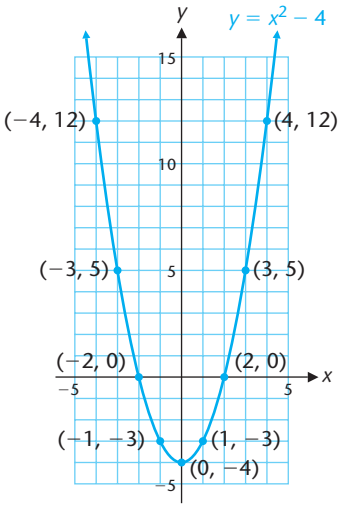
EXAMPLE 1 Graphing an Equation Using Point-by-Point Plotting

Sketch a graph of $y = x^2 - 4$.

SOLUTION

We make up a table of solutions—ordered pairs of real numbers that satisfy the given equation.

x	-4	-3	-2	-1	0	1	2	3	4
y	12	5	0	-3	-4	-3	0	5	12



› Figure 3

After plotting these solutions, if there are any portions of the graph that are unclear, we plot additional points until the shape of the graph is apparent. Then we join all these plotted points with a smooth curve, as shown in Figure 3. Arrowheads are used to indicate that the graph continues beyond the portion shown here with no significant changes in shape.

The resulting figure is called a *parabola.* Notice that if we fold the paper along the y axis, the right side will match the left side. We say that the graph is *symmetric with respect to the y axis* and call the y axis the *axis, or the axis of symmetry, of the parabola.* More is said about parabolas elsewhere in the text.

MATCHED PROBLEM 1

Sketch a graph of $y = 8 - x^2$ using point-by-point plotting.

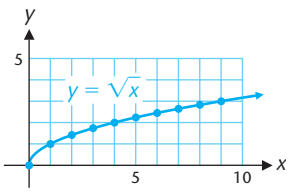
EXAMPLE 2 **Graphing an Equation Using Point-by-Point Plotting**

Sketch a graph of $y = \sqrt{x}$.

SOLUTION

Proceeding as before, we make up a table of solutions:

x	0	1	2	3	4	5	6	7	8	9
y	0	1	$\sqrt{2} \approx 1.4$	$\sqrt{3} \approx 1.7$	2	$\sqrt{5} \approx 2.2$	$\sqrt{6} \approx 2.4$	$\sqrt{7} \approx 2.6$	$\sqrt{8} \approx 2.8$	3



› Figure 4

For graphing purposes, the irrational numbers in the table were evaluated on a calculator and rounded to one decimal place. Plotting these points and connecting them with a smooth curve produces the graph in Figure 4.

Notice that we did not include any negative values of x in the table. If x is a negative real number, then $\sqrt{x}$ is not a real number. Because the coordinates of a point in a rectangular coordinate system must be real numbers, *when graphing an equation, we consider only those values of the variables that produce real solutions to the equation.* We have more to say about numbers of the form $\sqrt{x}$, where x is negative, elsewhere in this book.

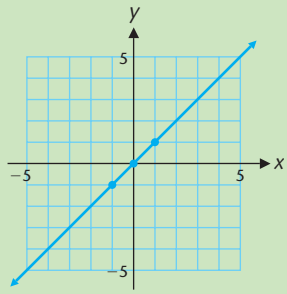
MATCHED PROBLEM 2

Sketch a graph of $y = 4 - \sqrt{x}$.

››› EXPLORE-DISCUSS 1

To graph the equation $y = -x^3 + 2x$, we use point-by-point plotting to obtain the graph in Figure 5.

x	y
-1	-1
0	0
1	1



› Figure 5

(A) Do you think this is the correct graph of the equation? If so, why? If not, why?

(B) Add points on the graph for $x = -2, -0.5, 0.5$, and 2.

(C) Now, what do you think the graph looks like? Sketch your version of the graph, adding more points as necessary.

(D) Write a short statement explaining any conclusions you might draw from parts A, B, and C.

As Explore-Discuss 1 illustrates, sometimes it can be difficult to determine the apparent shape of a graph by simply plotting a few points. One of the major objectives of this book is to develop mathematical tools that will help us analyze graphs.

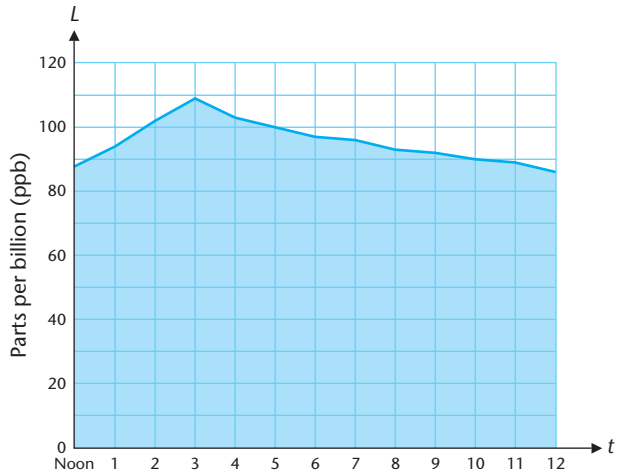
› Data Analysis

In applications, numeric data are often collected and presented in graphical form. A graph illustrates the relationship between two quantities, and estimating coordinates of points on the graph provides specific examples of this relationship, even if no equation for the graph is available. Example 3 illustrates the analysis of data presented in graphical form.

EXAMPLE 3 **Ozone Levels**

The ozone level during a 12-hour period in a suburb of Milwaukee, Wisconsin, on a particular summer day is given in Figure 6, where L is ozone in parts per billion and t is time in hours. Use this graph to estimate the following ozone levels to the nearest integer and times to the nearest quarter hour:

(A) The ozone level at 6 P.M.
(B) The highest ozone level and the time when it occurs
(C) The time(s) when the ozone level is 90 ppb

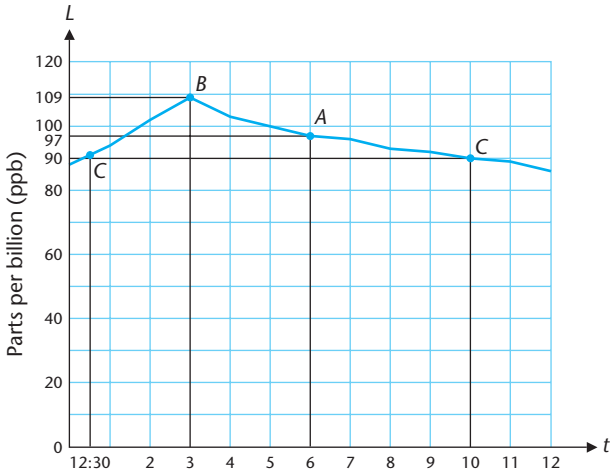


Source: Wisconsin Department of Natural Resources.

› **Figure 6**

SOLUTION

(A) The L coordinate of the point on the graph with t coordinate 6 is approximately 97 parts per billion (Fig. 7).
(B) The highest ozone level is approximately 109 parts per billion at 3 P.M.
(C) The ozone level is 90 parts per billion at about 12:30 P.M. and again at 10 P.M.



› **Figure 7**

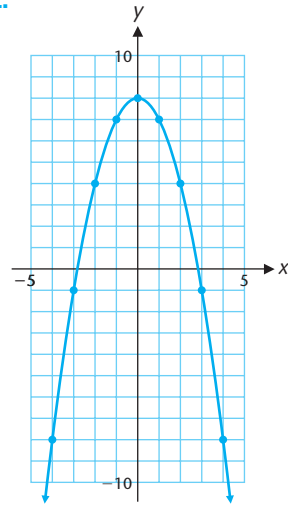
MATCHED PROBLEM 3

Use the graph in Figure 6 to estimate the following ozone level to the nearest integer and time(s) to the nearest quarter hour.

(A) The ozone level at 7 P.M.
(B) The time(s) when the ozone level is 100 parts per billion.

ANSWERS TO MATCHED PROBLEMS

1.



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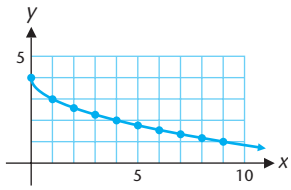


3. (A) 96 Parts per billion (B) 1:45 P.M. and 5 P.M.

B-2 Exercises

**Additional answers can be found in the Instructor Answer Appendix.*

A

In Problems 1–4, plot the given points in a rectangular coordinate system.

1. (5, 0), (3, −2), (−4, 2), (4, 4)

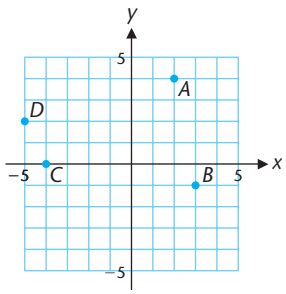
2. (0, 4), (−3, 2), (5, −1), (−2, −4)

3. (0, −2), (−1, −3), (4, −5), (−2, 1)

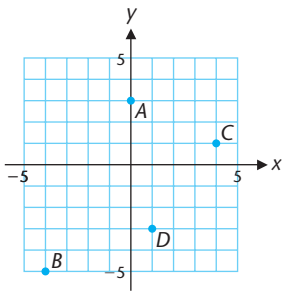
4. (−2, 0), (3, 2), (1, −4), (−3, 5)

In Problems 5–8 find the coordinates of points A, B, C, and D.

5.



6.



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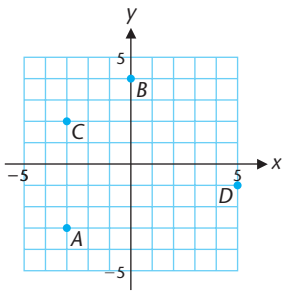

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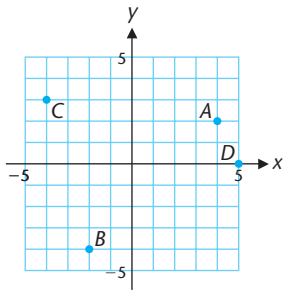


B

For each equation in Problems 9–14, make up a table of solutions using $x = -3, -2, -1, 0, 1, 2,$ and 3. Plot these solutions and graph the equation.

9. $y = x + 1$

10. $y = 2 - x$

11. $y = x^2 - 5$

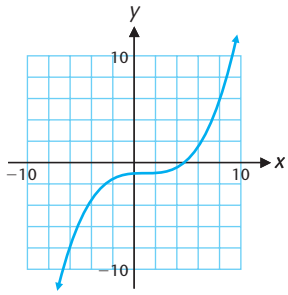
12. $y = 4 - x^2$

13. $y = 3 + x - 0.5x^2$

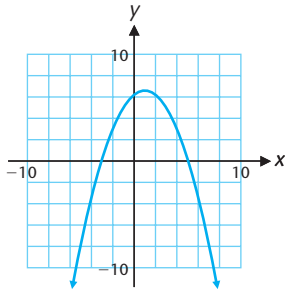
14. $y = 4 - x - 0.5x^2$

In Problems 15–18, use the graph to estimate to the nearest integer the missing coordinates of the indicated points. (Be sure you find all possible answers.)

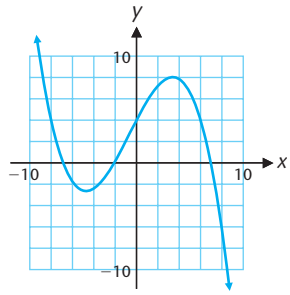
15. (A) (8, ?) 6 (B) (−5, ?) −5 (C) (0, ?) −1
(D) (?, 6) 8 (E) (?, −5) −5 (F) (?, 0) 5



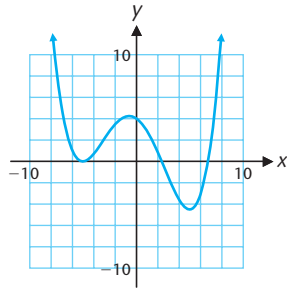
16. (A) (3, ?) 5 (B) (−5, ?) −8 (C) (0, ?) 6
(D) (?, 3) −2, 4 (E) (?, −4) −4, 6 (F) (?, 0) −3, 5



17. (A) (1, ?) 6 (B) (−8, ?) 4 (C) (0, ?) 4
(D) (?, −6) 8 (E) (?, 4) −8, 0, 6 (F) (?, 0) −7, −2, 7



18. (A) (6, ?) −3 (B) (−6, ?) 1 (C) (0, ?) 4
(D) (?, −2) 3, 6 (E) (?, 1) −6, −4, 2, 7 (F) (?, 0) −5, 2, 7



19. (A) Sketch a graph based on the solutions in the following table.

x	−2	0	2
y	−2	0	2

(B) Sketch a graph based on the solutions in the following table.

x	−1	0	1
y	2	0	−2

(C) Complete the following table for $y = x^3 - 3x$ and sketch a graph of the equation.

x	−2	−1	0	1	2
y					

(D) Write a short statement explaining any conclusions you might draw from parts A, B, and C.

20. (A) Sketch a graph based on the solutions in the following table.

x	-1	1	3
y	4	2	0

(B) Sketch a graph based on the solutions in the following table.

x	0	1	2
y	0	2	4

(C) Complete the following table for $y = 3x^2 - x^3$ and sketch a graph of the equation.

x	-1	0	1	2	3
y					

(D) Write a short statement explaining any conclusions you might draw from parts (A), (B), and (C).

In Problems 21–28, graph each equation using point-by-point plotting.

21. $y = x^{1/3}$ **22.** $y = x^{2/3}$

23. $y = x^3$ **24.** $y = x^4$

25. $y = \sqrt{x - 1}$ **26.** $y = \sqrt{5 - x}$

27. $y = \sqrt{1 + x^2}$ **28.** $y = x\sqrt{1 + x^2}$

C **29.** (A) Graph the triangle with vertices $A = (1, 1)$, $B = (7, 2)$, and $C = (4, 6)$.
(B) Now graph the triangle with vertices $A' = (1, -1)$, $B' = (7, -2)$, and $C' = (-4, 6)$ in the same coordinate system.
(C) How are these two triangles related? How would you describe the effect of changing the sign of the y coordinate of all the points on a graph?

30. (A) Graph the triangle with vertices $A = (1, 1)$, $B = (7, 2)$, and $C = (4, 6)$.
(B) Now graph the triangle with vertices $A' = (-1, 1)$, $B' = (-7, 2)$, and $C' = (-4, 6)$ in the same coordinate system.
(C) How are these two triangles related? How would you describe the effect of changing the sign of the x coordinate of all the points on a graph?

31. (A) Graph the triangle with vertices $A = (1, 1)$, $B = (7, 2)$, and $C = (4, 6)$.
(B) Now graph the triangle with vertices $A' = (-1, -1)$, $B' = (-7, -2)$, and $C' = (-4, -6)$ in the same coordinate system.
(C) How are these two triangles related? How would you describe the effect of changing the signs of the x and y coordinates of all the points on a graph?

32. (A) Graph the triangle with vertices $A = (1, 2)$, $B = (1, 4)$, and $C = (3, 4)$.
(B) Now graph $y = x$ and the triangle obtained by reversing the coordinates for each vertex of the original triangle: $A' = (2, 1)$, $B' = (4, 1)$, $C' = (4, 3)$.
(C) How are these two triangles related? How would you describe the effect of reversing the coordinates of each point on a graph?

APPLICATIONS

33. BUSINESS After extensive surveys, the marketing research department of a producer of popular cassette tapes developed the demand equation

$$n = 10 - p \qquad 5 \le p \le 10$$

where n is the number of units (in thousands) retailers are willing to buy per day at $\$p$ per tape. The company's daily revenue R (in thousands of dollars) is given by

$$R = np = (10 - p)p \qquad 5 \le p \le 10$$

Graph the revenue equation for the indicated values of p.

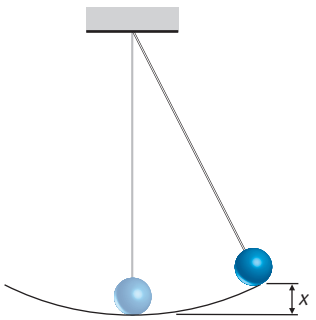
34. BUSINESS Repeat Problem 33 for the demand equation

$$n = 8 - p \qquad 4 \le p \le 8$$

35. PHYSICS The speed (in meters per second) of a ball swinging at the end of a pendulum is given by

$$v = 0.5\sqrt{2 - x}$$

where x is the vertical displacement (in centimeters) of the ball from its position at rest (see the figure).

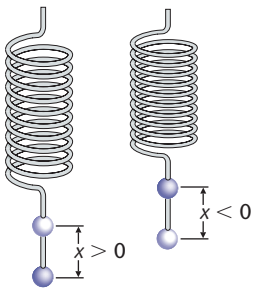


(A) Graph for $v = 0.5\sqrt{2 - x}$ for $0 \le x \le 2$.
(B) Describe the relationship between this graph and the physical behavior of the ball as it swings back and forth.

36. PHYSICS The speed (in meters per second) of a ball oscillating at the end of a spring is given by

$$v = 4\sqrt{25 - x^2}$$

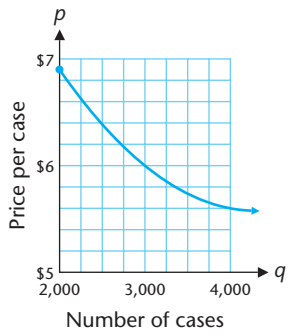
where x is the vertical displacement (in centimeters) of the ball from its position at rest (positive displacement measured downward—see the figure).



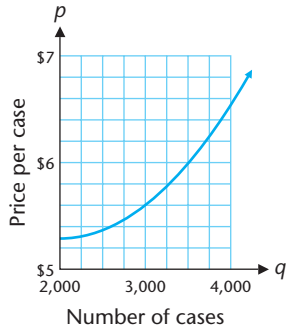
(A) Graph $v = 4\sqrt{25 - x^2}$ for $-5 \le x \le 5$.
(B) Describe the relationship between this graph and the physical behavior of the ball as it oscillates up and down.

DATA ANALYSIS

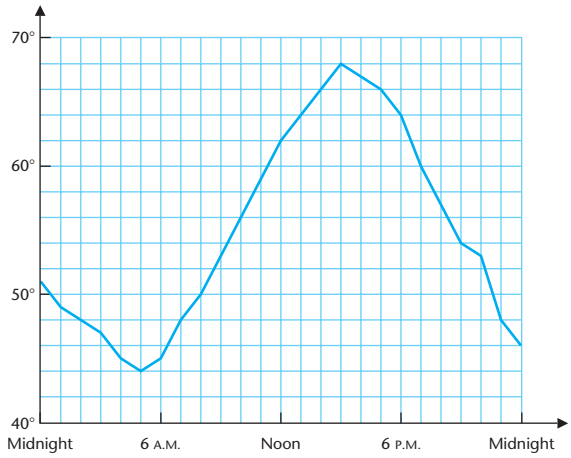
37. PRICE AND DEMAND The quantity of a product that consumers are willing to buy during some period depends on its price. The price p and corresponding weekly demand q for a particular brand of diet soda in a city are shown in the figure. Use this graph to estimate the following demands to the nearest 100 cases.
(A) What is the demand when the price is \$6.00 per case?
(B) Does the demand increase or decrease if the price is increased to \$6.30 per case? By how much?
(C) Does the demand increase or decrease if the price is decreased to \$5.70? By how much?
(D) Write a brief description of the relationship between price and demand illustrated by this graph.



38. PRICE AND SUPPLY The quantity of a product that suppliers are willing to sell during some period depends on its price. The price p and corresponding weekly supply q for a particular brand of diet soda in a city are shown in the figure. Use this graph to estimate the following supplies to the nearest 100 cases.
(A) What is the supply when the price is \$5.60 per case?
(B) Does the supply increase or decrease if the price is increased to \$5.80 per case? By how much?
(C) Does the supply increase or decrease if the price is decreased to \$5.40 per case? By how much?
(D) Write a brief description of the relationship between price and supply illustrated by this graph.



39. TEMPERATURE The temperature (in degrees Fahrenheit) during a spring day in the Midwest is given in the figure. Use this graph to estimate the following temperatures to the nearest degree and times to the nearest hour:
(A) The temperature at 9:00 A.M.
(B) The highest temperature and the time when it occurs
(C) The time(s) when the temperature is 49°



40. TEMPERATURE Use the figure for Problem 39 to estimate the following temperatures to the nearest degree and times to the nearest half hour:
(A) The temperature at 7:00 P.M.
(B) The lowest temperature and the time when it occurs
(C) The time(s) when the temperature is 52°

B-3 Basic Formulas in Analytic Geometry

- Distance Between Two Points
- Midpoint of a Line Segment
- Circles

Two basic problems studied in analytic geometry are

1. Given an equation, find its graph.
2. Given a figure (line, circle, parabola, ellipse, etc.) in a coordinate system, find its equation.

The first problem was discussed in the last section. In this section, we introduce some tools that are useful when studying the second problem.

Distance Between Two Points

Given two points P_1 and P_2 in a rectangular coordinate system, we denote the **distance between P_1 and P_2 by $d(P_1, P_2)$.** We begin with an example.

EXAMPLE 1 Distance Between Two Points

Find the distance between the points $P_1 = (1, 2)$ and $P_2 = (4, 6)$.

SOLUTION

Connecting the points P_1, P_2, and $P_3 = (4, 2)$ with straight line segments forms a right triangle (Fig. 1).

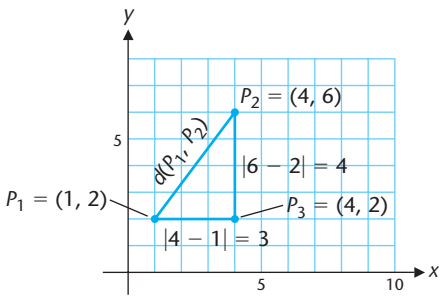


Figure 1

From the figure, we see that lengths of the legs of the triangle are

$$d(P_1, P_3) = |4 - 1| = 3$$

and

$$d(P_3, P_2) = |6 - 2| = 4$$

The length of the hypotenuse is $d(P_1, P_2)$, the distance we are seeking. Applying the Pythagorean theorem (see Appendix D) we have

$$\begin{aligned}[d(P_1, P_2)]^2 &= [d(P_1, P_3)]^2 + [d(P_3, P_2)]^2 \\ &= 3^2 + 4^2 \\ &= 9 + 16 \\ &= 25\end{aligned}$$

Therefore,

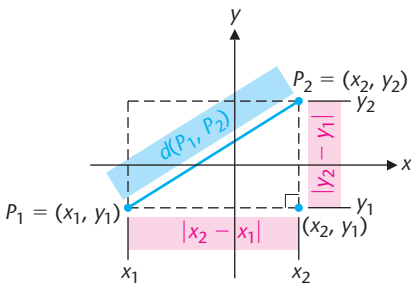
$$d(P_1, P_2 = \sqrt{25} = 5$$

MATCHED PROBLEM 1

Find the distance between the points $P_1 = (1, 2)$ and $P_2 = (13, 7)$.

The ideas used in Example 1 can be generalized to any two distinct points in the plane. If $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are two points in a rectangular coordinate system (Fig. 2), then

$$\begin{aligned}[d(P_1, P_2)]^2 &= |x_2 - x_1|^2 + |y_2 - y_1|^2 \\ &= (x_2 - x_1^2) + (y_2 - y_1)^2 \qquad \text{Because } |N|^2 = N^2.\end{aligned}$$



› **Figure 2** Distance between two points.

Therefore,

› THEOREM 1 Distance between $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

EXAMPLE 2 Using the Distance-Between-Two-Points Formula

Find the distance between the points $(-3, 5)$ and $(-2, -8)$.

SOLUTION

Let $(x_1, y_1) = (-3, 5)$ and $(x_2, y_2) = (-2, -8)$. Then,

$$\begin{aligned} d &= \sqrt{[(-2) - (-3)]^2 + [(-8) - 5]^2} \\ &= \sqrt{(-2 + 3)^2 + (-8 - 5)^2} = \sqrt{1^2 + (-13)^2} = \sqrt{1 + 169} = \sqrt{170} \end{aligned}$$

Notice that if we choose $(x_1, y_1) = (-2, -8)$ and $(x_2, y_2) = (-3, 5)$, then

$$d = \sqrt{[(-3) - (-2)]^2 + [5 - (-8)]^2} = \sqrt{1 + 169} = \sqrt{170}$$

so it doesn't matter which point we designate as P_1 or P_2.

MATCHED PROBLEM 2

Find the distance between the points $(6, -3)$ and $(-7, -5)$.

› Midpoint of a Line Segment

››› EXPLORE-DISCUSS 1

(A) Graph the line segment L joining the points $(1, 2)$ and $(7, 10)$.

(B) Find the average a of the x coordinates of these two points.

(C) Find the average b of the y coordinates of these two points.

(D) Plot the point (a, b). Is it on the line segment L?

(E) Find the distance between $(1, 2)$ and (a, b) and the distance between (a, b) and $(7, 10)$. How would you describe the point (a, b)?

The **midpoint** of the line segment between two points is the point on the line segment that is equidistant from each of the points. A formula for finding the midpoint is given in Theorem 2. The proof is discussed in the exercises.

› THEOREM 2 Midpoint Formula

The midpoint of the line segment joining $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

The point M is the unique point satisfying

$$d(P_1, M) = d(M, P_2) = \frac{1}{2}d(P_1, P_2)$$

Note that the coordinates of the midpoint are simply the averages of the respective coordinates of the two given points.

EXAMPLE 3 Using the Midpoint Formula

Find the midpoint M of the line segment joining $A = (-3, 2)$ and $B = (4, -5)$. Plot A, B, and M and verify that $d(A, M) = d(M, B) = \frac{1}{2}d(A, B)$.

SOLUTION

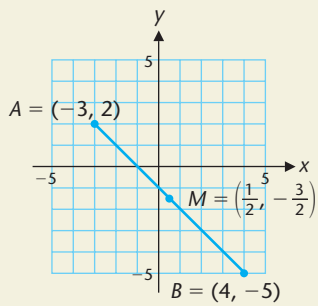
Finding the Midpoint

We use the midpoint formula with $(x_1, y_1) = (-3, 2)$ and $(x_2, y_2) = (4, -5)$ to obtain the coordinates of the midpoint M.

$$\begin{aligned} M &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{-3 + 4}{2}, \frac{2 + (-5)}{2}\right) \\ &= \left(\frac{1}{2}, \frac{-3}{2}\right) \\ &= (0.5, -1.5) \end{aligned}$$

Plotting and Verifying

Figure 3 shows the three points.



› Figure 3

The fraction form of M is probably more convenient for plotting the point. The decimal form is more convenient for computing distances.

Find the distances $d(A, M)$, $d(M, B)$, and $\frac{1}{2}d(A, B)$

$$\begin{aligned} d(A, M) &= \sqrt{(-3 - 0.5)^2 + [2 - (-1.5)]^2} \\ &= \sqrt{12.25 + 12.25} = \sqrt{24.5} \\ d(M, B) &= \sqrt{(0.5 - 4)^2 + [-1.5 - (-5)]^2} \\ &= \sqrt{12.25 + 12.25} = \sqrt{24.5} \\ d(A, B) &= \sqrt{(-3 - 4)^2 + [2 - (-5)]^2} \\ &= \sqrt{49 + 49} = \sqrt{98} \\ \frac{1}{2}d(A, B) &= \frac{1}{2}\sqrt{98} = \sqrt{\frac{98}{4}} = \sqrt{24.5} \end{aligned}$$

Because $d(A, M) = d(M, B) = \frac{1}{2}d(A, B)$, M is the midpoint of the line segment joining A and B.

MATCHED PROBLEM 3

Find the midpoint M of the line segment joining $A = (4, 1)$ and $B = (-3, -5)$. Plot A, B, and M and verify that $d(A, M) = d(M, B) = \frac{1}{2}d(A, B)$.

EXAMPLE 4 Using the Midpoint Formula

If $M = (1, 1)$ is the midpoint of the line segment joining $A = (-3, -1)$ and $B = (x, y)$, find the coordinates of B.

SOLUTIONS

Algebraic Solution

From the midpoint formula, we have

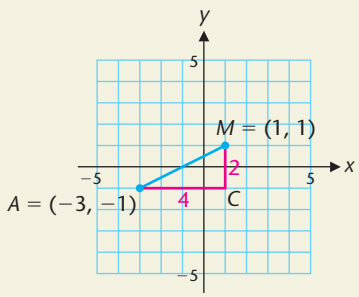
$$M = (1, 1) = \left(\frac{-3 + x}{2}, \frac{-1 + y}{2}\right)$$

We equate the corresponding coordinates and solve the resulting equations for x and y

$$\begin{aligned} 1 &= \frac{-3 + x}{2} & 1 &= \frac{-1 + y}{2} \\ 2 &= -3 + x & 2 &= -1 + y \\ 2 + 3 &= -3 + x + 3 \ * & 2 + 1 &= -1 + y + 1 \\ 5 &= x & 3 &= y \end{aligned}$$

Graphical Solution

Plot A and M and draw right triangle ACM as shown in Figure 4.



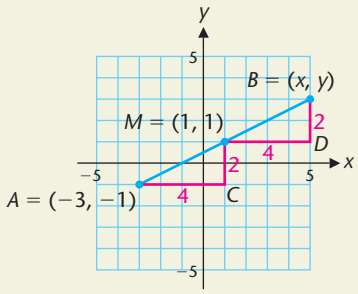
› Figure 4

*The dashed "think boxes" are used to enclose steps that may be performed mentally.

Therefore,

$$B = (5, 3)$$

Now form triangle MDB by shifting triangle ACM four units to the right and two units up (Fig. 5).



› Figure 5

Because triangles ACM and MDB are **congruent** (see Appendix D), $d(A, M) = d(M, B)$ and M must be the midpoint of the segment joining A and B. From the graph we can see that

$$x = 1 + 4 = 5 \qquad \text{and} \qquad y = 1 + 2 = 3$$

Therefore,

$$B = (5, 3)$$

MATCHED PROBLEM 4

If $M = (1, -1)$ is the midpoint of the line segment joining $A = (-1, -5)$ and $B = (x, y)$, find the coordinates of B.

› Circles

The distance-between-two-points formula would be helpful if its only use were to find actual distances between points, such as in Example 2. However, its more important use is in finding equations of figures in a rectangular coordinate system. We start with an example.

EXAMPLE 5 Equations and Graphs of Circles

Write an equation for the set of all points that are five units from the origin. Graph your equation.

SOLUTION

Writing the Equation

The distance between a point (x, y) and the origin is

$$d = \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2}$$

Therefore, an equation for the set of points that are five units from the origin is

$$\sqrt{x^2 + y^2} = 5$$

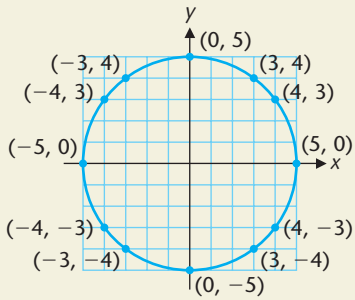
We square both sides of this equation to obtain an equation that does not contain any radicals.

$$x^2 + y^2 = 25$$

Graphing the Equation

We make up a table of solutions and plot the points in the table. Joining these points produces a familiar geometric object—a circle (Fig. 6).

x	y
-5	0
-4	± 3
-3	± 4
0	± 5
3	± 4
4	± 3
5	0



› Figure 6

MATCHED PROBLEM 5

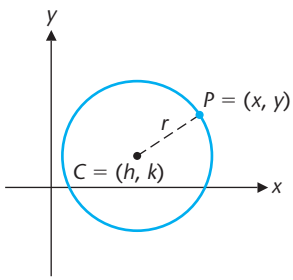
Write an equation for the set of all points that are 3 units from the origin. Graph your equation.

In Example 5, we began with a verbal description of a set of points, produced an algebraic equation that these points must satisfy, constructed a numerical table listing some of these points, and then drew a graphical representation of this set of points. The interplay between verbal, algebraic, numerical, and graphical concepts is one of the central themes of this book.

Now we generalize the ideas introduced in Example 5.

› **DEFINITION 1 Circle**

A **circle** is the set of all points in a plane equidistant from a fixed point. The fixed distance is called the **radius,** and the fixed point is called the **center.**



› Figure 7 Circle.

Let's find the equation of a circle with radius r $(r > 0)$ and center C at (h, k) in a rectangular coordinate system (Fig. 7). The circle consists of all points $P = (x, y)$ satisfying $d(P, C) = r$; that is, all points satisfying

$$\sqrt{(x - h)^2 + (y - k)^2} = r \qquad r > 0$$

or, equivalently,

$$(x - h)^2 + (y - k)^2 = r^2 \qquad r > 0$$

› THEOREM 3 Standard Equation of a Circle

For a circle with radius r and center at (h, k),

$$(x - h)^2 + (y - k)^2 = r^2 \quad r > 0$$

EXAMPLE 6 Equations and Graphs of Circles

Find the equation of a circle with radius 4 and center at $C = (-3, 6)$. Graph the equation.

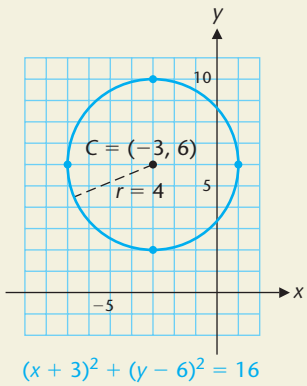
SOLUTION

Writing the Equation

$$C = (h, k) = (-3, 6) \text{ and } r = 4$$
$$(x - h)^2 + (y - k)^2 = r^2$$
$$[x - (-3)]^2 + (y - 6)^2 = 4^2$$
$$(x + 3)^2 + (y - 6)^2 = 16$$

Graphing the Equation

To graph the equation, plot the center and a few points on the circle, then draw a circle of radius 4 (Fig. 8).



$(x + 3)^2 + (y - 6)^2 = 16$

› Figure 8

MATCHED PROBLEM 6

Find the equation of a circle with radius 3 and center at $C = (3, -2)$. Graph the equation.

Refer to the solution of Example 6. If we square both binomials and collect like terms we obtain the following:

$$(x + 3)^2 + (y - 6)^2 = 16$$
$$x^2 + 6x + 9 + y^2 - 12y + 36 = 16$$
$$x^2 + 6x + y^2 - 12y + 29 = 0$$

With the exception of some degenerate cases, an equation of the form

$$x^2 + Ax + y^2 + By + C = 0$$

has a circle as its graph. To find the center and the radius, we use a technique called **completing the square,** which is based on the following equation:

$$\left(x + \frac{b}{2}\right)^2 = x^2 + bx + \left(\frac{b}{2}\right)^2$$

In the expanded form on the right, notice that the constant term $(b/2)^2$ is the square of one-half of the coefficient of the x term.

››› EXPLORE-DISCUSS 2

Replace ❐ and ❍ with numbers that make each equation true for all x.

(A) $(x + \square)^2 = x^2 + 4x + \bigcirc$

(B) $(x + \square)^2 = x^2 - 6x + \bigcirc$

(C) $(x + \square)^2 = x^2 + 5x + \bigcirc$

Example 7 illustrates the use of the completing-the-square technique.

EXAMPLE 7 Finding the Center and Radius of a Circle

Find the center and radius and sketch the graph of the circle with equation

$$x^2 - 10x + y^2 + 8y = 40$$

SOLUTION

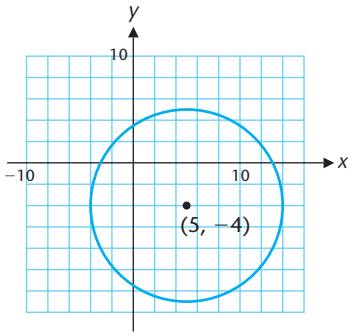
We apply the completing-the-square technique to the first two terms and then to the second two terms. Because we are dealing with an equation, we must be certain to add the required numbers to both sides of the equation.

$$x^2 - 10x + \qquad y^2 + 8y \qquad = 40$$

$$x^2 - 10x + \left(\frac{10}{2}\right)^2 + y^2 + 8y + \left(\frac{8}{2}\right)^2 = 40 + \left(\frac{10}{2}\right)^2 + \left(\frac{8}{2}\right)^2$$

$$x^2 - 10x + 25 + y^2 + 8y + 16 = 40 + 25 + 16$$

$$(x - 5)^2 + (y + 4)^2 = 81$$



› Figure 9

We now have the standard equation of a circle with radius 9 and center (5, −4). The graph is shown in Figure 9.

MATCHED PROBLEM 7

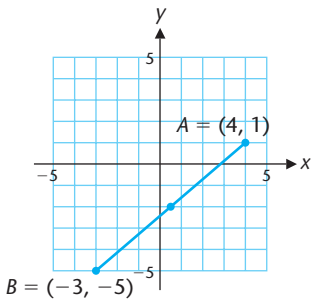
Find the center and radius and sketch the graph of the circle with equation

$$x^2 + 4x + y^2 - 6y = 12$$

ANSWERS TO MATCHED PROBLEMS

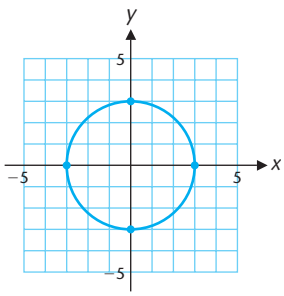
1. 13 **2.** $\sqrt{173}$

3. $M = (\frac{1}{2}, -2) = (0.5, -2)$; $d(A, M) = \sqrt{21.25} = d(M, B)$

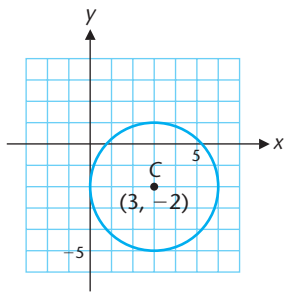


4. $B = (3, 3)$

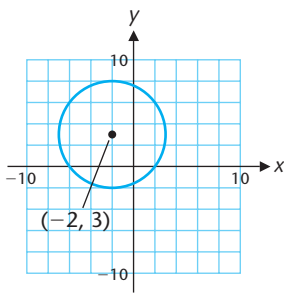
5. $x^2 + y^2 = 9$



6. $(x - 3)^2 + (y + 2)^2 = 9$



7. Circle with radius 5 and center (−2, 3)



B-3 Exercises

*Additional answers can be found in the Instructor Answer Appendix.

A

In Problems 1–8, find the distance between each pair of points and the midpoint of the line segment joining the points. Leave distance in radical form.

1. (1, 0), (4, 4)

2. (0, 1), (3, 5)

3. (0, −2), (5, 10)

4. (3, 0), (−2, −3)

5. (−6, −4), (3, 4)

6. (−5, 4), (6, −1)

7. (6, 6), (4, −2)

8. (5, −3), (−1, 4)

In Problems 9–16, write the equation of a circle with the indicated center and radius.

9. $C = (0, 0), r = 7$

10. $C = (0, 0), r = 5$

11. $C = (2, 3), r = 6$

12. $C = (5, 6), r = 2$

13. $C = (-4, 1), r = \sqrt{7}$

14. $C = (-5, 6), r = \sqrt{11}$

15. $C = (-3, -4), r = \sqrt{2}$

16. $C = (4, -1), r = \sqrt{5}$

In Problems 17–20, replace ❐ and ❍ with numbers that make the equation true for all x.

17. $(x + ❐)^2 = x^2 + 8x + ❍$

18. $(x + ❐)^2 = x^2 - 10x + ❍$

19. $(x + ❐)^2 = x^2 - 3x + ❍$

20. $(x + ❐)^2 = x^2 + x + ❍$

B

In Problems 21–26, write an equation for the given set of points. Graph your equation.

21. The set of all points that are two units from the origin.

22. The set of all points that are four units from the origin.

23. The set of all points that are one unit from (1, 0).

24. The set of all points that are one unit from (0, −1).

25. The set of all points that are three units from (−2, 1).

26. The set of all points that are two units from (3, −2).

27. Let M be the midpoint of A and B, where

$$A = (a_1, a_2), B = (1, 3), \text{ and } M = (-2, 6).$$

(A) Use the fact that −2 is the average of a_1 and 1 to find a_1.
(B) Use the fact that 6 is the average of a_2 and 3 to find a_2.
(C) Find $d(A, M)$ and $d(M, B)$.

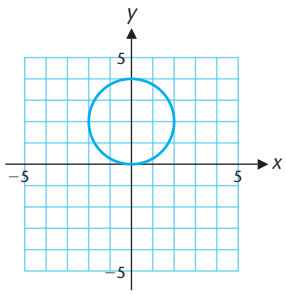
28. Let M be the midpoint of A and B, where

$$A = (-3, 5), B = (b_1, b_2), \text{ and } M = (4, -2).$$

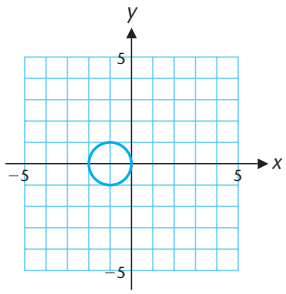
(A) Use the fact that 4 is the average of −3 and b_1 to find b_1.
(B) Use the fact that −2 is the average of 5 and b_2 to find b_2.
(C) Find $d(A, M)$ and $d(M, B)$.

In Problems 29–32, write a verbal description of the graph and then write an equation that would produce the graph.

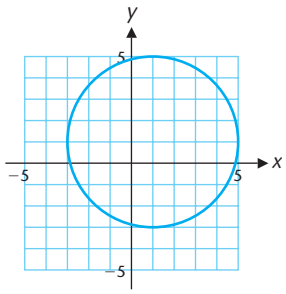
29.



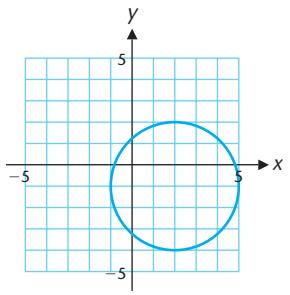
30.



31.



32.



In Problems 33–38, M is the midpoint of A and B. Find the indicated point. Verify that $d(A, M) = d(M, B) = \frac{1}{2}d(A, B)$.

33. $A = (-4.3, 5.2)$, $B = (9.6, -1.7)$, $M = ?$ (2.65, 1.75)

34. $A = (2.8, -3.5)$, $B = (-4.1, 7.6)$, $M = ?$ (−0.65, 2.05)

35. $A = (25, 10)$, $M = (-5, -2)$, $B = ?$ (−35, −14)

36. $M = (2.5, 3.5)$, $B = (12, 10)$, $A = ?$ (−7, −3)

37. $M = (-8, -6)$, $B = (2, 4)$, $A = ?$ (−18, −16)

38. $A = (-4, -2)$, $M = (-1.5, -4.5)$, $B = ?$ (1, −7)

In Problems 39–44, find the center and radius of the circle with the given equation.

39. $x^2 + y^2 - 2y = 0$ Center: (0, 1); radius: 1

40. $x^2 + 2x + y^2 = 0$ Center: (−1, 0); radius: 1

41. $x^2 - 2x + y^2 + 6y - 6 = 0$ Center: (1, −3); radius: 4

42. $x^2 + 4x + y^2 - 8y + 16 = 0$ Center: (−2, 4); radius: 2

43. $x^2 + x + y^2 + 3y + 2 = 0$ Center: (−0.5, −1.5); radius: $\sqrt{0.5}$

44. $x^2 - 5x + y^2 - 3y + 5 = 0$ Center: (2.5, 1.5); radius: $\sqrt{3.5}$

In Problems 45 and 46, show that the given points are the vertices of a right triangle (see the Pythagorean theorem in Appendix D). Find the length of the line segment from the midpoint of the hypotenuse to the opposite vertex.

45. (−3, 2), (1, −2), (8, 5) $\sqrt{32.5}$

46. (−1, 3), (3, 5), (5, 1) $\sqrt{10}$

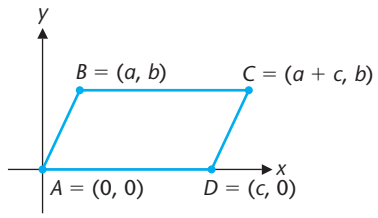
Find the perimeter (to two decimal places) of the triangle with the vertices indicated in Problems 47 and 48.

47. (−3, 1), (1, −2), (4, 3) 18.11

48. (−2, 4), (3, 1), (−3, −2) 18.62

C **49.** If $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ and $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$, show that $d(P_1, M) = d(M, P_2) = \frac{1}{2}d(P_1, P_2)$. (This is one step in the proof of Theorem 2).

50. A parallelogram *ABCD* is shown in the figure.
(A) Find the midpoint of the line segment joining A and C.
(B) Find the midpoint of the line segment joining B and D.
(C) What can you conclude about the diagonals of the parallelogram?



Find the equation of a circle that has a diameter with the end points given in Problems 51 and 52.

51. (7, −3), (1, 7) $(x - 4)^2 + (y - 2)^2 = 34$

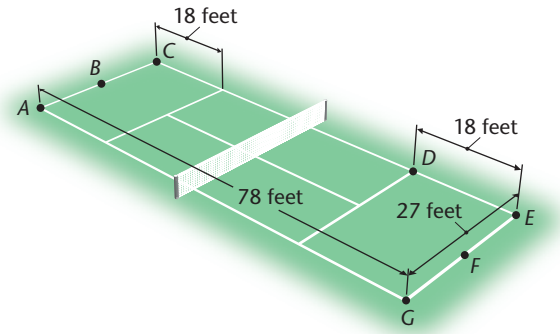
52. (−3, 2), (7, −4) $(x - 2)^2 + (y + 1)^2 = 34$

53. Find the equation of a circle with center (2, 2) whose graph passes through the point (3, −5). $(x - 2)^2 + (y - 2)^2 = 50$

54. Find the equation of a circle with center (−5, 4) whose graph passes through the point (2, −3). $(x + 5)^2 + (y - 4)^2 = 98$

APPLICATIONS

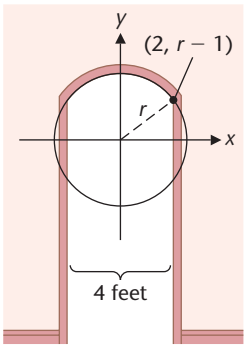
55. **SPORTS** A singles court for lawn tennis is a rectangle 27 feet wide and 78 feet long (see the figure). Points *B* and *F* are the midpoints of the end lines of the court.



(A) Sketch a graph of the court with *A* at the origin of your coordinate system, *C* on the positive *y* axis, and *G* on the positive *x* axis. Find the coordinates of points *A* through *G*.
(B) Find $d(B, D)$ and $d(F, C)$ to the nearest foot.

56. **SPORTS** Refer to Exercise 55. Find $d(A, D)$ and $d(C, G)$ to the nearest foot. 66 ft, 83 ft

57. ARCHITECTURE An arched doorway is formed by placing a circular arc on top of a rectangle (see the figure). If the doorway is 4 feet wide and the height of the arc above its ends is 1 foot, what is the radius of the circle containing the arc? [*Hint:* Note that $(2, r - 1)$ must satisfy $x^2 + y^2 = r^2$.]



Arched doorway

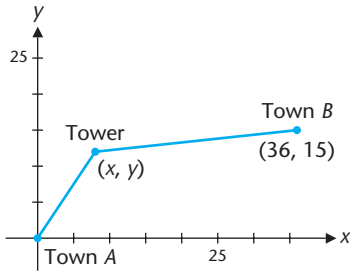
58. ENGINEERING The cross section of a rivet has a top that is an arc of a circle (see the figure). If the ends of the arc are 12 millimeters apart and the top is 4 millimeters above the ends, what is the radius of the circle containing the arc?

Rivet

***59.** CONSTRUCTION Town B is located 36 miles east and 15 miles north of town A (see the figure). A local telephone company wants to position a relay tower so that the distance from the tower to town B is twice the distance from the tower to town A.

(A) Show that the tower must lie on a circle, find the center and radius of this circle, and graph.

(B) If the company decides to position the tower on this circle at a point directly east of town A, how far from town A should they place the tower? Compute answer to one decimal place.



***60.** CONSTRUCTION Repeat Problem 59 if the distance from the tower to town A is twice the distance from the tower to town B.

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Special Topics

C

APPENDIX

OUTLINE

C-1 Significant Digits
C-2 Partial Fractions
C-3 Descartes' Rule of Signs

C-1 Significant Digits

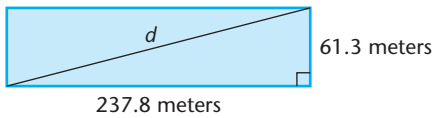
› Significant Digits
› Rounding Convention

› Significant Digits

Most calculations involving problems of the real world deal with numbers that are only approximate. It therefore seems reasonable to assume that a final answer should not be any more accurate than the least accurate number used in the calculation. This is an important point, because calculators tend to give the impression that greater accuracy is achieved than is warranted.

Suppose we wish to compute the length of the diagonal of a rectangular field from measurements of its sides of 237.8 meters and 61.3 meters. Using the Pythagorean theorem and a calculator, we find

$$d = \sqrt{237.8^2 + 61.3^2}$$
$$= 245.573\,878\ldots$$



The calculator answer suggests an accuracy that is not justified. What accuracy is justified? To answer this question, we introduce the idea of *significant digits.*

Whenever we write a measurement such as 61.3 meters, we assume that the measurement is accurate to the last digit written. Thus, the measurement 61.3 meters indicates that the measurement was made to the nearest tenth of a meter. That is, the actual width is between 61.25 meters and 61.35 meters. In general, the digits in a number that indicate the accuracy of the number are called **significant digits.** If all the digits in a number are nonzero, then they are all significant. Thus, the measurement 61.3 meters has three significant digits, and the measurement 237.8 meters has four significant digits.

What are the significant digits in the number 7,800? The accuracy of this number is not clear. It could represent a measurement with any of the following accuracies:

Between 7,750 and 7,850	Correct to the hundreds place
Between 7,795 and 7,805	Correct to the tens place
Between 7,799.5 and 7,800.5	Correct to the units place

To give a precise definition of significant digits that resolves this ambiguity, we use scientific notation.

› DEFINITION 1 Significant Digits

If a number x is written in scientific notation as

$$x = a \times 10^n \qquad 1 \le a < 10,\ n \text{ an integer}$$

then the number of significant digits in x is the number of digits in a.

Therefore,

7.8×10^3	has two significant digits
7.80×10^3	has three significant digits
7.800×10^3	has four significant digits

All three of these measurements have the same decimal representation (7,800), but each represents a different accuracy.

Definition 1 tells us how to write a number so that the number of significant digits is clear, but it does not tell us how to interpret the accuracy of a number that is not written in scientific notation. We will use the following convention for numbers that are written as decimal fractions:

› SIGNIFICANT DIGITS IN DECIMAL FRACTIONS

The number of significant digits in a number with no decimal point is found by counting the digits from left to right, starting with the first digit and ending with the last *nonzero* digit.

The number of significant digits in a number containing a decimal point is found by counting the digits from left to right, starting with the first *nonzero* digit and ending with the last digit.

Applying this rule to the number 7,800, we conclude that this number has two significant digits. If we want to indicate that it has three or four significant digits, we must use scientific notation.

EXAMPLE 1 Significant Digits in Decimal Fractions

Underline the significant digits in the following numbers:

(A) 70,007 (B) 82,000 (C) 5.600 (D) 0.0008 (E) 0.000 830

SOLUTIONS

(A) <u>70,007</u> (B) <u>82</u>,000 (C) <u>5.600</u> (D) 0.000<u>8</u> (E) 0.000 <u>830</u>

MATCHED PROBLEM 1

Underline the significant digits in the following numbers:

(A) 5,009 (B) 12,300 (C) 23.4000 (D) 0.00050 (E) 0.0012

Rounding Convention

In calculations involving multiplication, division, powers, and roots, we adopt the following convention:

ROUNDING CALCULATED VALUES

The result of a calculation is rounded to the same number of significant digits as the number used in the calculation that has the least number of significant digits.

Therefore, in computing the length of the diagonal of the rectangular field shown earlier, we write the answer rounded to three significant digits because the width has three significant digits and the length has four significant digits:

$$d = 246 \text{ meters}$$ Three significant digits

One Final Note: In rounding a number that is exactly halfway between a larger and a smaller number, we use the convention of making the final result even.

EXAMPLE 2 Rounding Numbers

Round each number to three significant digits.

(A) 43.0690 (B) 48.05 (C) 48.15 (D) $8.017\,632 \times 10^{-3}$

SOLUTIONS

(A) 43.1

(B) 48.0

(C) 48.2

Use the convention of making the digit before the 5 even if it is odd, or leaving it alone if it is even.

(D) 8.02×10^{-3}

MATCHED PROBLEM 2

Round each number to three significant digits.

(A) 3.1495 (B) 0.004 135 (C) 32,450 (D) $4.314\,764\,09 \times 10^{12}$

ANSWERS TO MATCHED PROBLEMS

1. (A) 5,009 (B) 12,300 (C) 23.4000 (D) 0.000 50 (E) 0.0012
2. (A) 3.15 (B) 0.004 14 (C) 32,400 (D) 4.31×10^{12}

C-1 Exercises

A

In Problems 1–12, underline the significant digits in each number.

1. 123,005
123,005
2. 3,400,002
3,400,002
3. 20,040
20,040
4. 300,600
300,600
5. 6.0
6.0
6. 7.00
7.00
7. 80.000
80,000
8. 900.0000
900.0000
9. 0.012
0.012
10. 0.0015
0.0015
11. 0.000 960
0.000 960
12. 0.000 700
0.000 700

B

In Problems 13–22, round each number to three significant digits.

13. 3.0780 3.08
14. 4.0240 4.02
15. 924,300 924, 000
16. 643,820
644,000
17. 23.65 23.6
18. 23.75 23.8
19. $2.816\,743 \times 10^{3}$
2.82×10^{3}
20. 56.114×10^{4} 56.1×10^{4}
21. $6.782\,045 \times 10^{-4}$
6.78×10^{-4}
22. $5.248\,102 \times 10^{-3}$ 5.25×10^{-3}

C

In Problems 23 and 24, find the diagonal of the rectangle with the indicated side measurements. Round answers to the number of significant digits appropriate for the given measurements.

23. 25 feet by 20 feet 30 feet

24. 2,900 yards by 1,570 yards 3,300 yards

C-2 Partial Fractions

- Basic Theorems
- Partial Fraction Decomposition

You have now had considerable experience combining two or more rational expressions into a single rational expression. For example, problems such as

$$\frac{2}{x+5}+\frac{3}{x-4}=\frac{2(x-4)+3(x+5)}{(x+5)(x-4)}=\frac{5x+7}{(x+5)(x-4)}$$

should seem routine. Frequently in more advanced courses, particularly in calculus, it is advantageous to be able to reverse this process—that is, to be able to express a rational expression as the sum of two or more simpler rational expressions called

partial fractions. As is often the case with reverse processes, the process of decomposing a rational expression into partial fractions is more difficult than combining rational expressions. Basic to the process is the factoring of polynomials, so many of the topics discussed in Chapter 3 can be put to effective use. Partial fraction decomposition is usually accomplished by solving a related system of linear equations. If you are familiar with basic techniques for solving linear systems discussed earlier in this book, such as Gauss–Jordan elimination, inverse matrix solutions, or Cramer's rule, you may use these as you see fit. However, all of the linear systems encountered in this section can also be solved by some special techniques developed here. Mathematically equivalent to the techniques mentioned, these special techniques are generally easier to use in partial fraction decomposition problems.

We confine our attention to rational expressions of the form $P(x)/D(x)$, where $P(x)$ and $D(x)$ are polynomials with real coefficients. In addition, we assume that the degree of $P(x)$ is less than the degree of $D(x)$. If the degree of $P(x)$ is greater than or equal to that of $D(x)$, we have only to divide $P(x)$ by $D(x)$ to obtain

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

where the degree of $R(x)$ is less than that of $D(x)$. For example,

$$\frac{x^4 - 3x^3 + 2x^2 - 5x + 1}{x^2 - 2x + 1} = x^2 - x - 1 + \frac{-6x + 2}{x^2 - 2x + 1}$$

If the degree of $P(x)$ is less than that of $D(x)$, then $P(x)/D(x)$ is called a **proper fraction.**

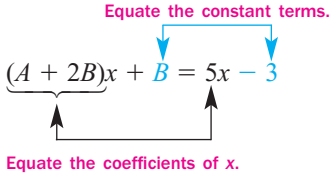
Basic Theorems

Our task now is to establish a systematic way to decompose a proper fraction into the sum of two or more partial fractions. Theorems 1, 2, and 3 take care of the problem completely.

THEOREM 1 Equal Polynomials

Two polynomials are equal to each other if and only if the coefficients of terms of like degree are equal.

For example, if



then

$$\begin{aligned} B &= -3 && \text{Substitute } B = -3 \text{ into the second equation to solve for } A. \\ A + 2B &= 5 \\ A + 2(-3) &= 5 \\ A &= 11 \end{aligned}$$

››› EXPLORE-DISCUSS 1

If

$$x + 5 = A(x + 1) + B(x - 3) \tag{1}$$

is a polynomial identity (that is, both sides represent the same polynomial), then equating coefficients produces the system

$$\begin{aligned} 1 &= A + B && \text{Equating coefficients of } x \\ 5 &= A - 3B && \text{Equating constant terms} \end{aligned}$$

(A) Solve this system graphically.

(B) For an alternate method of solution, substitute $x = 3$ in equation (1) to find A and then substitute $x = -1$ in equation (1) to find B. Explain why this method is valid.

The Linear and Quadratic Factors Theorem from Chapter 3 (page 323) underlies the technique of decomposing a rational function into partial fractions. We restate the theorem here.

› THEOREM 2 Linear and Quadratic Factors Theorem

For a polynomial of degree $n > 0$ with real coefficients, there always exists a factorization involving only linear and/or quadratic factors with real coefficients in which the quadratic factors have imaginary zeros.

The quadratic formula can be used to determine easily whether a given quadratic factor $ax^2 + bx + c$, with real coefficients, has imaginary zeros. If $b^2 - 4ac < 0$, then $ax^2 + bx + c$ has imaginary zeros. Otherwise its zeros are real. Therefore, $ax^2 + bx + c$ has imaginary zeros if and only if it cannot be factored as a product of linear factors with real coefficients.

Partial Fraction Decomposition

We are now ready to state Theorem 3, which forms the basis for partial fraction decomposition.

THEOREM 3 Partial Fraction Decomposition

Any proper fraction $P(x)/D(x)$ reduced to lowest terms can be decomposed into the sum of partial fractions as follows:

1. If $D(x)$ has a nonrepeating linear factor of the form $ax + b$, then the partial fraction decomposition of $P(x)/D(x)$ contains a term of the form

$$\frac{A}{ax + b} \qquad A \text{ a constant}$$

2. If $D(x)$ has a k-repeating linear factor of the form $(ax + b)^k$, then the partial fraction decomposition of $P(x)/D(x)$ contains k terms of the form

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_k}{(ax + b)^k} \qquad A_1, A_2, \ldots, A_k \text{ constants}$$

3. If $D(x)$ has a nonrepeating quadratic factor of the form $ax^2 + bx + c$ that has imaginary zeros, then the partial fraction decomposition of $P(x)/D(x)$ contains a term of the form

$$\frac{Ax + B}{ax^2 + bx + c} \qquad A, B \text{ constants}$$

4. If $D(x)$ has a k-repeating quadratic factor of the form $(ax^2 + bx + c)^k$, where $ax^2 + bx + c$ has imaginary zeros, then the partial fraction decomposition of $P(x)/D(x)$ contains k terms of the form

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$$
$$A_1, \ldots, A_k, B_1, \ldots, B_k \text{ constants}$$

Let's see how the theorem is used to obtain partial fraction decompositions in several examples.

EXAMPLE 1 Nonrepeating Linear Factors

Decompose into partial fractions $\dfrac{5x + 7}{x^2 + 2x - 3}$.

SOLUTION

We first try to factor the denominator. If it can't be factored in the real numbers, then we can't go any further. In this example, the denominator factors, so we apply part 1 from Theorem 3:

$$\frac{5x + 7}{(x - 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 3} \tag{2}$$

To find the constants A and B, we combine the fractions on the right side of equation (2) to obtain

$$\frac{5x + 7}{(x - 1)(x + 3)} = \frac{A(x + 3) + B(x - 1)}{(x - 1)(x + 3)}$$

Because these fractions have the same denominator, their numerators must be equal. Therefore,

$$5x + 7 = A(x + 3) + B(x - 1) \tag{3}$$

We could multiply the right side and find A and B by using Theorem 1, but in this case, it is easier to take advantage of the fact that equation (3) is an identity—that is, it must hold for all values of x. In particular, we note that if we let $x = 1$, then the second term of the right side drops out and we can solve for A:

$$\begin{aligned} 5 \cdot 1 + 7 &= A(1 + 3) + B(1 - 1) \\ 12 &= 4A \\ A &= 3 \end{aligned}$$

Similarly, if we let $x = -3$, the first term drops out and we find

$$\begin{aligned} -8 &= -4B \\ B &= 2 \end{aligned}$$

Hence,

$$\frac{5x + 7}{x^2 + 2x - 3} = \frac{3}{x - 1} + \frac{2}{x + 3} \tag{4}$$

as can easily be checked by adding the two fractions on the right. ◉

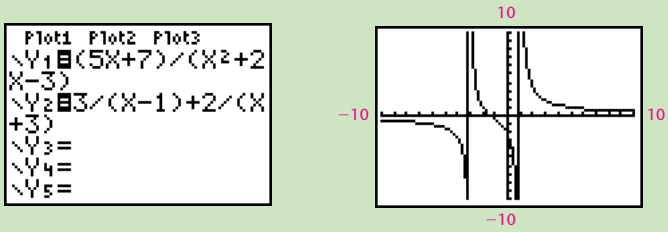
MATCHED PROBLEM 1

Decompose into partial fractions $\dfrac{7x + 6}{x^2 + x - 6}$.

◉

››› EXPLORE-DISCUSS 2

A graphing calculator can also be used to check a partial fraction decomposition. To check Example 1, we graph the left and right sides of equation (4) in a graphing calculator (Fig. 1). Discuss how the TRACE command on the graphing calculator can be used to check that the graphing calculator is displaying two identical graphs.



› Figure 1

EXAMPLE 2 Repeating Linear Factors

Decompose into partial fractions $\dfrac{6x^2 - 14x - 27}{(x + 2)(x - 3)^2}$.

SOLUTION

Using parts 1 and 2 from Theorem 3, we write

$$\frac{6x^2 - 14x - 27}{(x + 2)(x - 3)^2} = \frac{A}{x + 2} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2}$$
$$= \frac{A(x - 3)^2 + B(x + 2)(x - 3) + C(x + 2)}{(x + 2)(x - 3)^2}$$

Therefore, for all x,

$$6x^2 - 14x - 27 = A(x - 3)^2 + B(x + 2)(x - 3) + C(x + 2)$$

If $x = 3$, then

$-15 = 5C$

$C = -3$

If $x = -2$, then

$25 = 25A$

$A = 1$

There are no other values of x that will cause terms on the right to drop out. Because any value of x can be substituted to produce an equation relating A, B, and C, we let $x = 0$ and obtain

$$-27 = 9A - 6B + 2C \quad \text{Substitute } A = 1 \text{ and } C = -3.$$
$$-27 = 9 - 6B - 6$$
$$B = 5$$

Therefore,

$$\frac{6x^2 - 14x - 27}{(x + 2)(x - 3)^2} = \frac{1}{x + 2} + \frac{5}{x - 3} - \frac{3}{(x - 3)^2}$$

MATCHED PROBLEM 2

Decompose into partial fractions $\dfrac{x^2 + 11x + 15}{(x - 1)(x + 2)^2}$.

EXAMPLE 3 **Nonrepeating Linear and Quadratic Factors**

Decompose into partial fractions $\dfrac{5x^2 - 8x + 5}{(x - 2)(x^2 - x + 1)}$.

SOLUTION

Since the discriminant of the quadratic in the denominator is negative, this quadratic can't be factored further in the real numbers. Then, we use parts 1 and 3 from Theorem 3 to write

$$\begin{aligned}\frac{5x^2 - 8x + 5}{(x - 2)(x^2 - x + 1)} &= \frac{A}{x - 2} + \frac{Bx + C}{x^2 - x + 1} \\ &= \frac{A(x^2 - x + 1) + (Bx + C)(x - 2)}{(x - 2)(x^2 - x + 1)}\end{aligned}$$

Therefore, for all x,

$$5x^2 - 8x + 5 = A(x^2 - x + 1) + (Bx + C)(x - 2)$$

If $x = 2$, then

$$\begin{aligned}9 &= 3A \\ A &= 3\end{aligned}$$

If $x = 0$, then, using $A = 3$, we have

$$\begin{aligned}5 &= 3 - 2C \\ C &= -1\end{aligned}$$

If $x = 1$, then, using $A = 3$ and $C = -1$, we have

$$\begin{aligned}2 &= 3 + (B - 1)(-1) \\ B &= 2\end{aligned}$$

Hence,

$$\frac{5x^2 - 8x + 5}{(x - 2)(x^2 - x + 1)} = \frac{3}{x - 2} + \frac{2x - 1}{x^2 - x + 1}$$

MATCHED PROBLEM 3

Decompose into partial fractions $\dfrac{7x^2 - 11x + 6}{(x - 1)(2x^2 - 3x + 2)}$.

EXAMPLE 4 **Repeating Quadratic Factors**

Decompose into partial fractions $\dfrac{x^3 - 4x^2 + 9x - 5}{(x^2 - 2x + 3)^2}$

SOLUTION

Because $x^2 - 2x + 3$ can't be factored further in the real numbers, we proceed to use part 4 from Theorem 3 to write

$$\begin{aligned}\frac{x^3 - 4x^2 + 9x - 5}{(x^2 - 2x + 3)^2} &= \frac{Ax + B}{x^2 - 2x + 3} + \frac{Cx + D}{(x^2 - 2x + 3)^2} \\ &= \frac{(Ax + B)(x^2 - 2x + 3) + Cx + D}{(x^2 - 2x + 3)^2}\end{aligned}$$

Therefore, for all x,

$$x^3 - 4x^2 + 9x - 5 = (Ax + B)(x^2 - 2x + 3) + Cx + D$$

Because the substitution of carefully chosen values of x doesn't lead to the immediate determination of A, B, C, or D, we multiply and rearrange the right side to obtain

$$x^3 - 4x^2 + 9x - 5 = Ax^3 + (B - 2A)x^2 + (3A - 2B + C)x + (3B + D)$$

Now we use Theorem 1 to equate coefficients of terms of like degree:

$$\begin{aligned} A &= 1 \\ B - 2A &= -4 \\ 3A - 2B + C &= 9 \\ 3B + D &= -5 \end{aligned}$$

$$\underbrace{1x^3}_{Ax^3} + \underbrace{-4x^2}_{(B-2A)x^2} + \underbrace{+9x}_{(3A-2B+C)x} + \underbrace{-5}_{(3B+D)}$$

From these equations we easily find that $A = 1$, $B = -2$, $C = 2$, and $D = 1$. Now we can write

$$\frac{x^3 - 4x^2 + 9x - 5}{(x^2 - 2x + 3)^2} = \frac{x - 2}{x^2 - 2x + 3} + \frac{2x + 1}{(x^2 - 2x + 3)^2}$$

MATCHED PROBLEM 4

Decompose into partial fractions $\dfrac{3x^3 - 6x^2 + 7x - 2}{(x^2 - 2x + 2)^2}$.

ANSWERS TO MATCHED PROBLEMS

1. $\dfrac{4}{x-2} + \dfrac{3}{x+3}$
2. $\dfrac{3}{x-1} - \dfrac{2}{x+2} + \dfrac{1}{(x+2)^2}$
3. $\dfrac{2}{x-1} + \dfrac{3x-2}{2x^2-3x+2}$
4. $\dfrac{3x}{x^2-2x+2} + \dfrac{x-2}{(x^2-2x+2)^2}$

C-2 Exercises

*Additional answers can be found in the Instructor Answer Appendix.

A

In Problems 1–4, find A and B so that the right side is equal to the left. After cross-multiplying to produce a polynomial equation, solve each problem two ways (see Explore-Discuss 1). First, equate the coefficients of both sides to determine a linear system for A and B and solve this system. Second, solve for A and B by evaluating both sides for selected values of x.

1. $\dfrac{7x-14}{(x-4)(x+3)} = \dfrac{A}{x-4} + \dfrac{B}{x+3}$ $A = 2, B = 5$
2. $\dfrac{9x+21}{(x+5)(x-3)} = \dfrac{A}{x+5} + \dfrac{B}{x-3}$ $B = 6, A = 3$
3. $\dfrac{17x-1}{(2x-3)(3x-1)} = \dfrac{A}{2x-3} + \dfrac{B}{3x-1}$ $A = 7, B = -2$
4. $\dfrac{x-11}{(3x+2)(2x-1)} = \dfrac{A}{3x+2} + \dfrac{B}{2x-1}$ $B = -3, A = 5$

In Problems 5–10, find A, B, C, and D, so that the right side is equal to the left.

5. $\dfrac{3x^2+7x+1}{x(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$ $A = 1, B = 2, C = 3$
6. $\dfrac{x^2-6x+11}{(x+1)(x-2)^2} = \dfrac{A}{x+1} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2}$ $A = 2, B = -1, C = 1$
7. $\dfrac{3x^2+x}{(x-2)(x^2+3)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+3}$ $A = 2, B = 1, C = 3$
8. $\dfrac{5x^2-9x+19}{(x-4)(x^2+5)} = \dfrac{A}{x-4} + \dfrac{Bx+C}{x^2+5}$ $A = 3, B = 2, C = -1$
9. $\dfrac{2x^2+4x-1}{(x^2+x+1)^2} = \dfrac{Ax+B}{x^2+x+1} + \dfrac{Cx+D}{(x^2+x+1)^2}$ $A = 0, B = 2, C = 2, D = -3$
10. $\dfrac{3x^3-3x^2+10x-4}{(x^2-x+3)^2} = \dfrac{Ax+B}{x^2-x+3} + \dfrac{Cx+D}{(x^2-x+3)^2}$ $A = 3, B = 0, C = 1, D = -4$

B

In Problems 11–30, decompose into partial fractions.

11. $\dfrac{-x+22}{x^2-2x-8}$ $\dfrac{-4}{x+2} + \dfrac{3}{x-4}$
12. $\dfrac{-x-21}{x^2+2x-15}$ $\dfrac{-3}{x-3} + \dfrac{2}{x+5}$
13. $\dfrac{3x-13}{6x^2-x-12}$
14. $\dfrac{11x-11}{6x^2+7x-3}$
15. $\dfrac{x^2-12x+18}{x^3-6x^2+9x}$
16. $\dfrac{5x^2-36x+48}{x(x-4)^2}$
17. $\dfrac{5x^2+3x+6}{x^3+2x^2+3x}$
18. $\dfrac{6x^2-15x+16}{x^3-3x^2+4x}$
19. $\dfrac{2x^3+7x+5}{x^4+4x^2+4}$
20. $\dfrac{-5x^2+7x-18}{x^4+6x^2+9}$
21. $\dfrac{x^3-7x^2+17x-17}{x^2-5x+6}$ $x - 2 + \dfrac{3}{x-2} - \dfrac{2}{x-3}$
22. $\dfrac{x^3+x^2-13x+11}{x^2+2x-15}$ $x - 1 + \dfrac{3}{x+5} + \dfrac{1}{x-3}$

C

23. $\dfrac{4x^2 + 5x - 9}{x^3 - 6x - 9}$

24. $\dfrac{4x^2 - 8x + 1}{x^3 - x + 6}$

25. $\dfrac{x^2 + 16x + 18}{x^3 + 2x^2 - 15x - 36}$

26. $\dfrac{5x^2 - 18x + 1}{x^3 - x^2 - 8x + 12}$

$\dfrac{4}{x+3} + \dfrac{1}{x-2} + \dfrac{-3}{(x-2)^2}$

27. $\dfrac{-x^2 + x - 7}{x^4 - 5x^3 + 9x^2 - 8x + 4}$

$\dfrac{2}{x-2} - \dfrac{3}{(x-2)^2} - \dfrac{2x}{x^2 - x + 1}$

28. $\dfrac{-2x^3 + 12x^2 - 20x - 10}{x^4 - 7x^3 + 17x^2 - 21x + 18}$ $\dfrac{1}{x-3} + \dfrac{-2}{(x-3)^2} + \dfrac{-3x}{x^2 - x + 2}$

29. $\dfrac{4x^5 + 12x^4 - x^3 + 7x^2 - 4x + 2}{4x^4 + 4x^3 - 5x^2 + 5x - 2}$

30. $\dfrac{6x^5 - 13x^4 + x^3 - 8x^2 + 2x}{6x^4 - 7x^3 + x^2 + x - 1}$

$x - 1 + \dfrac{1}{2x+1} + \dfrac{-2}{x-1} + \dfrac{x-2}{3x^2 - 2x + 1}$

C-3 Descartes' Rule of Signs

- Variation in Sign
- Descartes' Rule of Signs

In this section we discuss *Descartes' rule of signs,* a theorem that gives information about the number of real zeros of a polynomial. When used in conjunction with the theorems discussed in Sections 3-3 and 3-4, Descartes' rule of signs can simplify the search for zeros.

Variation in Sign

When the terms of a polynomial with real coefficients are arranged in order of descending powers, we say that a **variation in sign** occurs if two successive terms have opposite signs. Missing terms (terms with 0 coefficients) are ignored. For a given polynomial, we are interested in the total number of variations in sign in both $P(x)$ and $P(-x)$.

EXAMPLE 1 Variations in Sign

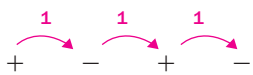
If $P(x) = 3x^4 - 2x^3 + 3x - 5$, find the number of variations in sign in $P(x)$ and in $P(-x)$.

SOLUTION

The signs of the coefficients in

$$P(x) = 3x^4 - 2x^3 + 3x - 5$$

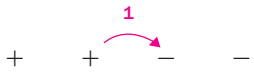
are



Therefore, $P(x)$ has three variations in sign. The signs of the coefficients in

$$P(-x) = 3x^4 + 2x^3 - 3x - 5$$

are



$P(-x)$ has one variation in sign.

MATCHED PROBLEM 1

If $P(x) = 2x^5 - x^4 - x^3 + x + 5$, find the number of variations in sign in $P(x)$ and in $P(-x)$

Descartes' Rule of Signs

The number of variations in sign for $P(x)$ and $P(-x)$ gives us useful information about the number of real zeros of a polynomial with real coefficients. In 1636 René Descartes (1596–1650), a French philosopher and mathematician, gave the first proof of a simplified version of a theorem that now bears his name. We state Theorem 1 without proof, because a proof is beyond the scope of this book.

THEOREM 1 Descartes' Rule of Signs

Given a polynomial $P(x)$ with real coefficients and nonzero constant term

1. **Positive Zeros.** The number of positive zeros of $P(x)$ is never greater than the number of variations in sign in $P(x)$ and, if less, then always by an even number.
2. **Negative Zeros.** The number of negative zeros of $P(x)$ is never greater than the number of variations in sign in $P(-x)$ and, if less, then always by an even number.

It is important to understand that when we refer to positive zeros and negative zeros we are referring to real zeros. There are no positive or negative imaginary numbers.

EXAMPLE 2 Combinations of Zeros

Let $P(x) = x^3 + x - 2$. How many real zeros does $P(x)$ have? How many imaginary zeros does $P(x)$ have?

SOLUTION

We apply Descartes' rule of signs

$$P(x) = x^3 + x - 2 \quad \text{One variation in sign}$$
$$P(-x) = -x^3 - x - 2 \quad \text{No variations in sign}$$

According to Descartes' rule of signs, $P(x)$ must have at most one positive zero and the number of positive zeros must differ from one by an even number. Because 0 does not differ from 1 by an even integer, 0 positive zeros is not a possibility, and $P(x)$ must have exactly one positive zero. Because $P(-x)$ has no variations in sign, $P(x)$ has no negative zeros. From the fundamental theorem of algebra (Section 3-4), $P(x)$ must have a total of three zeros. Therefore, the two remaining zeros must be imaginary.

MATCHED PROBLEM 2

Let $P(x) = x^4 + x^3 - 2$. How many real zeros does $P(x)$ have? How many imaginary zeros does $P(x)$ have?

››› EXPLORE-DISCUSS 1

Apply Descartes' rule of signs to each of the following polynomials to find the maximum possible number of real zeros. Then use a graphing utility to determine the exact number of real zeros and the exact number of imaginary zeros for each polynomial.

(A) $P(x) = x^4 - 4x^2 + 3$

(B) $Q(x) = x^4 - 4x^2 + 5$

Descartes' rule of signs enabled us to determine the exact number of real and imaginary zeros in Example 2. If there is more than zero or one variation in sign, then there can be more than one possible outcome. A table is a convenient way to summarize the various possibilities.

EXAMPLE 3 Possible Combinations of Zeros

Construct a table showing the possible combinations of positive, negative, and imaginary zeros of

(A) $P(x) = 3x^4 - 2x^3 + 3x - 5$ (B) $Q(x) = x^5 - 2x^4 + 5x^3 - 7x - 9$

SOLUTION

(A) $P(x)$ has three variations in sign and $P(-x)$ has one (see Example 1). Thus, $P(x)$ has either three positive zeros or one positive zero and exactly one negative zero. Because $P(x)$ must have a total of four zeros, any remaining zeros must be imaginary. The table summarizes the possible combinations of zeros, where + = positive, − = negative, I = imaginary. Note that the sum of each row is four, the degree of $P(x)$ and the total number of zeros.

+	−	I
3	1	0
1	1	2

(B) Apply Descartes' rule of signs,

$$Q(x) = x^5 - 2x^4 + 5x^3 - 7x - 9 \quad \text{Three variations in sign}$$
$$Q(-x) = -x^5 - 2x^4 - 5x^3 + 7x - 9 \quad \text{Two variations in sign}$$

Possible combinations of zeros are given in the table.

+	−	I
3	2	0
3	0	2
1	2	2
1	0	4

MATCHED PROBLEM 3

Construct a table showing the possible combinations of positive, negative, and imaginary zeros of

(A) $P(x) = x^4 - 3x + 1$ (B) $Q(x) = 4x^5 + 2x^4 - x^3 + x + 5$

ANSWERS TO MATCHED PROBLEMS

1. $P(x)$ has two variations in sign and $P(-x)$ has three.

2. $P(x)$ has two real zeros and two imaginary zeros.

3. (A)

+	−	I
2	0	2
0	0	4

(B)

+	−	I
2	3	0
2	1	2
0	3	2
0	1	4

C-3 Exercises

*Additional answers can be found in the Instructor Answer Appendix.

A *In Problems 1–6, find the number of variations in sign in $P(x)$ and $P(-x)$.*

1. $P(x) = x^3 + 2x + 7$

2. $P(x) = x^3 + 4x - 5$

3. $P(x) = x^3 - 3x^2 - 9$

4. $P(x) = x^3 + 6x^2 + 8$

5. $P(x) = x^4 + 2x^3 + 3x - 5$

6. $P(x) = x^4 - 4x^3 - 2x - 3$

In Problems 7–12, use Descartes' rule of signs to find the number of real zeros and the number of imaginary zeros for each polynomial. Check your answer by graphing the polynomial on a graphing utility.

7. $P(x) = x^3 + 2x - 4$

8. $Q(x) = x^3 + 3x + 6$

9. $f(x) = x^3 + 2x^2 + 1$

10. $g(x) = x^3 - 5x^2 - 7$

11. $s(x) = x^4 - 2x^3 - 7x - 8$

12. $r(x) = x^4 + 3x^3 + 4x - 9$

B *In Problems 13–22, construct a table showing the possible combinations of positive, negative, and imaginary zeros of each polynomial.*

13. $P(x) = x^3 - 3x^2 - 2x + 4$

14. $Q(x) = x^3 - 4x^2 + 3x + 5$

15. $t(x) = x^4 - 2x^3 - 4x^2 - 2x + 3$

16. $s(x) = x^4 + 2x^3 - 5x^2 + 3x - 6$

17. $f(x) = x^5 - x^4 + 3x^3 + 9x^2 - x + 5$

18. $g(x) = x^5 + x^4 - 4x^3 + 3x^2 - x - 1$

19. $P(x) = x^6 - 12$

20. $Q(x) = x^8 - 24$

21. $r(x) = x^7 + 32$

22. $w(x) = x^5 + 25$

In Problems 23–26, discuss the possible combinations of positive, negative, and imaginary zeros of $P(x) = x^2 + ax + b$ for the indicated values of a and b.

23. $a > 0, b > 0$

24. $a < 0, b > 0$

25. $a > 0, b < 0$

26. $a < 0, b < 0$

C *In Problems 27–32, construct a table showing the possible combinations of positive, negative, and imaginary zeros of each polynomial.*

27. $f(x) = x^6 - 3x^5 + 4x^4 + 3x^3 - 2x + 5$

28. $g(x) = x^6 + 4x^5 - 2x^4 - 6x^3 - 5x^2 + 7$

29. $s(x) = x^7 + 3x^5 + 4x^2 - 3x + 5$

30. $w(x) = x^7 + 2x^5 - 3x^2 + 2x - 7$

31. $P(x) = x^8 - x - 1$

32. $Q(x) = x^9 + x - 1$

In Problems 33–36, discuss the possible combinations of positive, negative, and imaginary zeros of $P(x) = x^3 + ax + b$ for the indicated values of a and b.

33. $a > 0, b > 0$

34. $a > 0, b < 0$

35. $a < 0, b > 0$

36. $a < 0, b > 0$

*Problem numbers that appear in blue indicate that require students to apply their reasoning and writing skills in the solution of the problem.

Geometric Formulas

D

APPENDIX

Similar Triangles

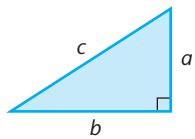
(A) Two triangles are similar if two angles of one triangle have the same measure as two angles of the other.

(B) If two triangles are similar, their corresponding sides are proportional:

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

Pythagorean Theorem

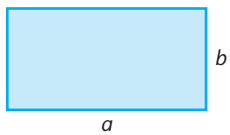
$$c^2 = a^2 + b^2$$



Rectangle

$A = ab$ Area

$P = 2a + 2b$ Perimeter

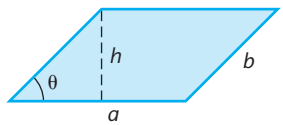


Parallelogram

h = Height

$A = ah = ab \sin \theta$ Area

$P = 2a + 2b$ Perimeter



Triangle

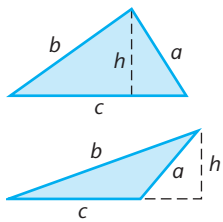
h = Height

$A = \frac{1}{2}hc$ Area

$P = a + b + c$ Perimeter

$s = \frac{1}{2}(a + b + c)$ Semiperimeter

$A = \sqrt{s(s - a)(s - b)(s - c)}$ Area—Heron's formula

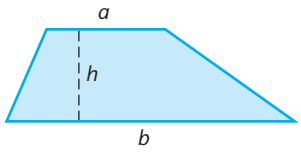


Trapezoid

Base a is parallel to base b.

h = Height

$A = \frac{1}{2}(a + b)h$ Area



Circle

R = Radius

D = Diameter

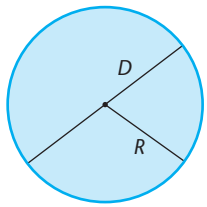
$D = 2R$

$A = \pi R^2 = \frac{1}{4}\pi D^2$ Area

$C = 2\pi R = \pi D$ Circumference

$\frac{C}{D} = \pi$ For all circles

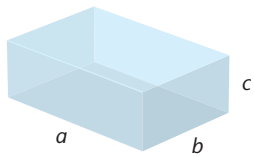
$\pi \approx 3.141\,59$



Rectangular Solid

$V = abc$ Volume

$T = 2ab + 2ac + 2bc$ Total surface area



Right Circular Cylinder

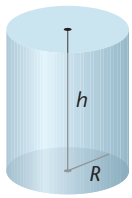
R = Radius of base

h = Height

$V = \pi R^2 h$ Volume

$S = 2\pi Rh$ Lateral surface area

$T = 2\pi R(R + h)$ Total surface area



Right Circular Cone

R = Radius of base

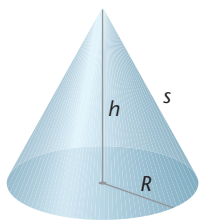
h = Height

s = Slant height

$V = \frac{1}{3}\pi R^2 h$ Volume

$S = \pi Rs = \pi R\sqrt{R^2 + h^2}$ Lateral surface area

$T = \pi R(R + s) = \pi R(R + \sqrt{R^2 + h^2})$ Total surface area



Sphere

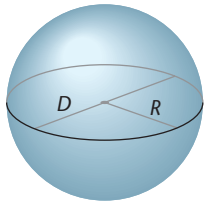
R = Radius

D = Diameter

$D = 2R$

$V = \frac{4}{3}\pi R^3 = \frac{1}{6}\pi D^3$ Volume

$S = 4\pi R^2 = \pi D^2$ Surface area

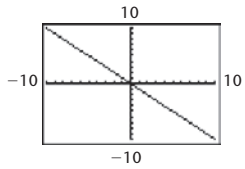


STUDENT ANSWER APPENDIX

Section 1-1

1. Yes **3.** No **5.** Yes **7.** Xmin = −7, Xmax = 6, Ymin = −9, Ymax = 14

11.



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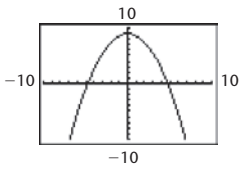

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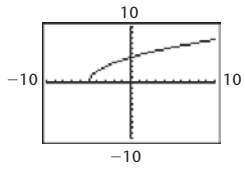
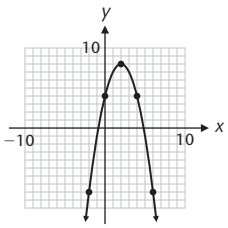


17. Answers can vary. One possible answer is Xmin = −20, Xmax = 10, Ymin = −2, Ymax = 10. **19.** Answers can vary. One possible answer is Xmin = −20, Xmax = 10, Ymin = 0, Ymax = 20.

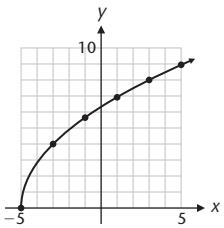
21.

x	$y = 4 + 4x - x^2$
−2	−8
0	4
2	8
4	4
6	−8



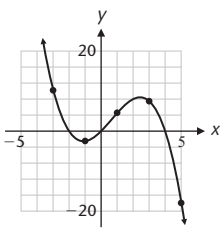
23.

x	$y = 2\sqrt{2x + 10}$
−5	0
−3	4
−1	5.7
1	6.9
3	8
5	8.9



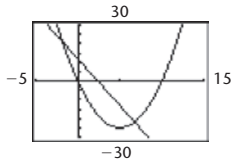
25.

x	$y = 0.5x(4 - x)(x + 2)$
−3	10.5
−1	−2.5
1	4.5
3	7.5
5	−17.5

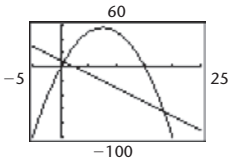


27. (A) −6.37 (B) 0.63 **29.** (A) 0.92 (B) −3.93

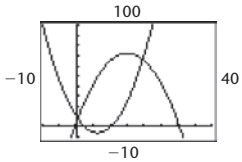
31. (6.77, −21.86), (−1.77, 20.86)



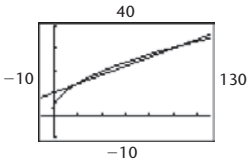
33. (0.54, 7.83), (18.46, −63.83)



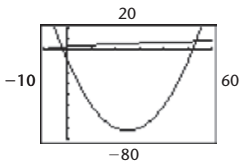
35. (0.36, 8.23), (19.64, 66. 05)



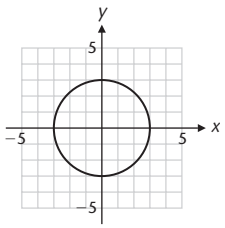
37. (18.84, 13.77), (106.16, 31.23)



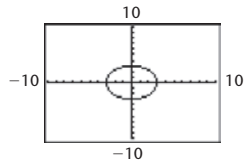
39. (−2.43, 2.75), (53.37, 7.96)



41. (A) The curve is a circle of radius 3 centered at the origin.



(B) This circle is distorted.



(C) ZDecimal produces a circle that fills the screen.
ZSquare produces a small circle.
ZoomFit produces another distorted circle.

43. (A) −7.99 (B) −5.85, 3.44, 12.41 (C) 14.60

45. (A) No solution (B) −8.81, 4.86 **47.** (A) −38.84, 11.16, 27.69 (B) −40, 20 (C) −41.07 **49.** (A) −17.84, 6.93 (B) −10 (C) No solution **51.** $x = 1.28, -4.98$ **53.** No solution **55.** $x = 2.03$ **57.** 1.4142 **61.** Four **63.** A 0.89-in. square or a 2.40-in. square can be cut out. Dimension for smaller square: 0.89″ × 9.22″ × 6.72″ Dimension for larger square: 2.40″ × 6.20″ × 3.70″ **65.** A 0.93-in. square or a 3.92-in. square can be cut out. Dimension for smaller square: 0.93″ × 10.14″ × 10.61″ Dimension for larger square: 3.92″ × 4.16″ × 6.12″ **67.** There are two solutions: radius = 20.93 ft and height = 29.07 ft or radius = 43.17 ft and height = 6.83 ft

69. (A)

x	17,800	15,600	13,600
y	20	25	30

(B) Demand decreases 2,000 cases (C) Demand increases 2,200 cases

71. (A)

y	20	25	30
R	356,000	390,000	408,000

(B) Revenue increases \$18,000 (C) Revenue decreases \$34,000
(D) The company should raise the price \$5 to increase the revenue.

Section 1-2

5. Function: $\{(-1, 3), (0, 2), (1, 1)\}$
7. Not a function: $\{(1, 3), (3, 5), (3, 7), (5, 7), (5, 9)\}$
9. Function: $\{(-1, 3), (0, 3), (1, 3), (2, 3)\}$
11. Domain $= \{2, 3, 4, 5\}$; Range $= \{4, 6, 8, 10\}$; function
13. Domain $= \{0, 5, 10\}$; Range $= \{-10, -5, 0, 5, 10\}$; not a function
15. Domain $= \{0, 1, 2, 3, 4, 5\}$; Range $= \{1, 2\}$; function
17. A function **19.** Not a function **21.** Not a function
23. (A) Function (B) Not a function **25.** Function
27. Not a function **29.** -8 **31.** -6 **33.** 1 **35.** 10 **37.** $-\frac{30}{17}$
39. 3 **41.** All real numbers; $(-\infty, \infty)$
43. $x \neq 4$; $(-\infty, 4) \cup (4, \infty)$ **45.** $t \geq 4$; $[4, \infty)$
47. $w \geq -\frac{7}{3}$; $\left[-\frac{7}{3}, \infty\right)$ **49.** All real numbers; $(-\infty, \infty)$
51. $v \neq 4, -4$; $(-\infty, -4) \cup (-4, 4) \cup (4, \infty)$
53. $x \geq -4, x \neq 1$; $[-4, 1) \cup (1, \infty)$
55. $t \geq 0, t \neq 9$; $[0, 9) \cup (9, \infty)$ **57.** $f(x) = 2x - 3$
59. $f(x) = 4x^2 - 2x + 9$ **61.** 3 **63.** $-6 - h$ **65.** $11 - 2h$
67. $g(x) = 3x + 1$ **69.** $F(x) = \frac{x}{8 + \sqrt{x}}$
71. Function f multiplies the domain element by 2 and then subtracts the product of 3 and the square of the domain element. **73.** Function F takes the square root of the sum of the fourth power of the domain element and 9. **75.** Function f multiplies the square of the input by 2, subtracts 4 times the input, and adds 6. $f(x) = 2x^2 - 4x + 6$ **77.** Function f multiplies the input by 4, subtracts 3 times the square root of the input, and adds 9. $f(x) = 4x - 3\sqrt{x} + 9$ **79.** (A) 3 (B) 3 **81.** (A) $2x + h$ (B) $x + a$ **83.** (A) $-6x - 3h + 9$ (B) $-3x - 3a + 9$
85. (A) $3x^2 + 3xh + h^2$ (B) $x^2 + ax + a^2$
87. The values of f are very close to 2 when x is close to 1.
89. (A)

x	0	5,000	10,000	15,000	20,000	25,000	30,000
$B(x)$	212	203	194	185	176	167	158

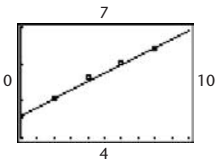
(B) The boiling point drops 9°F for each 5,000 foot increase in altitude.

91. (A) $s(0) = 0, s(1) = 16, s(2) = 64, s(3) = 144$ (B) $64 + 16h$
(C) Let $q(h) = \frac{s(2 + h) - s(2)}{h}$

h	-1	-0.1	-0.01	-0.001	0.001	0.01	0.1	1
$q(h)$	48	62.4	63.84	63.984	64.016	64.16	65.6	80

(D) $q(h)$, the average velocity from 2 to $2 + h$ seconds, approaches 64 ft/sec.
93. The rental charges are \$20 per day plus \$0.25 per mile driven.

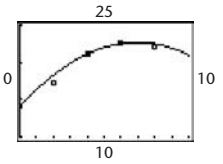
95. (A)



t	0	2	4	6	8
Data	4.59	5.08	5.66	6.03	6.41
$A(t)$	4.60	5.06	5.52	5.98	6.44

(B) The estimated price of admission is \$6.67 in 2006 and \$6.90 in 2007.

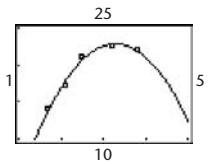
97. (A)



t	0	2	4	6	8
Data	14.0	17.3	21.2	22.5	22.0
$S(t)$	14.0	18.3	21.1	22.5	22.5

(B) Merck's estimated sales are \$22 billion in 2006 and \$20 billion in 2008.
(C) For the period 1997–2001, sales increased steadily by an average of approximately \$6 billion per year.

99. (A)



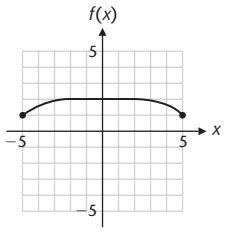
r	1.7	2.1	2.5	3.2	3.8
Data	14.0	17.3	21.2	22.5	22.0
$S(r)$	13.9	17.8	20.6	22.8	21.9

(B) Merck's estimated sales are \$7.4 billion if they spend \$1.2 billion on R&D, and \$20 billion if they spend \$4.2 billion on R&D.

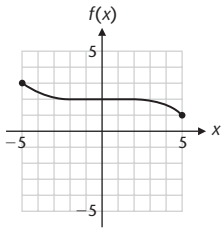
Section 1-3

7. (A) $[-4, 4)$ (B) $[-3, 3)$ (C) 0 (D) 0 (E) $[-4, 4)$ (F) None
9. (A) $(-\infty, \infty)$ (B) $[-4, \infty)$ (C) $-3, 1$ (D) -3 (E) $[-1, \infty)$ (F) $(-\infty, -1]$ (G) None (H) None **11.** (A) $(-\infty, 2) \cup (2, \infty)$ (The function is not defined at $x = 2$.) (B) $(-\infty, -1) \cup [1, \infty)$ (C) None (D) 1 (E) None (F) $(-\infty, -2] \cup (2, \infty)$ (G) $[-2, 2)$ (H) $x = 2$
13. $f(-4) = -3$; $f(0) = 0$; $f(4)$ is undefined
15. $h(-3) = 0$; $h(0) = -3$; $h(2) = 5$
17. $p(-2) = 1$; $p(2)$ is undefined; $p(5) = -4$

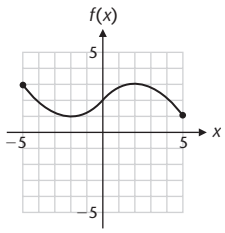
19. Increasing: $[-2, \infty)$, decreasing: $(-\infty, -2]$
21. Increasing: $(-\infty, -5]$, $[2, \infty)$, decreasing: $[-5, 2]$
23. Decreasing: $[-4, 3]$, constant: $(-\infty, -4]$, $[3, \infty)$
25. Increasing: $[-4, 0]$, $[4, \infty)$, decreasing: $(-\infty, -4]$, $[0, 4]$
27. x intercepts: -1.405, 6.405; y intercept: -9; local minimum: $f(2.5) = -15.25$ **29.** x intercept: 3.377; y intercept: 25; local minimum: $h(-1.155) = 21.921$; local maximum: $h(1.155) = 28.079$
31. x intercepts: ± 3.464; y intercept: 3.464; local minima: $m(-3.464) = 0$, $m(3.464) = 0$; local maximum: $m(0) = 3.464$
33. One possible answer:



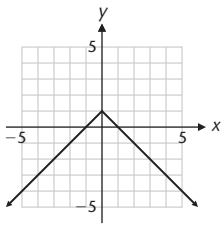
35. One possible answer:



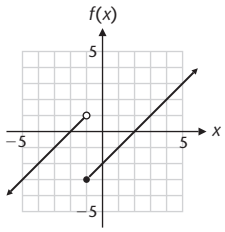
37. One possible answer:



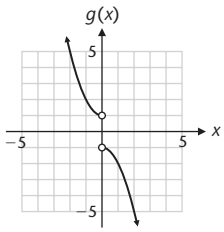
41. $f(-2) = -1$, $f(-1) = 0$, $f(1) = 0$, $f(2) = -1$



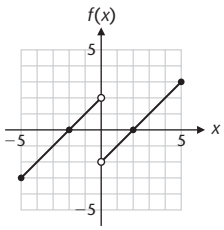
43. $f(-2) = 0$, $f(-1) = -3$, $f(1) = -1$, $f(2) = 0$



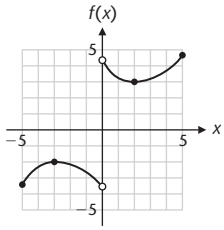
45. $f(-2) = 5$, $f(-1) = 2$, $f(1) = -2$, $f(2) = -5$



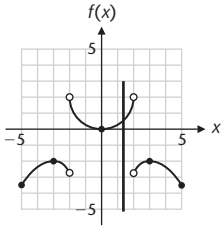
47. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$; y intercept: -30; x intercepts: -44.99, -0.82, 0.81 **49.** Domain: $(-\infty, \infty)$; range: $(-\infty, 10{,}200]$; y intercept: 200; x intercepts: -14.18, 14.18 **51.** Domain: $[0, \infty)$; range: $(-\infty, 16]$; y intercept: 0; x intercepts: 0, 64
53. Domain: $[-5, \infty)$; range: $[-134.02, \infty)$; y intercept: -111.80; x intercepts: -4.79, 14.94 **55.** The graph of f is rising and f is increasing on $(-\infty, 0.13]$ and $[7.87, \infty)$. The graph of f is falling and f is decreasing on $[0.13, 7.87]$. $f(0.13) = -9.81$ is a local maximum and $f(7.87) = -242.19$ is a local minimum. **57.** The graph of m is rising and m is increasing on $(-\infty, -12]$ and $[0, 12]$. The graph of m is falling and m is decreasing on $[-12, 0]$ and $[12, \infty)$. $m(-12) = 144$ and $m(12) = 144$ are local maxima and $m(0) = 0$ is a local minimum.
59. The graph of g is rising and g is increasing on $[-15, 2.5]$ and $[20, \infty)$. The graph of g is falling and g is decreasing on $(-\infty, -15]$ and $[2.5, 20]$. $g(2.5) = 306.25$ is a local maximum and $g(-15) = 0$ and $g(20) = 0$ are local minima. **61.** The graph of f decreases on $(-\infty, -2.15]$ to a local minimum value, $f(-2.15) = -36.62$, and then increases on $[-2.15, \infty)$.
63. The graph of h increases on $(-\infty, -4.64]$ to a local maximum value, $h(-4.64) \approx 281.93$, decreases on $[-4.64, 5.31]$ to a local minimum value, $h(5.31) \approx -211.41$, and then increases on $[5.31, \infty)$.
65. The graph of p decreases on $(-\infty, -3.77]$ to a local minimum value, $p(-3.77) \approx 0$, increases on $[-3.77, 0.50]$ to a local maximum value, $p(0.50) \approx 18.25$, decreases on $[0.50, 4.77]$ to a local minimum value, $p(4.77) \approx 0$, and then increases on $[4.77, \infty)$.
67. One possible answer:



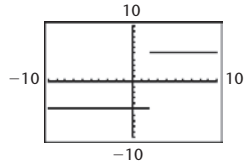
69. One possible answer:



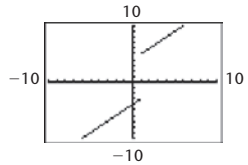
71. One possible answer:



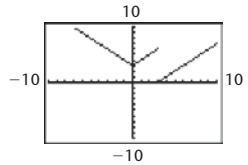
73. Domain: All real numbers except $x = 2$;
Range: $\{-5, 5\}$ (A set, not an interval);
Discontinuous at $x = 2$



75. Domain: All real numbers except $x = 1$;
Range: $(-\infty, -3) \cup (5, \infty)$;
Discontinuous at $x = 1$

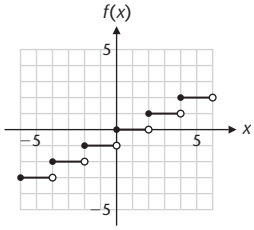


77. Domain: All real numbers except $x = 3$;
Range: $(0, \infty)$;
Discontinuous at $x = 3$



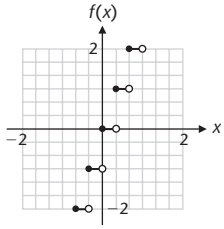
79. Domain: All real numbers
Range: All integers
Discontinuous at the even integers

$$f(x) = \begin{cases} \vdots & \vdots \\ -2 & \text{if } -4 \le x < -2 \\ -1 & \text{if } -2 \le x < 0 \\ 0 & \text{if } 0 \le x < 2 \\ 1 & \text{if } 2 \le x < 4 \\ 2 & \text{if } 4 \le x < 6 \\ \vdots & \vdots \end{cases}$$



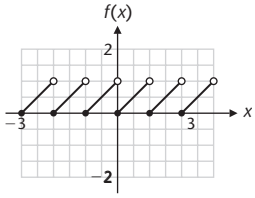
81. Domain: All real numbers
Range: All integers
Discontinuous at rational numbers of the form $\frac{k}{3}$, where k is an integer

$$f(x) = \begin{cases} \vdots & \vdots \\ -2 & \text{if } -\frac{2}{3} \le x < -\frac{1}{3} \\ -1 & \text{if } -\frac{1}{3} \le x < 0 \\ 0 & \text{if } 0 \le x < \frac{1}{3} \\ 1 & \text{if } \frac{1}{3} \le x < \frac{2}{3} \\ 2 & \text{if } \frac{2}{3} \le x < 1 \\ \vdots & \vdots \end{cases}$$

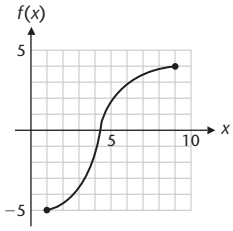


83. Domain: All real numbers; Range: [0, 1)
Discontinuous at all integers.

$$f(x) = \begin{cases} \vdots & \vdots \\ x + 2 & \text{if } -2 \le x < -1 \\ x + 1 & \text{if } -1 \le x < 0 \\ x & \text{if } 0 \le x < 1 \\ x - 1 & \text{if } 1 \le x < 2 \\ x - 2 & \text{if } 2 \le x < 3 \\ \vdots & \vdots \end{cases}$$

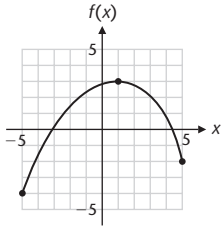


85. (A) One possible answer:



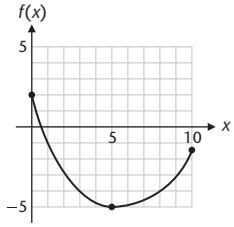
(B) The graph must cross the x axis exactly once.

87. (A) One possible answer:



(B) This graph crosses the x axis twice. To meet the conditions specified a graph must cross the x axis at least twice. If it crossed fewer times the function would have to be discontinuous somewhere. However, the graph could cross more times; in fact there is no upper limit on the number of times it can cross the x axis.

89. (A) One possible answer:



(B) The graph can cross the axis 0, 1, or 2 times.

91. The maximum revenue is \$25,714 when 857 car seats are sold.

93. The maximum volume is 654.98 cubic inches when the side of each square is 3.39 inches. **95.** The minimum cost is \$300,000 when the land portion of the pipe is 13.59 miles.

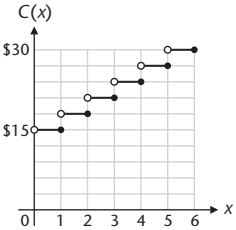
97.
$$\begin{aligned} f(4) &= 10[[0.5+0.4]] = 10(0) = 0 \\ f(-4) &= 10[[0.5-0.4]] = 10(0) = 0 \\ f(6) &= 10[[0.5+0.6]] = 10(1) = 10 \\ f(-6) &= 10[[0.5-0.6]] = 10(-1) = -10 \\ f(24) &= 10[[0.5+2.4]] = 10(2) = 20 \\ f(25) &= 10[[0.5+2.5]] = 10(3) = 30 \\ f(247) &= 10[[0.5+24.7]] = 10(25) = 250 \\ f(-243) &= 10[[0.5-24.3]] = 10(-24) = -240 \\ f(-245) &= 10[[0.5-24.5]] = 10(-24) = -240 \\ f(-246) &= 10[[0.5-24.6]] = 10(-25) = -250 \end{aligned}$$

f rounds numbers to the tens place

99. $f(x) = [\![100x + 0.5]\!]/100$

101. $$R(x) = \begin{cases} 32 & \text{if } 0 \le x \le 100 \\ 16 + 0.16x & \text{if } x > 100 \end{cases}$$

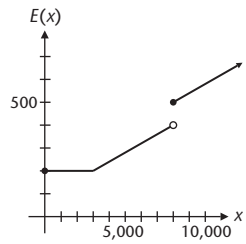
103. (A) $$C(x) = \begin{cases} 15 & 0 < x \le 1 \\ 18 & 1 < x \le 2 \\ 21 & 2 < x \le 3 \\ 24 & 3 < x \le 4 \\ 27 & 4 < x \le 5 \\ 30 & 5 < x \le 6 \end{cases}$$



(B) No, since $f(x) \ne C(x)$ at $x = 1, 2, 3, 4, 5$, or 6

105. $$E(x) = \begin{cases} 200 & \text{if } 0 \le x \le 3{,}000 \\ 80 + 0.04x & \text{if } 3{,}000 < x < 8{,}000 \\ 180 + 0.04x & \text{if } 8{,}000 \le x \end{cases}$$

Discontinuous at $x = 8{,}000$.
$E(5, 750) = \$310$; $E(9, 200) = \$548$



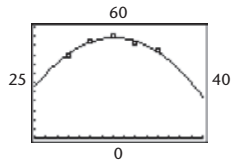
107. $T(x) = \begin{cases} 0.0535x & 0 \le x \le 19{,}890 \\ 0.0705x - 338.25 & 19{,}890 < x \le 65{,}330 \\ 0.0785x - 860.41 & x > 65{,}330 \end{cases}$

$T(10{,}000) = \$535$

$T(30{,}000) = \$1{,}776.75$

$T(100{,}000) = \$6{,}989.60$

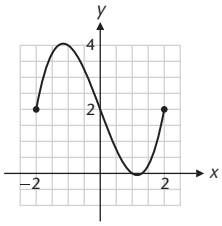
109. (A)



(B) 53,500 miles, 49,950 miles (C) The mileage increases to a maximum value of 55 thousand miles at a pressure of 32 lb/in.^{2} and then begins to decrease.

Section 1-4

5.



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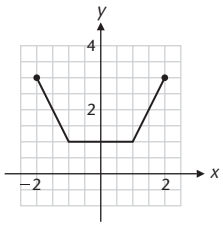
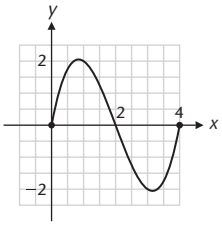
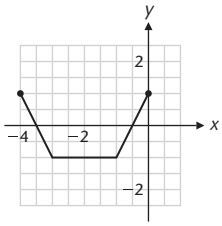


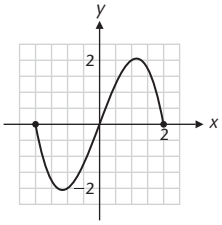
9.



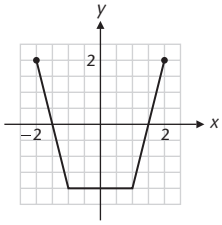
11.



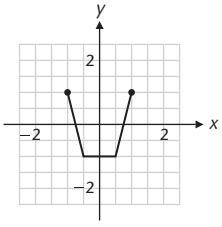
13.



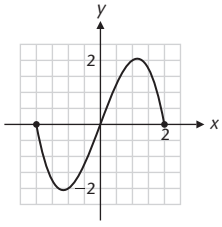
15.



17.



19.



21. odd **23.** even **25.** neither **27.** even **29.** neither

31. The graph of $y = x^2$ is shifted 2 units to the right; $y = (x - 2)^2$.

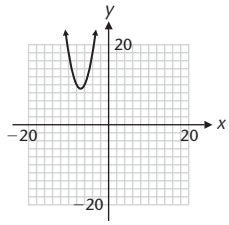
33. The graph of $y = x^3$ is shifted down 2 units; $y = x^3 - 2$.

35. The graph of $y = |x|$ is shrunk vertically by a factor of 0.25; $y = 0.25\,|x|$.

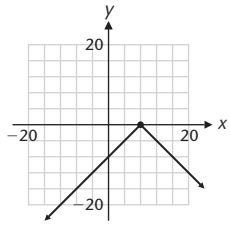
37. The graph of $y = x^3$ is reflected in the x axis (or the y axis); $y = -x^3$. **39.** $g(x) = \sqrt[3]{x + 4} - 5$ **41.** $g(x) = -0.5(6 + \sqrt{x})$

43. $g(x) = -2(x + 4)^2 - 2$ **45.** $g(x) = \sqrt{-\frac{1}{2}(x + 2)}$

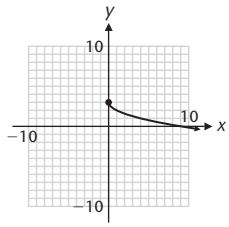
47. The graph of $y = x^2$ is shifted 7 units left and 9 units up.



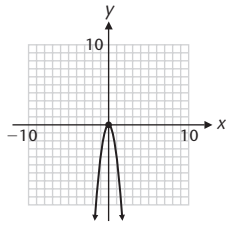
49. The graph of $y = |x|$ is shifted 8 units right and reflected in the x axis.



51. The graph of $y = \sqrt{x}$ is reflected in the x axis and shifted 3 units up.



53. The graph of $y = x^2$ is stretched vertically by a factor of 4 and reflected in the x axis.



55. $y = |x + 2| + 2$ **57.** $y = 4 - \sqrt{x}$ **59.** $y = 4 - (x - 1)^2$

61. $y = 0.5(x - 3)^3 + 1$ **63.** (A) f is a horizontal shrink of $y = \sqrt[3]{x}$ by a factor of 1/8. g is a vertical stretch of $y = \sqrt[3]{x}$ by a factor of 2.

(B) The graphs are identical. (C) $f(x) = \sqrt[3]{8x} = \sqrt[3]{8} \cdot \sqrt[3]{x} = 2\sqrt[3]{x}$

65. (A) The graphs are different; order is significant.

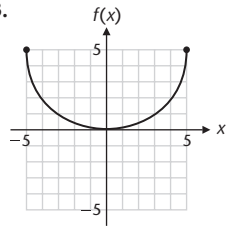
(B) i. $f(x) = -(x^2 - 5)$ ii. $f(x) = -x^2 - 5$

67. Reversing the order does not change the result.

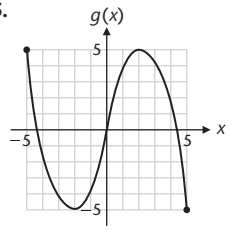
69. Reversing the order can change the result.

71. Reversing the order does not change the result.

73.

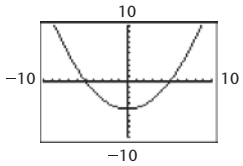


75.

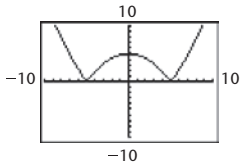


77. Conclusion: Any function can be written as the sum of two other functions, one even and the other odd.

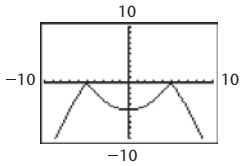
79. Graph of $f(x)$



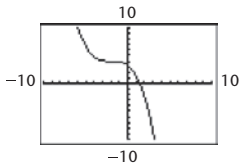
Graph of $|f(x)|$



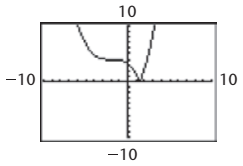
Graph of $-|f(x)|$



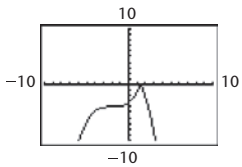
81. Graph of $f(x)$



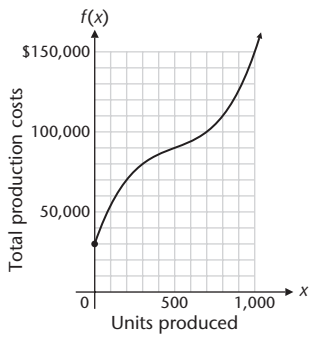
Graph of $|f(x)|$



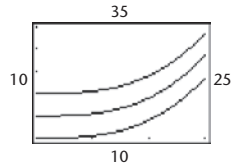
Graph of $-|f(x)|$



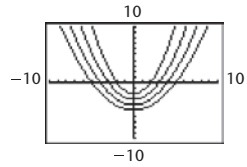
83. The graph of $y = |f(x)|$ is the same as the graph of $y = f(x)$ whenever $f(x) \geq 0$ and is the reflection of the graph of $y = f(x)$ with respect to the x axis whenever $f(x) < 0$. **85.** The graph of the function $C(x) = 30{,}000 + f(x)$ is the same as the given graph of the function $f(x)$ shifted up 30,000 units ($).



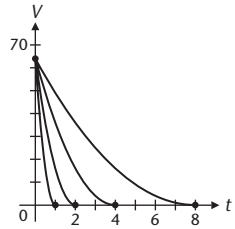
87. Each graph is a vertical translation of the graph of $y = 0.004(x - 10)^3$.



89. Each graph is a vertical shrink followed by a vertical translation of the graph of $y = x^2$.



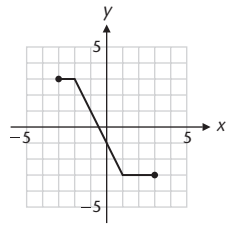
91. Each graph is a portion of the graph of a horizontal translation followed by a vertical shrink (except for $C = 8$) of the graph of $y = t^2$. Larger values of C correspond to a smaller opening.



Section 1-5

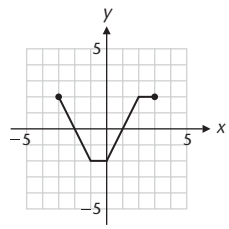
7.

x	−3	−2	−1	0	1	2	3
$f(x)$	1	0	−1	−2	−3	−2	−1
$g(x)$	2	3	2	1	0	−1	−2
$(f + g)(x)$	3	3	1	−1	−3	−3	−3



9.

x	−3	−2	−1	0	1	2	3
$fg(x)$	2	0	−2	−2	0	2	2



11. -2 **13.** 1 **15.** -2 **17.** 3

19. $(f \circ g)(-7) = 3$;
$(f \circ g)(0) = 9$;
$(f \circ g)(4) = -10$

21. $(f+g)(x) = 5x+1$; $(f-g)(x) = 3x-1$; $(fg)(x) = 4x^2 + 4x$
$\left(\frac{f}{g}\right)(x) = \frac{4x}{x+1}$; Domain $f+g$, $f-g$, fg: $(-\infty, \infty)$;
Domain of $f/g = (-\infty, -1) \cup (-1, \infty)$

23. $(f+g)(x) = 3x^2+1$; $(f-g)(x) = x^2-1$; $(fg)(x) = 2x^4 + 2x^2$;
$\left(\frac{f}{g}\right)(x) = \frac{2x^2}{x^2+1}$; Domain of each function: $(-\infty, \infty)$

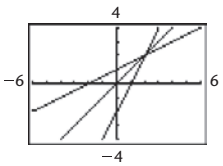
25. $(f+g)(x) = x^2+3x+4$; $(f-g)(x) = -x^2+3x+6$;
$(fg)(x) = 3x^3+5x^2-3x-5$; $\left(\frac{f}{g}\right)(x) = \frac{3x+5}{x^2-1}$;
Domain $f+g$, $f-g$, fg: $(-\infty, \infty)$
Domain of f/g: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

27. $(f \circ g)(x) = (x^2 - x + 1)^3$;
Domain: $(-\infty, \infty)$; $(g \circ f)(x) = x^6 - x^3 + 1$;
Domain: $(-\infty, \infty)$

29. $(f \circ g)(x) = |2x+4|$; Domain: $(-\infty, \infty)$;
$(g \circ f)(x) = 2|x+1| + 3$; Domain $(-\infty, \infty)$

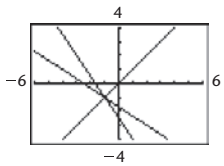
31. $(f \circ g)(x) = (2x^3+4)^{1/3}$; Domain: $(-\infty, \infty)$; $(g \circ f)(x) = 2x + 4$;
Domain: $(-\infty, \infty)$

33. $(f \circ g)(x) = (g \circ f)(x) = x$



Symmetric with respect
to the line $y = x$.

35. $(f \circ g)(x) = (g \circ f)(x) = x$



37. $(f+g)(x) = \sqrt{2-x} + \sqrt{x+3}$;
$(f-g)(x) = \sqrt{2-x} - \sqrt{x+3}$;
$(fg)(x) = \sqrt{6-x-x^2}$; $\left(\frac{f}{g}\right)(x) = \sqrt{\frac{2-x}{x+3}}$
The domain of the functions $f+g$, $f-g$, and fg is $[-3, 2]$.
The domain of $\frac{f}{g}$ is $(-3, 2]$.

39. $(f+g)(x) = 2\sqrt{x} - 2$; $(f-g)(x) = 6$; $(fg)(x) = x - 2\sqrt{x} - 8$;
$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}+2}{\sqrt{x}-4}$; The domain of $f+g$, $f-g$, and fg is $[0, \infty)$.
Domain of $\frac{f}{g} = [0, 16) \cup (16, \infty)$.

41. $(f+g)(x) = \sqrt{x^2+x-6} + \sqrt{7+6x-x^2}$;
$(f-g)(x) = \sqrt{x^2+x-6} - \sqrt{7+6x-x^2}$;
$(fg)(x) = \sqrt{-x^4+5x^3+19x^2-29x-42}$;
$\left(\frac{f}{g}\right)(x) = \sqrt{\frac{x^2+x-6}{7+6x-x^2}}$
The domain of the functions $f+g$, $f-g$, and fg is $[2, 7]$.
The domain of $\frac{f}{g}$ is $[2, 7)$.

43. $(f \circ g)(x) = \sqrt{x-4}$; Domain: $[4, \infty)$; $(g \circ f)(x) = \sqrt{x} - 4$;
Domain: $[0, \infty)$

45. $(f \circ g)(x) = \frac{1}{x} + 2$; Domain: $(-\infty, 0) \cup (0, \infty)$
$(g \circ f)(x) = \frac{1}{x+2}$; Domain: $(-\infty, -2) \cup (-2, \infty)$

47. $(f \circ g)(x) = \frac{1}{|x-1|}$; Domain: $(f \circ g)$: $(-\infty, 1) \cup (1, \infty)$
$(g \circ f)(x) = \frac{1}{|x|-1}$;
Domain of $g \circ f$: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

49. (d) **51.** (a)

53. One possible answer: $g(x) = 2x - 7$; $f(x) = x^4$;
$h(x) = (f \circ g)(x)$

55. One possible answer: $g(x) = 4 + 2x$; $f(x) = x^{1/2}$;
$h(x) = (f \circ g)(x)$

57. One possible answer: $f(x) = x^7$; $g(x) = 3x - 5$;
$h(x) = (g \circ f)(x)$

59. One possible answer: $f(x) = x^{-1/2}$; $g(x) = 4x + 3$;
$h(x) = (g \circ f)(x)$

63. $(f+g)(x) = 2x$; $(f-g)(x) = \frac{2}{x}$; $(fg)(x) = x^2 - \frac{1}{x^2}$;
$\frac{f}{(g)}(x) = \frac{x^2+1}{x^2-1}$. The domain of $f+g$, $f-g$, and fg is
$(-\infty, 0) \cup (0, \infty)$. The domain of $\frac{f}{g}$ is
$(-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, \infty)$.

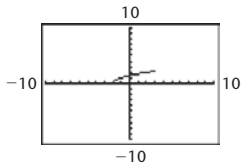
65. $(f+g)(x) = 2$; $(f-g)(x) = \frac{-2x}{|x|}$; $(fg)(x) = 0$; $\left(\frac{f}{g}\right)(x) = 0$
The domain of $f+g$, $f-g$, and fg is $(-\infty, 0) \cup (0, \infty)$.
Domain of $\frac{f}{g}$ is $(0, \infty)$.

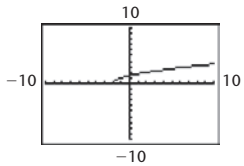
67. $(f \circ g)(x) = \sqrt{4-x^2}$; $(g \circ f)(x) = 4 - x$; Domain of
$f \circ g = [-2, 2]$; Domain of $g \circ f$ is $(-\infty, 4)$

69. $(f \circ g)(x) = \frac{6x-10}{x}$; $(g \circ f)(x) = \frac{x+5}{5-x}$; Domain of $f \circ g$ is
$(-\infty, 0) \cup (0, 2) \cup (2, \infty)$
Domain of $g \circ f$ is $(-\infty, 0) \cup (0, 5) \cup (5, \infty)$

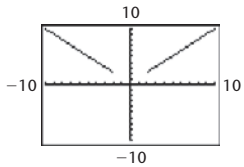
71. $(f \circ g)(x) = \sqrt{16-x^2}$; $(g \circ f)(x) = \sqrt{34-x^2}$; Domain of $f \circ g$
is $[-4, 4]$; Domain of $g \circ f$ is $[-5, 5]$

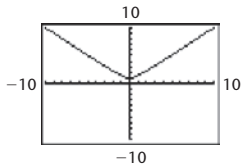
73. $(f \circ g)(x) = \sqrt{2+x}$; Domain of $f \circ g$ is $[-2, 3]$; The first graph
is correct.



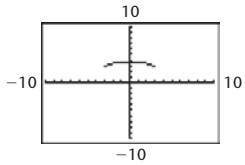


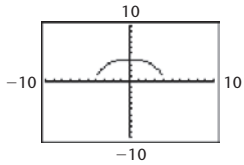
75. $(f \circ g)(x) = \sqrt{x^2+1}$; Domain of $f \circ g$ is $(-\infty, -2] \cup [2, \infty)$;
The first graph is correct.





77. $(f \circ g)(x) = \sqrt{16-x^2}$; Domain of $f \circ g$ is $[-3, 3]$; The first
graph is correct.





79. $P(p) = 4{,}400p - 200p^2 - 16{,}000$; maximum profit occurs when $p = \$11$. **81.** $V(t) = 0.016\pi t^{2/3}$

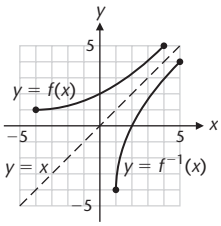
83. (A) $r(h) = \frac{1}{2}h$ (B) $V(h) = \frac{1}{12}\pi h^3$ (C) $V(t) = \frac{0.125}{12}\pi t^{3/2}$

Section 1-6

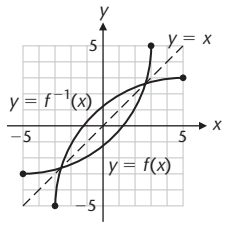
1. The original set and the reversed set are both one-to-one functions.
3. The original set is a function. The reversed set is not a function.
5. Neither set is a function. **7.** one-to-one **9.** not one-to-one
11. one-to-one **13.** not one-to-one **15.** one-to-one **17.** one-to-one
19. $f^{-1} = \{(3, 2), (4, 3), (5, 4), (6, 5)\}$
21. h^{-1} does not exist. **23.** $F^{-1} = \{(7, a), (11, c), (-9, e), (-13, g)\}$
31. one-to-one **33.** not one-to-one **35.** one-to-one **37.** one-to-one
39. not one-to-one **41.** one-to-one **43.** not one-to-one **45.** Yes
47. No **49.** Yes **51.** The function h multiplies the input by 3 and then subtracts 7. The inverse function adds 7 to the input and then divides by 3. The algebraic equation is $h^{-1}(x) = \frac{x + 7}{3}$.

53. The function m adds 11 to the input and then takes the cube root. The inverse function cubes the input and then subtracts 11. The algebraic equation is $m^{-1}(x) = x^3 - 11$. **55.** The function s multiplies the input by 3, adds 17, and then raises this expression to the fifth power. The inverse function takes the fifth root of the input, subtracts 17, and then divides by 3. The algebraic equation is $s^{-1}(x) = \frac{\sqrt[5]{x} - 17}{3}$.

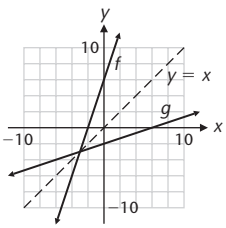
57. Domain of f^{-1}: $[1, 5]$
Range of f^{-1}: $[-4, 4]$;



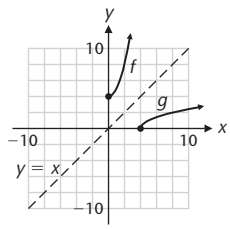
59. Domain of f^{-1}: $[-3, 5]$
Range of f^{-1}: $[-5, 3]$;



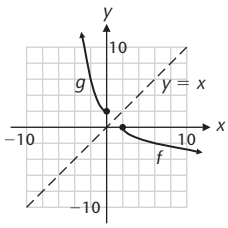
61.



63.



65.

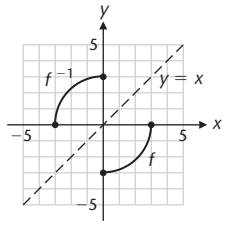


67. $f^{-1}(x) = \frac{1}{3}x$ **69.** $f^{-1}(x) = \frac{x + 3}{4}$ **71.** $f^{-1}(x) = 10x - 6$

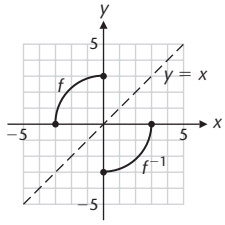
73. $f^{-1}(x) = \frac{x + 2}{x}$ **75.** $f^{-1}(x) = \frac{2x}{1 - x}$

77. $f^{-1}(x) = \frac{4x + 5}{3x - 2}$ **79.** $f^{-1}(x) = \sqrt[3]{x - 1}$

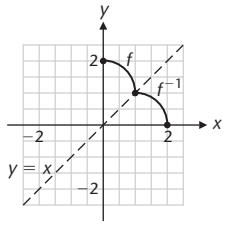
81. $f^{-1}(x) = (4 - x)^5 - 2$ **83.** $f^{-1}(x) = 16 - 4x^2$, $x \geq 0$
85. $f^{-1}(x) = (3 - x)^2 + 2$, $x \leq 3$
87. The x intercept of f is the y intercept of f^{-1} and the y intercept of f is the x intercept of f^{-1}. **89.** No; f is not one-to-one.
91. $f^{-1}(x) = 1 + \sqrt{x - 2}$ **93.** $f^{-1}(x) = -1 - \sqrt{x + 3}$
95. $f^{-1}(x) = \sqrt{9 - x^2}$;
Domain of f^{-1}: $[-3, 0]$;
Range of f^{-1}: $[0, 3]$



97. $f^{-1}(x) = -\sqrt{9 - x^2}$;
Domain of f^{-1}: $[0, 3]$;
Range of f^{-1}: $[-3, 0]$



99. $f^{-1}(x) = \sqrt{2x - x^2}$;
Domain of $f^{-1} = [1, 2]$;
Range of $f^{-1} = [0, 1]$



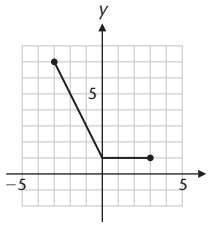
101. (A) $f^{-1}(x) = 2 - \sqrt{x}$ (B) $f^{-1}(x) = 2 + \sqrt{x}$
103. (A) $f^{-1}(x) = 2 - \sqrt{4 - x^2}$, $0 \leq x \leq 2$
(B) $f^{-1}(x) = 2 + \sqrt{4 - x^2}$, $0 \leq x \leq 2$
105. (A) $200 \leq q \leq 1{,}000$
(B) $p = \frac{15{,}000}{q} - 5$; domain: $200 \leq q \leq 1{,}000$; range: $10 \leq p \leq 70$
107. $R(x) = 50x - 0.025x^2$

Chapter 1 Review Exercises

1. Xmin $= -4$, Xmax $= 9$, Ymin $= -6$, Ymax $= 7$ *(1-1)*
2. (A) A one-to-one function with domain $\{1, 2, 3\}$ and range $\{1, 4, 9\}$. The inverse function is $\{(1, 1), (4, 2), (9, 3)\}$ with domain $\{1, 4, 9\}$ and range $\{1, 2, 3\}$. (B) Not a function. (C) A function that is not one-to-one. The domain is $\{-2, -1, 0, 1, 2\}$ and the range is $\{2\}$. (D) A one-to-one function with domain $\{-2, -1, 0, 1, 2,\}$ and range $\{-2, -1, 1, 2, 3\}$. The inverse function is $\{(2, -2), (3, -1), (-1, 0), (-2, 1), (1, 2)\}$ with domain $\{-2, -1, 1, 2, 3\}$ and range $\{-2, -1, 0, 1, 2\}$. *(1-2)*
3. (A) Not a function (B) A function (C) A function (D) Not a function *(1-2)* **4.** (A) -1 (B) 24 (C) 0 (D) 0 *(1-2)*

5.

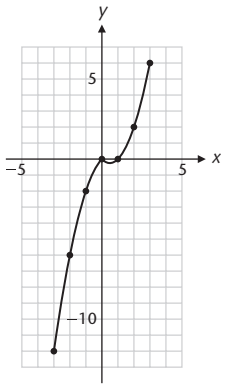
x	-3	-2	-1	0	1	2	3
$f(x)$	3	2	1	0	1	2	3
$g(x)$	-4	-3	-2	-1	0	1	2
$(f-g)(x)$	7	5	3	1	1	1	1



(1-5)

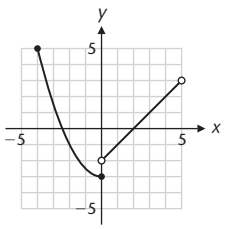
6.

x	-3	-2	-1	0	1	2	3
$(fg)(x)$	-12	-6	-2	0	0	2	6



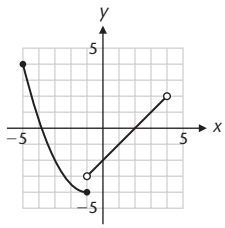
7. 2 *(1-5)* **8.** 1 *(1-5)* **9.** 0 *(1-5)* **10.** 2 *(1-5)* **11.** No *(1-6)* **12.** Yes *(1-6)* **13.** $(f \circ g)(-11) = 8$; $(f \circ g)(1) = 12$; $(f \circ g)(6) = 1$ *(1-5)* **14.** (A) Odd (B) Even (C) Neither *(1-4)* **15.** $f(-4) = 4$, $f(0) = -4$, $f(3) = 0$, $f(5)$ is not defined *(1-2, 1-3)* **16.** $x = -2, x = 1$ *(1-2, 1-3)* **17.** Domain: $[-4, 5)$; range: $[-4, 4]$ *(1-3)* **18.** Increasing: $[0, 5)$, decreasing: $[-4, 0]$ *(1-3)* **19.** $x = 0$ *(1-3)*

20.



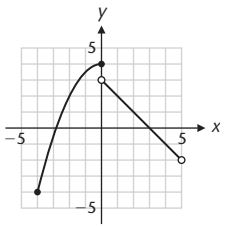
(1-4)

21.



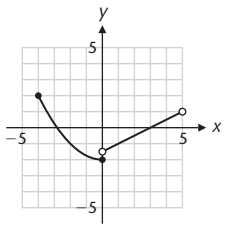
(1-4)

22.



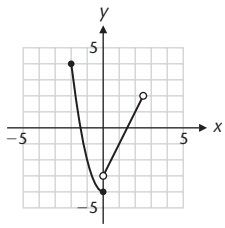
(1-4)

23.



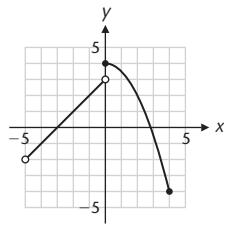
(1-4)

24.



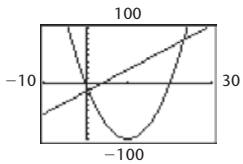
(1-4)

25.

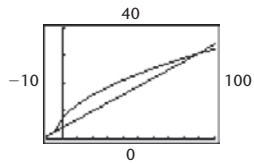


(1-4)

26. (A) g (B) m (C) n (D) f *(1-4)*
27. (A) $(f/g)(x) = (x^2 - 4)/(x + 3)$;
Domain of $f/g = (-\infty, -3) \cup (-3, \infty)$
(B) $(g/f)(x) = (x + 3)/(x^2 - 4)$;
Domain of $g/f = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
(C) $(f \circ g)(x) = x^2 + 6x + 5$; Domain of $f \circ g = (-\infty, \infty)$
(D) $(g \circ f)(x) = x^2 - 1$; Domain of $g \circ f = (-\infty, \infty)$ *(1-5)*
28. (A) 0 (B) 1 (C) 2 (D) 0 *(1-3)* **29.** (A) -2, 0 (B) -1 and 1 (C) No solution (D) $x = 3$ and $x < -2$ *(1-3)*
30. Domain $= (-\infty, \infty)$, range $= (-3, \infty)$ *(1-3)*
31. $[-2, -1]$, $[1, \infty)$ *(1-3)* **32.** $[-1, 1)$ *(1-3)*
33. $(-\infty, -2)$ *(1-3)* **34.** $x = -2, x = 1$ *(1-3)*
35. (0.64, −12.43), (23.36, 78.43) **36.** (−4.26, 2.72), (88.70, 30.61)

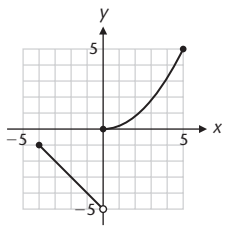


(1-1)



(1-1)

37. (A) −6.71, 5.67, 21.04 (B) 22.99 (C) −7.97, 8.28, 19.70 (D) −9.84 *(1-1, 1-3)* **38.** This equation defines a function. For any real number x, the number $y = 5 - 0.5x$ is the only number that corresponds to x. *(1-2)* **39.** This equation does not define a function. For example, the ordered pairs (2, 2) and (2, −2) both satisfy the equation. *(1-2)* **40.** (A) All real numbers (B) All real numbers except $t = 5$ (C) $w \geq 0$ or $[0, \infty)$ *(1-2)* **41.** $5 + 2h$ *(1-2)*
42. $f(x) = 4x^3 - \sqrt{x}$ *(1-2)* **43.** The function f multiplies the square of the domain element by 3, adds 4 times the domain element, and then subtracts 6. *(1-2)* **44.** x intercepts: 0, 3.30; y intercept: 0; local maximum: $g(1.31) = 5.15$; local minimum: $g(0) = 0$; domain: $[0, \infty)$; range: $(-\infty, 5.15]$ *(1-3)* **45.** x intercepts: −26.58, −3.58, 3.15; y intercept: −300; local maximum: $s(-18) = 2{,}616$; local minimum: $s(0) = -300$; domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$ *(1-3)*
46. (A)

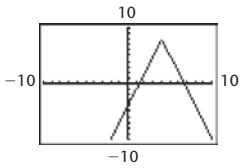


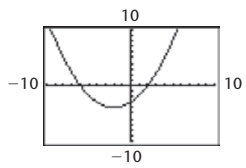
(B) Domain: $[-4, 5]$, range: $(-5, -1] \cup [0, 5]$ (C) $x = 0$ (D) Decreasing on $[-4, 0)$, increasing on $[0, 5]$ *(1-3)*
47. The graph of f increases on $(-\infty, -4.47]$ to a local maximum value, $f(-4.47) \approx 22.89$, decreases on $[-4.47, 4.47]$ to a local minimum value, $f(4.47) \approx -12.89$, and then increases on $[4.47, \infty)$. *(1-3)*

48. (A) Reflected across x axis (B) Shifted down 3 units (C) Shifted left 3 units (D) Shrunk horizontally by $\frac{1}{2}$ *(1-4)*

49. (A) $-(x-2)^2+4$ (B) $4-4\sqrt{x}$ *(1-4)*

50. $g(x)=8-3|x-4|$ **51.** $t(x)=0.25x^2+x-3$





(1-4) *(1-4)*

52. Yes *(1-6)*

53. The function k cubes the input and then adds 5. The inverse function subtracts 5 from the input and then takes the cube root of the result.

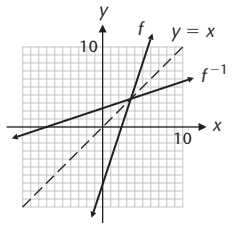
$$k^{-1}(x)=\sqrt[3]{x-5} \quad (1\text{-}6)$$

54. Domain: $x \geq 0, x \neq 9$ or $[0, 9) \cup (9, \infty)$ *(1-2)*

55. (A) $(f \circ g)(x)=\sqrt{|x|}-8$; $(g \circ f)(x)=|\sqrt{x}-8|$
(B) Domain of $f \circ g$ is the set of all real numbers.
The domain of $(g \circ f)$ is $[0, \infty)$ *(1-5)*

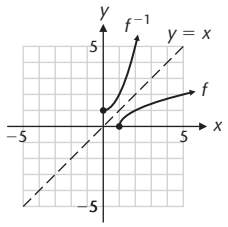
56. Functions f, h, and F are one-to-one. *(1-6)*

57. $f^{-1}(x)=(x+7)/3$;
Domain of f^{-1} = Range of $f^{-1}=(-\infty, \infty)$



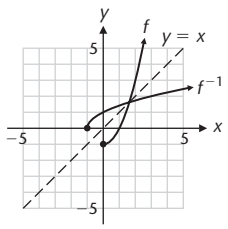
(1-6)

58. $f^{-1}(x)=x^2+1$;
Domain of $f^{-1}=[0, \infty)$;
Range of $f^{-1}=[1, \infty)$



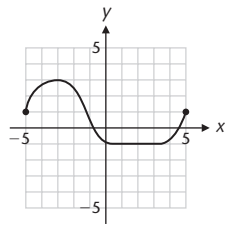
(1-6)

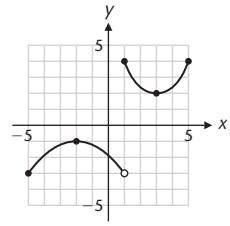
59. $f^{-1}(x)=\sqrt{x+1}$;
Domain of $f^{-1}=[-1, \infty)$;
Range of $f^{-1}=[0, \infty)$



(1-6)

60. (A) One possible answer: (B) One possible answer:

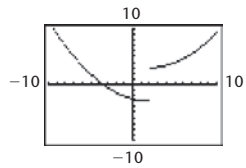




(1-3)

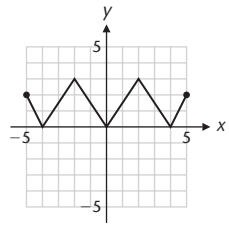
61. The function squares the input, multiples the result by 2, then subtracts 4 times the input and adds 5. $g(t)=2t^2-4t+5$ *(1-2)*

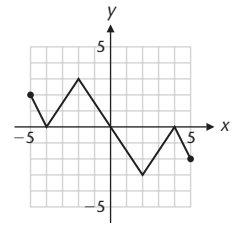
62. Domain: $x \neq 2$ or $(-\infty, 2) \cup (2, \infty)$; range: $y > -3$ or $(-3, \infty)$; discontinuous at $x=2$



(1-3)

63. (A) (B)





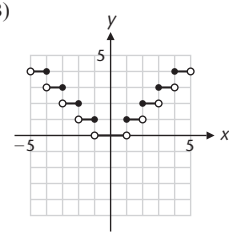
(1-4)

64. (A) $6x-5+3h$ (B) $3x+3a-5$ *(1-2)*

65. (A) The graph must cross the x axis exactly once. (B) The graph can cross the x axis once or not at all. *(1-3)*

66. (A) $f(x)=\begin{cases} 2 & \text{for } -3 < x \leq -2 \\ 1 & \text{for } -2 < x \leq -1 \\ 0 & \text{for } -1 < x < 1 \\ 1 & \text{for } 1 \leq x < 2 \\ 2 & \text{for } 2 \leq x < 3 \end{cases}$

(B)



(C) Range: Nonnegative integers (D) Discontinuous at all integers except 0 (E) Even *(1-3, 1-4)*

67. (A) [1, 3]
(B) $p=\dfrac{4{,}500}{p}-500$; Domain: [1, 3]; Range [1,000, 4,000]
(C) $R(p)=4{,}500-500p$
(D) $R(q)=\dfrac{9q}{1+0.002q}$ *(1-6)*

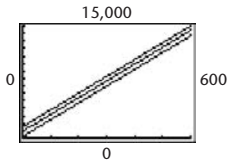
68. $P(p) = -6{,}500 + 600p - 10p^2$; the maximum profit is \$2,500 when the price is \$30. *(1-5)* **69.** (A) 11.3 sec (B) 155 m *(1-3)* **70.** (A) 3.4 seconds (B) 100.7 feet; 8 seconds *(1-3)* **71.** (A) The maximum volume is approximately 10,480 in.^{3} when the flap is 6.79 in. wide. (B) 4.22 in. or 9.71 in. *(1-3)*

72. (A)

Elevation	Oxygen Level (%)
0	21.6
2,000	20.1
4,000	18.5
6,000	17.0
8,000	15.5

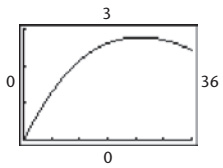
(B) The oxygen level in the air decreases as elevation increases. (C) -0.7 *(1-2)* **73.** The monthly charge is \$19, plus 1.2 cents for each minute used. *(1-2)*

74. (A)



(B) \$750 *(1-3)*

75. (A)



(B) The function increases on [0, 24.8] to a local maximum of 2.8 cc/sec, and then decreases on [24.8, 36]. *(1-3)*

76. (A)

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f(x)$	0	1	2	0	1	2	3	4	0	1	2	3	4	5	6	0

(B) $f(n^2) = 0$ (C) If $f(x) = 0$, then x is a perfect square integer. *(1-3)*

77. (A)

x	12	32	52	72	92
Data	5.41	4.81	4.51	4.00	3.75
$f(x)$	5.32	4.90	4.48	4.06	3.64

(B) 3.30 seconds *(1-2)*

78. $T(x) = \begin{cases} 0.02x & 0 \le x \le 3{,}000 \\ 0.03x - 30 & 3{,}000 < x \le 5{,}000 \\ 0.05x - 130 & 5{,}000 < x \le 17{,}000 \\ 0.0575x - 257.5 & x > 17{,}000 \end{cases}$

x	\$2,000	\$4,000	\$10,000	\$30,000
$T(x)$	\$40	\$90	\$370	\$1,467.50

(1-3)

Chapter 2

Section 2-1

7. Rise $= 3$, run $= 5$, slope $= \frac{3}{5} = 0.6$, $3x - 5y = -4$

9. Rise $= 2$, run $= 8$, slope $= \frac{2}{8} = 0.25$, $x - 4y = -8$

11. Rise $= -3$, run $= 5$, slope $= -\frac{3}{5} = -0.6$, $3x + 5y = -2$

13. x intercept: $x = -2$, y intercept: $y = 2$, slope $= 1$, $y = x + 2$

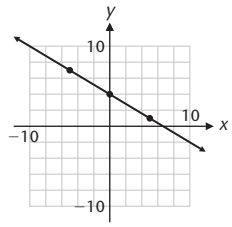
15. x intercept: $x = -2$, y intercept: $y = -4$, slope $= -2$, $y = -2x - 4$

17. x intercept: $x = 3$, y intercept: $y = -1$, slope $= \frac{1}{3}$, $y = \frac{1}{3}x - 1$

19. Not linear **21.** Linear **23.** Linear **25.** Linear **27.** Not linear

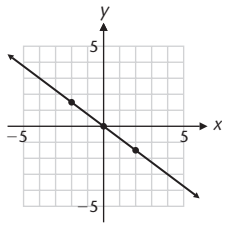
29. x intercept: $\frac{20}{3}$;

y intercept: 4; slope: $-\frac{3}{5}$



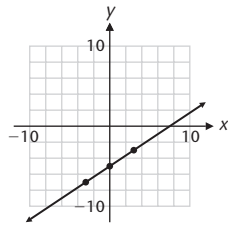
31. x intercept: 0;

y intercept: 0; slope: $-\frac{3}{4}$



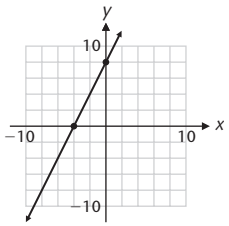
33. x intercept: $\frac{15}{2}$;

y intercept: -5; slope: $\frac{2}{3}$

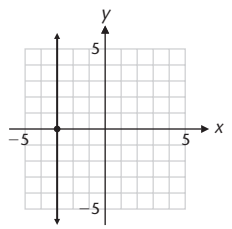


35. x intercept: -4;

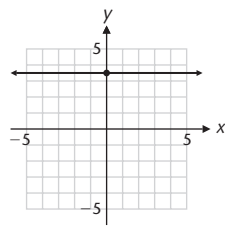
y intercept: 8; slope: 2



37. x intercept: -3;
y intercept: none; slope is not defined



39. x intercept: none;
y intercept: 3.5; slope: 0



41. $x = 2$ **43.** $y = -4$ **45.** $y = 4$ **47.** $x = 4$

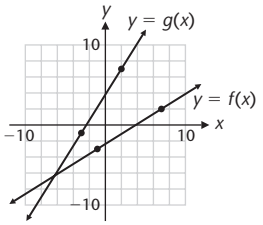
49. $y = x$; $x - y = 0$ **51.** $y = -\frac{2}{3}x - 4$; $2x + 3y = -12$

53. $y = -3x + 4$ **55.** $y = -\frac{2}{5}x + 2$ **57.** $y = -2x + 8$

59. $y = -\frac{4}{3}x + \frac{8}{3}$ **61.** $y = \frac{3}{4}x + 3$ **63.** $y = 3x + 13$

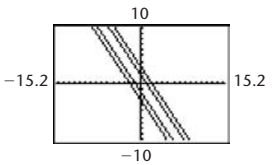
65. $y = 3x - 9$ **67.** $y = \frac{3}{2}x - \frac{15}{2}$ **69.** $y = 3x - 4$

73. (A) $f(x) = \frac{5}{8}x - \frac{19}{8}$ (B) $g(x) = \frac{8}{5}x + \frac{19}{5}$
(C) f and g are inverse functions.



75. 5/2 **77.** 3/2 **79.** 9/8

81. (A)



(B) Varying C produces a family of parallel lines.

85. (A) −2.13%/month; down (B) October 2001: 91%
January 2003: 59%

87. (A) 1.04 million/year; up (B) 1996: 25.95 million
2004: 34.27 million

89. $d = -23h + 145$; yes

91. (A) $F = \frac{9}{5}C + 32$ (B) 68°F, 30°C (C) $\frac{9}{5}$

93. (A) $V = -1{,}600t + 8{,}000$, $0 \le t \le 5$
(B) $V = \$3{,}200$ (C) −1,600; value decreases by \$1,600 per year

95. (A) $C(x) = 2{,}147 + 75x$
(B) The daily fixed costs are \$2,147 and the variable cost per club is \$75.

97. (A) $R = 0.00152C - 0.159$, $C \ge 210$ (B) $R = 0.236$
(C) Slope = 0.00152; coronary risk increases 0.00152 per unit increase in cholesterol above the 210 cholesterol level.

99. (A) $T = -5A + 70$ (B) 20°F
(C) Slope = −5; temperature decreases 5°F for each 1,000-foot increase in altitude.

Section 2-2

9. $x = c, x = f$ **11.** $x = b, x = e$ **13.** Identity
15. Conditional equation, $x = 0$ **17.** Contradiction **19.** $x = -3$
21. $x = 18$ **23.** $t = 9$ **25.** $x = 8$ **27.** All real numbers

29. $x = \frac{11}{2}$ **31.** $t = -\frac{5}{4}$ **33.** $m = 3$ **35.** No solution

37. $x = 7$ **39.** $x = \frac{6}{5}$ **41.** All real numbers except 3 **43.** $x = \frac{17}{5}$

45. No solution **47.** $w = \frac{1}{2}P - l$ **49.** $d = \frac{a_n - a_1}{n - 1}$

51. $f = \frac{d_1 d_2}{d_2 + d_1}$ **53.** $a = \frac{A - 2bc}{2b + 2c}$ **55.** $x = \frac{5y + 3}{2 - 3y}$

57. The graphs are identical for $x \ge 0$. For $x < 0$, each is the reflection of the other in the x axis. The equation is an identity only for $x \ge 0$.

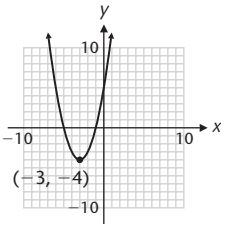
59. $L = 2W$ **61.** $W = \frac{1}{2}L$ **63.** $L = W + 3$ **65.** $L = W - 4$

67. (A) $y = 0.5x - 1.5$ (B) $y = 2x + 3$
The graphs are symmetric with respect to the line $y = x$. Each function is the inverse of the other.

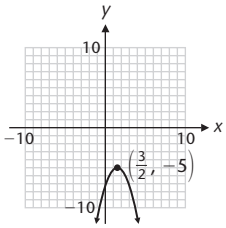
69. $x = 2$ **71.** All real numbers except for 0 and 1
73. 27, 28, 29 **75.** 8, 10, 12, and 14 **77.** 10 in. by 20 in.
79. $C(x) = 124 + 0.12x$, 1,050 doughnuts
81. \$19,750 **83.** 0.27 m/sec **85.** (A) 216 mi (B) 225 mi
87. 90 miles **89.** 10 gallons **91.** 11.25 liters
93. (A) $T = 30 + 25(x - 3)$ (B) 330°C (C) 13 km
95. $y = 468x + 6{,}832$; tuition should be \$11,044 in 2008, and reach \$15,000 in 2017.
97. Men's: $y = -0.103x + 51.509$; women's: $y = -0.145x + 58.286$; the models indicate that women will catch up with men in about 2129.
99. Supply: $y = 4.95x - 4.22$, demand: $y = -7.65x + 25.7$, equilibrium price: \$2.37/bu

Section 2-3

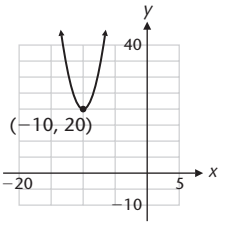
7. Vertex: (−3, −4); axis: $x = -3$



9. Vertex: $\left(\frac{3}{2}, -5\right)$; axis: $x = \frac{3}{2}$



11. Vertex: (−10, 20); axis: $x = -10$



13. The graph is shifted 2 units right and 1 unit up.
15. The graph is reflected in the x axis, then shifted 1 unit left.
17. The graph is shifted 2 units right and 3 units down.
19. k **21.** m **23.** h
25. $x^2 + 10x + 25 = (x + 5)^2$
27. $x^2 - 7x + \frac{49}{4} = \left(x - \frac{7}{2}\right)^2$ **29.** $x^2 + \frac{7}{5}x + \frac{49}{100} = \left(x + \frac{7}{10}\right)^2$

31. $f(x) = (x - 2)^2 + 1$; vertex: (2,1); axis: $x = 2$
33. $h(x) = -(x + 1)^2 - 2$; vertex: (−1, −2); axis: $x = -1$
35. $m(x) = 2(x - 3)^2 + 4$; vertex: (3, 4); axis: $x = 3$

37. $f(x) = \frac{1}{2}(x + 3)^2 - 8$; vertex: (−3, −8); axis: $x = -3$

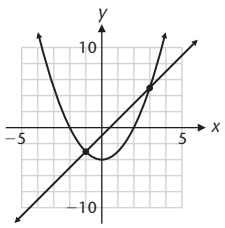
39. $f(x) = 2(x - 6)^2 + 18$; vertex: (6, 18); axis: $x = 6$
41. Vertex: (−4, −8); The graph is symmetric about the axis, $x = -4$. It decreases until reaching a minimum at (−4, −8), then increases. The range is $[-8, \infty)$.

43. Vertex: $\left(-\frac{7}{2}, \frac{65}{4}\right)$; The graph is symmetric about the axis, $x = -\frac{7}{2}$. It increases until reaching a maximum at $\left(-\frac{7}{2}, \frac{65}{4}\right)$, then decreases. The range is $\left(-\infty, \frac{65}{4}\right]$.

45. Vertex: $\left(\frac{9}{4}, \frac{19}{4}\right)$; The graph is symmetric about the axis, $x = \frac{9}{4}$. It decreases until reaching a minimum at $\left(\frac{9}{4}, \frac{19}{4}\right)$, then increases. The range is $\left[\frac{19}{4}, \infty\right)$.

47. Vertex: $\left(\frac{5}{2}, \frac{149}{2}\right)$; The graph is symmetric about the axis, $x = \frac{5}{2}$. It increases until reaching a maximum at $\left(\frac{5}{2}, \frac{149}{2}\right)$, then decreases. The range is $\left(-\infty, \frac{149}{2}\right]$.

49. $y = -2x^2 + 16x - 24$ **51.** $y = -0.5x^2 - 4x + 4$
53. $y = 5x^2 + 50x + 100$ **55.** $y = 2x^2 - 4x - 2$
57. $y = -0.5x^2 - x + 3.5$ **59.** $y = x^2 - 2x - 3$
61. $y = -0.5x^2 + 2x + 2.5$
65. All the graphs are translations of the graph of $y = x^2$.
71. $y = 2x - 1$



73. (A) $1 + h$
(B)

h	slope $= 1 + h$
1	2
0.1	1.1
0.01	1.01
0.001	1.001

; The slope seems to be approaching 1.
75. The minimum product is -225 for the numbers 15 and -15. There is no maximum product.
77. 26 employees; \$322,800
79. (A) 2003 (B) The domain values should be whole numbers.
81. (A) $A(x) = -x^2 + 50x + 5{,}000$, $0 \le x \le 100$
(B) $x = 25$ (C) 75 ft × 75 ft
83. After 25 sec **85.** 100 ft
87. (A) $d(t) = -16t^2 + 176t$, $0 \le t \le 11$ (B) 1.68 sec; 9.32 sec
89. (A) $h(x) = -0.14x^2 + 14$, $-10 \le x \le 10$ (B) No
(C) 11.76 ft (D) 7.56 ft
91. (A) $p = d(x) = -0.129x + 9.27$ (B) \$4.64

Section 2-4

7. Imaginary, complex **9.** Imaginary, pure imaginary, complex
11. Real, complex **13.** -1 **15.** -16 **17.** -3
19. -24 or $-24 + 0i$ **21.** $7 + 5i$ **23.** $5 + 3i$ **25.** $2 + 4i$
27. $5 + 9i$ **29.** $4 - 3i$ **31.** $-12 - 6i$ **33.** $15 - 3i$ **35.** $-4 - 33i$
37. 65 or $65 + 0i$ **39.** $7 - 24i$
41. $\frac{2}{5} - \frac{1}{5}i$ **43.** $\frac{3}{13} + \frac{11}{13}i$ **45.** $5 + 3i$ **47.** 4 **49.** $4i$ **51.** $4i$
53. -4 **55.** $7 - 5i$ **57.** $-3 + 2i$ **59.** $8 + 25i$
61. $\frac{5}{7} - \frac{2}{7}i$ **63.** $\frac{2}{13} + \frac{3}{13}i$ **65.** $-\frac{2}{5}i$ or $0 - \frac{2}{5}i$ **67.** $\frac{3}{2} - \frac{1}{2}i$
69. $-6i$ or $0 - 6i$ **73.** $i^{18} = -1$, $i^{32} = 1$, $i^{67} = -i$
75. $x = 3, y = -2$ **77.** $x = 2, y = 3$ **79.** $0.6 + 1.2i$
81. $1.5 + 0.5i$ **87.** $(a + bi)(a - bi) = a^2 + b^2$
89. $(a + c) + (b + d)i$ **91.** $a^2 + b^2$ or $(a^2 + b^2) + 0i$
93. $(ac - bd) + (ad + bc)i$
95. $i^{4k} = (i^4)^k = (i^2 \cdot i^2)^k = [(-1)(-1)]^k = 1^k = 1$
99. (1) Definition of addition; (2) Commutative (+) property for R; (3) Definition of addition

Section 2-5

7. $x = 8, -\frac{3}{2}$ **9.** $x = 5, -2$ **11.** $u = 0, 2$
13. $y = \frac{2}{3}$ (double root) **15.** $x = \frac{3}{2}, 4$ **17.** $x = 3 \pm 2\sqrt{3}$
19. $t = 2 \pm 2i$ **21.** $m = -1 \pm 2i\sqrt{2}$ **23.** $d = -5, 2.5$
25. $v = \frac{1}{2} \pm \frac{1}{2}i$ **27.** $y = -\frac{3}{8} \pm \frac{3\sqrt{15}}{8}i$ **29.** $x = 5 \pm 2\sqrt{7}$
31. $x = 2 \pm 2i$ **33.** $x = \frac{2 \pm \sqrt{2}}{2}$ **35.** $x = \frac{1}{5} \pm \frac{3}{5}i$
37. $x = -1 \pm \sqrt{2}$ **39.** Two real zeros **41.** Two imaginary zeros
43. One real zero **45.** One real zero **47.** Two real zeros
49. Two imaginary zeros **51.** $x = 3 \pm 2\sqrt{3}$
53. $y = \frac{3 \pm \sqrt{3}}{2}$ **55.** $x = \frac{1 \pm \sqrt{7}}{3}$ **57.** $x = -\frac{5}{4}, \frac{2}{3}$
59. $x = \frac{3 \pm \sqrt{13}}{2}$ **61.** $t = \sqrt{\frac{2s}{g}}$ **63.** $I = \frac{E + \sqrt{E^2 - 4RP}}{2R}$
65. $x = \frac{\sqrt{7} \pm i}{2}$ **67.** $x = \sqrt{3}$ **69.** $x = \frac{-\sqrt{3} \pm \sqrt{19}}{2}$
71. $x = \frac{5}{2} \pm \frac{i\sqrt{11}}{2}$ **73.** $x = 3$ **75.** $x = \frac{7}{2} \pm \frac{\sqrt{13}}{2}$
77. $x = 0, \frac{2}{3}$ **79.** $x = -\frac{5}{4} \pm \frac{i\sqrt{15}}{4}$
81. If $c < 4$ there are two distinct real roots, if $c = 4$ there is one real double root, and if $c > 4$ there are two distinct imaginary roots.
83. $x = -i, -2i$ **85.** $x = \sqrt{2} - i, -\sqrt{2} - i$
87. $x = 1, -\frac{1}{2} \pm \frac{1}{2}i\sqrt{3}$ **89.** No
93. The $\pm$ in front still yields the same two numbers even if a is negative. **95.** 8, 13 **97.** 12, 14 **99.** At 8:06 a.m. **101.** 2.19 ft
103. 51.9 ft × 23.1 ft
105. (A) $A(w) = 400w - 2w^2$, $0 \le w \le 200$ (B) $50 \le w \le 150$
(C) No, the maximum cross-sectional area is 20,000 square feet when $w = 100$ feet. **107.** 52 mi
109. (A) $y = -0.000587x^2 + 0.0302x + 0.945$
(B) 2010 (C) 1.16 gallons
111. (A) $y = -0.328x^2 + 18.3x + 339$ (B) 2004 (C) 353 billion
113. (A) $y = 0.0476x^2 + 0.0452x - 0.357$ (B) 64 mph
115. (A) $y = -0.00217x^2 + 0.140x + 1.90$
(B) Speed: 41.5 mph; mileage: 3.97 mpg; time: 2.41 hours; gas used: 25.2 gallons; cost: \$82.10

Section 2-6

1. T **3.** F **5.** F **7.** F **9.** $2u^2 - 4u = 0$, $u = x^{-3}$

11. Not of quadratic type **13.** $\frac{10}{9} + 4u - 7u^2 = 0$, $u = \frac{1}{x^2}$

19. $x = 8$ **21.** No solution **23.** $x = 22$ **25.** No solution

27. $y = \pm 2, \pm i\sqrt{2}$ **29.** $x = \pm\frac{1}{2}i$ **31.** $x = 2, 4$ **33.** $n = 8$

35. $x = 0, 4$ **37.** $x = \frac{1}{8}, -8$ **39.** $m = 3, -2, \frac{1}{2} \pm \frac{\sqrt{7}}{2}i$

41. No solution **43.** $y = 1$ **45.** $x = 2$ **47.** $x = -\frac{3}{2} + \frac{1}{2}i$

49. $n = -\frac{3}{4}, \frac{1}{5}$ **51.** $y = \pm 3, \pm 1$ **53.** $y = 1, 16$

55. $m = 3, 7, 2, 8$ **59.** $x = -2$ **61.** $y = \pm\sqrt{\frac{3 \pm \sqrt{3}}{2}}$ (four roots)

63. $m = 9, 16$ **65.** $t = 4, 81$ **67.** $x = -4, 39{,}596$

69. $x = \left(\frac{4}{5 \pm \sqrt{17}}\right)^5 \approx 0.016203, 1974.98$

71. 5.3 in. by 8.5 in. **73.** 2,277 ft

75. (A) $A = w\sqrt{256 - w^2}$, $0 < w < 16$ (B) 13.1 in. by 9.1 in.
(C) The maximum area is 128 in.^{2} when the beam is a square with side 11.3 in. **77.** 1.65 ft or 3.65 ft

Section 2-7

5. (c, f) **7.** $(-\infty, b] \cup [e, \infty)$ **9.** $(-\infty, c] \cup [f, \infty)$
11. $(-\infty, b) \cup (e, \infty)$ **13.** $|x - 3| < 5$ **15.** $|y + 1| > 6$
17. $|a - 3| \le 5$ **19.** $|d + 2| \ge 4$
21. The distance from y to the origin is no more than 7 units.
23. The distance from w to the origin is greater than 7 units
25. $(-\infty, 5)$; $x < 5$ **27.** $(2, \infty)$; $t > 2$ **29.** $(3, \infty)$; $m > 3$

31. $(-4, 3)$; $-4 < x < 3$ **33.** $(-\infty, 0] \cup \left[\frac{7}{2}, \infty\right)$; $x \le 0$ or $x \ge \frac{7}{2}$

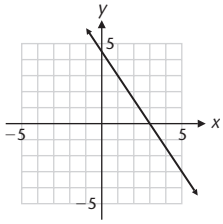
35. $(2, 8)$; $2 < s < 8$ **37.** $(-\infty, 2) \cup (8, \infty)$; $s < 2$ or $s > 8$
39. $(-2, 3]$; $-2 < t \le 3$ **41.** $[-30, 18)$; $-30 \le x < 18$
43. $(-\infty, -14)$; $q < -14$ **45.** $(-5, 2)$; $-5 < x < 2$
47. $(-\infty, 3) \cup (7, \infty)$; $x < 3$ or $x > 7$ **49.** $[0, 8]$; $0 \le x \le 8$
51. No solution **53.** $(-\infty, -3] \cup [7, \infty)$; $x \le -3$ or $x \ge 7$
55. $(-\infty, 0.7) \cup (4.3, \infty)$; $x < 0.7$ or $x > 4.3$

57. $[-1.6, 2.1]$; $-1.6 \le x \le 2.1$ **59.** $\left[1, \frac{11}{3}\right]$; $1 \le x \le \frac{11}{3}$
61. $(-\infty, -1) \cup (5, \infty)$; $t < -1$ or $t > 5$
63. $(-\infty, -11] \cup [-6, \infty)$; $u \le -11$ or $u \ge -6$ **65.** $>$ **67.** $>$
69. The distance between x and 3 is between 0 and 0.1; $(2.9, 3) \cup (3, 3.1)$
71. The distance between x and c is between 0 and $2c$; $(-c, c) \cup (c, 3c)$
73. Positive: $\left(-\infty, -\frac{3}{2}\right), (4, \infty)$; negative: $\left(-\frac{3}{2}, 4\right)$
75. Positive: $(-\infty, -8.77), (-0.23, \infty)$; negative: $(-8.77, -0.23)$
77. $x^2 \ge 0$ is one of many examples
79. $|A - 12.436| < 0.001$, $(12.435, 12.437)$
81. (A) $x > 40{,}625$ (B) $x = 40{,}625$
83. (B) $x > 52{,}000$ (C) Raise wholesale price \$3.50 to \$66.50
85. $33 < x < 122$ **87.** $16°C$ *to* $27°C$
89. (A) $d(t) = -16t^2 + 128t$ (B) $3 < t < 5$
91. $T(x) = 28x + 2$, $5.286 < x < 7.071$
93. $y = -0.00738x + 28.2$, $5{,}180 < x < 7{,}890$
95. $y = -0.0623x + 877$, $3{,}640 < x < 8{,}460$
97. (A) $p = d(x) = -0.0000727x + 3.50$, $0 \le x \le 48{,}100$
(B) $R(x) = -0.0000727x^2 + 3.50x$, $0 \le x \le 48{,}100$
$C(x) = 20{,}000 + 0.5x$, $x \ge 0$
(C) The company will break even at the sales levels of 8,360 gallons or 32,900 gallons. Any sales between these two levels will produce a profit. Any sales less than 8,360 or greater than 32,900 will result in a loss.
(D) The maximum profit is \$10,900 when the sales are 20,600 gallons.

Chapter 2 Review Exercises

1. Rise $= -2$, run $= 5$, slope $= -\frac{2}{5} = -0.4$, $2x + 5y = 7$ *(2–1)*

2. slope: $-\frac{3}{2}$ *(2–1)*



3 $2x + 3y = 12$ *(2–1)* **4.** $y = -\frac{2}{3}x + 2$ *(2–1)*

5. Vertical: $x = -3$, slope not defined; horizontal: $y = 4$, slope $= 0$ *(2–1)*

6. (A) $x = 21$ (B) $x = \frac{30}{11}$ *(2–2)*

7. (A) $f(x) = -(x + 1)^2 + 4$ (B) It is the same as the graph of $y = x^2$ reflected in the x axis, shifted left 1 unit, and up 4 units.
(C) $x = -3, 1$ *(2–3, 2–5)*

8. (A) $f(x) = \left(x - \frac{3}{2}\right)^2 - \frac{17}{4}$

(B) It is the same as the graph of $y = x^2$ shifted right $\frac{3}{2}$ units, and down $\frac{17}{4}$ units. (C) $x = \frac{3 \pm \sqrt{17}}{2}$ *(2–3, 2–5)*

9. Vertex: $(\frac{3}{2}, \frac{11}{4})$; maximum *(2–3)* **10.** (A) $3 - 6i$ (B) $15 + 3i$
(C) $2 + i$ (D) $-1 - 3i$ *(2–4)* **11.** $x = -\frac{17}{6}$ *(2–2)*

12. $x = -\frac{5}{2} \pm \frac{\sqrt{23}}{2}$ *(2–5)* **13.** $x = 2$ *(2–1, 2–5)*

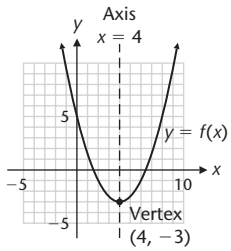
14. $x = \pm\frac{\sqrt{14}}{2}$ *(2–5)* **15.** $x = 0, 2$ *(2–5)* **16.** $x = \frac{1}{2}, 3$ *(2–5)*

17. $m = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ *(2–5)* **18.** $y = \frac{3 \pm \sqrt{33}}{4}$ *(2–5)*

19. $x = 3$ *(2–6)* **20.** $x = 2, 3$ *(2–6)* **21.** $x = -2, 6$ *(2–6)*
22. $x \ge 1$; $[1, \infty)$ *(2–7)* **23.** $(-5, 4)$; $-5 < x < 4$ *(2–7)*
24. $x < -2$ or $x > 6$; $(-\infty, -2) \cup (6, \infty)$ *(2–7)*
26. $3x + 2y = -6$ *(2–1)* **27.** (A) $y = -2x - 3$
(B) $y = \frac{1}{2}x + 2$ *(2–1)* **28.** $-14 < y < -4$; $(-14, -4)$ *(2–7)*

29. $x \le 2.5$ or $5.5 \le x$ *(2–7)* **30.** $x \le 0.15$ or $x \ge 3.4$ *(2–7)*
31. The distance from y to 5 is less than or equal to 2 units; $3 \le y \le 7$; $[3, 7]$ *(2–7)* **32.** The distance from t to -6 is greater than 9 units; $t < -15$ or $t > 3$; $(-\infty, -15) \cup (3, \infty)$ *(2–7)*
33. Two real zeros *(2–5)* **34.** One real zero *(2–5)*
35. Two imaginary zeros *(2–5)*

36. (A)



(B) Increasing: $[4, \infty)$; decreasing $(-\infty, 4]$; range: $[-3, \infty)$
(C) The minimum is $f(4) = -3$. *(2–3)*

37. $y = 2x + 2$; $f(x) = -0.5x^2 + x + 1.5$ *(2–1, 2–3)*

38. (A) $5 + 4i$ (B) $-i$ *(2–4)*

39. (A) $-1 + i$ (B) $\frac{4}{13} - \frac{7}{13}i$ (C) $\frac{5}{2} - 2i$ (D) -20 *(2–4)*

40. $x = \frac{-5 \pm \sqrt{5}}{2}$ *(2–5)* **41.** $u = 1 \pm i\sqrt{2}$ *(2–5)*

42. $\frac{1}{2} - \frac{3}{2}i$ *(2–6)* **43.** $x = -\frac{27}{8}, 64$ *(2–6)*

44. $m = \pm 3i, \pm 2$ *(2–6)* **45.** $y = \frac{9}{4}, 3$ *(2–6)*

46. (A) $y = 2x - 5$ (B) $y = 0.5x + 2.5$
The graphs are symmetric with respect to the line $y = x$.
Each function is the inverse of the other. *(2–2)*

49. If $c < 9$ there are two distinct real roots, if $c = 9$ there is one real double root, and if $c > 9$ there are two distinct imaginary roots. *(2–5)*

50. $M = \frac{P}{1 - dt}$ *(2–2)* **51.** $I = \frac{E \pm \sqrt{E^2 - 4PR}}{2R}$ *(2–5)*

52. True for all real b and all negative a. *(2–7)*

53. $\frac{a}{b}$ is less than 1. *(2–7)*

54. $6 - d < x < 6 + d, x \neq 6, (6 - d, 6) \cup (6, 6 + d)$ *(2–7)*

55. 1 *(2–4)* **56.** Perpendicular *(2–1)* **57.** $2 - 3i$ *(2–5)*

58. $b = 10, c = 29$ *(2–5)* **59.** $x = 1, 243$ *(2–6)*

60. $x = -1; \frac{1 \pm i\sqrt{3}}{2}$ *(2–5)* **61.** 47, 48, 49 *(2–2)*

62. 4, 6, 8 *(2–2)* **63.** $b = 5h$ *(2–2)* **64.** $h = \frac{1}{4}b$ *(2–2)*

65. (A) 3.4 inches/hour (B) 11.1 inches *(2–1)*

66. (A) $y = 2{,}281{,}250x + 60{,}000{,}000$ (B) Mid-January 2008 *(2–1)*

67. 18.6 hours *(2–3)* **68.** 12.5 in. by 30.0 in. *(2–5)*

69. 196 ft *(2–5)* **70.** (A) 4,750 calculators; $7,437.50 (B) 2,614 or 6,886 calculators (C) None (2–3) **71.** 3,240 or 9,260 calculators *(2–7)*

72. Profit: $3{,}240 < x < 9{,}260$

Loss: $0 \leq x < 3{,}240$ or $x > 9{,}260$ *(2–7)*

73. (A) $V = -1{,}250t + 12{,}000$ B. $V = \$5{,}750$ *(2–1)*

74. (A) $R = 1.6C$ (B) $R = \$168$ *(2–1)*

75. $E(x) = \begin{cases} 400 & \text{if } x \leq 3{,}000 \\ 0.1x + 100 & \text{if } x > 3{,}000 \end{cases}$

$E(2{,}000) = 400$; $E(5{,}000) = 600$ *(2–1)*

76. (A) $A(x) = 60x - \frac{3}{2}x^2$ (B) $0 < x < 40$ (C) $x = 20, y = 15$ *(2–3)*

77. (A) $H = 0.7(220 - A)$ (B) $H = 140$ beats per minute (C) $A = 40$ years old *(2–1)* **78.** 20 cm by 24 cm *(2–5)*

79. $B = 14.58$ ft or 6.58 ft *(2–6)* **80.** 6.6 ft *(2–5)*

81. (A) $y = 0.0615x^2 - 2.88x + 48.0$ (B) 2008 *(2–3)*

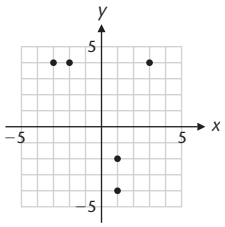
82. (A) $y = -3.10x + 54.6$ (B) 45.3% *(2–2)*

83. Supply: $y = 0.0000258x + 0.0533$; demand:
$y = -0.0000221x + 1.63$; equilibrium price: $0.90 *(2–2)*

84. (A) $R(x) = -0.0000221x^2 + 1.63x$, $0 \leq x \leq 73{,}800$
$C(x) = 15{,}000 + 0.2x$, $x \geq 0$
(B) The company will break even at sales levels of 13,200 pounds and 51,500 pounds. It will make a profit for any sales level between these two break-even levels and a loss for any sales level less than 13,200 pounds or greater than 51,500 pounds.
(C) The maximum profit is $8,130 when 32,400 pounds of broccoli are sold. *(2–5, 2–7)* **85.** (A) $y = -0.00174x^2 + 0.0865x + 0.980$
(B) Speed: 31.5 mph; mileage: 1.98 mpg; time: 3.17 hours; gas used: 50.5 gallons; cost: $178 *(2–5)*

Chapters 1 & 2 Cumulative Review

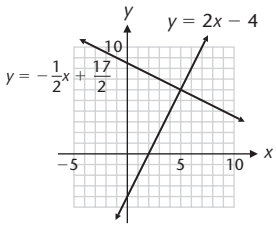
1. (A)



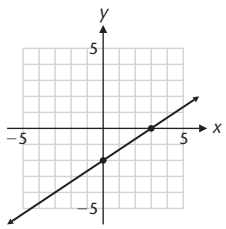
(B) Xmin $= -3$, Xmax $= 3$, Ymin $= -4$, Ymax $= 4$
(C) No *(1–1, 1–2)*

2. (A) $y = 2x - 4$ (B) $y = -\frac{1}{2}x + \frac{17}{2}$

(C) *(2–1)*



3. Slope: $\frac{2}{3}$; y intercept: -2; x intercept: 3 *(2–1)*

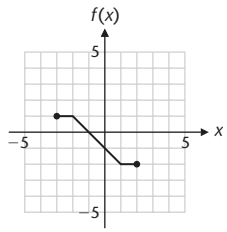


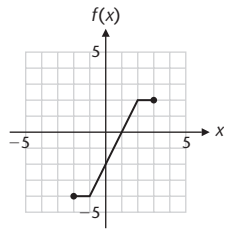
4. (A) 2 (B) 4 (C) $\frac{-2}{5}$ *(1–2)*

5. (A) Vertically stretched by a factor of 2 (B) Shifted right 2 units
(C) Shifted down 2 units *(1–4)*

6. Domain: $[-2, 3]$; range: $[-1, 2]$ *(1–2)* **7.** Neither (1–4)

8. (A) (B) *(1–4)*





9. No solution *(2–2)* **10.** $x = -\frac{9}{2}, -3$ *(2–5)*

11. $x = \frac{5}{2}$ *(2–2)* **12.** $x = 0, -4$ *(2–5)* **13.** $x = \pm\sqrt{5}$ *(2–5)*

14. $x = 3 \pm \sqrt{7}$ *(2–5)* **15.** $x = 3$ *(2–6)* **16.** $y \geq 5$; $[5, \infty)$ *(2–7)*

17. $-3 < x \leq 5/3$; $(-3, 5/3]$ *(2–7)*

18. $-5 < x < 9$; $(-5, 9)$ *(2–7)*

19. $x \leq -5$ or $x \geq 2$; $(-\infty, -5] \cup [2, \infty)$ *(2–7)*

20. (A) $f(x) = (x - 2)^2 - 5$ (B) It is the same as the graph of $y = x^2$ shifted to the right 2 units and down 5 units.
(C) $x = 2 \pm \sqrt{5}$ *(2–3, 2–5)*

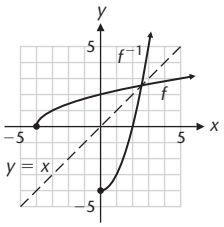
21. (A) $7 - 10i$ (B) $23 + 7i$ (C) $1 - i$ *(2–4)*

22. (A)All real numbers $(-\infty, \infty)$ (B) $\{-2\} \cup [1, \infty)$ (C) 1
(D) $[-3, -2]$ and $[2, \infty)$ (E) $-2, 2$ *(1-4)*

23. $(f \circ g)(x) = \frac{x}{3 - x}$; domain: $(-\infty, 0) \cup (0, 3) \cup (3, \infty)$ *(1–5)*

24. $f^{-1}(x) = \frac{x - 5}{2}$ or $\frac{1}{2}x - \frac{5}{2}$ *(1–6)*

25. (A) $f^{-1}(x) = x^2 - 4$, domain: $x \geq 0$
(B) Domain of $f = [-4, \infty)$ = range of f^{-1}; range of $f = [0, \infty)$ = domain of f^{-1}
(C)



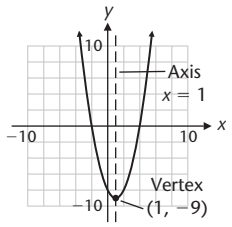
(1-6)

26. Only f is one-to-one *(1–6)* **27.** (A) $y = -\frac{3}{2}x - 8$

(B) $y = \frac{2}{3}x + 5$ *(2–1)* **28.** Range: $[-9, \infty)$;

minimum: $f(1) = -9$; y intercept: $f(0) = -8$;

x intercepts: $x = 4$ and $x = -2$



(2-3)

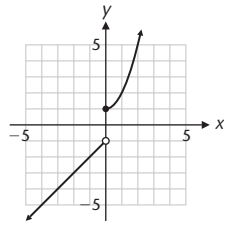
29. $x < \frac{3}{2}$ or $x > 3$; $(-\infty, \frac{3}{2}) \cup (3, \infty)$ *(2–7)*

30. $x < -3.1$ or $x > 5.1$; $(-\infty, -3.1) \cup (5.1, \infty)$ *(2–7)*

31. (A) $0 + 0i$ or 0 (B) $\frac{6}{5}$ (C) $-i$ *(2–4)*

32. (A) $3 + 18i$ (B) $-2.9 + 10.7i$ (C) $-4 - 6i$ *(2–4)*

33. Domain: all real numbers; range: $(-\infty, -1) \cup [1, \infty)$; discontinuous at $x = 0$



(1-3)

34. The graph of $y = |x|$ is vertically shrunk by $\frac{1}{2}$, reflected in the x axis, shifted 2 units to the right and 3 units up;
$y = -\frac{1}{2}|x - 2| + 3$. *(1–4)* **35.** $y = (x + 2)^2 - 3$ *(2–3)*

36. $y = 3 \pm i\sqrt{5}$ *(2–6)* **37.** $x = \frac{27}{8}, -\frac{1}{8}$ *(2–6)*

38. $u = \pm 2i, \pm\sqrt{3}$ *(2–6)* **39.** $t = \frac{9}{4}$ *(2–6)*

40. $x = \frac{4}{3}i$ *(2–6)*

41. If $b < -2$ or $b > 2$ there are two distinct real roots; if $b = -2$ or $b = 2$, there is one real double root; and if $-2 < b < 2$ there are two distinct imaginary roots. *(2–5)*

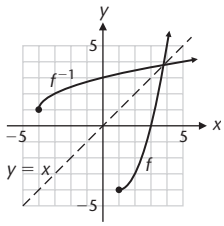
44. $-5 - 2h$ *(1–3)* **45.** $y = 2\sqrt[3]{x + 1} - 1$ *(1–4)*

46. (A) $h = \frac{A - 2\pi r^2}{2\pi r}$ (B) $r = -\frac{h}{2} + \sqrt{\frac{h^2}{4} + \frac{A}{2\pi}}$

The negative root is discarded since r must be positive. *(2–2, 2–5)*

47. (A) Domain g: $[-2, 2]$
(B) $\left(\frac{f}{g}\right)(x) = \frac{x^2}{\sqrt{4 - x^2}}$; domain of f/g is $(-2, 2)$
(C) $(f \circ g)(x) = 4 - x^2$; domain of $f \circ g$ is $[-2, 2]$. *(1–5)*

48. (A) $f^{-1}(x) = 1 + \sqrt{x + 4}$ (B) Domain of $f^{-1} = [-4, \infty)$; range of f^{-1} = domain of $f = [1, \infty)$
(C)



(1-6)

49. 0 *(2–4)* **50.** All a and b such that $a < b$. *(2–2)*

51. $\frac{a^2 - b^2}{a^2 + b^2} + \frac{2ab}{a^2 + b^2}i$ *(2–4)* **52.** $x = \frac{\sqrt{2} \pm i}{3}$ *(2–5)*

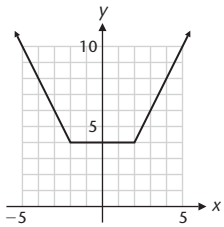
53. $x = \pm 2i, \pm 3i$ *(2–6)* **54.** $x = -1.5 + 0.5i$ *(2–6)*

55. $x = \left(\frac{6}{1 \pm \sqrt{13}}\right)^5$ *(2–6)*

57. (A) $x + 0.5h - 3$ (B) $0.5x + 0.5a - 3$ *(1–2)*

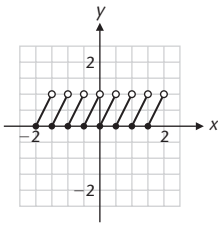
59. $f(x) = \begin{cases} -2x & \text{if} \quad x < -2 \\ 4 & \text{if} \quad -2 \leq x \leq 2 \\ 2x & \text{if} \quad x > 2 \end{cases}$

Domain: all real numbers; range: $[4, \infty)$ *(1–3)*



60. $f(x) = \begin{cases} \vdots & \vdots \\ 2x + 2 & \text{if } -1 \le x < -\frac{1}{2} \\ 2x + 1 & \text{if } -\frac{1}{2} \le x < 0 \\ 2x & \text{if } 0 \le x < \frac{1}{2} \\ 2x - 1 & \text{if } \frac{1}{2} \le x < 1 \\ 2x - 2 & \text{if } 1 \le x < \frac{3}{2} \\ 2x - 3 & \text{if } \frac{3}{2} \le x < 2 \\ \vdots & \vdots \end{cases}$

Domain: All real numbers; range: $[0, 1)$; discontinuous at $x = k/2$, k an integer



(1-3)

61. $x = 2, -1 \pm i\sqrt{3}$ *(2–5)*

62. (A) $y = 400{,}000 - 13{,}200\,x$ *(2-1)* (B) 30.3 hours *(2-1)*
(C) $100{,}000 < 400{,}000 - 13{,}200x < 200{,}000$; between 15.2 and 22.7 hours after the pumps were started *(2–7)*

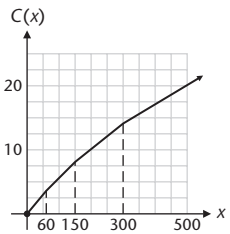
63. 8,800 books *(2–2)* **64.** $|p - 200| \le 10$ *(2–2)*

65. (A) Profit: $\$5.5 < p < \$8$ or ($5.5, $8)
(B) Loss: $\$0 \le p < \$5.5$ or $p > \$8$. $[\$0, \$5.5) \cup (\$8, \infty)$ *(2–3, 2–7)*

66. (A) $v = -2{,}000t + 20{,}000$ (B) $t = -0.0005v + 10$; $d^{-1}(v)$ provides the time it takes the equipment to depreciate to a given value v. *(1–6)* **67.** 40 mi from A to B and 75 mi from B to C, or 75 mi from A to B and 40 mi from B to C *(2–5)*

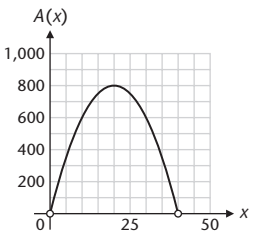
68. $x = -900p + 4{,}571$; 1,610 bottles *(2–2)*

69. $c(x) = \begin{cases} 0.6x & \text{if } 0 \le x \le 60 \\ 0.5x + 0.6 & \text{if } 60 < x \le 150 \\ 0.40x + 2.1 & \text{if } 150 < x \le 300 \\ 0.03x + 5.1 & \text{if } 300 < x \end{cases}$



(1-3)

70. (A) $A(x) = 80x - 2x^2$ (B) $0 < x < 40$
(C) 20 feet by 40 feet



(1-3, 2-3)

71. (A) $f(1) = 1; f(2) = 0; f(3) = 1; f(4) = 0$

(B) $f(n) = \begin{cases} 1 & \text{if } n \text{ is an odd integer} \\ 0 & \text{if } n \text{ is an even integer} \end{cases}$ *(1–3)*

72. (A) 30,000 bushels (B) The demand decreases to 20,000 bushels
(C) The demand increases to 40,000 bushels

(E)

q	20	25	30	35	40
p	340	332	325	320	315

$y = 0.0229x^2 - 2.61x + 383$ *(1–1, 2–1)*

73. (A) $y = 0.124x^2 - 5.84x + 309$
(B) 1970 level: 2017; 1945 level: 2029
(C) Consumption decreases from 1970 to 1990, then slowly increases. *(2–5)*

74. (A) $y = 0.0481x^2 + 0.0690x + 2.21$ (B) 66.6 mph *(2–5)*

75. (A) $y = -0.00149x^2 + 0.0627x + 1.21$
(B) Speed: 26.9 mph; mileage: 1.82 mpg; time: 7.43 hours; gas used: 110 gallons; cost: $360 *(2–5)*

Chapter 3

Section 3-1

5. T **7.** F **9.** F **11.** c **13.** d

15. Zeros: -1, 3; turning point: (1, 4); $P(x) \to -\infty$ as $x \to \infty$ and $P(x) \to -\infty$ as $x \to -\infty$

17. Zeros: -2, 1; turning points: $(-1, 4)$, (1, 0); $P(x) \to \infty$ as $x \to \infty$ and $P(x) \to -\infty$ as $x \to -\infty$

19. The graph does not increase or decrease without bound as $x \to \infty$ and as $x \to -\infty$.

21. The graph has an infinite number of turning points.

23. Polynomial, degree 3 **25.** Not a polynomial

27. Not a polynomial **29.** Polynomial, degree 3

31. Zeros: -3, 0, 3, $2i$, $-2i$; x intercepts: -3, 0, 3

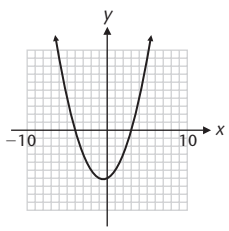
33. Zeros: -5, $3i$, $-3i$, $4i$, $-4i$; x intercept: -5

35. (A) $P(x) \to \infty$ as $x \to \infty$ and $P(x) \to -\infty$ as $x \to -\infty$; three intercepts and two local extrema
(B) x intercepts: -0.86, 1.68, 4.18; local maximum: $P(0.21) \approx 6.21$; local minimum: $P(3.12) \approx -6.06$

37. (A) $P(x) \to -\infty$ as $x \to \infty$ and $P(x) \to \infty$ as $x \to -\infty$; three intercepts and two local extrema
(B) x intercept: 4.47; local minimum: $P(-0.12) \approx 4.94$; local maximum: $P(2.79) \approx 17.21$

39. (A) $P(x) \to \infty$ as $x \to \infty$ and as $x \to -\infty$; four intercepts and three local extrema
(B) x intercepts: none; local minimum: $P(-1.87) \approx 5.81$; local maximum: $P(-0.28) \approx 12.43$; local minimum: $P(1.41) \approx 4.59$

41.



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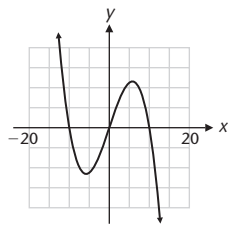
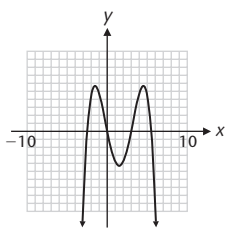
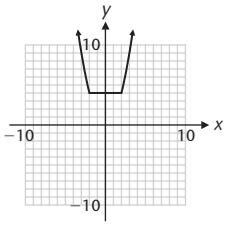


45.



47. $p(x) = x(x-2)(x+3) = x^3 + x^2 - 6x$
49. $p(x) = x(2x+1)(x-4) = 2x^3 - 7x^2 - 4x$
51. $p(x) = (x-2)(x-3)(x-4) = x^3 - 9x^2 + 26x - 24$
53. $p(x) = x(x-i)(x+i) = x^3 + x$
55. $p(x) = x^3$
57. No such polynomial exists.
59. ∞ **61.** $-\infty$ **63.** $-\infty$ **65.** $-\infty$
67. x intercepts: -12.69, -0.72, 4.41; local maximum: $P(2.07) \approx 96.07$; local minimum: $P(-8.07) \approx -424.07$
69. x intercepts: -16.06, 0.50, 15.56; local maximum: $P(-9.13) \approx 65.86$; local minimum: $P(9.13) \approx -55.86$
71. x intercepts: -16.15, -2.53, 1.56, 14.12; local minimum: $P(-11.68) \approx -1{,}395.99$; local maximum: $P(-0.50) \approx 95.72$; local minimum: $P(9.92) \approx -1{,}140.27$
73. x intercepts: 1, 1.09; local minimum: $P(1.05) \approx -0.20$; local maximum: $P(6.01) \approx 605.01$; local minimum: $P(10.94) \approx 9.70$
75. (A) 1; 3 (B) 0; 4
77. No; $f(x) = x^2 + x$ has even degree, but $f(1) \neq f(-1)$.
79. (A) (B) $\{c \mid -2 \le c \le 2\}$



81. It is possible.
83. (A) $R(x) = 0.0004x^3 - x^2 + 569x$
(B) 364 air conditioners; price: \$258; max revenue: \$93,911
85. (A) $V = 28x + 28x^2 + 8x^3$ (B) 0.097 ft
87. (A) $y = 0.0178x^3 + 0.0775x^2 + 3.42x + 23.2$
(B) \$2,613.0 billion (or \$2.613 trillion)
(C) $y = 1.29x^2 - 17.0x + 62.8$; cubic model predicts more rapid long-term increase.
89. (A) Cubic model: $y = -0.000151x^3 + 0.0109x^2 - 0.207x + 10.7$; quartic model: $y = 0.000\,00920x^4 - 0.00116x^3 + 0.0449x^2 - 0.558x + 11.0$ (B) Cubic model predicts 5.9 marriage rate, quartic predicts 7.9. (C) The quartic model fits the data points better, but the cubic model provides a more realistic prediction. The quartic model predicts a change in direction not supported by the data.
(D) The data decreases, then increases, then decreases again; quadratic models can only change direction once.

Section 3-2

7. $2m + 1$
9. $4x - 5 + \dfrac{11}{2x+1}$ **11.** $x^2 + x + 1$
13. $2y^2 - 5y + 13 - \dfrac{27}{y+2}$ **15.** $x + 5 + \dfrac{3}{x-2}$
17. $4x - 2 - \dfrac{3}{x+3}$ **19.** $2x^2 + 4x + 5 + \dfrac{11}{x-2}$
21. 4 **23.** 3 **25.** -6 **27.** Yes **29.** Yes **31.** $2x^2 + x + 3$
33. $x^2 - 2x - 5 - \dfrac{4}{2x-1}$ **35.** $x - 4 + \dfrac{-5x-7}{x^2-2}$
37. $3x^2 - 3x - 5 + \dfrac{8x+15}{x^2+x+2}$ **39.** $3x^3 - 3x^2 + 3x - 4$
41. $x^4 - x^3 + x^2 - x + 1$ **43.** $3x^3 - 7x^2 + 21x - 67 + \dfrac{200}{x+3}$
45. $2x^5 - 3x^4 - 15x^3 + 2x + 10$ **47.** $4x^3 - 6x - 2 + \dfrac{2}{x+\frac{1}{2}}$
49. $4x^2 - 2x - 4$ **51.** $3x^3 - 0.8x^2 + 1.68x - 2.328 + \dfrac{0.0688}{x-0.4}$
53. $3x^4 - 0.4x^3 + 5.32x^2 - 4.256x - 3.5952 - \dfrac{0.12384}{x+0.8}$
55. -20 **57.** $\frac{25}{8}$ **59.** $1 - 8i$ **61.** $x = -3, 2$ **63.** $x = -1, \frac{2}{3}$
65. $x = -2i, 2i$ **67.** $x = 2 - i, 2 + i$ **69.** $x^2 + (-3 + i)x - 3i$
71. (A) -5 (B) $-40i$ (C) 0 (D) 0 **73.** $k = -4$ **75.** $k = 16$
77. (A) In both cases the coefficient of x is a_2, the constant term is $a_2r + a_1$, and the remainder is $(a_2r + a_1)r + a_0$.
(B) The remainder expanded is $a_2r^2 + a_1r + a_0 = P(r)$.
79. $P(-2) = 81$; $P(1.7) = 6.2452$ **83.** No

Section 3-3

1. $x = 4$: no; $x = -3$: yes **3.** $x = 2$: yes; $x = -4$: no
5. $x = 1$: no; $x = -3$: no **7.** -5.372, 0.372
9. -1.752, 0.432, 1.320 **17.** $[-2, -1] \cup \{1\} \cup [3, \infty)$
19. $(-2, -1) \cup (3, \infty)$ **21.** $(-\infty, -5.372) \cup (0.372, \infty)$
23. $(-\infty, -1.752] \cup [0.432, 1.320]$
25. Upper bound: 2; lower bound: -2
27. Upper bound: 3; lower bound: -2
29. Upper bound: 2; lower bound: -3
31. (A) Upper bound: 3; lower bound: -1 (B) 2.25
33. (A) Upper bound: 3; lower bound: -4 (B) -3.51, 2.12
35. (A) Upper bound: 2; lower bound: -3 (B) -2.09, 0.75, 1.88
37. (A) Upper bound: 1; lower bound: -1 (B) 0.83
39. (A) $P(3) < 0$ and $P(4) > 0$ (B) five intervals; 3.2
41. (A) $P(-2) < 0$ and $P(-1) > 0$ (B) six intervals; -1.4
43. (A) $P(3) < 0$ and $P(4) > 0$ (B) four intervals; 3.1
45. (A) $P(-1) < 0$ and $P(0) > 0$ (B) five intervals; -0.5
47. No; P does not change sign at $x = 1$.
49. No; P does not change sign at $x = 3$.
51. -1.83, 3.83 **53.** -1.24, 2, 3.24
55. -0.22, 2, 2.22 **57.** $(-7, -3) \cup (\frac{5}{2}, \infty)$
59. $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$ **61.** $(-\infty, -3] \cup [0, 3]$
63. $(-\infty, -3] \cup [-2, 0]$
65. $[-2.507, 1.222] \cup [2.285, \infty)$ **67.** (2.484, 4.873)
69. $(-\infty, -3.101] \cup [-2.259, 0.259] \cup [1.101, \infty)$
71. (A) Three real zeros; yes; A third degree polynomial can have at most three zeros.
(B) Yes (C) All entries in the quotient row are negative.
73. (A) Upper bound: 30; lower bound: -10 (B) -1.29, 0.31, 24.98
75. (A) Upper bound: 30; lower bound: -40
(B) -36.53, -2.33, 2.40, 24.46
77. (A) Upper bound: 20; lower bound: -10 (B) -7.47, 14.03
79. (A) Upper bound: 30; lower bound: -20
(B) -17.66, 2.5 (double zero), 22.66
81. (A) Upper bound: 40; lower bound: -40 (B) -30.45, 9.06, 39.80
83. Yes **87.** (A) 81 or 460 toasters (B) Between 176 and 396 toasters
89. $x^4 - 3x^2 - 2x + 4 = 0$; (1, 1) and (1.659, 2.752)
91. $4x^3 - 84x^2 + 432x - 600 = 0$; 2.319 in. or 4.590 in.
93. $x^3 - 15x^2 + 30 = 0$; 1.490 ft
95. $y = 0.0142x^3 - 0.219x^2 + 1.55x + 42.7$; after 2010
97. $y = 0.420x + 69.2$; after 2013
99. $y = 0.000\,281x^3 - 0.0220x^2 + 0.853x + 14.1$; 2024

Section 3-4

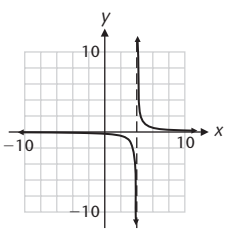
1. -8 (multiplicity 3), 6 (multiplicity 2); degree of $P(x)$ is 5.
3. -4 (multiplicity 3), 3 (multiplicity 2); -1; degree of $P(x)$ is 6.
5. $2i$ (multiplicity 3), $-2i$ (multiplicity 4), -2 (multiplicity 5), 2 (multiplicity 5); degree of $P(x)$ is 17.
7. -9 (multiplicity 2), -3, 0, 3, $3i$, $-3i$; degree of $P(x)$ is 7.

13. False **15.** $P(x) = (x - 3)^2(x + 4)$; degree 3
17. $P(x) = (x + 7)^3[x - (-3 + \sqrt{2})][x - (-3 - \sqrt{2})]$; degree 5
19. $P(x) = [x - (2 - 3i)][x - 2(2 + 3i)](x + 4)^2$; degree 4
21. $(x + 2)(x - 1)(x - 3)$; degree 3 **23.** $(x + 2)^2(x - 1)^2$; degree 4
25. $(x + 3)(x + 2)x(x - 1)(x - 2)$; degree 5 **27.** $\pm 1, \pm 2, \pm 3, \pm 6$
29. $\pm 1, \pm 2, \pm 4, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}$
31. $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{1}{6}, \pm\frac{1}{12}$
33. (A) $(x^2 + 1)(x^2 + 4)$ (B) $(x + i)(x - i)(x + 2i)(x - 2i)$
35. (A) $(x - 1)(x^2 + 25)$ (B) $(x - 1)(x + 5i)(x - 5i)$
37. $P(x) = (x + 1)(x + 4)(x + 4)$
39. $P(x) = (x - 1)(x + 1)(x - i)(x + i)$
41. $P(x) = (2x - 1)[x - (4 + 5i)][x - (4 - 5i)]$
43. 2, 3, -5 **45.** 0, 2, $-\frac{2}{5}, \frac{1}{2}$ **47.** 2 (double), $\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$
49. $\pm i\sqrt{5}, \pm i\sqrt{6}$ **51.** -1 (double), $-\frac{1}{3}, 2 \pm i$
53. $\frac{1}{2}, 1 \pm \sqrt{2}$ **55.** -2 (double), $\pm\sqrt{5}$ **57.** $\pm 2, 1 \pm \sqrt{2}$
59. $\pm i, \pm 3i$ **61.** $\pm 1, \frac{3}{2}, \pm i$ **63.** $P(x) = (x + 2)(3x + 2)(2x - 1)$
65. $P(x) = (x + 4)[x - (1 + \sqrt{2})][x - (1 - \sqrt{2})]$
67. $P(x) = (x - 2)(x + 1)(2x - 1)(2x + 1)$
69. $x^2 - 8x + 41$ **71.** $x^2 - 6x + 25$ **73.** $x^2 - 2ax + a^2 + b^2$
75. (D) The other zeros are $2 - i$ and 3. **77.** -1 and $3 + i$
79. $5i$ and 3 **81.** $2 - i, \sqrt{2}, -\sqrt{2}$
83. $\sqrt{6}$ is a zero of $P(x) = x^2 - 6$, but $P(x)$ has no rational zeros.
85. $\sqrt[3]{5}$ is a zero of $P(x) = x^3 - 5$, but $P(x)$ has no rational zeros.
87. 2 **89.** 3 **91.** $\frac{1}{3}, 6 \pm 2\sqrt{3}$ **93.** $\frac{3}{2}, -\frac{5}{2}, \pm 4i$
95. $\frac{3}{2}$ (double), $4 \pm \sqrt{6}$ **97.** (A) 3 (B) $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$
99. maximum of n; minimum of 1
101. No, since $P(x)$ is not a polynomial with real coefficients (the coefficient of x is the imaginary number $2i$).
107. 2 feet **109.** 0.5×0.5 inches or 1.59×1.59 inches

Section 3-5

7. $g(x)$ **9.** $h(x)$ **11.** (A) ∞ (B) $-\infty$ (C) 2 (D) 2
13. (A) $-\infty$ (B) ∞ (C) 2 (D) 2
15. Domain: $(-\infty, -1) \cup (-1, \infty)$; x intercept: 2
17. Domain: $(-\infty, -4) \cup (-4, 4) \cup (4, \infty)$; x intercepts: -1, 1
19. Domain: $(-\infty, -3) \cup (-3, 4) \cup (4, \infty)$; x intercepts: -2, 3
21. Domain: all real numbers; x intercept: 0
23. Domain: $(-\infty, 0) \cup (0, \infty)$; x intercepts: -2, 2
25. Vertical asymptote: $x = 4$; horizontal asymptote: $y = 2$
27. Vertical asymptotes: $x = -4$, $x = 4$; horizontal asymptote: $y = \frac{2}{3}$
29. No vertical asymptotes; horizontal asymptote: $y = 0$
31. Vertical asymptotes: $x = -1$, $x = \frac{5}{3}$; no horizontal asymptote
33. Vertical asymptote: $x = 0$; horizontal asymptote: $y = 4$
35. The graph has more than one horizontal asymptote.
37. The graph has a sharp corner at (0, 0).
39. The graph of f is the same as the graph of g except that f has a hole at (0, 2). **41.** The graph of f is the same as the graph of g except that f has a hole at $(-2, -\frac{1}{6})$.

43.



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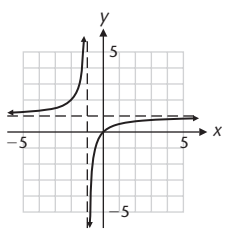

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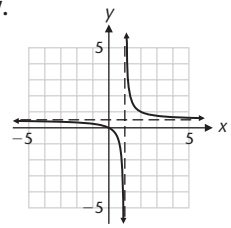

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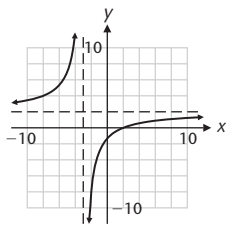

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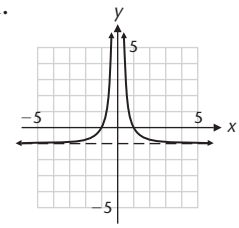
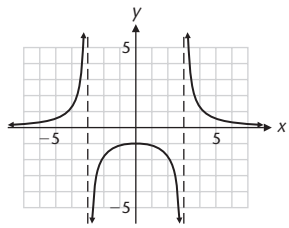


53.



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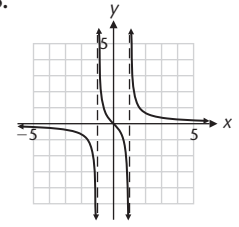

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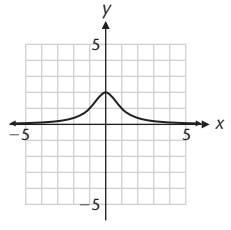

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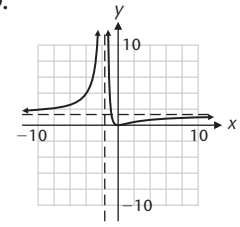

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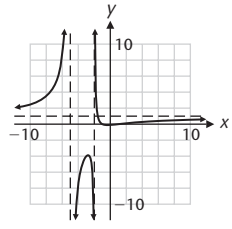

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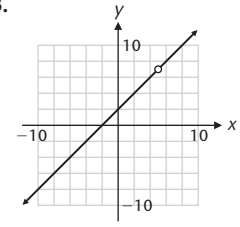
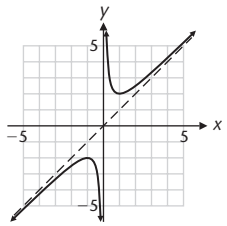


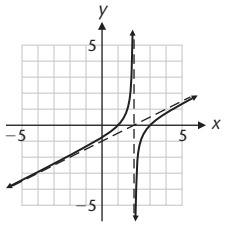
65. $f(x) = \dfrac{3(x^2 - 1)(x^2 - 4)}{x^4 + 1}$ **67.** $f(x) = \dfrac{(2x + 5)(x - 10) + 100}{x - 10}$
69. [0, 2) **71.** $(-4, \frac{2}{5}) \cup (4, \infty)$ **73.** $[-2, 0) \cup [10, \infty)$
75. $(-\frac{3}{2}, -1) \cup (0, 1) \cup (\frac{3}{2}, \infty)$ **77.** $(-6.541, -2) \cup (-0.459, \infty)$
79. $(-3, -2.110) \cup (7.110, \infty)$
81. $(-\infty, 0) \cup (0, 0.595) \cup [8.405, \infty)$ **83.** (5, 7.429)
85. $(-\infty, -2.333] \cup (-1, 0)$
87. Vertical asymptote: $x = 1$; slant asymptote: $y = 2x + 2$
89. Slant asymptote: $y = x$
91. Vertical asymptote: $x = 0$; slant asymptote: $y = 2x - 3$
93. $f(x) \to 5$ as $x \to \infty$ and $f(x) \to -5$ as $x \to -\infty$; the lines $y = 5$ and $y = -5$ are horizontal asymptotes.
95. $f(x) \to 4$ as $x \to \infty$ and $f(x) \to -4$ as $x \to -\infty$; the lines $y = 4$ and $y = -4$ are horizontal asymptotes.

97.



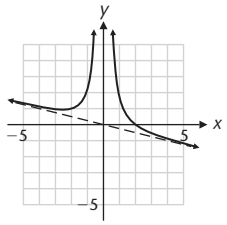
Vertical asymptote: $x = 0$
Slant asymptote: $y = x$

99.



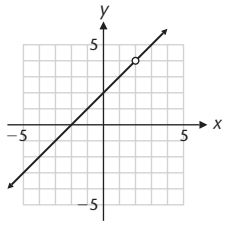
Vertical asymptote: $x = 2$
Slant asymptote:
$y = \frac{1}{2}x - 1$

101.

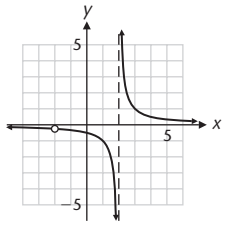


Vertical asymptote: $x = 0$
Slant asymptote:
$y = -\frac{1}{4}x$

103. Domain: $x \neq 2$, or $(-\infty, 2) \cup (2, \infty)$

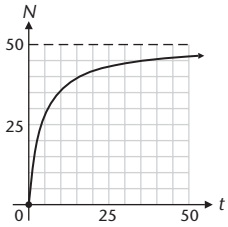


105. Domain: $x \neq 2, -2$ or $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$



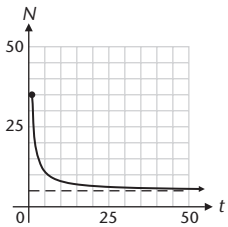
Vertical asymptote: $x = 2$
Horizontal asymptote: $y = 0$

107. As $t \to \infty$, $N \to 50$



An employee will eventually be able to assemble 50 components per day.

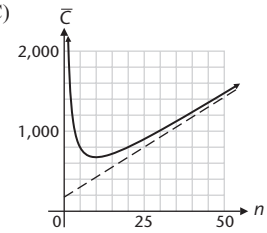
109. As $t \to \infty$, $N \to 5$



The average retention in the long run approaches five symbols.

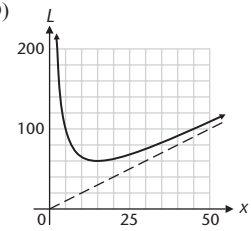
111. (A) $\overline{C}(n) = 25n + 175 + \dfrac{2{,}500}{n}$ (B) 10 yr

(C)



113. (A) $L(x) = 2x + \dfrac{450}{x}$ (B) (0, 112.5) (C) 15 ft by 15 ft

(D)



Section 3-6

7. $F = \dfrac{k}{x}$ **9.** $R = kST$ **11.** $L = km^3$ **13.** $A = kc^2 d$ **15.** $P = kx$

17. $h = \dfrac{k}{\sqrt{s}}$ **19.** $R = k\dfrac{m}{d^2}$ **21.** $D = k\dfrac{xy^2}{z}$ **23.** Decreases

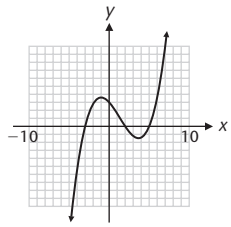
25. Increases **27.** $u = \dfrac{3\sqrt{10}}{2}$ **29.** $L = \dfrac{81}{4}$ **31.** $Q = 144$ **33.** Direct

35. Inverse **37.** $t = \dfrac{k}{T}$ **39.** $L = k\dfrac{wh^2}{x}$ **41.** $N = k\dfrac{F}{d}$ **45.** 83 lb

47. 20 amperes **49.** The new horsepower must be 8 times the original.
51. No effect **53.** 1.47 hr **55.** 20 days
57. The length would be quadrupled. **59.** 540 lb
61. (A) $\Delta S = kS$ (B) 10 oz (C) 8 candlepower
63. The volume would be 8 times the original. **65.** 32 times/s

Chapter 3 Review

1. (A) Not a polynomial (B) Polynomial, degree 3 *(3-1)*
2. Zeros: -1, 3; turning points: $(-1, 0)$, $(1, 2)$, $(3, 0)$;
$P(x) \to \infty$ as $x \to \infty$ and $P(x) \to \infty$ as $x \to -\infty$ *(3-1)*
3. $2x^3 + 3x^2 - 1 = (x + 2)(2x^2 - x + 2) - 5$ *(3-2)*
4. $P(3) = -8$ *(3-2)* **5.** 2, -4, -1 *(3-1)*
6.



(3-1)
7. $P(x) = (x - 5)(x + 3)(x - 2) = x^3 - 4x^2 - 11x + 30$ *(3-1)*
8. $1 - i$ is a zero. *(3-4)*
9. (A) $P(x) = (x + 2)x(x - 2) = x^3 - 4x$ (B) $P(x) \to \infty$ as $x \to \infty$ and $P(x) \to -\infty$ as $x \to -\infty$ *(3-1)*
10. Lower bounds: -2, -1; upper bound: 4 *(3-3)*

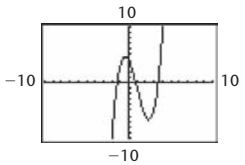
11. $P(1) = -5$ and $P(2) = 1$ are of opposite sign. *(3-3)*
12. $\pm 1, \pm 2, \pm 3, \pm 6$ *(3-4)* **13.** $-1, 2, 3$ *(3-4)*

14. (A) Domain is $(-\infty, -4) \cup (-4, \infty)$; x intercept is $\frac{3}{2}$
(B) Domain is $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$; x intercept is 0 *(3-5)*
15. (A) Horizontal asymptote: $y = 2$; vertical asymptote: $x = -4$
(B) Horizontal asymptote: $y = 0$; vertical asymptotes: $x = -2$, $x = 3$ *(3-5)* **16.** $F = k\sqrt{x}$ *(3-6)* **17.** $G = kxy^2$ *(3-6)*

18. $H = \dfrac{k}{z^3}$ *(3-6)* **19.** $R = kx^2y^2$ *(3-6)* **20.** $S = \dfrac{k}{u^2}$ *(3-6)*

21. $T = k\dfrac{v}{w}$ *(3-6)* **22.** It will increase. *(3-6)*

23. Cleanliness will decrease. *(3-6)* **24.** The graph does not increase or decrease without bound as $x \to \infty$ and as $x \to -\infty$. *(3-1)*
25. (A) The graph of $P(x)$ has three x intercepts and two turning points; $P(x) \to \infty$ as $x \to \infty$ and $P(x) \to -\infty$ as $x \to -\infty$



(B) 3.53 *(3-1)*

26. $P(x) = \left(x - \frac{1}{4}\right)(8x^3 - 12x^2 - 16x - 8) + 5$; $P\left(\frac{1}{4}\right) = 5$ *(3-2)*

27. $x^2 - 9x + 18 + \dfrac{-41x + 1}{x^2 + 2x}$ *(3-2)* **28.** -4 *(3-2)*

29. $P(x) = [x - (1 + \sqrt{2})][x - (1 - \sqrt{2})]$ *(3-2)*
30. Yes, since $P(-1) = 0$, $x - (-1) = x + 1$ must be a factor. *(3-2)*

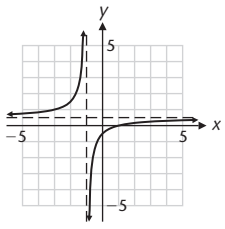
31. $x = -1, \frac{7}{2}$ *(3-2)* **32.** $4, -\frac{1}{2}, -2$ *(3-4)*

33. $(x - 4)(2x + 1)(x + 2)$ *(3-4)* **34.** No rational zeros *(3-4)*

35. $-1, \frac{1}{2}$, and $\dfrac{1 \pm i\sqrt{3}}{2}$ *(3-4)*

36. $(x + 1)(2x - 1)\left(x - \dfrac{1 + i\sqrt{3}}{2}\right)\left(x - \dfrac{1 - i\sqrt{3}}{2}\right)$ *(3-4)*

37. deg $P(x) = 9$; 1 (multiplicity 3), -1 (multiplicity 4), i, $-i$ *(3-4)*
38. (A) $(x - 2)(x + 2)(x^2 + 9)$
(B) $(x - 2)(x + 2)(x - 3i)(x + 3i)$ *(3-4)*
39. (A) -0.24 (double zero); 2; 4.24 (double zero)
(B) -0.24 can be approximated with the MAXIMUM command; 2 can be approximated with the bisection method; 4.24 can be approximated with the MINIMUM command *(3-3)*
40. (A) Upper bound: 7; lower bound: -5 (B) 4 intervals (C) -4.67, 6.62 *(3-3)*
41. (A) Domain is $(-\infty, -1) \cup (-1, \infty)$; x intercept: $x = 1$; y intercept: $y = -\frac{1}{2}$ (B) vertical asymptote: $x = -1$; horizontal asymptote: $y = \frac{1}{2}$
(C) *(3-5)*



42. $[-\frac{5}{3}, 0] \cup [3, \infty)$ *(3-3)* **43.** (A) $(-\infty, -2.562) \cup (1, 1.562)$
(B) $(-\infty, -2.414) \cup (0.414, 2)$ *(3-3)*
44. The graph is discontinuous at $x = 0$, but $x = 0$ is not a vertical asymptote. *(3-5)* **45.** $B = 2$ (3-6) **46.** $D = 120$ (3-6)
47. $P(x) = [x - (1 + i)][x^2 + (1 + i)x + (3 + 2i)] + 3 + 5i$ *(3-2)*

48. $P(x) = \left(x + \frac{1}{2}\right)^2(x + 3)(x - 1)^3$; the degree is 6. *(3-4)*

49. $P(x) = (x + 5)[x - (2 - 3i)][x - (2 + 3i)]$; the degree is 3. *(3-4)*

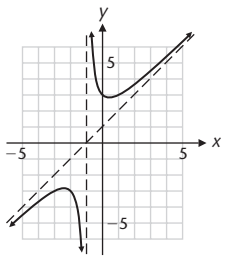
50. $\dfrac{1}{2}, \pm 2, 1 \pm \sqrt{2}$ *(3-4)*

51. $(x - 2)(x + 2)(2x - 1)[x - (1 - \sqrt{2})][x - (1 + \sqrt{2})]$ *(3-4)*
52. Zeros: 0.91, 1; local minimum: $P(-8.94) \approx 9.7$; local maximum: $P(-4.01) \approx 605.01$; local minimum: $P(0.95) \approx -0.20$ *(3-3)*
53. Since $P(x)$ changes sign three times, the minimal degree is 3. *(3-1)*
54. $P(x) = a(x - r)(x^2 - 2x + 5)$ and since the constant term, $-5ar$, must be an integer, r must be a rational number. *(3-4)*

55. (A) 3 (B) $-\dfrac{3}{2} \pm \dfrac{3i}{2}\sqrt{3}$ *(3-4)*

56. (A) Upper bound: 30; lower bound: -30 (B) -23.54, 21.57 *(3-3)*

57.



(3-5)

58. $y = 2$ and $y = -2$ *(3-5)*
59. (A) $(-\infty, -1.879) \cup [-1.732, 0.347) \cup (1.532, 1.732]$
(B) $(-1.879, -1.843) \cup (0.347, 0.379) \cup (1.420, 1.532) \cup (5.044, \infty)$ *(3-5)* **60.** 3; none of the candidates for rational zeros (± 1, ± 2, and ± 4) are actually zeros. *(3-4)*

61. $f(x) = \dfrac{5x(x + 3)(x - 2)}{(x + 1)^2(x - 4)}$ *(3-5)*

62. (A) 2.7 hours (B) Between 10.9 and 17.5 hours *(3-3)*
63. F would be $\frac{1}{2}$ the original. *(3-6)*

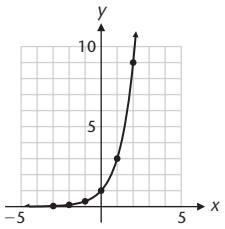
64. $v = \dfrac{k\sqrt{T}}{\sqrt{w}}$ *(3-6)* **65.** 20 days *(3-6)* **66.** \$9.00 *(3-6)*

67. $2x^3 - 32x + 48 = 0$, 4×12 ft or 5.2×9.2 ft *(3-3)*
68. 4.789 ft *(3-3)* **69.** 1.450 in. or 4.465 in. *(3-3)*
70. $(-2, 4)$, $(-1.562, 2.440)$, $(1, 1)$, $(2.562, 6.564)$ *(3-3)*
71. (A) $y = -0.103x^3 + 0.849x^2 - 2.56x + 222$
(B) 339 refrigerators (C) 36 ads *(3-1)*
72. (B) $y = -0.00424x^3 + 0.507x^2 - 19.3x + 297$
(C) 25 yr, 40 yr, 54 yr *(3-1)*

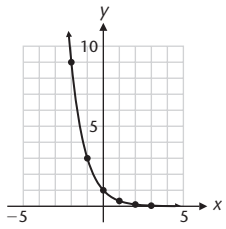
Chapter 4

Section 4-1

7. (A) g (B) n (C) f (D) m
9. 16.24 **11.** 7.524 **13.** 1.649 **15.** 4.469
17.



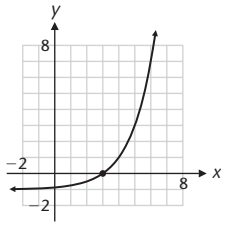
19.

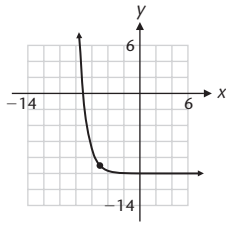


21. 10^{2x+3} **23.** 3^{2x-1} **25.** $\dfrac{4^{3xz}}{5^{3yz}}$ **27.** e^{3x-1} **29.** (B) e

31. The graph of $y = 2^x$ is shifted 3 units right and 1 unit down.

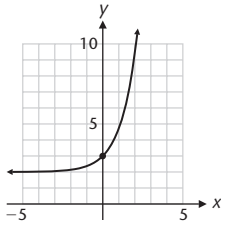
33. The graph of $y = (\frac{1}{3})^x$ is shifted 5 units left and 10 units down.

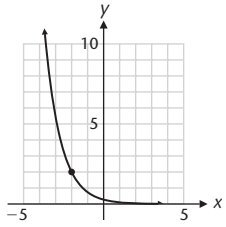




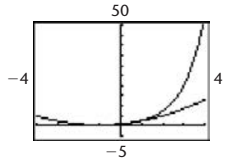
35. The graph of $y = e^x$ is shifted 2 units up.

37. The graph of $y = e^x$ is reflected through the y axis, shifted to the left 2 units, and stretched vertically by a factor of 2.

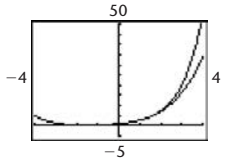




39. $x = 2$ **41.** $x = -1, 3$ **43.** $x = \dfrac{2}{3}$ **45.** $x = \dfrac{1}{2}, 1$ **47.** $x = 0, \dfrac{3}{2}$
49. $x = 0$ **51.** $x = 0, 5$ **53.** $a = 1$ or $a = -1$
59. The graph of a nonconstant polynomial has no horizontal asymptote.
61. $\dfrac{e^{-2x}(-2x - 3)}{x^4}$ **63.** $2e^{2x} + 2e^{-2x}$
65. No local extrema; no x intercept; y intercept: 2.14; horizontal asymptote: $y = 2$
67. Local minimum: $m(0) = 1$; no x intercepts; y intercept: 1; no horizontal asymptotes
69. Local maximum: $s(0) = 1$; no x intercept; y intercept: 1; horizontal asymptote: x axis
71. No local extrema; no x intercept; y intercept: 50; horizontal asymptotes: x axis and $y = 200$
73. Local minimum: $m(0.91) \approx 2.67$; x intercepts: -0.55; horizontal asymptote: $y = 2$
75. Local minimum: $f(0) = 1$; no x intercepts; no horizontal asymptotes
77. $f(x) \to 2.718 \approx e$ as $x \to 0$
79.



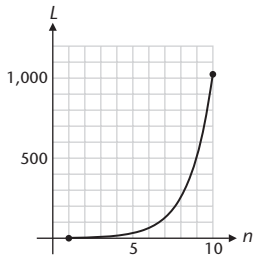
81.



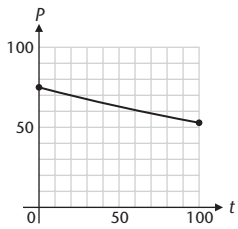
83. As $x \to \infty$, $f_n(x) \to 0$; the line $y = 0$ is a horizontal asymptote. As $x \to -\infty$, $f_1(x) \to -\infty$ and $f_3(x) \to -\infty$, while $f_2(x) \to \infty$. As $x \to -\infty$, $f_n(x) \to \infty$ if n is even and $f_n(x) \to -\infty$ if n is odd.
85. $2^{1.4} = 2.6390$; $2^{1.41} = 2.6574$; $2^{1.414} = 2.6648$; $2^{1.4142} = 2.6651$; $2^{1.41421} = 2.6651$; $2^{1.414214} = 2.6651$; $2^{\sqrt{2}} \approx 2.6651$
87. (A) \$4,225.92 (B) \$12,002.71 **89.** (A) \$10,691.81 (B) \$36,336.69 **91.** Yes, after 6,217 days **93.** No **95.** \$12,197.09
97. Gill Savings: \$1,230.60; Richardson Savings and Loan: \$1,231.00; USA Savings: \$1,229.03 **99.** \$24,602

Section 4-2

5. $A = 200(2)^{t/5}$ **7.** $A = 2{,}000e^{0.02t}$ **9.** $A = 100\left(\dfrac{1}{2}\right)^{t/6}$
11. $A = 4e^{-0.124t}$
13.



15. (A) 76 flies (B) 570 flies **17.** (A) 2,252,800 (B) 407,800,360
19. (A) 19 pounds (B) 7.9 pounds **21.** 7.3 billion **23.** 2008
25.



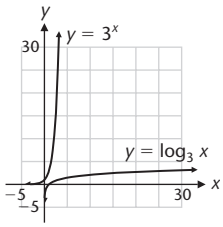
27. (A) 62% (B) 39% **29.** (A) 47.7 million people (B) 56.0 million people **31.** $T = 50°F$ **33.** q approaches 0.000 9 coulombs, the upper limit for the charge on the capacitor. **35.** (A) 25 deer, 37 deer (B) 10 years (C) A approaches 100 deer, the upper limit for the number of deer the island can support.
37. $y = 14{,}910(0.8163)^x$; estimated purchase price: \$14,910; estimated value after 10 years: \$1,959
39. (A) $y = \dfrac{842}{1 + 2.08e^{-0.187x}}$ (B) 836.0 billion kilowatt-hours

Section 4-3

7. $81 = 3^4$ **9.** $0.001 = 10^{-3}$ **11.** $\frac{1}{36} = 6^{-2}$ **13.** $\log_4 8 = \frac{3}{2}$
15. $\log_{32} \frac{1}{2} = -\frac{1}{5}$ **17.** $\log_{2/3} \frac{8}{27} = 3$
19.

x	$y = 3^x$
-3	$\frac{1}{27}$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9
3	27

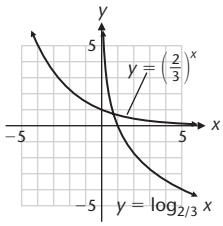
x	$y = \log_3 x$
$\frac{1}{27}$	-3
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2
27	3



21.

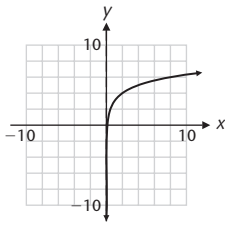
x	$y = (\frac{2}{3})^x$
-3	$\frac{27}{8}$
-2	$\frac{9}{4}$
-1	$\frac{3}{2}$
0	1
1	$\frac{2}{3}$
2	$\frac{4}{9}$
3	$\frac{8}{27}$

x	$y = \log_{2/3} x$
$\frac{27}{8}$	-3
$\frac{9}{4}$	-2
$\frac{3}{2}$	-1
1	0
$\frac{2}{3}$	1
$\frac{4}{9}$	2
$\frac{8}{27}$	3

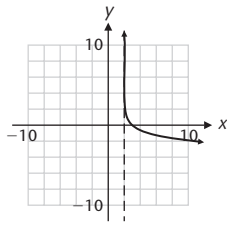


23. 0 **25.** 1 **27.** 4 **29.** -2 **31.** 3 **33.** -1 **35.** 5 **37.** $\frac{1}{3}$
39. 4.6923 **41.** 3.9905 **43.** 1.3181 **45.** 2.9759 **47.** 200,800
49. 0.0006648 **51.** 47.73 **53.** 0.6760 **55.** $x = 4$ **57.** $y = 2$
59. $b = 4$ **61.** b is any positive real number except 1. **63.** $x = 2$
65. $y = -2$ **67.** $b = 100$ **69.** $x = \frac{1}{16}$ **71.** $y = \frac{3}{4}$ **73.** 4.959
75. 7.861 **77.** 2.280 **79.** $\log x - \log y$ **81.** $4 \log x + 3 \log y$
83. $\ln\left(\frac{x}{y}\right)$ **85.** $\ln\left(\frac{x^2y^5}{z}\right)$ **87.** 1 **89.** -10

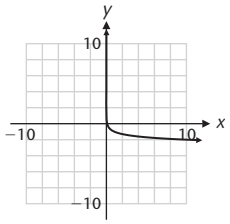
91. The graph of g is the same as the graph of f shifted upward 3 units; g is increasing. Domain: $(0, \infty)$; vertical asymptote: $x = 0$



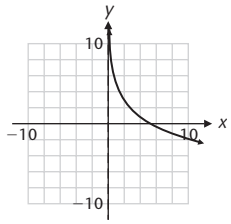
93. The graph of g is the same as the graph of f shifted 2 units to the right; g is decreasing. Domain: $(2, \infty)$; vertical asymptote: $x = 2$



95. The graph of g is the same as the graph of f reflected through the x axis and shifted downward 1 unit; g is decreasing. Domain: $(0, \infty)$; vertical asymptote: $x = 0$

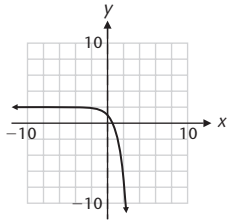


97. The graph of g is the same as the graph of f reflected through the x axis, stretched vertically by a factor of 3, and shifted upward 5 units; g is decreasing. Domain: $(0, \infty)$; vertical asymptote: $x = 0$

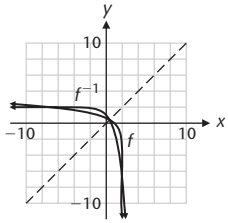


99. $f^{-1}(x) = 5^x$ **101.** $f^{-1}(x) = 3^{x/4} - 3$
103. (A) $f^{-1}(x) = 2 - 3^x$

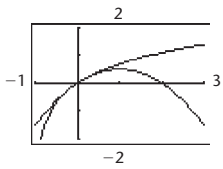
(B)



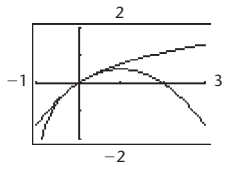
(C)



107.



109.



Section 4-4

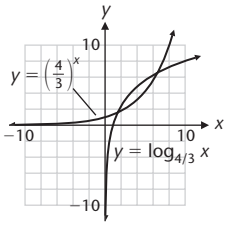
5. (A) 0 decibels (B) 120 decibels **7.** 30 decibels **9.** 8.6
11. 1,000 times as powerful **13.** 6.5 **15.** 11.3 **17.** 7.67 km/s
19. (A) 8.3, basic (B) 3.0, acidic **21.** 6.3×10^{-6} moles per liter
23. (A) $m = 6$ (B) 100 times brighter
25. (A) $y = -525.1 + 142.8 \ln x$; 2003: 136.7 bushels per acre; 2010: 146.1 bushels per acre

Section 4-5

7. $x = 1.46$ **9.** $x = 0.321$ **11.** $x = 1.29$ **13.** $x = 3.50$
15. $x = 1.80$ **17.** $x = 2.07$ **19.** $x = 16$ **21.** $x = -1, 9$
23. $x = 20$ **25.** $x = 5$ **27.** $x = \dfrac{11}{9}$ **29.** $x = \dfrac{4}{3}$ **31.** $x = 3$
33. $x = 14.2$ **35.** $x = -1.83$ **37.** $x = 1.46$ **39.** $x = -1.85$
41. $x = 11.7$ **43.** $x = \pm 1.21$ **45.** $x = 5$ **47.** $x = 2 + \sqrt{3}$
49. $x = \dfrac{1 + \sqrt{89}}{4}$ **51.** $x = 1, e^2, e^{-2}$ **53.** $x = e^e$ **55.** $x = 100, 0.1$
57. (B) 2 **59.** (B) 2 **61.** $r = \dfrac{1}{t} \ln \dfrac{A}{P}$ **63.** $I = I_0(10^{D/10})$
65. $I = I_0[10^{(6-M)/2.5}]$ **67.** $t = -\dfrac{L}{R} \ln\left(1 - \dfrac{RI}{E}\right)$
69. $x = \ln (y \pm \sqrt{y^2 - 1})$ **71.** $x = \dfrac{1}{2} \ln \dfrac{1 + y}{1 - y}$
73. $x = 0.38$ **75.** $x = 0.55$ **77.** $x = 0.57$ **79.** $x = 0.85$
81. $x = 0.43$ **83.** $x = 0.27$ **85.** 5 years to the nearest year
87. $r = 0.0916$ or 9.16% **89.** (A) 3.67% per year
(B) $P = 10.5e^{0.0367t}$; in 2022 **91.** 61 years **93.** 2.58 hours
95. 10,010 years **97.** The year 1001 **99.** 7.52 seconds
101. $k = 0.40$, 2.9 hours **103.** (A) 7.94×10^{14} (B) 2.76 days
105. 0.426 years, or about 155 days

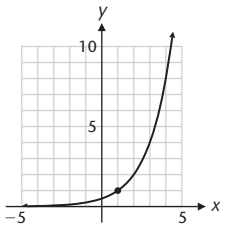
Chapter 4 Review

1. (A) m (B) f (C) n (D) g *(4-1, 4-3)* **2.** $\log m = n$ *(4-3)*
3. $\ln x = y$ *(4-3)* **4.** $x = 10^y$ *(4-3)* **5.** $y = e^x$ *(4-3)*
6.

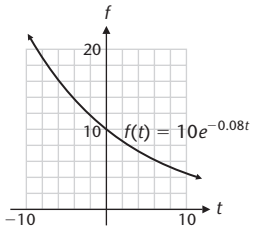


(4-1, 4-3)

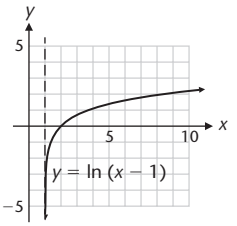
7. 7^{2x} *(4-1)* **8.** e^{2x^2} *(4-1)* **9.** $x = 8$ *(4-3)*
10. $x = 5$ *(4-3)* **11.** $x = 3$ *(4-3)* **12.** $x = 1.24$ *(4-5)*
13. $x = 11.9$ *(4-5)* **14.** $x = 0.984$ *(4-3)* **15.** $x = 103$ *(4-3)*
16. 1.145 *(4-3)* **17.** Not defined *(4-3)* **18.** 2.211 *(4-3)*
19. 11.59 *(4-1)* **20.** $\log \dfrac{xy^3}{\sqrt{z}}$ *(4-3)* **21.** $3 \ln x - \ln y$ *(4-3)*
22. $x = 4$ *(4-5)* **23.** $x = 2$ *(4-5)* **24.** $x = 3, -1$ *(4-5)*
25. $x = 1$ *(4-5)* **26.** $x = \ln 2 / \ln 3$ *(4-5)* **27.** $x = \ln 7$ *(4-5)*
28. $x = 3, -3$ *(4-5)* **29.** $x = -2$ *(4-5)* **30.** $x = \dfrac{1}{3}$ *(4-5)*
31. $x = 64$ *(4-5)* **32.** $x = e$ *(4-5)* **33.** $x = 33$ *(4-5)*
34. $x = 1$ *(4-5)* **35.** $x = 41.8$ *(4-1)* **36.** $x = 1.95$ *(4-3)*
37. $x = 0.0400$ *(4-3)* **38.** $x = -6.67$ *(4-3)* **39.** $x = 1.66$ *(4-3)*
40. $x = 2.32$ *(4-5)* **41.** $x = 3.92$ *(4-5)* **42.** $x = 92.1$ *(4-5)*
43. $x = 2.11$ *(4-5)* **44.** $x = 0.881$ *(4-5)* **45.** $x = 300$ *(4-5)*
46. $x = 2$ *(4-5)* **47.** $x = 1$ *(4-5)* **48.** $x = \dfrac{3 + \sqrt{13}}{2}$ *(4-5)*
49. $x = 1, 10^3, 10^{-3}$ *(4-5)* **50.** $x = 10^e$ *(4-5)* **51.** $e^{-x} - 1$ *(4-1)*
52. $2 - 2e^{-2x}$ *(4-1)*
53.



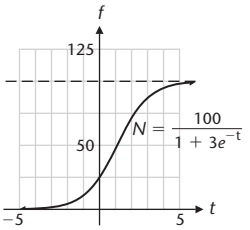
Domain $= (-\infty, \infty)$; range $= (0, \infty)$; y intercept: 0.5; horizontal asymptote: $y = 0$ *(4-1)*
54.



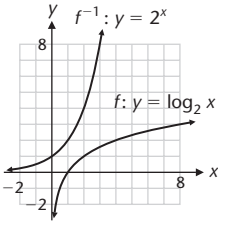
Domain $= (-\infty, \infty)$; range $= (0, \infty)$; y intercept: 10; horizontal asymptote: $y = 0$ *(4-1)*
55.



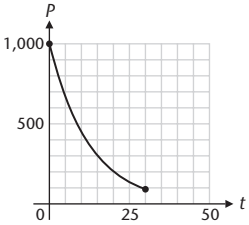
Domain $= (1, \infty)$; range $= (-\infty, \infty)$; x intercept: 2; vertical asymptote: $x = 1$ *(4-3)*
56.



Domain $= (-\infty, \infty)$; range $= (0, 100)$; y intercept: 25; horizontal asymptote: $y = 0$ and $y = 100$ *(4-1)*
57. Reflected through $y = x$: $y = \ln x$; reflected through the x axis: $y = -e^x$; reflected through the y axis: $y = e^{-x}$. *(4-1)*
58. 0.018, 2.187 *(4-5)* **59.** (1.003, 0.010), (3.653, 4.502) *(4-5)*
60. $I = I_0(10^{D/10})$ *(4-5)* **61.** $x = \pm\sqrt{-2 \ln (\sqrt{2\pi}y)}$ *(4-5)*
62. $I = I_0(e^{-kx})$ *(4-5)*
63. $n = -\dfrac{\ln\left(1 - \dfrac{Pi}{r}\right)}{\ln (1 + i)}$ *(4-5)*
64. (A) $y = e^{-x/3}$ is decreasing while $y = 4 \ln (x + 1)$ is increasing without bound. (B) 0.258 *(4-5)* **65.** $y = ce^{-5t}$ *(4-5)*
66.



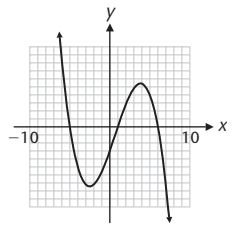
Domain $f = (0, \infty) =$ Range f^{-1}
Range $f = (-\infty, \infty) =$ Domain f^{-1} *(4-3)*
67. If $\log_1 x = y$, then we would have to have $1^y = x$; that is, $1 = x$ for arbitrary positive x, which is impossible. *(4-3)*
69. 23.4 years *(4-2)* **70.** 23.1 years *(4-2)*
71. 37,100 years *(4-2)* **72.** (A) $N = 2^{2t}$ (or $N = 4^t$)
(B) 15 days *(4-2)* **73.** 1.5×10^{26} dollars *(4-2)*
74. (A) (B) 0



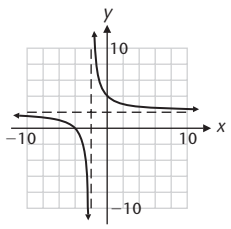
(4-1)

75. 6.6 *(4-4)* **76.** $10^{16.85}$ or 7.08×10^{16} joules *(4-4)*
77. The level of the louder sound is 50 decibels more. *(4-4)*
78. $k = 0.00942$, $d = 489$ feet *(4-4)* **79.** 3 years *(4-2)*
80. (A) $y = 43.3(1.09)^x$; 2010: \$569 billion; 2020: \$1,360 billion
(B) 2015 *(4-2)*
81. (A) $y = -21{,}800 + 5{,}153 \ln x$; 1996: 1,720 million bushels; 2010: 2,422 million bushels *(4-4)*

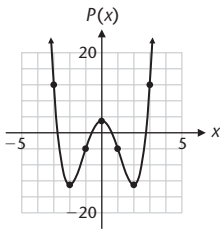
Chapter 3&4 Cumulative Review
1. (A) $P(x) = (x + 1)^2(x - 1)(x - 2)$
(B) $P(x) \to \infty$ as $x \to \infty$ and as $x \to -\infty$ *(3-1)*
2.



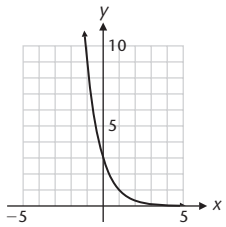
(3-1)
3. (A) m (B) g (C) n (D) f *(4-1)*
4. $3x^3 + 5x^2 - 18x - 3 = (x + 3)(3x^2 - 4x - 6) + 15$ *(3-2)*
5. $-2, 3, 5$ *(3-1)* **6.** $P(1) = -5$ and $P(2) = 5$ are of opposite sign. *(3-3)* **7.** $1, 2, -4$ *(3-4)* **8.** (A) $x = \log y$ (B) $x = e^y$ *(4-3)*
9. (A) $8e^{3x}$ (B) e^{5x} *(4-1)* **10.** (A) 9 (B) 4 (C) $\frac{1}{2}$ *(4-3)*
11. (A) 0.371 (B) 11.4 (C) 0.0562 (D) 15.6 *(4-3)*
12. $E = k\frac{p}{x^3}$ *(3-6)* **13.** $F = k\frac{q_1 q_2}{r^2}$ *(3-6)*
14. The graph of a nonconstant polynomial has no horizontal asymptote. *(3-1)* **15.** The graph does not approach the horizontal asymptote as $x \to -\infty$. *(3-5)* **16.** $f(x) = 3 \ln x - \sqrt{x}$ *(4-3)*
17. The function f multiplies the base e raised to power of one-half the domain element by 100 and then subtracts 50. *(4-1)*
18. (A) Domain: $x \neq -2$; x intercept: $x = -4$; y intercept: $y = 4$
(B) Vertical asymptote: $x = -2$; horizontal asymptote: $y = 2$
(C)



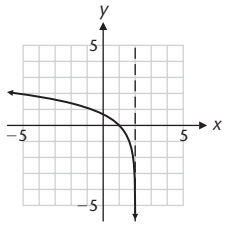
(3-5)
19. $0, -4, \pm 2i$; 0 and -4 are x intercepts *(3-1)*
20. $[-4, 0]$ *(3-3)* **21.** $P\left(\frac{1}{2}\right) = \frac{5}{2}$ *(3-2)*
22. $-2/3$ and 4 *(3-2)* **23.** (B) $x + 1$ *(3-2)*
24. (A) The graph of $P(x)$ has four x intercepts and three turning points; $P(x) \to \infty$ as $x \to \infty$ and as $x \to -\infty$



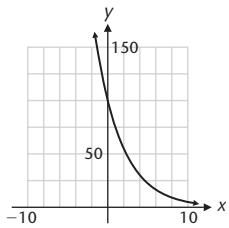
(3-1)
(B) 2.76
25. (A) -0.56 and 3.56 must be double zeros and 2 must be a simple zero. *(3-3)* **26.** (A) Upper bound: 4; lower bound: -6
(B) Four intervals (C) $-5.68, 3.80$ *(3-3)*
27. $3, 1 \pm \frac{1}{2}i$ *(3-4)*
28. $P(x) = (x + 1)(x + 4)(x^2 - 3) = (x + 1)(x + 4)(x - \sqrt{3})(x + \sqrt{3})$. The four zeros are $-1, -4, \pm\sqrt{3}$. *(3-4)*
29. $x = 4, -2$ *(4-5)* **30.** $x = \ln 6/\ln 3$ *(4-5)*
31. $x = \frac{1}{2}, -1$ *(4-5)* **32.** $x = 2.5$ *(4-5)* **33.** $x = 10$ *(4-5)*
34. $x = \frac{1}{27}$ *(4-5)* **35.** $x = 5$ *(4-5)* **36.** $x = 7$ *(4-5)*
37. $x = 5$ *(4-5)* **38.** $x = e^{1/10}$ *(4-5)* **39.** $x = 1, e^{1/2}$ *(4-5)*
40. $x = 3.38$ *(4-5)* **41.** $x = 4.26$ *(4-5)* **42.** $x = 2.32$ *(4-5)*
43. $x = 3.67$ *(4-5)* **44.** $x = 0.549$ *(4-5)*
45.



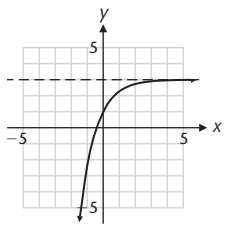
Domain: $(-\infty, \infty)$; range: $(0, \infty)$; y intercept: 3; horizontal asymptote: $y = 0$ *(4-1)*
46.



Domain: $(-\infty, 2)$; range: $(-\infty, \infty)$; x intercept: 1; y intercept: ln 2; vertical asymptote: $x = 2$ *(4-3)*
47.

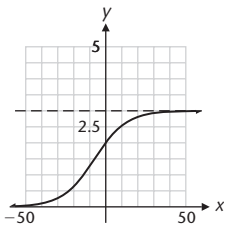


Domain: $(-\infty, \infty)$; range: $(0, \infty)$; y intercept: 100; horizontal asymptote: $y = 0$ *(4-1)*
48.



Domain: $(-\infty, \infty)$; range: $(-\infty, 3)$; x intercept: -0.41; y intercept: 1; horizontal asymptote: $y = 3$ *(4-1)*

49.



Domain: $(-\infty, \infty)$; range: $(0, 3)$; y intercept: 2; horizontal asymptotes: $y = 0$ and $y = 3$ *(4-1)*

50. Reflected in the line $y = x$: $y = e^x$; reflected in the x axis: $y = -\ln x$; reflected in the y axis: $y = \ln(-x)$ *(4-3)*

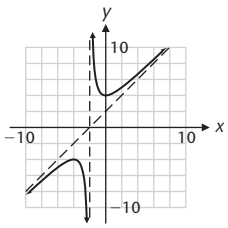
51. (A) For $x > 0$, $y = e^{-x}$ decreases from 1 to 0, while $\ln x$ increases from $-\infty$ to ∞. Consequently, the graphs can intersect at exactly one point. (B) 1.31 *(4-3)*

52. (A) $(x^2 + 3)(x^2 + 6)$
(B) $(x + i\sqrt{3})(x - i\sqrt{3})(x + i\sqrt{6})(x - i\sqrt{6})$ *(3-4)*

53. (A) $(x + 5)(x - 5)(x^2 + 2)$
(B) $(x + 5)(x - 5)(x + i\sqrt{2})(x - i\sqrt{2})$ *(3-4)*

54. $G = 19.6$ *(3-6)* **55.** $H = 48$ *(3-6)*

56.



(3-5)

Vertical asymptote: $x = -2$;
Slant asymptote: $y = x + 2$

57. $(-\infty, -1] \cup [0, 1] \cup (2, \infty)$ *(3-5)*

58. Zeros: 2.97, 3; local minimum: $P(2.98) \approx -0.02$; local maximum: $P(7.03) \approx 264.03$; local minimum: $P(10.98) \approx 15.98$ *(3-3)*

59. $P(x) = (x + 1)^2x^3(x - 3 - 5i)(x - 3 + 5i)$; degree 7 *(3-4)*

60. Yes, for example:
$P(x) = (x + i)(x - i)(x + \sqrt{2})(x - \sqrt{2}) = x^4 - x^2 - 2$ *(3-4)*

61. (A) Upper bound: 20; lower bound: -30 (B) -26.98, -6.22, 7.23, 16.67 *(3-3)*

62. 2, -1 (double), and $2 \pm i\sqrt{2}$;
$P(x) = (x - 2)(x + 1)^2(x - 2 - i\sqrt{2})(x - 2 + i\sqrt{2})$ *(3-4)*

63. -2 (double), -1.88, 0.35, 1.53 *(3-4)*

64. $f(x) = \dfrac{3(x - 5)(x - 8)}{(x - 1)^2}$ *(3-5)* **65.** $n = \dfrac{\ln\left(1 + \frac{4i}{P}\right)}{\ln(1 + i)}$ *(4-5)*

66. $y = Ae^{5x}$ *(4-5)* **67.** $x = \ln\left(y + \sqrt{y^2 + 2}\right)$ *(4-5)*

68. $(-\infty, -1) \cup (-0.535, 1) \cup (1.869, \infty)$ *(3-5)* **69.** (A) 6.1 hours (B) 4.3 or 7.6 hours (C) 1.5 to 8.9 hours *(3-1, 3-3)*

70. (A) 390 seconds (B) $y = 90$; the student's best time will eventually approach 90 seconds. (C) It occurs at $w = -2$; only positive values of w are relevant. *(3-5)*

71. $x = 2$ feet and $y = 2$ feet, or $x = 1.28$ feet and $y = 4.88$ feet *(3-4)* **72.** 1.79 feet by 3.35 feet *(3-4)*

73. $t^2 = kd^3$ *(3-6)* **74.** 1.2 miles per second *(3-6)*

75. (A) 69.8 million (B) 148 million *(3-6)* **76.** 10.2 years *(4-1)*

77. 9.90 years *(4-1)* **78.** 63.1 times as powerful *(4-4)*

79. 6.31×10^{-4} w/m^2 *(4-4)* **80.** (A) 78.9 years (B) 78.0 years (C) 79.1 years (D) 79.1 years *(3-1, 4-2)* **81.** Quadratic *(3-1, 4-2)*

Chapter 5

Section 5-1

7. 40° **9.** 270° **11.** 405° **13.** 6 **15.** 2.5 **17.** 0.05 **19.** $\frac{\pi}{4}$

21. $\frac{3\pi}{2}$ **23.** $\frac{13\pi}{6}$ **25.** $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$

27. $-\frac{\pi}{4}, -\frac{\pi}{2}, -\frac{3\pi}{4}, -\pi$ **29.** $\frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}, 2\pi$

31. 60°, 120°, 180°, 240°, 300°, 360°

33. −90°, −180°, −270°, −360° **35.** 36°, 72°, 108°, 144°, 180°

37. T **39.** T **41.** F **43.** 38.683° **45.** 5.859° **47.** 354.141°

49. 27°36′ **51.** 3°2′31″ **53.** 403°13′23″ **55.** 1.117

57. 1.892 **59.** 0.234 **61.** 53.29° **63.** 64.74° **65.** −134.65°

67. Quadrant III **69.** Quadrant II **71.** Quadrant III

73. Quadrantal angle **75.** Quadrant IV **77.** Quadrant IV

79. Quadrant II **81.** Quadrantal angle **83.** Quadrant II

85. Quadrant III

87. A central angle of radian measure 1 is an angle subtended by an arc of the same length as the radius of the circle.

89. 510° **91.** 280° **93.** −870°, −510°

95. $\frac{9\pi}{4}, \frac{17\pi}{4}$ **97.** $\frac{5\pi}{6}, \frac{17\pi}{6}, \frac{29\pi}{6}$ **99.** $-\frac{3\pi}{2}, \frac{\pi}{2}$

105. 20.94 radians/second; 62.83 feet/second

107. $\frac{7\pi}{4}$ radians **109.** 200 radians

111. $\frac{\pi}{26} = 0.12$ radians **113.** 12

115. Front wheel: 138.9 radians/second; back wheel: 92.6 radians/second

117. 1,102 nautical miles **119.** 510 knots; 8.5° per hour

121. 24,000 miles

123. The 7.5° angle and θ have a common side. (An extended vertical pole in Alexandria will pass through the center of the Earth.) The sun's rays are essentially parallel when they arrive at the Earth. Thus, the other two sides of the angles are parallel, because a sun ray to the bottom of the well, when extended, will pass through the center of the Earth. From geometry we know that the alternate interior angles made by a line intersecting two parallel lines are equal. Therefore, $\theta = 7.5°$.

125. 865,000 miles **127.** 33 feet

Section 5-2

7. $(0, -1)$ **9.** $(1, 0)$ **11.** $(1/\sqrt{2}, 1/\sqrt{2})$ **13.** $(\sqrt{3}/2, 1/2)$

15. $(1/2, -\sqrt{3}/2)$ **17.** $(1/2, -\sqrt{3}/2)$ **19.** $(-1/\sqrt{2}, -1/\sqrt{2})$

21. $(-1/\sqrt{2}, -1/\sqrt{2})$ **23.** -1 **25.** 1 **27.** $\sqrt{2}$ **29.** $1/\sqrt{3}$

31. $-\sqrt{3}/2$ **33.** $2/\sqrt{3}$ **35.** $-1/\sqrt{2}$ **37.** 1

39. $\frac{12}{13}$ **41.** $\frac{13}{5}$ **43.** $-\frac{15}{8}$ **45.** $\frac{17}{8}$ **47.** 0 **49.** Undefined **51.** 0

53. $\frac{3}{4}$ **55.** $\frac{41}{9}$ **57.** $-\frac{8}{15}$ **59.** $-\frac{41}{9}$ **61.** $-\frac{1}{\sqrt{5}}$

63. Quadrants II and III **65.** Quadrants I and II

67. Quadrants II and IV **69.** −0.5573 **71**. −14.60 **73.** 1.000

75. 0.8138 **77.** 0.5290 **79.** 0.4226 **81.** −1.573 **83.** 0.8439

85. −0.3363 **87.** 0.9174 **89.** F **91.** F **93.** T **95.** F **97.** F

99. F **101.** T **103.** T

105. Zeros: none; turning points: $(\pi, -1)$, $(2\pi, 1)$, $(3\pi, -1)$

107. Zeros: $0, \pi, 2\pi, 3\pi, 4\pi$; turning points: none

109. a, $-$; b, $\div$ **111.** a, $-$; b, $-$ **113.** a, $+$; b, $-$ **115.** a, $-$; b, $-$

117. a, $+$; b, $+$ **119.** 0; $2k\pi$, k any integer

121. $3\pi/4$; $3\pi/4 - 2k\pi$, k any integer

123. $W(x)$ represents the coordinates of a point on a unit circle that is $|x|$ units from (1,0), in a counterclockwise direction if x is positive and in a clockwise direction if x is negative. $W(x + 4\pi)$ has the same coordinates as $W(x)$, because we return to the same point every time we go around the unit circle any integer multiple of 2π units (the circumference of the circle) in either direction.
125. T **127.** F **129.** T **131.** (A) $\sin 0.4 = 0.4$ (B) $\cos 0.4 = 0.9$ (C) $\tan 0.4 = 0.4$
133. (A) $\sec 2.2 = -2$ (B) $\tan 5.9 = -0.4$ (C) $\cot 3.8 = 1$
135. $\sin x < 0$ in Quadrants III and IV; $\cot x < 0$ in Quadrants II and IV; therefore, both are true in Quadrant IV.
137. $\cos x < 0$ in Quadrants II and III; $\sec x > 0$ in Quadrants I and IV; therefore, it is not possible to have both true for the same value of x.
139. None **141.** $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ **143.** $\frac{\pi}{2}$ and $\frac{3\pi}{2}$
149. 75 square meters **151.** $12\sqrt{3} \approx 20.78$ square inches
153. $a_1 = 0.5$, $a_2 = 1.377583$, $a_3 = 1.569596$, $a_4 = 1.570796$, $a_5 = 1.570796$; $\frac{\pi}{2} \approx 1.570796$

Section 5-3

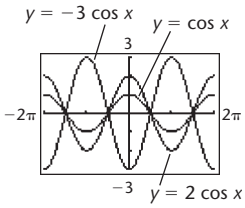
7. $\frac{8}{17}$ **9.** $\frac{17}{8}$ **11.** $\frac{8}{15}$
13. $\cos\theta$ **15.** $\sec\theta$ **17.** $\cot\theta$ **19.** 60.55° **21.** 82.90° **23.** 37.09°
25. $\alpha = 72.2°$, $a = 3.28$, $b = 1.05$
27. $\alpha = 46°40'$, $b = 116$, $c = 169$ **29.** $\beta = 67°0'$, $b = 127$, $c = 138$
31. $\beta = 36.79°$, $a = 31.85$, $c = 39.77$
33. $\beta = 54°40'$, $\alpha = 35°20'$, $c = 10.4$
35. $\beta = 52°30'$, $\alpha = 37.30'$, $a = 7.67$
37. F **39.** T **41.** F **43.** 26.6° **45.** 78.7° **47.** 63.4°
49. ± 0.36 **51.** ± 5.67 **53.** ± 0.11
55. (A) $\cos\theta = OA/1 = OA$ (B) Angle $OED = \theta$; $\cot\theta = DE/1 = DE$ (C) $\sec\theta = OC/1 = OC$
57. (A) As θ approaches 90°, $OA = \cos\theta$ approaches 0. (B) As θ approaches 90°, $DE = \cot\theta$ approaches 0. (C) As θ approaches 90°, $OC = \sec\theta$ increases without bound.
59. (A) As θ approaches 0°, $AD = \sin\theta$ approaches 0. (B) As θ approaches 0°, $CD = \tan\theta$ approaches 0. (C) As θ approaches 0°, $OE = \csc\theta$ increases without bound.
63. 228 feet **65.** 127.5 feet **67.** 2,225 miles **69.** 44°
71. 9.8 meters per second^2
73. (B)

θ	$C(\theta)$
10°	$368,222
20°	$363,435
30°	$360,622
40°	$360,146
50°	$360,050

75. 0.77 meter

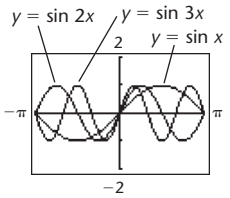
Section 5-4

7. 2π, π, 2π **9.** (A) 1 unit (B) Indefinitely far (C) Indefinitely far
11. (A) -2π, $-\pi$, 0, π, 2π (B) $-3\pi/2$, $-\pi/2$, $\pi/2$, $3\pi/2$ (C) No x intercepts
13. (A) None (B) $-3\pi/2$, $-\pi/2$, $\pi/2$, $3\pi/2$ (C) -2π, $-\pi$, 0, π, 2π
15. (A) No vertical asymptotes (B) $-3\pi/2$, $-\pi/2$, $\pi/2$, $3\pi/2$ (C) -2π, $-\pi$, 0, π, 2π
17. (A) A shift of $\pi/2$ to the left will transform the cosecant graph into the secant graph. [The answer is not unique—see part B.]
(B) The graph of $y = -\csc(x - \pi/2)$ is a $\pi/2$ shift to the right and a reflection in the x axis of the graph of $y = \csc x$. The result is the graph of $y = \sec x$.
19. 60° **21.** $\frac{\pi}{6}$ **23.** $\frac{\pi}{3}$ **25.** 10°
27. $\cos\theta = -\frac{4}{5}$, $\tan\theta = -\frac{3}{4}$, $\sec\theta = -\frac{5}{4}$, $\cot\theta = -\frac{4}{3}$, $\csc\theta = \frac{5}{3}$
29. $\sin\theta = -\frac{2}{3}$, $\sec\theta = -\frac{3}{\sqrt{5}}$, $\tan\theta = \frac{2}{\sqrt{5}}$, $\cot\theta = \frac{\sqrt{5}}{2}$, $\csc\theta = -\frac{3}{2}$
31. $\sin\theta = -\frac{1}{\sqrt{5}}$, $\cos\theta = -\frac{2}{\sqrt{5}}$, $\csc\theta = -\sqrt{5}$, $\sec\theta = -\frac{\sqrt{5}}{2}$, $\cot\theta = 2$
33. $\sin\theta = \frac{-\sqrt{99}}{10}$, $\cos\theta = \frac{1}{10}$, $\tan\theta = -\sqrt{99}$, $\csc\theta = -\frac{10}{\sqrt{99}}$, $\cot\theta = -\frac{1}{\sqrt{99}}$
35. $\sin\theta = \frac{4}{5}$, $\csc\theta = \frac{5}{4}$, $\cos\theta = \frac{3}{5}$, $\sec\theta = \frac{5}{3}$, $\tan\theta = \frac{4}{3}$, $\cot\theta = \frac{3}{4}$
37. $\sin\theta = \frac{\sqrt{3}}{2}$, $\csc\theta = \frac{2}{\sqrt{3}}$, $\cos\theta = -\frac{1}{2}$, $\sec\theta = -2$, $\tan\theta = -\sqrt{3}$, $\cot\theta = -\frac{1}{\sqrt{3}}$
39. $\sin\theta = \frac{15}{113}$, $\cos\theta = \frac{112}{113}$, $\tan\theta = \frac{15}{112}$, $\csc\theta = \frac{113}{15}$, $\sec\theta = \frac{113}{112}$, $\cot\theta = \frac{112}{15}$
41. $\sin\theta = -\frac{2}{\sqrt{13}}$, $\cos\theta = \frac{3}{\sqrt{13}}$, $\tan\theta = -\frac{2}{3}$, $\csc\theta = -\frac{\sqrt{13}}{2}$, $\sec\theta = \frac{\sqrt{13}}{3}$, $\cot\theta = -\frac{3}{2}$
43. Even **45.** Even **47.** Odd **49.** Odd
51. 120° or $\frac{2\pi}{3}$ radians **53.** 210° or $\frac{7\pi}{6}$ radians
55. 240° or $\frac{4\pi}{3}$ radians
57. Tangent and secant, because $\tan\theta = b/a$ and $\sec\theta = r/a$ and $a = 0$ if $P = (a, b)$ is on the vertical axis (division by zero is not defined).
59. 150°, 210°
61. $\frac{\pi}{4}$, $\frac{5\pi}{4}$ **63.** F **65.** T **67.** F **69.** F **71.** T
73. $f(x) = b$, b any real number
75. $f(x) = b$, b any real number
77. (A)



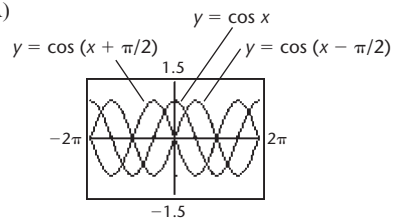
(B) No
(C) 1 unit; 2 units; 3 units
(D) The deviation of the graph from the x axis is changed by changing A. The deviation appears to be $|A|$.

79. (A)



(B) 1; 2; 3
(C) n

81. (A)



(B) The graph of $y = \cos x$ is shifted $|C|$ units to the right if $C < 0$ and $|C|$ units to the left if $C > 0$.

83. For each case, the number is not in the domain of the function and an error message of some type will appear.

85. (A) For x close to 0, the two graphs are almost identical.
(B)

x	−0.3	−0.2	−0.1	0.0	0.1	0.2	0.3
$\sin x$	−0.296	−0.199	−0.100	0.000	0.100	0.199	0.296

87. (A) 1.75 radians (B) (−0.713, 3.936)
89. 2π units **91.** k, $0.866k$, $0.5k$
95. (A) 3.31371, 3.14263, 3.14160, 3.14159 (B) $\pi = 3.1415926\ldots$
97. (A) 44.07; −0.32 (B) $y = -0.33x - 1.28$.

Section 5-5

7. $A = 3, P = 2\pi$ **9.** $A = \frac{1}{2}, P = 2\pi$ **11.** $A = 1, P = \dfrac{2\pi}{3}$

13. Period $= \dfrac{\pi}{4}$ **15.** Period $= \dfrac{1}{8}$ **17.** Period $= 4\pi$

19. $A = 1, P = \dfrac{2\pi}{\pi} = 2$; zeros: $-2, -1, 0, 1, 2$

21. Period $= 2\pi$; zeros: $\pi, 3\pi$

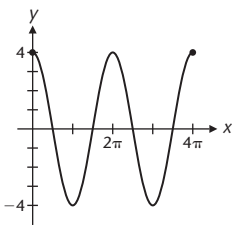
23. $A = 3, P = \dfrac{2\pi}{\pi} = \pi$; turning points: $\left(-\dfrac{\pi}{2}, -3\right), (0, 3), \left(\dfrac{\pi}{2}, -3\right)$

25. Period $= 2$; turning points: (0, 2) (1, −2), (2, 2)

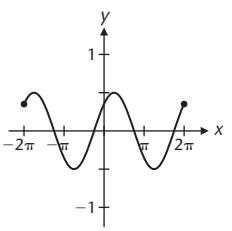
27. $y = 3 \sin 4x$ **29.** $y = -10 \sin \pi x$

31. $y = 5 \cos \frac{1}{4}x$ **33.** $y = -0.5 \cos \dfrac{\pi x}{4}$

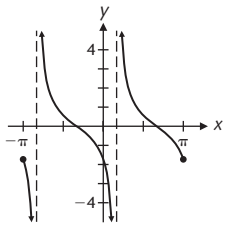
35. Amplitude: 4: Period: 2π; phase shift: 0



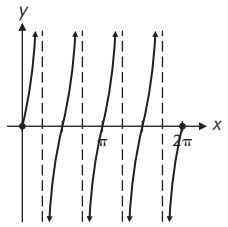
37. Amplitude: $\frac{1}{2}$; period: 2π; phase shift: $-\dfrac{\pi}{4}$



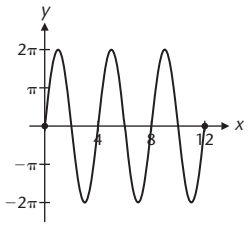
39. Period: π; phase shift: $\dfrac{\pi}{6}$



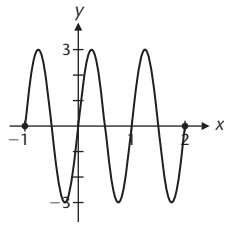
41. Period: $\dfrac{\pi}{2}$; phase shift: 0



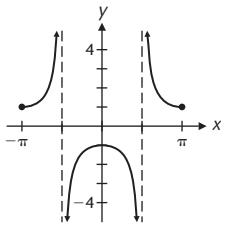
43. Amplitude: 2π; period: 4; phase shift: 0



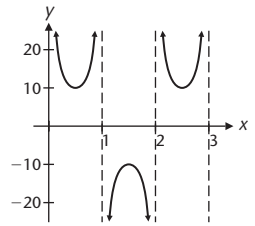
45. Amplitude: 3; period: 1; phase shift: $-\frac{1}{2}$



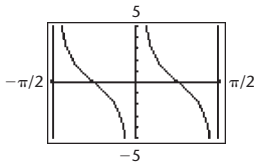
47. Period: 2π; phase shift: $-\pi$



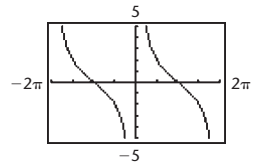
49. Period: 2; phase shift: 0



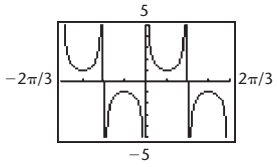
51. T **53.** F **55.** T **57.** T **59.** T **61.** F **63.** $y = \cos 2x$
65. $y = 1 - \cos 2x$
67. $y = 2 \cot 2x$



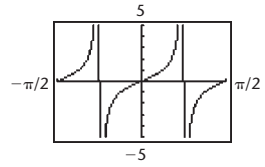
69. $y = \cot (x/2)$



71. $y = \csc 3x$



73. $y = \tan 2x$



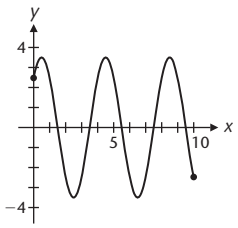
75. $y = -4 \sin\left(\dfrac{\pi}{2}x - \dfrac{\pi}{2}\right)$

77. $y = \dfrac{1}{2} \cos\left(\dfrac{x}{4} - \dfrac{3\pi}{4}\right)$

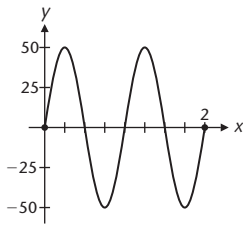
79. $y = 3 \cos\left(\dfrac{\pi}{6}x + \dfrac{\pi}{6}\right)$

81. $y = -2 \sin\left(\dfrac{\pi}{6}x - \dfrac{\pi}{6}\right)$

83. $A = 3.5, P = 4$, phase shift $= -0.5$



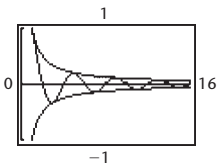
85. $A = 50, P = 1$, phase shift $= 0.25$



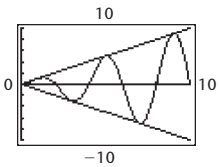
87. $y = 2 \sin (x + 0.785)$ **89.** $y = 2 \sin (x - 0.524)$

91. $y = 5 \sin (2x - 0.284)$

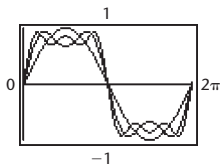
93. The amplitude is decreasing with time. This is often referred to as a *damped sine wave*. Example are a car's vertical motion, which is damped by the suspension system after the car goes over a bump, and the slowing down of a pendulum that is released away from the vertical line of suspension (air resistance and friction).



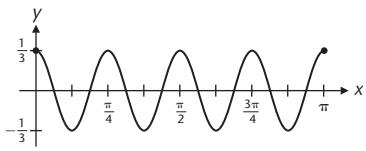
95. The amplitude is increasing with time. In physical and electrical systems this is referred to as *resonance*. Some example are the swinging of a bridge during high winds and the movement of tall buildings during an earthquake. Some bridges and buildings are destroyed when the resonance reaches the elastic limits of the structure.



97.

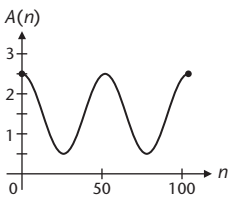


99. $A = \dfrac{1}{3}, P = \dfrac{2\pi}{8} = \dfrac{\pi}{4}$

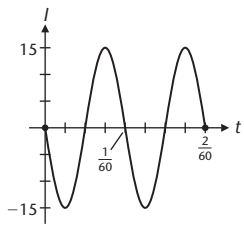


101. $y = -8 \cos 4\pi t$

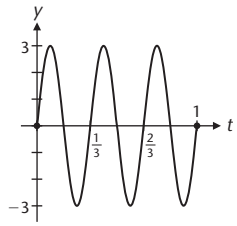
103. The graph shows the seasonal changes of sulfur dioxide pollutant in the atmosphere; more is produced during winter months because of increased heating.



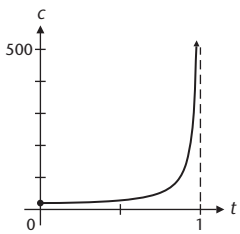
105. (A) $A = 15, P = \frac{1}{60}$, phase shift $= -\frac{1}{240}$ (C) $\frac{1}{240}$ (B)



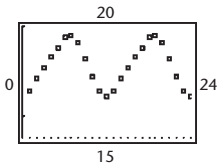
107. $A = 3, P = \frac{1}{3}$



109. (A) $c = 20 \sec (\pi t/2), [0, 1)$ (C) The length of the light beam starts at 20 feet and increases slowly at first, then increases rapidly without end. (B)

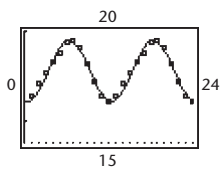


111. (A)

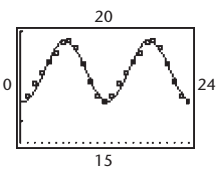


(B) $y = 18.22 + 1.37 \sin\left(\dfrac{\pi x}{6} - 1.75\right)$

(C)



(D)



Section 5-6

7. $\dfrac{\pi}{2}$ **9.** $\dfrac{\pi}{3}$ **11.** $\dfrac{\pi}{3}$ **13.** $\dfrac{\pi}{4}$ **15.** 0 **17.** $\dfrac{\pi}{6}$ **19.** 1.144 **21.** 1.561

23. Not defined **25.** $\dfrac{\pi}{4}$ **27.** $\dfrac{\pi}{3}$ **29.** 25 **31.** 2.3 **33.** $\frac{1}{2}$ **35.** $-\sqrt{2}$

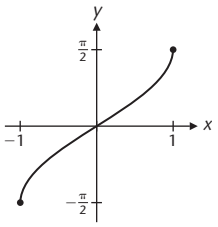
37. 0 **39.** $2\pi/3$ **41.** -1.472 **43.** -0.9810 **45.** 2.645 **47.** $-45°$

49. $-60°$ **51.** $180°$ **53.** $43.51°$ **55.** $-21.48°$ **57.** $-89.93°$

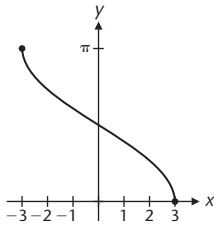
59. $\sin^{-1} (\sin 2) = 1.416 \neq 2$. For the identity $\sin^{-1} (\sin x) = x$ to hold, x must be in the restricted domain of the sine function; that is, $-\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$. The number 2 is not in the restricted domain.

61. T **63.** F **65.** T **67.** T

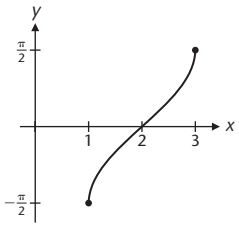
69.



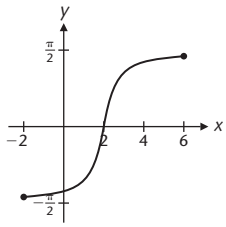
71.



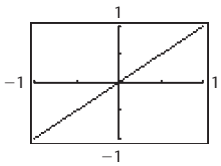
73.



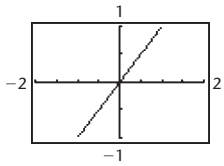
75.



77. (A)



(B) The domain of $\cos^{-1}$ is restricted to $-1 \le x \le 1$; therefore no graph will appear for other x.



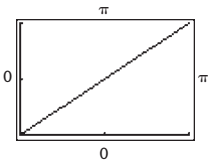
79. $\sqrt{1 - x^2}$ **81.** $\dfrac{1}{\sqrt{1 + x^2}}$

83. $f^{-1}(x) = -2 + \sin^{-1} x, -1 \le x \le 1$

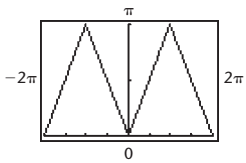
85. $f^{-1}(x) = \cos^{-1}\left(\dfrac{x - 5}{2}\right), 3 \le x \le 7$

87. $f^{-1}(x) = 3 + \cos^{-1}\left(\dfrac{x - 4}{2}\right), 2 \le x \le 6$

89. (A)

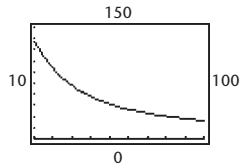


(B) The domain for $\cos x$ is $(-\infty, \infty)$ and the range is $[-1, 1]$, which is the domain for $\cos^{-1} x$. Therefore, $y = \cos^{-1}(\cos x)$ has a graph over the interval $(-\infty, \infty)$, but $\cos^{-1}(\cos x) = x$ only on the restricted domain of $\cos x$, $[0, \pi]$.

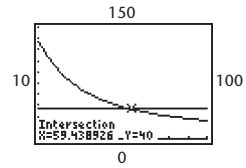


91. 75.38°; 24.41°

93. (A)

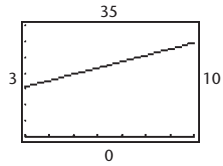


(B) 59.44 mm

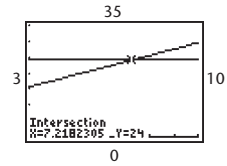


95. 21.59 inches

97. (A)



(B) 7.22 inches



99. (B) 76.10 feet

Chapter 5 Review Exercises

1. 2.5 radians *(5-1)* **2.** 7.5 centimeters *(5-1)*

3. $\alpha = 54.8°$, $a = 16.5$ feet $b = 11.6$ feet *(5-3)*

4. (A) $\dfrac{\pi}{3}$ (B) 60° (C) $\dfrac{\pi}{6}$ (D) 30° *(5-4)*

5. (A)III, IV (B) II, III (C) II, IV *(5-2)*

6. (A) $\frac{12}{13}$ (B) $-\frac{13}{5}$ (C) $-\frac{12}{5}$ *(5-2)* **7.** (A) $-\frac{3}{5}$ (B) $\frac{5}{4}$ (C) $-\frac{4}{3}$ *(5-4)*

8.

θ°	θ rad	sin θ	cos θ	tan θ	csc θ	sec θ	cot θ
0°	0	0	1	0	ND*	1	ND
30°	π/6	1/2	$\sqrt{3}/2$	$1/\sqrt{3}$	2	$2/\sqrt{3}$	$\sqrt{3}$
45°	π/4	$1/\sqrt{2}$	$1/\sqrt{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	π/3	$\sqrt{3}/2$	1/2	$\sqrt{3}$	$2/\sqrt{3}$	2	$1/\sqrt{3}$
90°	π/2	1	0	ND	1	ND	0
180°	π	0	−1	0	ND	−1	ND
270°	3π/2	−1	0	ND	−1	ND	0
360°	2π	0	1	0	ND	1	ND

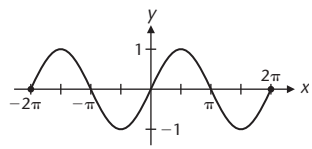
*ND = not defined *(5-1, 5-2)*

9. (A) 2π (B) 2π (C) π *(5-4)*

10. (A) Domain = $(-\infty, \infty)$, range = $[-1, 1]$

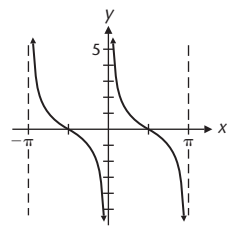
(B) Domain is set of all real number except $x = \dfrac{2k + 1}{2}\pi$, k an integer, range = all real numbers *(5-4)*

11.



(5-4)

12.



(5-4)

13. The central angle in a circle subtended by an arc of half the length of the radius. *(5-1)*

14. If the graph of $y = \sin x$ is shifted $\dfrac{\pi}{2}$ units to the left, the result will be the graph of $y = \cos x$. *(5-4)*

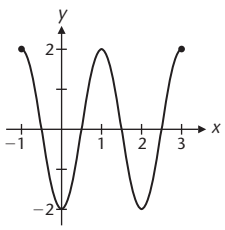
15. 132.878° *(5-1)* **16.** 788°30′16″ *(5-1)* **17.** 1.125 *(5-1)*
18. 78.50° *(5-1)*
19. $\alpha = 49.7°$; $\beta = 40.3°$; $c = 20.6$ centimeters *(5-3)*
20. (A) II (B) Quadrantal (C) III **21.** (A) and (C) *(5-1)*
22. (B) and (C) *(5-2)*
23. (A) $\frac{\pi}{2}, \frac{3\pi}{2}$ (B) 0, π (C) 0, π *(5-4)*

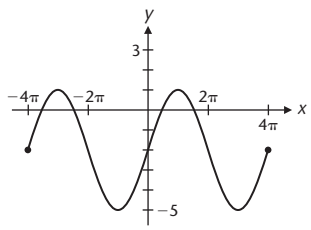
24. Because the coordinates of a point on a unit circle are given by $P = (a, b) = (\cos x, \sin x)$, we evaluate $P = (\cos(-8.305), \sin(-8.305))$—using a calculator set in radian mode—to obtain $P = (-0.436, -0.900)$. Note that $x = -8.305$, because P is moving clockwise. The quadrant in which $P = (a, b)$ lies can be determined by the signs of a and b. In this case, P is in the third quadrant, because a is negative and b is negative. *(5-1, 5-2)*

25. $\sin\theta = -\frac{4}{\sqrt{17}}$, $\cos\theta = -\frac{1}{\sqrt{17}}$, $\csc\theta = -\frac{\sqrt{17}}{4}$, $\sec\theta = -\sqrt{17}$, $\cot\theta = \frac{1}{4}$ *(5-4)*

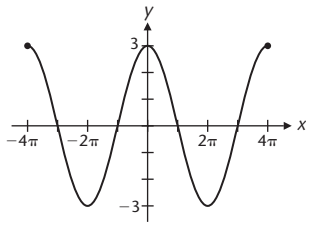
26. Zeros: $-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$; turning points: $(-\pi, -3)$, $(0, 3)$, $(\pi, -3)$ *(5-2)*

27. 0 *(5-2)* **28.** Not defined *(5-2)* **29.** 0 *(5-6)*
30. $-\frac{1}{\sqrt{2}}$ or $-\frac{\sqrt{2}}{2}$ *(5-2)* **31.** $\frac{\pi}{4}$ *(5-6)*
32. $\frac{2}{\sqrt{3}}$ or $\frac{-2\sqrt{3}}{3}$ *(5-2)* **33.** $\frac{\pi}{3}$ *(5-6)* **34.** $-\frac{1}{2}$ *(5-2)*
35. $-\frac{\pi}{4}$ *(5-6)* **36.** $-\frac{1}{\sqrt{3}}$ or $-\frac{\sqrt{3}}{3}$ *(5-2)* **37.** $-\frac{\pi}{6}$ *(5-6)*
38. $\frac{5\pi}{5}$ *(5-6)* **39.** 0.33 *(5-6)* **40.** $-\sqrt{2}$ *(5-6)* **42.** $-\frac{4}{3}$ *(5-6)*
43. 0.4431 *(5-2)* **44.** −15.17 *(5-2)* **45.** −2.077 *(5-2)*
46. −0.9750 *(5-5)* **47.** Not defined *(5-5)* **48.** 1.557 *(5-6)*
49. 1.096 *(5-6)* **50.** Not defined *(5-6)*
51. (A) $\theta = -30°$ (B) $\theta = 120°$ *(5-6)*
52. (A) $\theta = 151.20°$ (B) $\theta = 82.28°$ *(5-6)*
53. $\cos^{-1}[\cos(-2)] = 2$. For the identity $\cos^{-1}(\cos x) = x$ to hold, x must be in the restricted domain of the cosine function; that is, $0 \le x \le \pi$. The number -2 is not in the restricted domain. *(5-6)*
54. $A = 2, P = 2$ *(5-5)* **55.** *(5-5)*

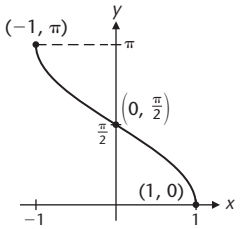




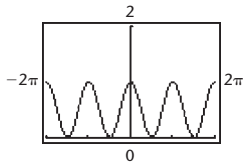
56. $y = 6\cos 2x$; $-\frac{\pi}{2} \le x \le \pi$ *(5-5)*
57. $y = -0.5\sin \pi x$; $-1 \le x \le 2$ *(5-5)*
58. If the graph of $y = \tan x$ is shifted $\frac{\pi}{2}$ units to the right and reflected in the x axis, the result will be the graph of $y = \cot x$. *(5-4)*
59. (A) $\cos x$ (B) $\tan^2 x$ *(5-4)*
60. *(5-5)*



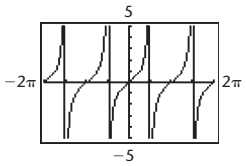
61. $A = 2$; $P = 4$; phase shift $= \frac{2}{3}$ *(5-5)*
62. Domain $= [-1, 1]$; range $= [0, \pi]$ *(6-6)*



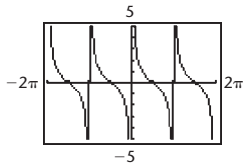
63. $y = \frac{1}{2}\cos 2x + \frac{1}{2}$ *(5-5)*



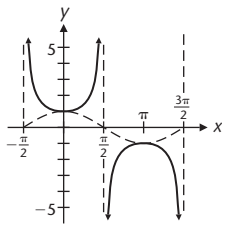
64. (A) $y = \tan x$



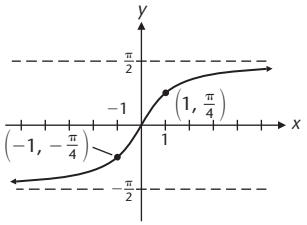
(B) $y = \cot x$ *(5-5)*



65. (A) Even (B) Neither *(5-4)* **66.** F *(5-3)*
67. T *(5-3)* **68.** (A) 2.5 radians (B) (−6.41, 4.79) *(5-1, 5-2)*
69. (A) $\frac{2\pi}{3}$ (B) $\frac{5\pi}{4}$ *(5-2)*
70. *(5-4)*



71. Domain = all real numbers; range = $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *(5-6)*



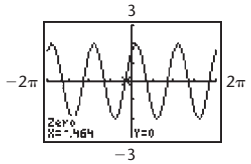
72. Phase shift = $-\frac{1}{2}$; period $P = 1$ *(5-5)*
73. Phase shift = $\frac{\pi}{2}$; period = 4π *(5-5)*
74. (A) Sine has origin symmetry. (B) Cosine has y axis symmetry. (C) Tangent has origin symmetry. *(5-4)*

75. $\dfrac{1}{\sqrt{1-x^2}}$ *(5-6)*

76. For each case, the number is not in the domain of the function and an error message of some type will appear. *(5-2, 5-6)*

77. $y = 2\sin\left(\pi x + \dfrac{\pi}{4}\right)$ *(5-5)*

78. $y = 2\sin(2x + 0.928)$ *(5-5)*



79. $\pm\frac{\pi}{6}, \pm\frac{\pi}{2}, \pm\frac{5\pi}{6}$ *(5-5)* **80.** $\frac{2\pi}{5}$ radians *(5-1)*

81. 28.3 centimeters *(5-2)*
82. 8.38 radians/second; 167.55 feet/second *(5-1)*
83. $I = 30\cos 120\pi t$ *(5-5)*

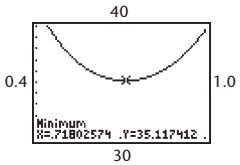
84. (A) $L = 10\csc\theta + 15\sec\theta$; $0 < \theta < \dfrac{\pi}{2}$

(B)

θ (radians)	0.4	0.5	0.6	0.7	0.8	0.9	1.0
L (feet)	42.0	38.0	35.9	35.1	35.5	39.6	39.6

To the nearest foot, 35 feet is the length of the longest log that can make the corner.

(C) Length of longest log that can make the corner is 35.1 feet.

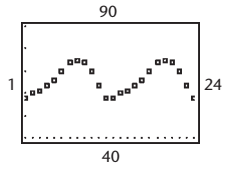


(D) Length L increases without bound. *(5-2, 5-3)*

85. (A) $R(t) = 4 - 3\cos\dfrac{\pi}{6}t$.

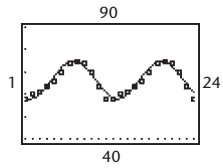
(B) The graph shows the seasonal changes in soft drink consumption. Most is consumed in August and the least in February. *(5-5)*

86. (A)



(B) $y = 66.5 + 8.5\sin\left(\dfrac{\pi}{6}x - 2.4\right)$

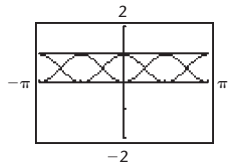
(C) *(5-5)*



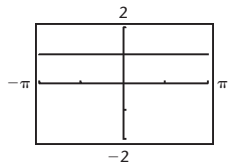
Chapter 6

Section 6-1

41.



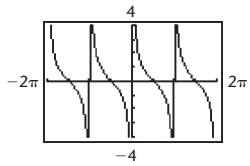
43.



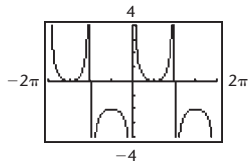
45. Yes **47.** No **49.** Yes **51.** No **53.** No **55.** No **57.** Yes **59.** Yes
91. Not an identity **93.** An identity **95.** Not an identity
97. An identity **99.** An identity **101.** Not an identity

109. (A) $\cot x = \dfrac{\cos x}{\sin x}$ (B) $\sin^2 x + \cos^2 x = 1$ (C) $\csc x = \dfrac{1}{\sin x}$

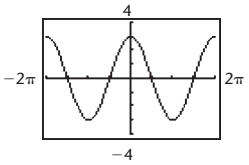
111. $g(x) = \cot x$



113. $g(x) = -1 + \csc x$



115. $g(x) = 3\cos x$



117. III, IV **119.** I, II **121.** All quadrants **123.** I, IV
125. $a\cos x$ **127.** $a\sec x$

Section 6-2

15. Yes **17.** No **19.** No **21.** Yes **27.** $\frac{1}{2}(\cos x - \sqrt{3}\sin x)$
29. $\sin x$ **31.** $\dfrac{\tan x + \sqrt{3}}{1 - \sqrt{3}\tan x}$ **33.** $\dfrac{\sqrt{3}-1}{2\sqrt{2}}$ **35.** $\dfrac{\sqrt{3}-1}{2\sqrt{2}}$
37. $\dfrac{1+\sqrt{3}}{1-\sqrt{3}}$ **39.** $\dfrac{-1-\sqrt{3}}{2\sqrt{2}}$ **41.** $\dfrac{\sqrt{3}}{2}$ **43.** 1

45. $\sin(x-y) = \dfrac{-3-4\sqrt{8}}{15}$; $\tan(x+y) = \dfrac{4\sqrt{8}-3}{4+3\sqrt{8}}$

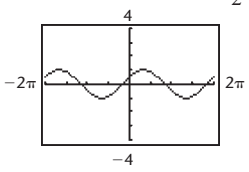
47. $\sin(x-y)=\dfrac{-2}{\sqrt{5}}$; $\tan(x+y)=\frac{2}{11}$

63. -0.3685, -0.3685; 0.9771, 0.9771

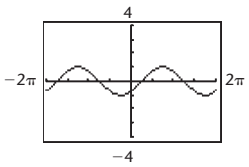
65. -0.4429, -0.4429; -2.682, -2.682

67. If $x=0$ and $y=0$, both sides are defined, but are not equal.

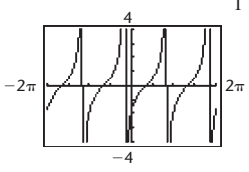
69. $y_1=\sin(x+\pi/6)$; $y_2=\dfrac{\sqrt{3}}{2}\sin x+\dfrac{1}{2}\cos x$



71. $y_1=\cos(x-3\pi/4)$; $y_2=-\dfrac{\sqrt{2}}{2}\cos x+\dfrac{\sqrt{2}}{2}\sin x$

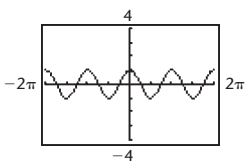


73. $y_1=\tan(x+2\pi/3)$; $y_2=\dfrac{\tan x-\sqrt{3}}{1+\sqrt{3}\tan x}$



75. $\frac{24}{25}$ **77.** $-\frac{1}{2}$ **79.** $xy+\sqrt{1-x^2}\sqrt{1-y^2}$

83. $y_1=\cos 1.2x\cos 0.8x-\sin 1.2x\sin 0.8x$; $y_2=\cos 2x$



89. (C) 3,510 feet

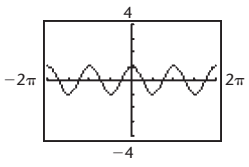
Section 6-3

7. $\frac{1}{2}=\frac{1}{2}$ **9.** $-\sqrt{3}=-\sqrt{3}$ **11.** $1=1$

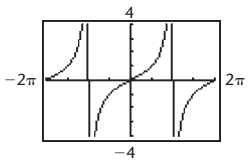
13. $\dfrac{\sqrt{2-\sqrt{2}}}{2}$ **15.** $\dfrac{\sqrt{2-\sqrt{2}}}{2}$ **17.** $\dfrac{1}{\sqrt{3}-2}$ **19.** $\dfrac{\sqrt{2+\sqrt{2}}}{2}$

21. $\dfrac{\sqrt{2-\sqrt{3}}}{2}$ **23.** $\dfrac{1}{\sqrt{3}-2}$

25.



27.



55. No **57.** Yes **59.** No

61. $\sin 2x=-\frac{24}{25}$, $\cos 2x=\frac{7}{25}$, $\tan 2x=-\frac{24}{7}$

63. $\sin 2x=-\frac{120}{169}$, $\cos 2x=\frac{119}{169}$, $\tan 2x=-\frac{120}{119}$

65. $\sin 2x=-\dfrac{4\sqrt{5}}{9}$, $\cos 2x=-\dfrac{1}{9}$, $\tan 2x=4\sqrt{5}$

67. $\sin 2x=\dfrac{4}{5}$, $\cos 2x=-\dfrac{3}{5}$, $\tan 2x=-\dfrac{4}{3}$

69. $\sin\dfrac{x}{2}=\sqrt{\dfrac{3+2\sqrt{2}}{6}}$, $\cos\dfrac{x}{2}=-\sqrt{\dfrac{3-2\sqrt{2}}{6}}$, $\tan\dfrac{x}{2}=-3-2\sqrt{2}$

71. $\sin\dfrac{x}{2}=-\dfrac{2\sqrt{5}}{5}$, $\cos\dfrac{x}{2}=\dfrac{\sqrt{5}}{5}$, $\tan\dfrac{x}{2}=-2$

73. $\sin\dfrac{x}{2}=\dfrac{5}{\sqrt{34}}$, $\cos\dfrac{x}{2}=-\dfrac{3}{\sqrt{34}}$, $\tan\dfrac{x}{2}=-\dfrac{5}{3}$

75. $\sin\dfrac{x}{2}=\sqrt{\dfrac{10-3\sqrt{10}}{20}}$, $\cos\dfrac{x}{2}=\sqrt{\dfrac{10+3\sqrt{10}}{20}}$,

$\tan\dfrac{x}{2}=\sqrt{\dfrac{10-3\sqrt{10}}{10+3\sqrt{10}}}$

77. (A) 2θ is a second quadrant angle, because θ is a first quadrant angle and $\tan 2\theta$ is negative for 2θ in the second quadrant and not for 2θ in the first. (B) Construct a reference triangle for 2θ in the second quadrant with $(a, b)=(-3,4)$. Use the Pythagorean theorem to find $r=5$. Therefore, $\sin 2\theta=4/5$ and $\cos 2\theta=-3/5$. (C) The double-angle identities $\cos 2\theta=1-2\sin^2\theta$ and $\cos 2\theta=2\cos^2\theta-1$. (D) Use the identities in part C in the form

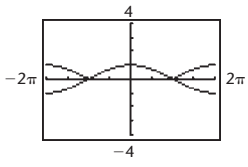
$$\sin\theta=\sqrt{\frac{1-\cos 2\theta}{2}}\quad\text{and}\quad\cos\theta=\sqrt{\frac{1+\cos 2\theta}{2}}$$

The positive radicals are used because θ is in quadrant one.
(E) $\sin\theta=2\sqrt{5}/5$; $\cos\theta=\sqrt{5}/5$

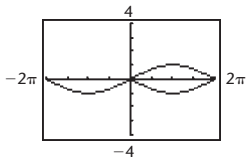
79. (A) $-0.72335=-0.72335$ (B) $-0.58821=-0.58821$

81. (A) $-3.2518=-3.2518$ (B) $0.89279=0.89279$

83. $y_1=y_2$ for $[-\pi,\pi]$

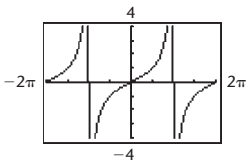


85. $y_1=y_2$ for $[-2\pi, 0]$

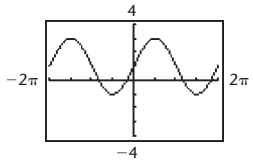


91. $-\frac{7}{25}$ **93.** $-\frac{24}{7}$ **95.** $\frac{\sqrt{5}}{5}$

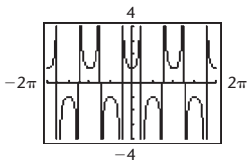
97. $\tan\dfrac{x}{2}$



99. $1+2\sin x$



101. $\sec 2x$



103. $x=\frac{224}{17}\approx 13.176\ m$; $\theta=28.955°$

105. (A) $d=\dfrac{v_0^2\sin 2\theta}{32\ \text{ft/sec}^2}$ (B) $\theta=45°$

107. (B) Table 1

n	10	100	1,000	10,000
A_n	2.93893	3.13953	3.14157	3.14159

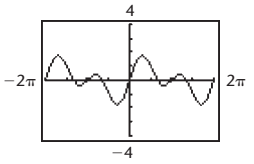
(C) A_n appears to approach π, the area of the circle with radius 1.
(D) A_n will not exactly equal the area of the circumscribing circle for any n no matter how large n is chosen; however, A_n can be made as close to the area of the circumscribing circle as we like by making n sufficiently large.

Section 6-4

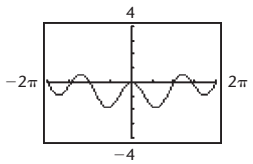
7. $\frac{1}{2}\sin 4m + \frac{1}{2}\sin 2m$ **9.** $\frac{1}{2}\cos 2u - \frac{1}{2}\cos 4u$ **11.** $2\sin 2t\cos t$
13. $2\sin 7w\sin 2w$ **15.** $\frac{\sqrt{3}-2}{4}$ **17.** $-\frac{\sqrt{2}}{4}$ **19.** $\frac{\sqrt{2}-1}{4}$
21. $\frac{\sqrt{2}}{4}$ **23.** $-\frac{(2+\sqrt{3})}{4}$ **25.** $\frac{\sqrt{2}+\sqrt{3}}{4}$ **27.** $-\frac{\sqrt{6}}{2}$ **29.** $-\frac{\sqrt{2}}{2}$
31. $\frac{\sqrt{2}}{2}$ **33.** $\frac{\sqrt{6}}{2}$

37. Let $x = u + v$ and $y = u - v$ and solve the resulting system for u and v in terms of x and y, then substitute the results into the first identity. The second identity will result after a small amount of algebraic manipulation.
55. Yes **57.** No **59.** Yes
61. (A) $-0.34207 = -0.34207$ (B) $-0.05311 = -0.05311$
63. (A) $-0.19115 = -0.19115$ (B) $-0.46541 = -0.46541$

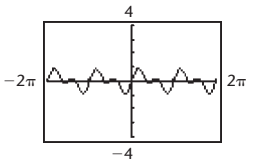
65. $y_2 = 2\sin\frac{3x}{2}\cos\frac{x}{2}$



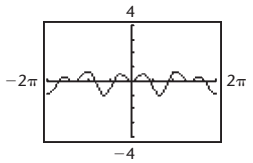
67. $y_2 = -2\sin x\sin 0.7x$



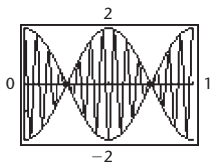
69. $y_2 = \frac{1}{2}(\sin 4x + \sin 2x)$



71. $y_2 = \frac{1}{2}(\cos 1.6x - \cos 3x)$

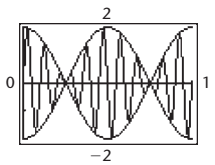


75. (A)



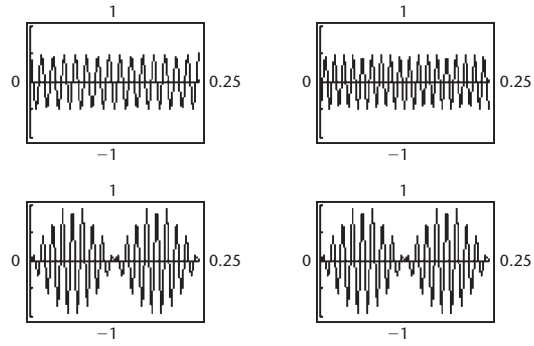
(B) $y_1 = \cos(30\pi x) + \cos(26\pi x)$; graph same as part A

77. (A)



(B) $y_1 = \sin(22\pi x) + \sin(18\pi x)$; graph same as part A

79. (B)



Section 6-5

3. $\frac{3\pi}{2}$ **5.** $\frac{3\pi}{2} + 2k\pi$, k any integer **7.** 45°, 225° **9.** 30°, 150°
11. $30° + k(360°)$, $150° + k(360°)$, k any integer **13.** 240°, 300°
15. $240° + k(360°)$, $300° + k(360°)$, k any integer
17. $\frac{\pi}{3}, \frac{4\pi}{3}$ **19.** $\frac{\pi}{3} + k\pi$, k any integer **21.** 1.1279, 5.1553
23. 74.0546° **25.** $3.5075 + 2k\pi$, $5.9172 + 2k\pi$, k any integer
27. 0.3376 **29.** 2.7642 **31.** $k(180°)$, $135° + k(180°)$, k any integer
33. $0, 2\pi/3, \pi, 4\pi/3$ **35.** $4\pi/3$ **37.** 210°, 330° **39.** 60°, 180°, 300°
41. $\pi/3, \pi, 5\pi/3$ **43.** All x such that $0 \le x < 2\pi$ **45.** No solution
47. 41.81° **49.** 1.911 **51.** 0.3747, 2.767
53. $0.3747 + 2k\pi$, $2.767 + 2k\pi$, k any integer **55.** 0.3747, 2.7669
57. $0.3747 + 2k\pi$, $2.769 + 2k\pi$, k any integer **59.** (−1.1530, 1.1530)
61. [0.5424, 5.3778], [5.9227, ∞) **63.** 1.8183
65. $\tan^{-1}(-5.377)$ has exactly one value, −1.387; the equation $\tan x = -5.377$ has infinitely many solutions, which are found by adding $k\pi$, k any integer, to each solution in one period of $\tan x$.
67. $0, 3\pi/2$ **69.** π **71.** 0.1204, 0.1384
73. (A) 0.3183 is the largest zero. $y = 0$ is a horizontal asymptote for the graph of f. (B) Infinitely many zeros exist between 0 and b, for any b, however small. The exploration graphs suggest this conclusion, which is reinforced by the following reasoning: Note that for each interval $(0, b]$, however small, as x tends to zero through positive numbers, $1/x$ increases without bound, and as $1/x$ increases without bound, $\sin(1/x)$ will cross the x axis an unlimited number of times. The function f does not have a smallest zero because, between 0 and b, no matter how small b is, there is always an unlimited number of zeros.
75. 0.009235 seconds **77.** 50.77° **81.** 2.267 radians
83. (A) 12.4575 millimeters (B) 2.6496 millimeters
85. $(r, \theta) = (0, 0°), (0, 180°), (0, 360°)$ **87.** $\theta = 45°$

Chapter 6 Review Exercises

5. $\frac{1}{2}\sin 8\alpha + \frac{1}{2}\sin 2\alpha$ *(6-4)*
6. $-2\sin 6x\sin x$ *(6-4)*
7. $\cos x$ *(6-2)*
8. $135° + k360°$, $225° + k360°$, k any integer *(6-5)*
9. $k\pi$ or $\frac{\pi}{4} + k\pi$, k any integer *(6-5)*
10. $x = \begin{cases} 0.7878 + 2k\pi \\ 2.3538 + 2k\pi \end{cases}$ k any integer *(6-5)*
11. $x = \begin{cases} 75.1849° + k360° \\ 284.8151° + k360° \end{cases}$ k any integer *(6-5)*
12. −1.4032 *(6-5)* **13.** 3.1855 *(6-5)*
14. (A) Not an identity (B) An identity *(6-1)*

24. $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ or $2+\sqrt{3}$ *(6-2)*

25. $\frac{\sqrt{2}+\sqrt{6}}{4}$ *(6-2)* **26.** $\frac{\sqrt{2+\sqrt{3}}}{2}$ *(6-3)*

27. $-\frac{\sqrt{2+\sqrt{2}}}{2}$ *(6-3)* **28.** $\frac{-2-\sqrt{3}}{4}$ *(6-4)*

29. $-\frac{\sqrt{6}}{2}$ *(6-4)* **30.** $\frac{1+\sqrt{2}}{4}$ *(6-4)*

31. $\frac{\sqrt{2}}{2}$ *(6-4)* **32.** No *(6-1)* **33.** Yes *(6-2)* **34.** No *(6-2)*

35. No *(6-2)* **36.** $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ *(6-5)* **37.** 0°, 120° *(6-5)*

38. $x = 0 + 2k\pi$, $x = \pi + 2k\pi$, $x = \frac{\pi}{6} + 2k\pi$, $x = \frac{5\pi}{6} + 2k\pi$, k any integer. The first two can also be written together as $x = k\pi$, k any integer. *(6-5)*

39. $x = 0 + 2k\pi$, $x = \pi + 2k\pi$, $x = \frac{\pi}{6} + 2k\pi$, $x = \frac{11\pi}{6} + 2k\pi$, k any integer. The first two can also be written together as $x = k\pi$, k any integer. *(6-5)*

40. $120° + k360°$, $240° + k360°$, k any integer *(6-5)*

41. $14.34° + k180°$ *(6-5)*

42. $x = \begin{cases} 0.6259 + 2k\pi \\ 2.516 + 2k\pi \end{cases}$ k any integer *(6-5)*

43. 1.178, 2.749 *(6-5)* **44.** 1.4903 *(6-5)*

45. $(-\infty, 1.4903)$ *(6-5)* **46.** $-0.6716, 0.6716$ *(6-5)*

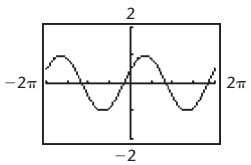
47. $[-0.6716, 0.6716]$ *(6-5)*

48. (A) Yes (B) Conditional equation, because the equation is false for $x = 1$ and $y = 1$, for example, and both sides are defined at $x = 1$ and $y = 1$. *(6-1)*

49. $\sin^{-1} 0.3351$ has exactly one value, whereas the equation $\sin x = 0.3351$ has infinitely many solutions. *(6-5)*

50. (A) Not an identity *(6-1)* (B) An identity

51. $y_2 = \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x$ *(6-2)*



52. (A) $0, \frac{2\pi}{3}, \frac{4\pi}{3}$ (B) 0, 2.0944 and 4.1888 *(6-5)*

53. 0.149 and -2.233 *(6-5)*

54. (A) $\frac{3}{\sqrt{10}}$ or $\frac{3\sqrt{10}}{10}$ (B) $\frac{7}{25}$ *(6-3)* **55.** $-\frac{24}{25}$ *(6-3)*

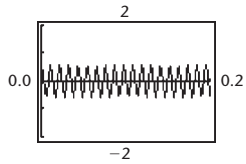
56. $\frac{24}{25}$ *(6-2)* **57.** (A) $0, \frac{\pi}{3}, \frac{2\pi}{3}$ (B) 0, 1.0472, 2.0944 *(6-5)*

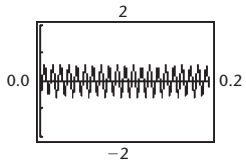
58. (A) 0.6817, 1.3183 (B) As x increases without bound, $\frac{1}{x-1}$ tends to 0 through positive numbers and $\sin\frac{1}{x-1}$ tends to 0 through positive numbers. $y = 0$ is a horizontal asymptote for the graph of f. (C) The exploratory graphs are left to the student. There are infinitely many zeros in any interval containing $x = 1$. The number $x = 1$ is not a zero because $\sin\frac{1}{x-1}$ is not defined at $x = 1$. *(6-5)*

59. $x = \sqrt{27}$; $x = 5.196$ centimeters, $\theta = 30.000°$ *(6-3)*

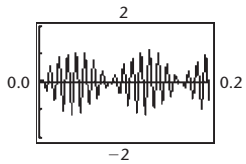
60. 0.00346 seconds *(6-5)*

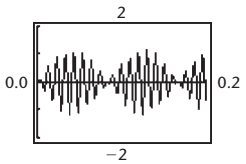
61. (B) $y = 0.6\cos 184\pi t$ $y = -0.6\cos 208\pi t$





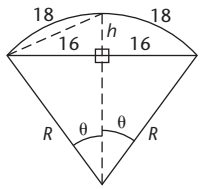
$y = 0.6\cos 184\pi t - 0.6\cos 208\pi t$ $y = 1.2\sin 12\pi t \sin 196\pi t$





(6-4)

62. Height = 7.057 feet, radius = 21.668 feet



From the figure, $R\theta = 18$ and $\sin\theta = \frac{16}{R}$. From these two equations, solving each for R in terms of θ and setting the results equal to each other, we obtain the desired trigonometric equation. *(6-5)*

Chapter 7

Section 7-1

9. $\gamma = 79°$, $a = 41$ feet, $b = 20$ feet

11. $\beta = 40°$, $a = 16$ kilometers, $c = 5.8$ kilometers

13. $\alpha = 49°$, $a = 53$ yards, $b = 66$ yards

15. $\beta = 81°$, $b = 16$ centimeters, $c = 12$ centimeters

17. 1 triangle; the case where α is acute and $a = 2 = h$

19. 1 triangle; the case where α is acute and $a \geq b$ ($a = 6$, $b = 4$)

21. 0 triangles; the case where α is acute and $0 < a < h$ ($a = 1$, $h = 2$)

23. 2 triangles; the case where α is acute and $h < a < b$ ($h = 2$, $a = 3$, $b = 4$)

25. $\beta = 49.5°$, $a = 20.0$ feet, $c = 4.81$ feet

27. $\gamma = 58.1°$, $a = 140$ meters, $c = 129$ meters

29. No solution

31. Triangle I: $\beta = 158.8°$, $\gamma = 5.3°$, $c = 7.55$ inches; triangle II: $\beta = 21.2°$, $\gamma = 142.9°$, $c = 49.3$ inches

33. Triangle I: $\alpha = 116.6°$, $\gamma = 24.5°$, $c = 19.8$ inches; triangle II: $\alpha = 63.4°$, $\gamma = 77.7°$, $c = 46.7$ inches

34. Triangle I: $\alpha = 124.0°$, $\gamma = 28.7°$, $c = 141$ centimeters; triangle II: $\alpha = 56.0°$, $\gamma = 96.7°$, $c = 292$ centimeters

35. No solution **37.** $\alpha = 22°10'$, $\gamma = 128°20'$, $c = 89.9$ millimeters

39. $\beta = 90°$, $\gamma = 60°$, $c = 50$ feet **41.** $k = 25.2$ sin $42.3° = 17.0$

43. Left side: 16.204; right side: 16.073

45. 4.06 miles, 2.47 miles **47.** 353 feet **49.** 5.8 inches, 3.1 inches

51. 4.42×10^7 kilometers, 2.39×10^8 kilometers **53.** 159 feet

55. 1,100 feet **57.** $R = 7.76$ millimeters, $s = 13.4$ millimeters

Section 7-2

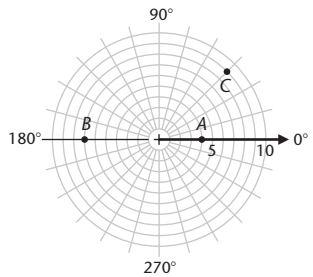
7. Angle γ is acute. A triangle can have at most one obtuse angle. Because α is acute, then, if the triangle has an obtuse angle it must be the angle opposite the longer of the two sides, b and c. Therefore, γ, the angle opposite the shorter of the two sides, c, must be acute.
9. $a = 6.03$ yards, $\beta = 56.6°$, $\gamma = 52.2°$
11. $c = 14.0$ millimeters, $\alpha = 20°40'$, $\beta = 39°0'$
13. If the triangle has an obtuse angle, then it must be the angle opposite the longest side; in this case, β.
15. $\alpha = 23.0°$, $\beta = 94.9°$, $\gamma = 62.1°$
17. $\alpha = 67.3°$, $\beta = 54.6°$, $\gamma = 58.1°$
19. No solution, because $\alpha + \gamma > 180°$
21. $b = 23.1$ inches, $\alpha = 46.1°$, $\gamma = 29.4°$
23. $\beta = 10.8°$, $a = 22.5$ meters, $b = 5.01$ meters
25. $\alpha = 30.7°$, $\gamma = 110.9°$, $c = 21.0$ inches
27. $\alpha = 49.1°$, $\beta = 102.9°$, $\gamma = 28.0°$
29. Triangle I: $\beta = 109.7°$, $\alpha = 11.9°$, $a = 1.58$ meters; triangle II: $\beta = 70.3°$, $\alpha = 51.3°$, $a = 5.99$ meters
31. No solution **33.** Triangle I: $\gamma = 140.5°$, $\alpha = 25.9°$, $a = 40.1$ meters; triangle II: $\gamma = 39.5°$, $\alpha = 126.9°$, $a = 73.5$ meters
35. 168 square feet **37.** 240 square yards **39.** 320 square meters
41. 30 square inches **43.** 25,500 square feet **45.** 9.3 square kilometers **47.** 2.8 square meters **49.** 53,900 square yards
51. 68 square inches **53.** F **55.** T **61.** 120 yards **63.** 100.6°
65. 5.81 feet **67.** 121 miles **69.** 74.1 meters **71.** 0.284 radians
73. $\alpha = 31°50'$, $\beta = 50°10'$, $\gamma = 98°0'$ **75.** $\angle CAB = 33°$
77. 24,800 miles

Section 7-3

7. $\langle 6, 5\rangle$ **9.** $\langle -7, 14\rangle$ **11.** $\langle -6, 7\rangle$ **13.** $\langle -5, -8\rangle$ **15.** 10
17. 13 **19.** 25 **21.** $\sqrt{2}$ **23.** $|\mathbf{u} + \mathbf{v}| = 77$ grams, $\alpha = 15°$
25. $|\mathbf{u} + \mathbf{v}| = 23$ knots, $\alpha = 6°$
27. $|\mathbf{u}| + 12$ kilograms, $|\mathbf{v}| = 6.0$ kilograms
29. $|\mathbf{u}| + 109$ miles per hour, $|\mathbf{v}| = 160$ miles per hour
31. (A) $\langle 1, 4\rangle$ (B) $\langle 3, -2\rangle$ (C) $\langle 14, -1\rangle$
33. (A) $\langle -2, 1\rangle$ (B) $\langle -6, -3\rangle$ (C) $\langle -10, -1\rangle$ **35.** $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j}$
37. $\mathbf{v} = 3\mathbf{i}$ **39.** $\mathbf{v} = -5\mathbf{i} - 2\mathbf{j}$ **41.** $5\mathbf{i} + 2\mathbf{j}$ **43.** $-16\mathbf{j}$ **45.** $-8\mathbf{j}$
47. $\left\langle \frac{4}{5}, \frac{3}{5}\right\rangle$ **49.** $\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$ **51.** $\langle -1, 0\rangle$ **53.** $\left\langle \frac{5}{6}, \frac{\sqrt{11}}{6}\right\rangle$
55. F **57.** T **59.** F **61.** F **63.** F **75.** 288°, 7.6 knots
77. 3,900 pounds at 72° **79.** (A) 388 pounds (B) 4,030 pounds
81. To the right **83.** 760 pounds to the left; 761 pounds to the right
85. 518 pounds to the left, 390 pounds to the right
87. This corresponds to a tension force of 462 pounds in member CB. This corresponds to a compression force of 231 pounds in member AB.
89. AB = a compression of 2,360 pounds;
BC = a tension of 2,000 pounds

Section 7-4

7.



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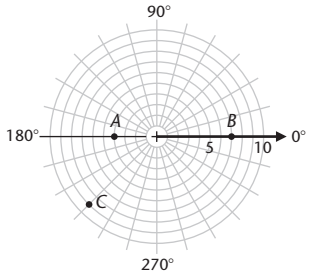

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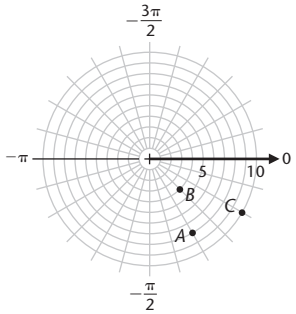

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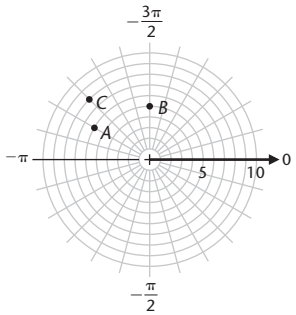
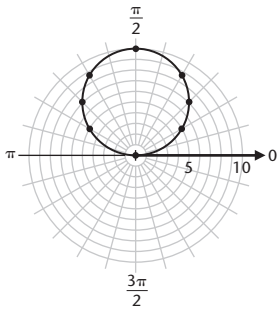


15. $(5, -\pi/4)$: The polar axis is rotated $\pi/4$ radians clockwise (negative direction) and the point is located five units from the pole along the positive polar axis. $(5, 7\pi/4)$: The polar axis is rotated $7\pi/4$ radians counterclockwise (positive direction) and the point is located 5 units from the pole along the positive polar axis. $(-5, -5\pi/4)$: The polar axis is rotated $5\pi/4$ radians clockwise (negative direction) and the point is located five units from the pole along the negative polar axis.

17.



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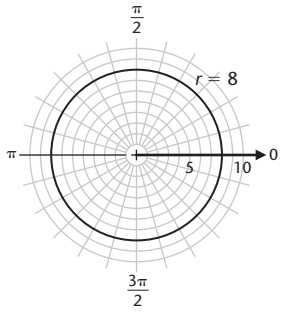

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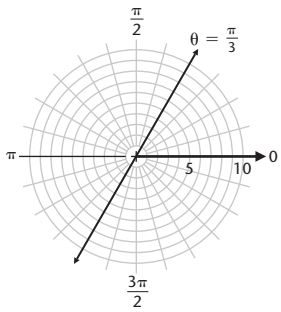
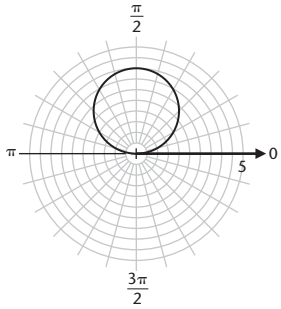


23. (5.196, 3.000) **25.** (1.848, −0.765) **27.** (2.078, 3.688)
29. (7.9, 64°) **31.** (26, −32°) **33.** (7.61, −164.4°)

35.



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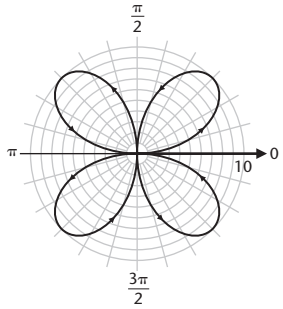

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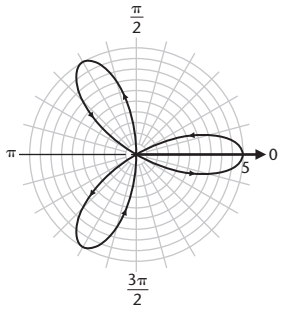

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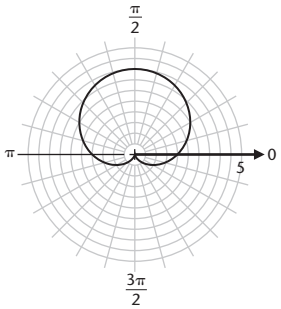

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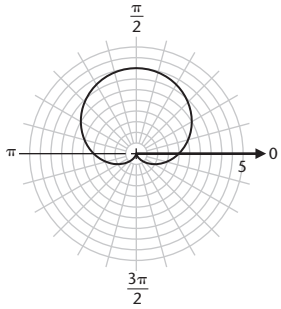

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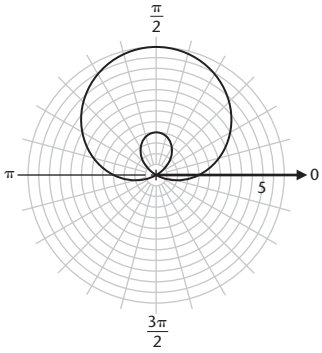

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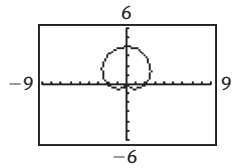
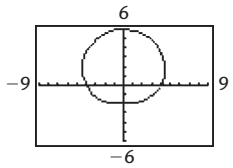
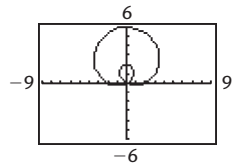
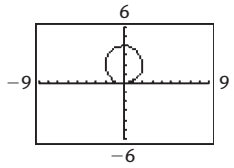
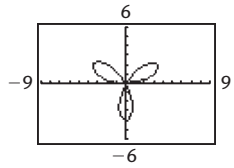




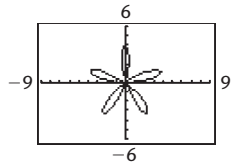


47. (A)

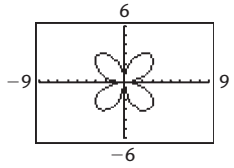


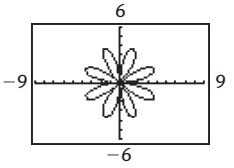


(B) 7 (C) n

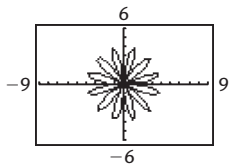


49. (A)





(B) 16 (C) $2n$



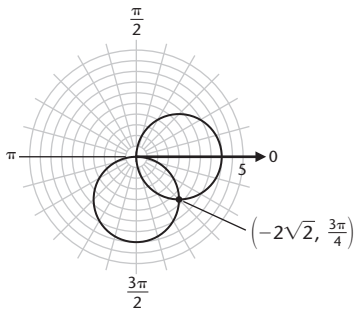
51. $r = 5 \sin \theta$ **53.** $\tan \theta = 1$ or $\theta = \dfrac{\pi}{4}$

55. $r = \dfrac{4 \cos \theta}{\sin^2 \theta} = 4 \cot \theta \csc \theta$ **57.** $3x - 4y = -1$

59. $x^2 + y^2 = -2y$ **61.** $y = x$

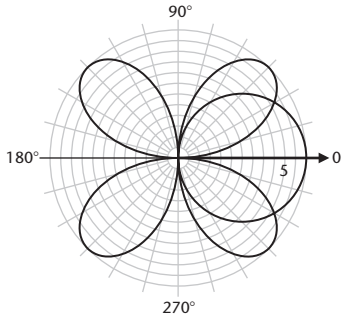
63. For each n, there are n large petals and n small petals. For n odd, the small petals are within the large petals; for n even, the small petals are between the large petals.

65.



$(r, \theta) = \left(-2\sqrt{2}, \frac{3\pi}{4}\right)$ [*Note:* (0, 0) is not a solution of the system even though the graphs cross at the origin.]

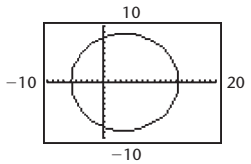
67.

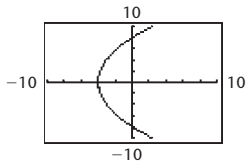


$(r, \theta) = (0, 90°), (0, 270°), (3\sqrt{3}, 30°), (-3\sqrt{3}, 150°)$ [*Note:* (0, 0) is not a solution of the system even though the graphs cross at the origin.]

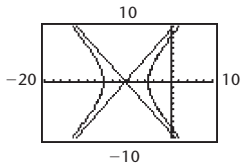
69. 3.368 units **71.** 6 knots, 13 knots, 12 knots, 9 knots

73. (A) Ellipse (B) Parabola

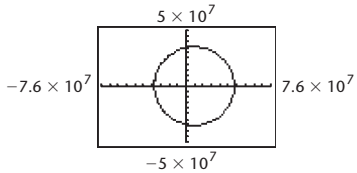




(C) Hyperbola



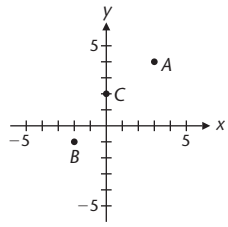
75. (A) Aphelion: 4.34×10^7 miles; perihelion: 2.85×10^7 miles



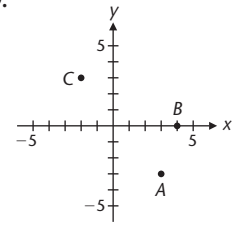
(B) Faster at perihelion. Because the distance from the sun to Mercury is less at perihelion than at aphelion, the planet must move faster near perihelion for the line joining Mercury to the sun to sweep out equal areas in equal intervals of time.

Section 7-5

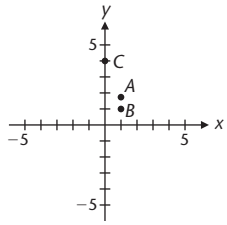
7.



9.



11.



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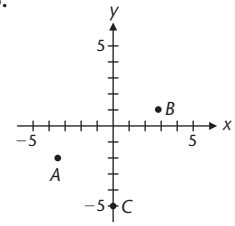
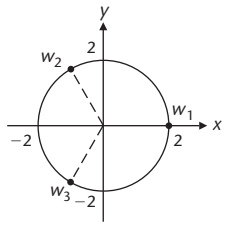
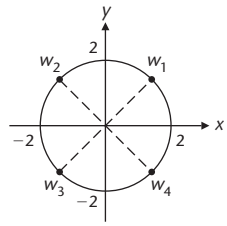


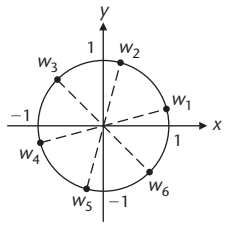
15. $2e^{30°i}$ **17.** $\sqrt{3}e^{(-\pi/2)i}$ **19.** $1 + i\sqrt{3}$ **21.** $3\sqrt{3} + 3i$
23. $e^{250°i}$; $e^{140°i}$ **25.** $14e^{113°i}$; $3.5e^{51°i}$ **27.** $10e^{135°i}$; $2.5e^{(-31°)i}$
29. $36.42e^{4.35i}$; $0.26e^{(-0.83i)}$ **31.** $e^{100°i}$ **33.** $8e^{90°i}$ **35.** $8e^{60°i}$
37. $8e^{180°i}$ **39.** $-8 + 8\sqrt{3}i$ **41.** 16 **43.** 1
45. $w_1 = 2e^{10°i}$, $w_2 = 2e^{130°i}$, $w_3 = 2e^{250°i}$
47. $w_1 = 3e^{15°i}$, $w_2 = 3e^{105°i}$,
$w_3 = 3e^{195°i}$, $w_4 = 3e^{285°i}$
49. $w_1 = 2^{1/10}e^{(-9°)i}$, $w_2 = 2^{1/10}e^{63°i}$, $w_3 = 2^{1/10}e^{135°i}$,
$w_4 = 2^{1/10}e^{207°i}$, $w_5 = 2^{1/10}e^{279°i}$
51. $w_1 = 2e^{0°i}$, $w_2 = 2e^{120°i}$,
$w_3 = 2e^{240°i}$



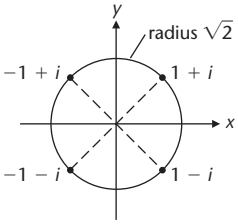
53. $w_1 = 2e^{45°i}$, $w_2 = 2e^{135°i}$,
$w_3 = 2e^{225°i}$, $w_4 = 2e^{315°i}$



55. $w_1 = 1e^{15°i}$, $w_2 = 1e^{75°i}$,
$w_3 = 1e^{135°i}$, $w_4 = 1e^{195°i}$,
$w_5 = 1e^{255°i}$, $w_6 = 1e^{315°i}$



57. (A) $(1 + i)^4 + 4 = -4 + 4 = 0$. There are three other roots. (B) The four roots are equally spaced around the circle. Because there are four roots, the angle between successive roots on the circle is $360°/4 = 90°$.



(C) $(-1 + i)^4 + 4 = -4 + 4 = 0$;
$(-1 - i)^4 + 4 = -4 + 4 = 0$; $(1 - i)^4 + 4 = -4 + 4 = 0$

59. $x_1 = 4e^{60°i} = 2 + 2\sqrt{3}i$, $x_2 = 4e^{180°i} = -4$,
$x_3 = 4e^{300°i} = 2 - 2\sqrt{3}i$

61. $x_1 = 3e^{0°i} = 3$, $x_2 = 3e^{120°i} = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i$,

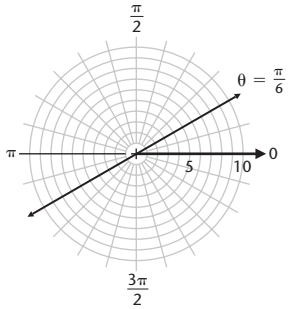
$x_3 = 3e^{240°i} = -\frac{3}{2} - \frac{3\sqrt{3}}{2}i$

63. T **65.** F **67.** T **69.** T **71.** F

77. $x_1 = 2e^{0°i}$, $x_2 = 2e^{72°i}$, $x_3 = 2e^{144°i}$, $x_4 = 2e^{216°i}$, $x_5 = 2e^{288°i}$
79. $w_1 = 1e^{36°i}$, $w_2 = 1e^{108°i}$, $w_3 = 1e^{180°i}$, $w_4 = 1e^{252°i}$, $w_5 = 1e^{324°i}$
81. $P(x) = (x - 2i)(x + 2i)[x - (-\sqrt{3} + i)][x - (-\sqrt{3} - i)]$
$[x - (\sqrt{3} + i)][x - (\sqrt{3} - i)]$

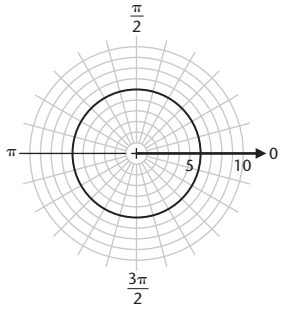
Chapter 7 Review Exercises

1. 1 *(7-1)* **2.** 0 *(7-1)* **3.** 2 *(7-1)*
4. Angle β is acute. A triangle can have at most one obtuse angle. Because α is acute, then, if the triangle has an obtuse angle, it must be the angle opposite the longer of the two sides, *b* and *c*. The, b, the angle opposite the shorter of the two sides, b, must be acute. *(7-2)*
5. $\gamma = 75°$, $a = 47$ meters, $b = 31$ meters *(7-1)*
6. $a = 4.00$ feet, $\beta = 36°$, $\gamma = 129°$ *(7-1, 7-2)*
7. $\beta = 19°$, $\alpha = 40°$, $a = 8.2$ centimeters *(7-1)*
8. $|\mathbf{u} + \mathbf{v}| = 170$ miles per hour, $\theta = 19°$ *(7-3)*
9. $\langle 3, -7 \rangle$ *(7-3)* **10.** $\sqrt{34}$ *(7-3)*
11.



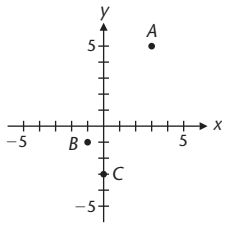
(7-4)

12.



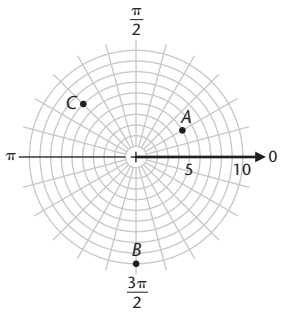
(7-4)

13.



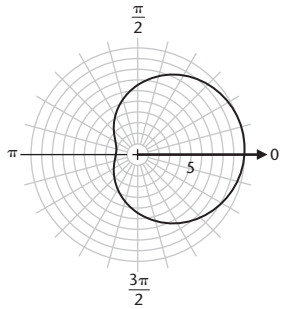
(7–5)

14. $(-10, -210°)$: The polar axis is rotated 210° clockwise (negative direction) and the point is located 10 units from the pole along the negative polar axis. $(-10, 150°)$: The polar axis is rotated 150° counterclockwise (positive direction) and the point is located 10 units from the pole along the negative polar axis. (10, 330°): The polar axis is rotated 330° counterclockwise and the point is located 10 units from the pole along the positive polar axis. *(7-4)*
15.



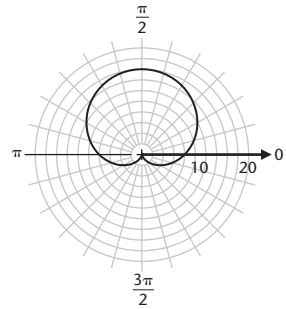
(7-5)

16. (A) $2e^{(-60°)i}$ (B) $2\sqrt{3} - 2i$ *(7-5)* **17.** (B) (A) 1 *(7-5)*
18. $8 + 8i\sqrt{3}$ *(7-5)* **19.** If the triangle has an obtuse angle, then it must be the angle opposite the longest side; in this case, α. *(7-2)*
20. $b = 10.5$ centimeters, $\alpha = 27.2°$, $\gamma = 37.4$ *(7-2)*
21. No Solution (7-1)
22. Two solutions. Obtuse case;
$\beta = 133.9°$, $\gamma = 19.7°$, $c = 39.6$ kilometers *(7-1)*
23. $\alpha = 41.1°$, $\beta = 74.2°$, $\gamma = 64.7°$ *(7-1, 7-2)*
24. The sum of all of the force vectors must be the zero vector for the object to remain at rest. *(7-3)*
25. $|\mathbf{u} + \mathbf{v}| = 98.0$ kilograms, $\alpha = 17.1°$ *(7-3)*
26. (A) $\mathbf{u} = -3\mathbf{i} + 9\mathbf{j}$ (B) $\mathbf{v} = -2\mathbf{j}$ *(7-3)*
27. (A) $\langle -4, 7 \rangle$ (B) $\langle -14, 13 \rangle$ *(7-3)*
28. (A) $-2\mathbf{i} - 4\mathbf{j}$ (B) -10j *(7-3)*
29. $\mathbf{u} = \left\langle \frac{-1}{\sqrt{10}}, \frac{-3}{\sqrt{10}} \right\rangle$ *(7-3)*
30. 790 square feet *(7-2)* **31.** 133 square meters *(7-2)*
32. 1,200 square yards *(7-2)* **33.** 410 square inches *(7-2)*
34.



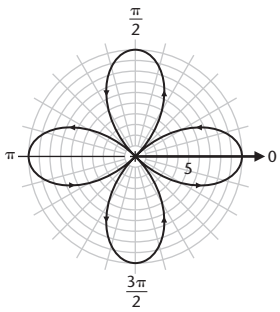
(7-4)

35.



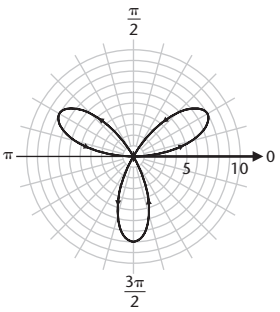
(7-4)

36.



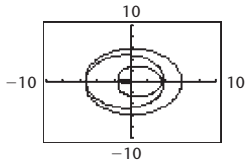
(7-4)

37.



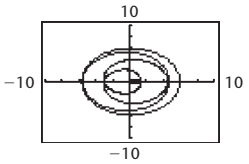
(7-4)

38.



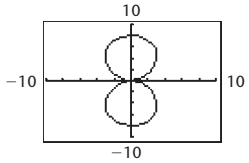
(7-4)

39.

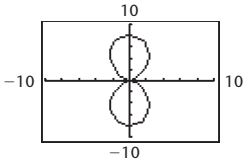


(7-4)

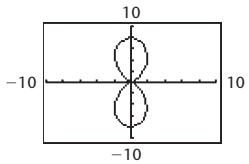
40. $n = 1$



$n = 2$

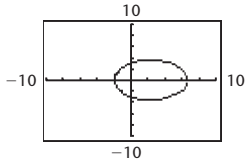


$n = 3$

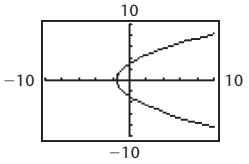


2 leaves for all n *(7-4)*

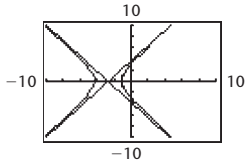
41. (A) Ellipse



(B) Parabola

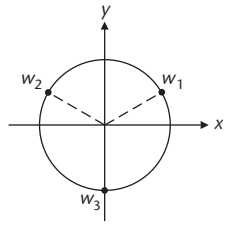


(C) Hyperbola

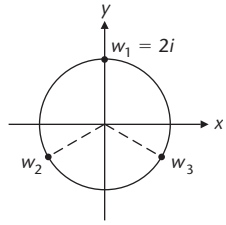


(7-4)

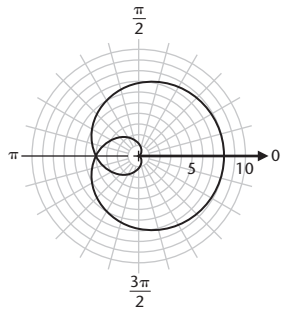
42. $r^2 = 6r\cos\theta$ or $r = 6\cos\theta$ *(7-4)* **43.** $x^2 + y^2 = 5x$ *(7-5)*
44. $z_1 = \sqrt{2}e^{135°i}$, $z_2 = 2e^{(-120°)i}$, $z_3 = 5e^{0°i}$ *(7-5)*
45. $z_1 = 1 + i$, $z_2 = (-3\sqrt{3}/2) - (3/2)i$, $z_3 = -1 - i\sqrt{3}$ *(7-5)*
46. (A) $32e^{44°i}$ (B) $2e^{6°i}$ *(7-5)*
47. (A) $-8 - 8\sqrt{3}i$ (B) $-8 - 13.86i$ *(7-5)*
48. $w_1 = (\sqrt{3}/2) + (1/2)i$,
$w_2 = (-\sqrt{3}/2) + (1/2)i$, $w_3 = -i$ *(7-5)*



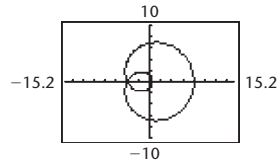
49. $2e^{50°i}$, $2e^{170°i}$, $2e^{290°i}$ *(7-5)*
50. $(4e^{15°i})^2 = 16e^{30°i} = 8\sqrt{3} + 8i$ *(7-5)*
51. (5.76, −26.08°) *(7-4)* **52.** (−5.30, −2.38) *(7-4)*
53. $5.26e^{127.20°i}$ *(7-5)* **54.** $-7.27 - 2.32i$ *(7-5)*
55. (A) There are a total of three cube roots and they are spaced equally around a circle of radius 2.



(B) $w_2 = -\sqrt{3} - i$, $w_3 = \sqrt{3} - i$
(C) The cube of each cube root is $-8i$. *(7-5)*
56. $k = 44.6 \sin 23.4°$ *(7-1)*
59. (A)



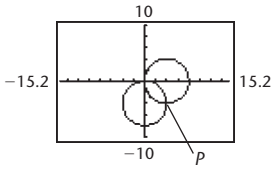
(B)



(7-4)

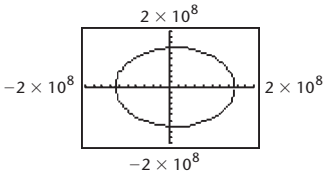
60. (A) The coordinates of P represent a simultaneous solution.
(B) $r = -4\sqrt{2}$, $\theta = 3\pi/4$

(C) The two graphs go through the pole at different values of θ. *(7-4)*



61. $1, -1, i, -i, \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$ *(7-5)*

62. $P(x) = (x + 2i)[x - (-\sqrt{3} + i)][x - (\sqrt{3} + i)]$ *(7-5)*
63. 438 miles *(7-3)* **64.** 438 miles per hour at 83° *(7-3)*
65. 86°, 464 miles per hour *(7-3)* **66.** 0.6 miles *(7-1)*
67. 177 pounds at 15.2° relative to **v** *(7-3)*
68. 19 kilograms at 204° relative to **u** *(7-3)* **69.** 5,740 pounds *(7-3)*
70. (A) Distance at aphelion: 1.56×10^8 miles; distance at perihelion: 1.29×10^8 miles

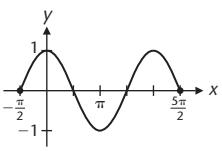


(B) Distance at aphelion: 1.56×10^8 miles; distance at perihelion: 1.29×10^8 miles *(7-4)*

Chapters 5–7 Cumulative Review

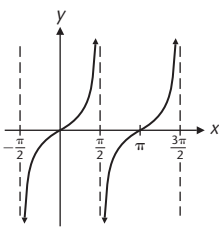
1. 1.86 meters *(5-1)*
2. θ = 57.3°, 14.5 centimeters, 7.83 centimeters *(5-3)*
3. (A) I, II (B) I, IV (C) I, III *(5-2)*
4. (A) $-\frac{3}{5}$ (B) $\frac{5}{4}$ (C) $-\frac{4}{3}$ *(5-4)* **5.** (A) $\frac{\pi}{4}$ (B) 65° (C) 30° *(5-4)*
6. (A) Domain: all real numbers; range: $-1 \le y \le 1$; period: 2π
(B) Domain: all real numbers; range: $-1 \le y \le 1$; period: 2π
(C) Domain: all real numbers except $x = \frac{\pi}{2} + k\pi$, k an integer; range: all real numbers; period: π *(5-4)*

7.



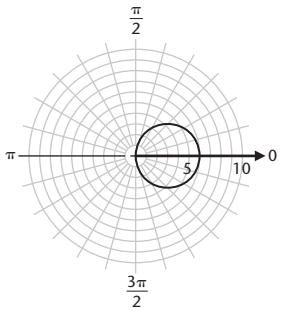
(5-4)

8.



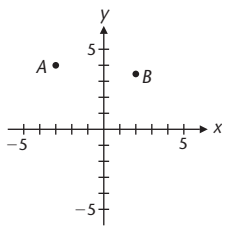
(5-4)

9. The central angle of a circle subtended by an arc of twice the length of the radius. *(5-1)*
10. If the graph of $y = \cos x$ is shifted $\pi/2$ units to the right, the result will be the graph of $y = \sin x$. *(5-4)*
15. (A) Not an identity (B) An identity *(6-1)*
16. Angle α is acute. A triangle can have at most one obtuse angle. Because β is acute, then, if the triangle has an obtuse angle it must be the angle opposite the longer of the two sides, a and c. Therefore, α, the angle opposite the shorter of the two sides, a, must be acute. *(7-2)*
17. 0.3245, 2.8171 *(6-5)* **18.** −76.2154° *(6-5)*
19. $b = 22$ feet, $\alpha = 28°$, $\gamma = 31°$ *(7-1, 7-2)* **20.** $\langle 6, -3 \rangle$ *(7-3)*
21. (5, −30°): The polar axis is rotated 30° clockwise (negative direction) and the point is located 5 units from the pole along the positive polar axis. (−5, −210°): The polar axis is rotated 210° clockwise (negative direction) and the point is located 5 units from the pole along the negative polar axis. (5, 330°): The polar axis is rotated 330° counterclockwise (positive direction) and the point is located 5 units from the pole along the positive polar axis. *(7-4)*
22.



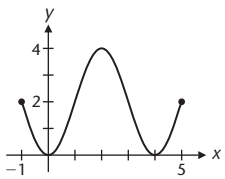
(7-4)

23.



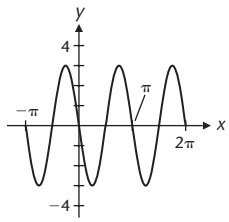
(7-5)

24. $4\sqrt{3} + 4i$ *(7-5)* **25.** $-\frac{7\pi}{6}$, 870° *(5-2)*
26. 75.06° *(5-1)* **27.** (A) and (C) *(5-2)* **28.** $-\frac{1}{2}$ *(5-2)*
29. Not defined *(5-2)* **30.** −1 *(5-2)* **31.** $\frac{2}{\sqrt{3}}$ *(5-2)*
32. π *(5-6)* **33.** Not defined *(5-6)* **34.** $\frac{2\pi}{3}$ *(5-6)*
35. 0.55 *(5-6)* **36.** $\frac{3}{5}$ *(5-6)* **37.** $\frac{1}{\sqrt{5}}$ *(5-6)*
38. (A) 9.871 (B) −3.748 (C) −1.559 (D) Not defined *(5-2, 5-6)*
39. *(5-5)*

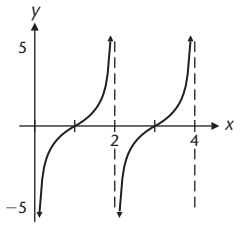


40. (A) 150° (B) −19.755° *(5-6)*
41. $\sin^{-1}(\sin 3) = 0.142$. For the identity $\sin^{-1}(\sin x) = x$ to hold, x must be in the restricted domain of the sine function; that is, $-\pi/2 \le x \le \pi/2$. The number 3 is not in the restricted domain. *(5-6)*
42. Because the coordinates of a point on a unit circle are given by $P = (a, b) = (\cos x, \sin x)$, we evaluate $P = (\cos(11.205), \sin(11.205))$—using a calculator set in radian mode—to obtain $P = (0.208, -0.978)$. The quadrant in which $P = (a, b)$ lies can be determined by the signs of a and b. In this case, P is in the fourth quadrant, because a is positive and b is negative. *(5-1, 5-2)*
43. The equation has infinitely many solutions [$x = \tan^{-1}(-24.5) + k\pi$, k any integer]; $\tan^{-1}(-24.5)$ has a unique value (−1.530 to three decimal places). *(5-6)*
44. $y = 3 + 2\sin \pi x$ *(5-5)*

45. $A = 3$, $P = \pi$, phase shift $= \dfrac{\pi}{2}$ *(5-5)*

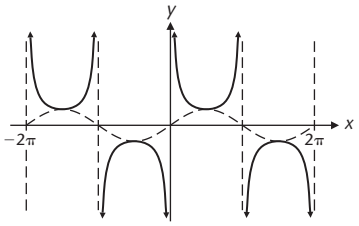


46. Period $= 2$, phase shift $= 1$



(5-5)

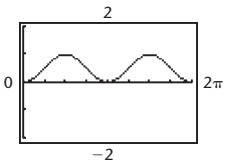
47.



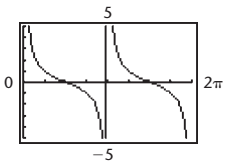
(5-4)

48. If the graph of $y = \cot x$ is shifted to the left $\pi/2$ units and reflected in the x axis, the result will be the graph of $y = \tan x$. *(5-4)*

49. $y = \frac{1}{2} - \frac{1}{2}\cos 2x$ *(5-5)*



50. $y = \cot x$ *(5-5)*



51. (A) Yes (B) Conditional, because both sides are defined as $x = \pi/2$, for example, but $\pi/2$ is not a solution *(6-1)*

58. (A) Not an identity (B) An identity *(6-1)*

59. 0 *(6-2)*

60. $\sin 2x = -\dfrac{24}{25}$, $\cos\dfrac{x}{2} = \sqrt{\dfrac{1}{10}}$ or $\dfrac{\sqrt{10}}{10}$ *(6-3)*

61. 1,500 square feet *(7-2)* **62.** 14.1 square meters *(7-2)*

63. 30°, 150°, 270° *(6-5)*

64. $x = k\pi, \dfrac{\pi}{3} + 2k\pi, -\dfrac{\pi}{3} + 2k\pi$, k any integer *(6-5)*

65. (A) $\pi/2$, $3\pi/2$, $7\pi/6$, $11\pi/6$ **66.** $x = 0.926$ *(6-5)*

67. $\gamma = 107.2°$, $\alpha = 25.0°$, $\beta = 47.8°$ *(7-1, 7-2)*

68. No solution *(7-1)*

69. $\beta = 120.7°$, $\gamma = 6.4°$, $c = 4.81$ inches *(7-1)*

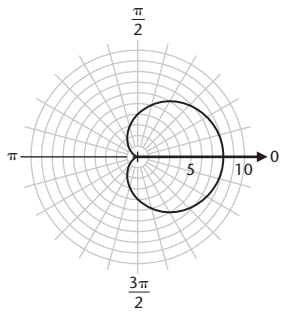
70. β must be acute. A triangle can have at most one obtuse angle, and because γ is actute, the obtuse angle, if present, must be opposite the longer of the two sides a and b. *(7-2)*

71. $|\mathbf{u} + \mathbf{v}| = 35.6$ pounds, $\alpha = 16.3°$ *(7-1, 7-2, 7-3)*

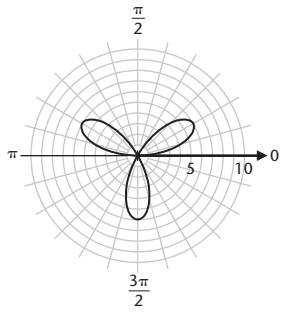
72. (A) $\langle 1, 3\rangle$ (B) $3\mathbf{i} + \mathbf{j}$ *(7-3)* **73.** $r = 8\sin\theta$ *(7-4)*

74. $x^2 + y^2 = -4x$ *(7-4)*

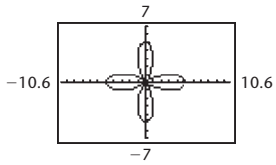
75. *(7-4)*



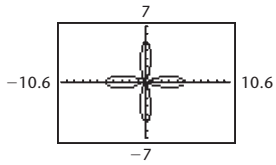
76. *(7-4)*



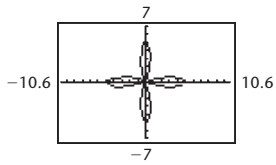
77. $n = 1$



$n = 2$

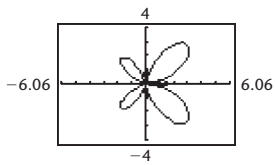


$n = 3$



4 leaves for all n *(7-4)*

78. *(7-4)*

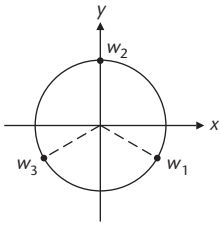


79. $(4.23, -131.07°)$ *(7-4)* **80.** $(-3.68, 5.02)$ *(7-4)*

81. $\sqrt{3} - i$ *(7-5)* **82.** $z = 2e^{120°i}$ *(7-5)*

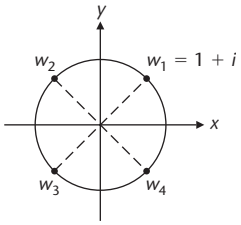
83. $64 + 0i = 64$ *(7-5)*

84. $w_1 = \frac{\sqrt{3}}{2} - \frac{1}{2}i, w_2 = i, w_3 = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$ *(7-5)*



85. $5.82e^{(-146.99°)i}$ *(7-5)* **86.** $-6.70 + 1.94i$ *(7-5)*

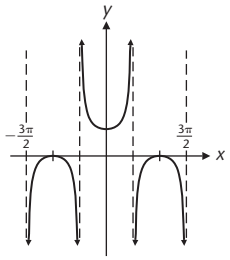
87. (A) There are a total of four fourth roots and they are spaced equally around a circle of radius $\sqrt{2}$.



(B) $w_2 = -1 + i, w_3 = -1 - i, w_4 = 1 - i$
(C) The fourth power of each fourth root is -4. *(7-5)*

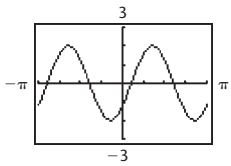
88. $a = \cos 1.2 = 0.362, b = \sin 1.2 = 0.932$ *(5-2)*

89. *(5-4)*



90. $y = 3\cos(2\pi x - \pi/4)$; amplitude = 3, period = 1, phase shift = 1/8 *(5-5)*

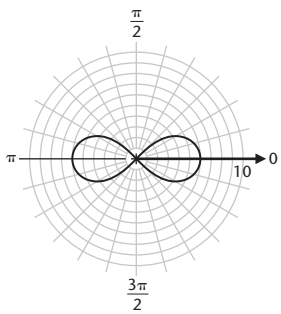
91. $y = 2\sin(2x - 0.644)$ *(5-5)*

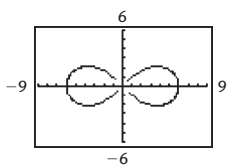


92. $\frac{1}{\sqrt{1-x^2}}$ *(5-6)* **93.** $\frac{24}{25}$ *(5-6, 6-3)* **94.** (A) $\frac{2}{\sqrt{5}}$ or $\frac{2\sqrt{5}}{5}$

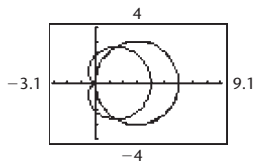
(B) $-\frac{7}{25}$ *(6-3)* **95.** (A) $\pi/3, 5\pi/3$ (B) 1.0472, 5.2360 *(6-5)*

96. (A) (B) *(7-4)*





97. (A)



(B) 6
(C) $(3, \pi/3), (3, 5\pi/3)$
(D) The points on r_2 and r_1 arrive at the intersection points for different values of θ, except for the two found in part C. *(7-4)*

98. $P(x) = (x - i)[x - (\sqrt{3}/2 - i/2)][x - (-\sqrt{3}/2 - i/2)]$ *(7-5)*

99. $\frac{2\pi}{73}$ radians *(5-1)* **100.** 1,088 meters *(5-3)*

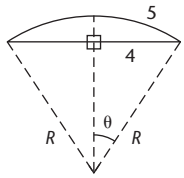
101. 5.88 inches *(5-3, 7-2)* **102.** 76° *(7-2)*

103. $I = 50\cos 220\pi t$ *(5-5)*

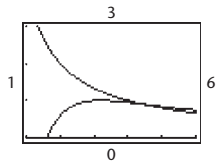
104. 274 miles per hour at 117° *(7-3)*

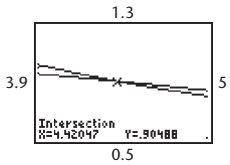
105. Both have a tension of 234 pounds *(7-4)*

106. (A) Add the perpendicular bisector of the chord as shown in the figure. Then, $\sin\theta = 4/R$ and $\theta = 5/R$. Substituting the second into the first, we obtain $\sin 5/R = 4/R$.

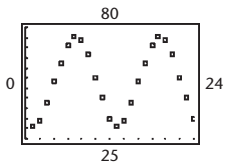


(B) R cannot be isolated on one side of the equation.
(C) Plot $y_1 = \sin 5/R$ and $y_2 = 4/R$ in the same viewing window and solve for R at the point of intersection using the INTERSECT command (see figure). $R = 4.420$ centimeters. *(6-5)*

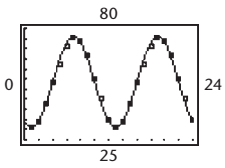




107. (A)



(B) $y = 53.5 + 22.5\sin(\pi x/6 - 2.1)$
(C)

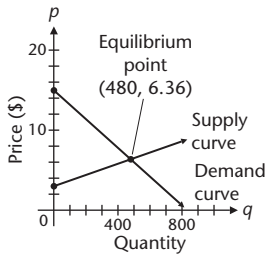


(5-5)

Chapter 8

Section 8-1

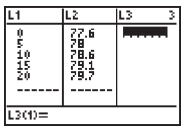
7. b, no solution **9.** d, (1, −3) **11.** (5, 2) **13.** (2, −3)
15. No solution (parallel lines) **17.** $x = 8, y = 19$
19. $x = 6, y = 2$ **21.** $x = 2, y = -1$ **23.** $x = 5, y = 2$
25. $m = 1, n = -2/3$ **27.** $x = 2{,}500, y = 200$ **29.** $u = 1.1, v = 0.3$
31. $x = -5/4, y = 5/3$ **33.** $x = 1.12, y = 2.41$
35. $x = -2.24, y = -3.31$ **37.** $q = x + y - 5, p = 3x + 2y - 12$
39. $x = \dfrac{dh - bk}{ad - bc}, y = \dfrac{ak - ch}{ad - bc}, ad - bc \neq 0$
41. Airspeed = 330 mph, wind rate = 80 mph
43. 2.475 km **45.** 40 mL 50% solution, 60 mL 80% solution
47. 5,200 **49.** $7,200 at 10% and $4,800 at 15%
51. Mexico plant: 75 hr; Taiwan plant: 50 hr
53. Mix A: 80 g; Mix B: 60 g
55. (A) Supply: 143 T-shirts; demand: 611 T-shirts
(B) Supply: 714 T-shirts; demand: 389 T-shirts
(C) Equilibrium price: $6.36; equilibrium quantity: 480 T-shirts
(D)

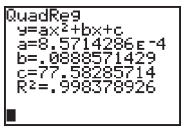


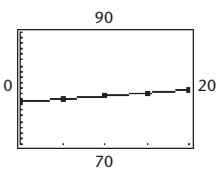
57. (A) $p = 0.001q + 0.15$ (B) $p = -0.002q + 1.89$
(C) Equilibrium price = $0.73; equilibrium quantity = 580 bushels
59. (A) $a = 196, b = -16$ (B) 196 ft (C) 3.5 s
61. 40 s, 24 s, 120 ml

Section 8-2

5. (−1, −1, −2) **7.** No solution
9. $\{(3s + 2, s, -2s - 1) \mid s \text{ any real number}\}$ **11.** (−3, −2, 5)
13. (2, 0, −1) **15.** $\{(-2s + 5, s, 3s - 4) \mid s \text{ any real number}\}$
17. (−1, 2, −4) **19.** (5, 2, −3) **21.** No solution
23. $\{(s/3 - 4/3, 2s/3 - 8/3, s) \mid s \text{ any real number}\}$
25. $\{(-4s + 7, 2s - 4, s) \mid s \text{ any real number}\}$
27. $a = 2, b = -4, c = -7$ **29.** $a = -4, b = 6, c = -12$
31. 880 lawn movers, 160 snowblowers, 280 chain saws
33. Michigan plant: 19 days; New York plant: 16 days; Ohio plant: 14 days
35. $30,000 in treasury bonds, $20,000 in municipal bonds, $50,000 in corporate bonds
37. Peanuts: 750 lb; cashews: 250 lb; walnuts: 500 lb
39. $y = 0.01x^2 + x + 75$; 450 million
41. $y = 0.004x^2 + 0.06x + 77.6$; 1995–2000: 79.4 yr; 2000–2005: 80.4 yr.
43.

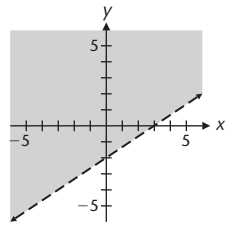






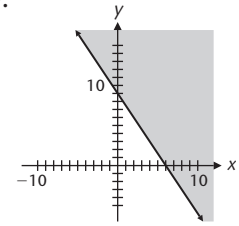
Section 8-3

5.



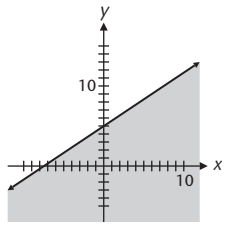
The solution region is the set of points (x, y) that are below the graph of the line $2x - 3y = 6$.

7.



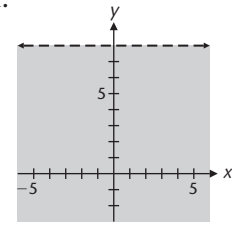
The solution region is the set of points (x, y) that are on or above the graph of the line $3x + 2y = 18$.

9.



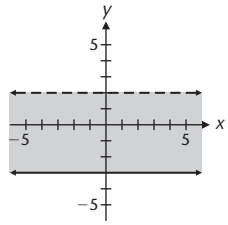
The solution region is the set of points (x, y) that are on or below the graph of the line $y = \frac{2}{3}x + 5$.

11.



The solution region is the set of points (x, y) that are below the graph of the horizontal line $y = 8$.

13.

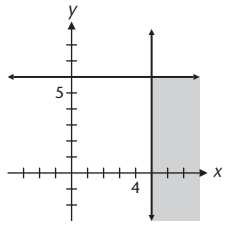


The solution region is the set of points (x, y) that are below the graph of the horizontal line $y = 2$ and on or above the graph of the horizontal line $y = -3$.

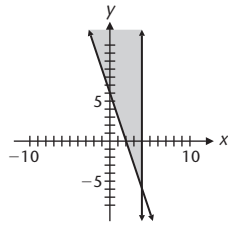
15. Region IV

17. Region I

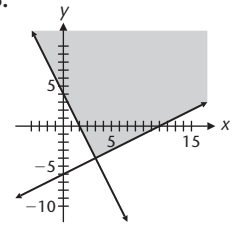
19.



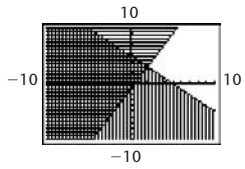
21.



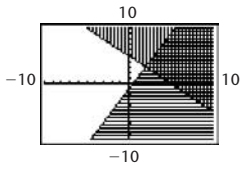
23.



25.

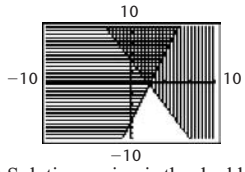


(A) Solution region is the double-shaded region.

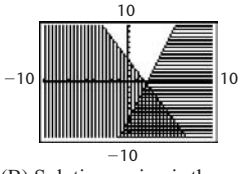


(B) Solution region is the unshaded region.

27.



(A) Solution region is the double-shaded region.

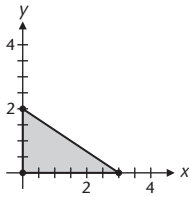


(B) Solution region is the unshaded region.

29. Region IV; corner points: (6, 4), (8, 0), and (18, 0)

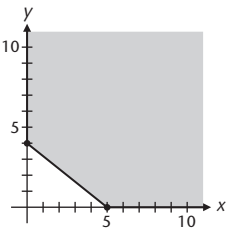
31. Region I; corner points: (0, 16), (6, 4), and (18, 0)

33.



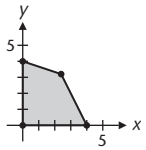
Bounded; corner points: (0, 0), (0, 2), (3, 0)

35.



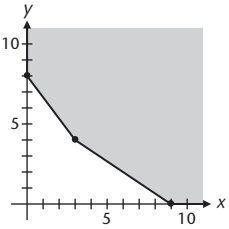
Unbounded; corner points: (0, 4) and (5, 0)

37.



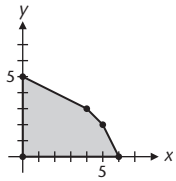
Bounded; corner points: $(0, 4), (0, 0), \left(\frac{12}{5}, \frac{16}{5}\right), (4, 0)$

39.



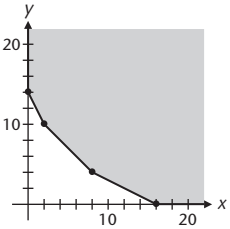
Unbounded; corner points: (9, 0), (0, 8), and (3, 4)

41.



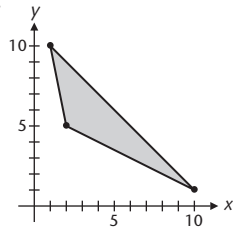
Bounded; corner points: (6, 0), (4, 3), (5, 2), (0, 0), and (0, 5)

43.



Unbounded; corner points: (0, 14), (2, 10), (8, 4), (16, 0)

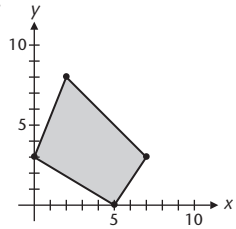
45.



Bounded; corner points: (2, 5), (10, 1), (1, 10)

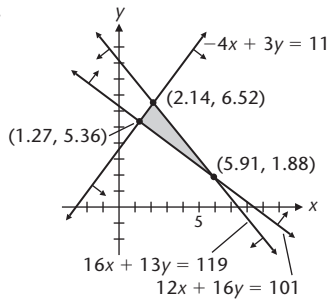
47. The feasible region is empty.

49.



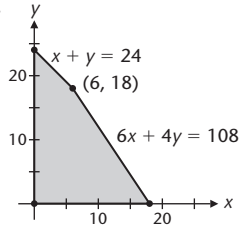
Bounded; corner points: (0, 3), (5, 0), (7, 3), (2, 8)

51.



Bounded; corner points: (0, 3), (5, 0), (7, 3), (2, 8)

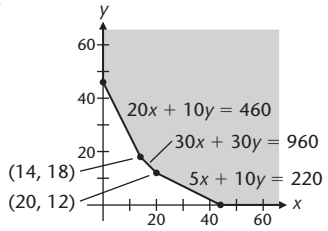
53.



$$\begin{aligned} 6x + 4y &\le 108 \\ x + y &\le 24 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

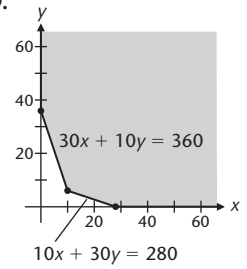
55. (A) All production schedules in the feasible region that are on the graph of $50x + 60y = 1{,}100$ will result in a profit of \$1,100.
(B) There are many possible choices. For example, producing 5 trick and 15 slalom skies will produce a profit of \$1,150. The graph of the line $50x + 60y = 1{,}150$ includes all the production schedules in the feasible region that result in a profit of \$1,150.

57.



$$\begin{aligned} 20x + 10y &\ge 460 \\ 30x + 30y &\ge 960 \\ 5x + 10y &\ge 220 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

59.



$10x + 30y \geq 280$
$30x + 10y \geq 360$
$x \geq 0$
$y \geq 0$

Section 8-4

9. Maximum value of z on S is 16 at (7, 9).
11. Maximum value of z on S is 84 at both (0, 12) and (7, 9).
13. Maximum value of z on S is 25 at (7, 9).
15. Maximum value of z on S is 70 at (10, 0).
17. Minimum value of z on T is 32 at (0, 8).
19. Minimum value of z on T is 36 at both (12, 0) and (4, 3).
21. Minimum value of z on T is 10 at (4, 3).
23. Minimum value of z on T is 32 at both (0, 8) and (4, 3).
25. Maximum value of z on S is 18 at (4, 3).
27. Minimum value of z on S is 12 at (4, 0).
29. Maximum value of z on S is 52 at (4, 10).
31. Minimum value of z on S is 44 at (4, 4).
33. The minimum value of z on S is 1,500 at (60, 0). The maximum value of z on S is 3,000 at (60, 30) and (120, 0). (Multiple optimal solutions)
35. The minimum value of z on S is 300 at (0, 20). The maximum value of z on S is 1,725 at (60, 15).
37. Max $P = 5{,}507$ at $x_1 = 6.62$ and $x_2 = 4.25$

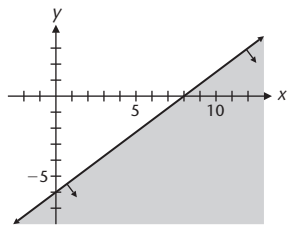
39. (A) $a > 2b$ (B) $\frac{1}{3}b < a < 2b$ (C) $a < \frac{1}{3}b$ or $b > 3a$

(D) $a = 2b$ (E) $b = 3a$
41. (A) 6 trick skis, 18 slalom skis; \$780
(B) The maximum profit decreases to \$720 when 18 trick and no slalom skis are produced.
(C) The maximum profit increases to \$1,080 when no trick and 24 slalom skis are produced.
43. 9 model A trucks and 6 model 5 trucks to realize the minimum cost of \$279,000
45. (A) 40 tables, 40 chairs; \$4,600
(B) The maximum profit decreases to \$3,800 when 20 tables and 80 chairs are produced.
47. (A) Max $P =$ \$450 when 750 gallons are produced using the old process exclusively.
(B) The maximum profit decreases to \$380 when 400 gallons are produced using the old process and 700 gallons using the new process.
(C) The maximum profit decreases to \$288 when 1,440 gallons are produced using the new process exclusively.
49. The nitrogen will range from a minimum of 940 pounds when 40 bags of brand A and 100 bags of brand B are used, to a maximum of 1,190 pounds when 140 bags of brand A and 50 pounds of brand B are used.

Chapter 8 Review

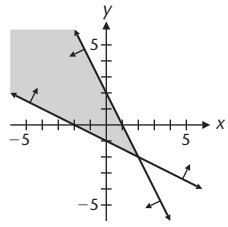
1. $x = 3, y = 3$ *(8-1)* **2.** $x = 3, y = -2$ *(8-1)*
3. $x = 2, y = -1$ *(8-1)* **4.** $x = 1.1875, y = 1.625$ *(8-1)*

5.



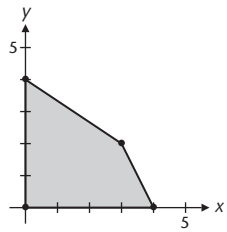
(8-3)

6.



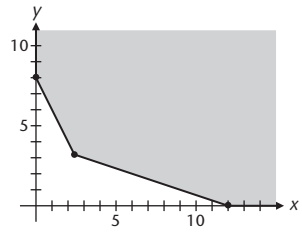
(8-3)

7. (2, 3) *(8-1)* **8.** No solution (inconsistent) *(8-1)*
9. Infinitely many solutions $\left(t, \frac{4s + 8}{3}\right)$ for any real number s *(8-1)*
10. (3, −1, −2) *(8-2)* **11.** (2, 0, −1) *(8-2)*
12. $2x - 3y = 12$; $2x - 3y \leq 12$ *(8-3)*
13. $4x + y = 8$; $4x + y \geq 8$ *(8-3)*
14. Min $z = 18$ at (0, 6); max $z = 42$ at (6, 4) *(8-4)*
15. $x = 2, y = 1, z = -1$ *(8-2)*
16. $x = -5s - 12, y = 3s + 7, z = s$
is a solution for every real number s. There are infinitely many solutions. *(8-2)*
17. No solution *(8-2)* **18.** No solution *(8-2)*
19. Bounded; corner: (0, 4), (0, 0), (4, 0), and (3, 2)



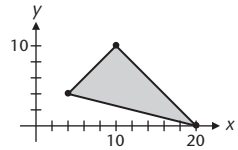
(8-3)

20. Unbounded; corner: (0, 8), (12, 0), and $\left(\frac{12}{5}, \frac{16}{5}\right)$



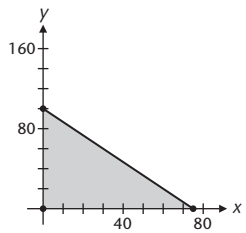
(8-3)

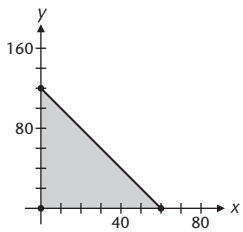
21. Bounded; corner points: (4, 4), (10, 10), (20, 0)



(8-3)

22. Max $z = 45$ at (4, 2) *(8-4)*
23. Min $z = 75$ at (3, 5) and (15, 0) (multiple optimal solutions) (15)
24. Min $z = 44$ at (4, 3); max $z = 82$ at (2, 9) *(8-4)*
25. $x_1 = 1{,}000, x_2 = 4{,}000, x_3 = 2{,}000$ *(8-2)*
26. Max $z = 26{,}000$ at (600, 400) *(8-4)*
27. $\frac{1}{2}$-lb packages: 48, $\frac{1}{3}$-lb packages: 72 *(8-1)*
28. 6 m by 8 m *(8-1)*
29. 40 g mix *A*, 60 g mix *B*, 30 g mix *C* *(8-2)*
30.





(A) $4x + 3y \le 300$
$x, y \ge 0$

(B) $2x + y \le 120$
$x, y \ge 0$ *(8-3)*

31. (A) Maximum profit is $P = \$7{,}800$ when 80 regular and 30 competition sails are produced.
(B) The maximum profit increases to \$8,750 when 70 competition and no regular sails are produced.
(C) The maximum profit decreases to \$7,200 when no competition and 120 regular sails are produced. *(8-4)*
32. (A) The minimum cost is $C = \$13$ when 100 grams of mix *A* and 150 grams of mix *B* are used.
(B) The minimum cost decreases to \$9 when 50 grams of mix *A* and 275 grams of mix *B* are used.
(C) The minimum cost increases to \$28.75 when 250 grams of mix *A* and 75 grams of mix *B* are used. *(8-4)*

Chapter 9

Section 9-1

11. No **13.** Yes **15.** No **17.** Yes
19. $x_1 = -2, x_2 = 3, x_3 = 0$
21. $x_1 = 2t + 3, x_2 = -t - 5, x_3 = t$, t any real number
23. No solution
25. $x_1 = 2s + 3t - 5, x_2 = s, x_3 = -3t + 2,$
$x_4 = t$, s and t any real numbers

27. $\left[\begin{array}{cc|c} 1 & 0 & -7 \\ 0 & 1 & 3 \end{array}\right]$ **29.** $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$

31. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -\frac{5}{3} \\ 0 & 1 & -2 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$ **33.** $x_1 = -2, x_2 = 3, x_3 = 1$
35. $x_1 = 0, x_2 = -2, x_3 = 2$
37. $x_1 = 2t + 3, x_2 = t - 2,$
$x_3 = t$, t any real number
39. $x_1 = 1, x_2 = 2$ **41.** No solution
43. $x_1 = 2t + 4, x_2 = t + 1, x_3 = t$, t any real number
45. $x_1 = s + 2t - 1, x_2 = s, x_3 = t$, s and t any real numbers
47. No solution **49.** $x_1 = 2.5t - 4, x_2 = \text{t}, x_3 = -5$ for t any real number **51.** $x_1 = 1, x_2 = -2, x_3 = 1$
53. (A) Dependent with two parameters (B) Dependent with one parameter (C) Independent (D) Impossible
59. $x_1 = 2s - 3t + 3, x_2 = s + 2t + 2, x_3 = s, x_4 = t$ for s and t any real numbers
61. $x_1 = -0.5, x_2 = 0.2, x_3 = 0.3, x_4 = -0.4$
63. $x_1 = 2s - 1.5t + 1, x_2 = s, x_3 = -t + 1.5,$
$x_4 = 0.5t - 0.5, x_5 = t$ for s and t any real numbers
65. $x_1 = (3t - 100)$ 15¢ stamps, $x_2 = (145 - 4t)$ 20¢ stamps,
$x_3 = t$ 35¢ stamps, where $t = 34, 35,$ or 36
67. $x_1 = (6t - 24)$ 500-cc containers of 10% solution; $x_2 = (48 - 8t)$ 500-cc containers of 20% solution; $x_3 = t$ 1,000-cc containers of 50% solution, where $t = 4, 5,$ or 6
69. $a = 3, b = 2, c = 1$ **71.** $a = -2, b = -4, c = -20$
73. (A) $x_1 = 20$ one-person boats, $x_2 = 220$ two-person boats, $x_3 = 100$ four-person boats (B) $x_1 = (t - 80)$ one-person boats, $x_2 = (-2t + 420)$ two-person boats, $x_3 = t$ four-person boats, $80 \le t \le 210$, t an integer
(C) No solution; no production schedule will use all the labor-hours in all departments.
75. (A) $x_1 = 8$ ounces of food *A*, $x_2 = 2$ ounces of food *B*, $x_3 = 4$ ounces of food *C* (B) No solution (C) $x_1 = 8$ ounces of food *A*, $x_2 = -2t + 10$ ounces of food *B*, $x_3 = t$ ounces of food *C*, $0 \le t \le 5$
77. $10 - t$ barrels of mix *A*, $t - 5$ barrels of mix *B*, $25 - 2t$ barrels of mix *C*, and t barrels of mix *D*, where t is an integer satisfying $5 \le t \le 10$

Section 9-2

11. $\begin{bmatrix} 2 & 5 \\ 4 & -6 \end{bmatrix}$ **13.** $\begin{bmatrix} 3 & 2 \\ -2 & 8 \\ 12 & -5 \end{bmatrix}$ **15.** Not defined

17. $\begin{bmatrix} 3 & -5 & 6 \\ 1 & 1 & 8 \end{bmatrix}$ **19.** $\begin{bmatrix} 12 & -16 & 28 \\ -8 & 36 & 20 \end{bmatrix}$ **21.** [41]

23. $\begin{bmatrix} 3 \\ -13 \end{bmatrix}$ **25.** $\begin{bmatrix} 13 & 8 \\ 26 & 48 \end{bmatrix}$ **27.** $\begin{bmatrix} 16 & -18 \\ -10 & 18 \end{bmatrix}$

29. $[-14]$ **31.** $\begin{bmatrix} -20 & 10 \\ -12 & 6 \end{bmatrix}$ **33.** [11] **35.** $\begin{bmatrix} 3 & -2 & -4 \\ 6 & -4 & -8 \\ -9 & 6 & 12 \end{bmatrix}$

37. Not defined **39.** $\begin{bmatrix} -6 & 7 & -11 \\ 4 & 18 & -4 \end{bmatrix}$

41. $\begin{bmatrix} -3 & 6 & 8 \\ -18 & 12 & 10 \\ 4 & 6 & 24 \end{bmatrix}$ **43.** $\begin{bmatrix} 5 & -11 & 15 \\ 4 & -7 & 3 \\ 0 & 10 & 4 \end{bmatrix}$ **45.** $\begin{bmatrix} -0.2 & 1.2 \\ 2.6 & -0.6 \\ -0.2 & 2.2 \end{bmatrix}$

47. $\begin{bmatrix} -31 & 16 \\ 61 & -25 \\ -3 & 77 \end{bmatrix}$ **49.** Not defined **51.** $\begin{bmatrix} -2 & 25 & -15 \\ 26 & -25 & 45 \\ -2 & 45 & -25 \end{bmatrix}$

53. $\begin{bmatrix} -26 & -15 & -25 \\ -4 & -18 & 4 \\ 2 & 43 & -19 \end{bmatrix}$

55. $B^n \to \begin{bmatrix} 0.25 & 0.75 \\ 0.25 & 0.75 \end{bmatrix}, AB^n \to [0.25 \quad 0.75]$

57. $a = -1, b = 1, c = 3, d = -5$ **59.** $x = 1, y = 2$
61. $a^2 + bc = 0$ **63.** $a = -b$ and $c = -d$ **65.** $x = -5, y = 4$
67. $a = 3, b = 1, c = 1, d = -2$ **69.** All are true

71.

	Guitar	Banjo	
	\$33	\$26	Materials
	\$57	\$77	Labor

73.

	Basic car	Air	AM/FM radio	Cruise control
Model *A*	$3,330	$77	$42	$27
Model *B*	$2,125	$93	$95	$50
Model *C*	$1,270	$113	$121	$52

(Markup)

75. (A) $11.80 (B) $30.30

(C) *MN* gives the labor costs per boat at each plant.

(D)

	Plant I	Plant II	
$MN =$	$11.80	$13.80	One-person boat
	$18.50	$21.60	Two-person boat
	$26.00	$30.30	Four-person boat

77. (A) $A^2 = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

There is one way to travel from Baltimore to Atlanta with one intermediate connection; there are two ways to travel from Atlanta to Chicago with one intermediate connection. In general, the elements in A^2 indicate the number of different ways to travel from the *i*th city to the *j*th city with one intermediate connection.

(B) $A^3 = \begin{bmatrix} 2 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$

There is one way to travel from Denver to Baltimore with two intermediate connections; there are two ways to travel from Atlanta to El Paso with two intermediate connections. In general, the elements in A^3 indicate the number of different ways to travel from the *i*th city to the *j* th city with two intermediate connections.

(C) $A + A^2 + A^3 + A^4 = \begin{bmatrix} 2 & 3 & 2 & 5 & 2 \\ 1 & 1 & 4 & 2 & 1 \\ 4 & 1 & 3 & 2 & 4 \\ 1 & 1 & 4 & 2 & 1 \\ 1 & 1 & 1 & 3 & 1 \end{bmatrix}$

It is possible to travel from any origin to any destination with at most three intermediate connections.

79. (A) $3,550 (B) $6,000 (C) *NM* gives the total cost per town.

(D)

	Cost/town	
$NM =$	$3,550	Berkeley
	$6,000	Oakland

(E)

	Telephone call	House call	Letter
$[1 \quad 1] N =$	[3,000	1,300	13,000]

(F)

	Total contacts	
$N \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} =$	6,500	Berkeley
	10,800	Oakland

81. (A) $\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & 1 & 2 & 3 & 1 & 2 \\ 1 & 0 & 2 & 3 & 2 & 2 \\ 1 & 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 3 & 2 & 0 \end{bmatrix}$

(C) $BC = \begin{bmatrix} 9 \\ 10 \\ 6 \\ 4 \\ 9 \\ 11 \end{bmatrix}$ where $C = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(D) Frank, Bart, Aaron and Elvis (tie), Charles, Dan

Section 9-3

7. $\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$ **9.** $\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$ **11.** $\begin{bmatrix} -2 & 1 & 3 \\ 2 & 4 & -2 \\ 5 & 1 & 0 \end{bmatrix}$

13. $\begin{bmatrix} -2 & 1 & 3 \\ 2 & 4 & -2 \\ 5 & 1 & 0 \end{bmatrix}$ **15.** Yes

17. No **19.** Yes **21.** No **23**. Yes

25. $\begin{bmatrix} 1 & -9 \\ 0 & 1 \end{bmatrix}$ **27.** $\begin{bmatrix} -5 & -2 \\ 2 & 1 \end{bmatrix}$ **29.** $\begin{bmatrix} -3 & -7 \\ -2 & -5 \end{bmatrix}$

31. Does not exist **33.** $\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$

35. $\begin{bmatrix} 0 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 0 \end{bmatrix}$ **37.** $\begin{bmatrix} -19 & 9 & -7 \\ 15 & -7 & 6 \\ -2 & 1 & -1 \end{bmatrix}$ **39.** $\begin{bmatrix} 1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ 2 & 1 & 4 \end{bmatrix}$

41. Does not exist

43. $\begin{bmatrix} -9 & -15 & 10 \\ 4 & 5 & -4 \\ -1 & -1 & 1 \end{bmatrix}$

45. A^{-1} exists if and only if all the elements on the main diagonal are nonzero.

47. In both parts, $A^{-1} = A$ and $A^2 = I$.

49. In both parts, $(A^{-1})^{-1} = A$.

51. (A) $(AB)^{-1} = \begin{bmatrix} 29 & -41 \\ -12 & 17 \end{bmatrix}$, $A^{-1}B^{-1} = \begin{bmatrix} 23 & -33 \\ -16 & 23 \end{bmatrix}$,

$B^{-1}A^{-1} = \begin{bmatrix} 29 & -41 \\ -12 & 17 \end{bmatrix}$

(B) $(AB)^{-1} = \begin{bmatrix} 0.7 & -0.1 \\ -1.8 & 0.4 \end{bmatrix}$, $A^{-1}B^{-1} = \begin{bmatrix} 0.1 & 0 \\ -0.4 & 1 \end{bmatrix}$,

$B^{-1}A^{-1} = \begin{bmatrix} 0.7 & -0.1 \\ -1.8 & 0.4 \end{bmatrix}$

53. 14 5 195 74 97 37 181 67 49 18 121 43 103 41

55. GREEN EGGS AND HAM

57. 21 56 55 25 58 46 97 94 48 75 45 58 63 45 59 48 64 80 44 69 68 104 123 72 127

59. LYNDON BAINES JOHNSON

Section 9-4

11. $\begin{aligned} 2x_1 - x_2 &= 3 \\ x_1 + 3x_2 &= -2 \end{aligned}$ **13.** $\begin{aligned} -2x_1 + x_3 &= 3 \\ x_1 + 2x_2 + x_3 &= -4 \\ x_2 - x_3 &= 2 \end{aligned}$

15. $\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

17. $\begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$

19. $x_1 = -8$ and $x_2 = 2$ **21.** $x_1 = 0$ and $x_2 = 4$

23. $x_1 = 3, x_2 = -2$ **25.** $x_1 = 11, x_2 = 4$

27. $x_1 = -20, x_2 = 9$; $x_1 = 18, x_2 = -7$; $x_1 = 19, x_2 = -8$

29. $x_1 = 8, x_2 = 5$; $x_1 = 4, x_2 = 4$; $x_1 = -18, x_2 = -12$

31. (A) $x_1 = -3, x_2 = -4, x_3 = -2$ (B) $x_1 = 4, x_2 = 5, x_3 = 1$
(C) $x_1 = 2, x_2 = -1, x_3 = -1$

33.(A) $x_1 = -19, x_2 = 17, x_3 = -3$
(B) $x_1 = -104, x_2 = 82, x_3 = -11$
(C) $x_1 = 100, x_2 = -78, x_3 = 10$

35. $x_1 = 2.5 + 2t, x_2 = t$, t any real number **37.** No solution

39. $x_1 = 13t + 3, x_2 = 8t + 1, x_3 = t$, t any real number

41. $X = (A - B)^{-1}C$ **43.** $X = (I - A)^{-1}C$

45. $X = (3I - A)^{-1}C$

47. (A) $x_1 = 1, x_2 = 0$ (B) $x_1 = -2{,}000, x_2 = 1{,}000$
(C) $x_1 = 2{,}001, x_2 = -1{,}000$

49. Concert 1: 6,000 \$4 tickets and 4,000 \$8 tickets; Concert 2: 5,000 \$4 tickets and 5,000 \$8 tickets; Concert 3: 3,000 \$4 tickets and 7,000 \$8 tickets

51. (A) $I_1 = 4, I_2 = 6, I_3 = 2$
(B) $I_1 = 3, I_2 = 7, I_3 = 4$
(C) $I_1 = 7, I_2 = 8, I_3 = 1$

53. (A) $a = 1, b = 4, c = -3$
(B) $a = -2, b = 5, c = 1$
(C) $a = 11, b = -46, c = 43$

55. Diet 1: 60 oz mix A and 80 oz mix B; Diet 2: 20 oz mix A and 60 oz mix B; Diet 3: 0 oz mix A and 100 oz mix B

Section 9-5

7. 7 **9.** -17 **11.** 9.79

13. $\begin{vmatrix} 4 & 6 \\ -2 & 8 \end{vmatrix}$ **15.** $\begin{vmatrix} 5 & -1 \\ 0 & -2 \end{vmatrix}$

17. $(-1)^{1+1}\begin{vmatrix} 4 & 6 \\ -2 & 8 \end{vmatrix} = 44$ **19.** $(-1)^{2+3}\begin{vmatrix} 5 & -1 \\ 0 & -2 \end{vmatrix} = 10$

21. 10 **23.** -21 **25.** -40

27. $(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$ **29.** $(-1)^{4+3}\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix}$

31. 22 **33.** -12 **35.** 0 **37.** 6 **39.** 60 **41.** 114 **43.** False **45.** True

47. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$; interchanging the rows of this determinant changes the sign.

49. $\begin{vmatrix} ka & b \\ kc & d \end{vmatrix} = k\begin{vmatrix} a & b \\ c & d \end{vmatrix}$; multiplying a column of this determinant by a number k multiplies the value of the determinant by k.

51. $\begin{vmatrix} kc + a & kd + b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$; adding a multiple of one row to the other row does not change the value of the determinant.

55. $49 = (-7)(-7)$ **57.** $f(x) = x^2 - 4x + 3$; 1, 3

59. $f(x) = x^3 + 2x^2 - 8x$; -4, 0, 2

Section 9-6

3. Theorem 1 **5.** Theorem 1 **7.** Theorem 2 **9.** Theorem 3
11. Theorem 5

13. $x = 0$ **15.** $x = 5$ **17.** -10 **19.** 10 **21.** -10 **23.** 25
25. -12 **27.** Theorem 1 **29.** Theorem 2 **31.** Theorem 5
33. $x = 5, y = 0$ **35.** $x = -3, y = 10$ **37.** -28 **39.** 106
41. 0 **43.** 6 **45.** 14

47. Expand the left side of the equation using minors.

49. Expand both sides of the equation and compare.

51. This follows from Theorem 4.

53. Expand the determinant about the first row to obtain $(y_1 - y_2)x - (x_1 - x_2)y + (x_1y_2 - x_2y_1) = 0$.

55. If the determinant is 0, then the area of the triangle formed by the three points is zero. The only way this can happen is if the three points are on the same line; that is, the points are colinear.

Section 9-7

5. $x = 5, y = -2$ **7.** $x = 1, y = -1$

9. $x = -\frac{6}{5}, y = \frac{3}{5}$ **11.** $x = \frac{2}{17}, y = -\frac{20}{17}$

13. $x = 6{,}400, y = 6{,}600$ **15.** $x = 760, y = 760$

17. $x = 2, y = -2, z = -1$

19. $x = \frac{4}{3}, y = -\frac{1}{3}, z = \frac{2}{3}$ **21.** $x = -9, y = -\frac{7}{3}, z = 6$

23. $x = \frac{3}{2}, y = -\frac{7}{6}, z = \frac{2}{3}$

25. If $a = \frac{3}{2}$ and $b = \frac{15}{4}$, there are an infinite number of solutions. If $a = \frac{3}{2}$ and $b \neq \frac{15}{4}$, there are no solutions. If $a \neq \frac{3}{2}$, there is one solution.

27. $x = 4$ **29.** $y = 2$

31. $z = \frac{5}{2}$ **33.** Since $D = 0$, the system either has no solution or infinitely many. Since $x = 0, y = 0, z = 0$ is a solution, the second case must hold.

37. (A) $R = 200p + 300q - 6p^2 + 6pq - 3q^2$
(B) $p = -0.3x - 0.4y + 180, q = -0.2x - 0.6y + 220$, $R = 180x + 220y - 0.3x^2 - 0.6xy - 0.6y^2$

Review Exercises

1. $\left[\begin{array}{cc|c} 3 & -6 & 12 \\ 1 & -4 & 5 \end{array}\right]$ *(9-1)* **2.** $\left[\begin{array}{cc|c} 1 & -4 & 5 \\ 1 & -2 & 4 \end{array}\right]$ *(9-1)*

3. $\left[\begin{array}{cc|c} 1 & -4 & 5 \\ 0 & 6 & -3 \end{array}\right]$ *(9-1)*

4. $\begin{aligned} x_1 &= 4 \\ x_2 &= -7 \end{aligned}$ The solution is $(4, -7)$ *(9-1)*

5. $\begin{aligned} x_1 - x_2 &= 4 \\ 0 &= 1 \end{aligned}$ No solution *(9-1)*

6. $x_1 - x_2 = 4$
$x_1 = t + 4, x_2 = t$ is the solution, for t any real number *(9-1)*

7. $\begin{bmatrix} 4 & 8 \\ -12 & 18 \end{bmatrix}$ *(9-2)* **8.** $[-11]$ *(9-2)*

9. $[-15 \quad 19]$ *(9-2)* **10.** $\begin{bmatrix} 16 \\ -6 \end{bmatrix}$ *(9-2)*

11. $\begin{bmatrix} 3 & 3 \\ -4 & 9 \end{bmatrix}$ *(9-2)* **12.** Not defined *(9-2)*

13. Not defined *(9-2)* **14.** $\begin{bmatrix} 13 & -29 \\ 20 & -24 \end{bmatrix}$ *(9-2)*

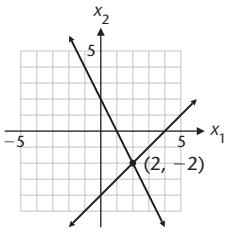
15. $[-5 \quad 18]$ *(9-2)* **16.** $\begin{bmatrix} 2 & 7 \\ -1 & -4 \end{bmatrix}$ *(9-3)*

17. (A) $x_1 = -1, x_2 = 3$ (B) $x_1 = 1, x_2 = 2$
(C) $x_1 = 8, x_2 = -10$ *(9-4)*

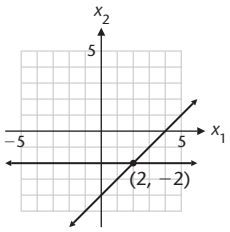
18. -17 *(9-5)* **19.** 0 *(9-5, 9-6)*

20. $x = 2, y = -1$ *(9-7)* **21.** (A) -2 (B) 6 (C) 2 *(9-6)*

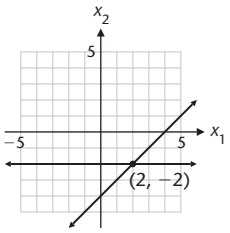
22. $x_1 = 2, x_2 = -2$; each pair of lines has the same intersection point. *(9-1)*



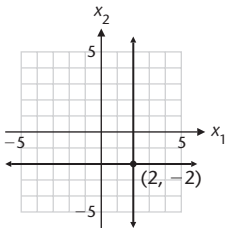
$x_1 - x_2 = 4$
$2x_1 + x_2 = 2$



$x_1 - x_2 = 4$
$3x_2 = -6$



$x_1 - x_2 = 4$
$x_2 = -2$



$x_1 = 2$
$x_2 = -2$

23. $x_1 = -1, x_2 = 3$ *(9-1)* **24.** $x_1 = -1, x_2 = 2, x_3 = 1$ *(9-1)*

25. $x_1 = 2, x_2 = 1, x_3 = -1$ *(9-1)*

26. $x_1 = -5t - 12, x_2 = 3t + 7, x_3 = t$ is a solution for every real number t. There are infinitely many solutions. *(9-1)*

27. No solution *(9-1)*

28. $x_1 = -\frac{3}{7}t - \frac{4}{7}, x_2 = \frac{5}{7}t + \frac{9}{7}, x_3 = t$ is a solution for every real number t. There are infinitely many solutions. *(9-1)*

29. $\begin{bmatrix} 7 & 16 & -9 \\ 28 & 40 & -30 \\ -21 & -8 & 17 \end{bmatrix}$ *(9-2)* **30.** $\begin{bmatrix} 22 & 19 \\ 38 & 42 \end{bmatrix}$ *(9-2)*

31. $\begin{bmatrix} 12 & 24 & -6 \\ 0 & 0 & 0 \\ -8 & -16 & 4 \end{bmatrix}$ *(9-2)* **32.** $[16]$ *(9-2)*

33. Not defined *(9-2)* **34.** $\begin{bmatrix} 63 & -24 & -39 \\ -42 & 16 & 26 \end{bmatrix}$ *(9-2)*

35. $\begin{bmatrix} -1 & 1 & 1 \\ -2 & 3 & 2 \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$ *(9-3)*

36. (A) $x_1 = 2, x_2 = 1, x_3 = -1$ (B) $x_1 = 1, x_2 = -2, x_3 = 1$
(C) $x_1 = -1, x_2 = 2, x_3 = -2$ *(9-4)*

37. $-\frac{11}{12}$ *(9-4)* **38.** 35 *(9-5, 9-6)*

39. $y = \frac{10}{5} = 2$ *(9-7)*

40. (A) A unique solution (B) Either no solution or an infinite number *(9-4)* **41.** No *(9-4)*

42. (A) A unique solution (B) No solution (C) An infinite number of solutions *(9-1)* **43.** $X = (A - C)^{-1}B$ *(9-4)*

44. $\begin{bmatrix} -\frac{11}{12} & -\frac{1}{12} & 5 \\ \frac{10}{12} & \frac{2}{12} & -4 \\ \frac{1}{12} & -\frac{1}{12} & 0 \end{bmatrix}$ or $\frac{1}{12}\begin{bmatrix} -11 & -1 & 60 \\ 10 & 2 & -48 \\ 1 & -1 & 0 \end{bmatrix}$ *(9-3)*

45. $x_1 = 1{,}000, x_2 = 4{,}000, x_3 = 2{,}000$ *(9-4)* **46.** 42 *(9-6)*

47. $\begin{vmatrix} u + kv & v \\ w + kx & x \end{vmatrix} = (u + kv)x - (w + kx)v$

$= ux + kvx - wv - kvx = ux - wv = \begin{vmatrix} u & v \\ w & x \end{vmatrix}$ *(9-6)*

48. Theorem 4 in Section 9.6 implies that both points satisfy the equation. All other points on the line through the given points will also satisfy the equation. *(9-6)*

49. $x_1 = 40$ grams of mix A, $x_2 = 60$ grams of mix B, $x_3 = 30$ grams mix C *(9-1)*

50. (A) $x_1 = 22$ nickels, $x_2 = 8$ dimes
(B) $x_1 = 3t + 22$ nickels, $x_2 = 8 - 4t$ dimes, $x_3 = t$ quarters, $t = 0$, 1, or 2 *(9-1)*

51. (A) 60 tons at Big Bend, 20 tons at Saw Pit
(B) 30 tons at Big Bend, 50 tons at Saw Pit
(C) 40 tons at Big Bend, 40 tons at Saw Pit *(9-4)*

52. (A) \$27
(B) Elements in LH give the total cost of manufacturing each product at each plant.
(C)

	North Carolina	South Carolina
$LH =$	\$46.35	\$41.00
	\$30.45	\$27.00

(9-2)

53. (A) $\begin{bmatrix} 1{,}600 & 1{,}730 \\ 890 & 720 \end{bmatrix}$ Desks / Stands

(B) $\begin{bmatrix} 200 & 160 \\ 80 & 40 \end{bmatrix}$

(C) $\begin{bmatrix} 3{,}150 \\ 1{,}550 \end{bmatrix}$ Desks / Stands

Total production of each item in January *(9-2)*

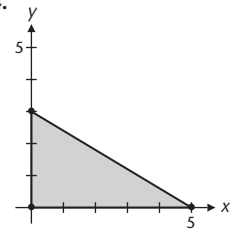
54. GRAPHING UTILITY *(9-3)*

Cumulative Review

1. $x = 2, y = -1$ *(8-1)* **2.** $(-1, 2)$ *(8-1)*

3. No solution *(8-1)*

4. *(8-4)*



5. Maximum: 33; Minimum: 10 *(8-5)*

6. (A) $\begin{bmatrix} 0 & -3 \\ 3 & -9 \end{bmatrix}$ (B) Not defined (C) $[3]$ (D) $\begin{bmatrix} 1 & 7 \\ 4 & -7 \end{bmatrix}$
(E) $[-1, 8]$ (F) Not defined *(9-2)*

7. -10 *(9-5)* **8.** (A) $x_1 = 3, x_2 = -4$ (B) $x_1 = 2t + 3, x_2 = t$, t any real number (C) No solution *(9-1)*

9. (A) $\left[\begin{array}{cc|c} 1 & 1 & 3 \\ -1 & 1 & 5 \end{array}\right]$ (B) $\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 4 \end{array}\right]$

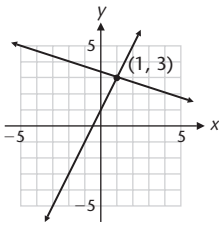
(C) $x_1 = -1, x_2 = 4$ *(9-1, 9-2)*

10. (A) $\begin{bmatrix} 1 & -3 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$ (B) $A^{-1} = \begin{bmatrix} -5 & 3 \\ -2 & 1 \end{bmatrix}$

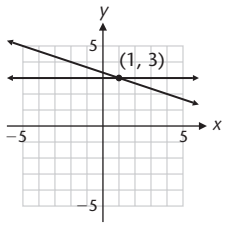
(C) $x_1 = 13, x_2 = 5$ (D) $x_1 = -11, x_2 = -4$ *(9-4)*

11. (A) 2 (B) $x = \frac{1}{2}, y = 0$ *(9-7)*

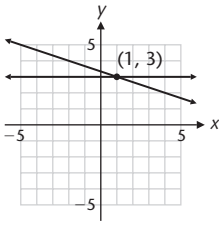
12. $x_1 = 1, x_2 = 3$; each pair of lines has the same intersection point. *(9-1)*



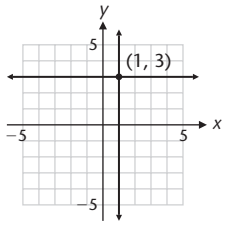
$x_1 + 3x_2 = 10$
$2x_1 - x_2 = -1$



$x_1 + 3x_2 = 10$
$-7x_2 = -21$



$x_1 + 3x_2 = 10$
$x_2 = 3$



$x_1 = 1$
$x_2 = 3$

13. (1.53, 3.35) *(8-1)* **14.** (1, 0, −2) *(9-1)*

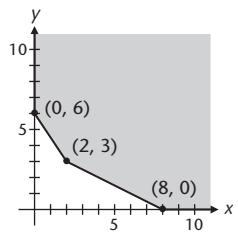
15. No solution *(9-1)*

16. $(t - 3, t - 2, t)$ t any real number *(9-1)*

17. (A) $[-3]$ (B) $\begin{bmatrix} 1 & 2 & -1 \\ -1 & -2 & 1 \\ 2 & 4 & -2 \end{bmatrix}$ *(9-2)*

18. (A) $\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$ (B) Not defined *(9-2)*

19. *(9-5)*



20. 63 *(8-5)*

21. (A) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 6 & 3 \\ 2 & 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$ (B) $A^{-1} = \begin{bmatrix} -3 & 2 & 0 \\ 2 & -2 & 1 \\ -2 & 3 & -2 \end{bmatrix}$

(C) $(7, -5, 6)$ (D) $(-6, 3, -2)$ *(9-4)*

22. (A) $D = 1$ (B) $z = 32$ *(9-6, 9-7)*

23. (A) Infinite number of solutions (B) No solution
(C) Unique solution *(9-1)*

24. $A = I$, the $n \times n$ identity *(9-4)* **25.** L, M, and P *(9-1)*

29. True *(9-2)* **30.** True *(9-1)* **31.** False *(9-2)*

32. False *(9-2)* **33.** True *(9-3)* **34.** True *(9-3)*

35. True *(9-5)* **36.** True *(9-5)*

37. $8,000 at 8% and $4,000 at 14% *(8-1)*

38. 60 grams of mix A, 50 grams of mix B, 40 grams of mix C *(9-1)*

39. 1 model A truck, 6 model B trucks, and 5 model C trucks; or 3 model A trucks, 3 model B trucks, and 6 model C trucks; or 5 model A trucks and 7 model C trucks. *(9-1)*

40. (A) Manufacturing 400 standard and 200 deluxe day packs produces a maximum weekly profit of $5,600.
(B) The maximum weekly profit increases to $6,000 when 0 standard and 400 deluxe day packs are manufactured.
(C) The maximum weekly profit increases to $6,000 when 600 standard and 0 deluxe day packs are manufactured. *(8-5)*

41. (A) $M\begin{bmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 82.25 \\ 83 \\ 92 \\ 83.75 \\ 82 \end{bmatrix}$ Ann, Bob, Carol, Dan, Eric

(B) $M\begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 83 \\ 84.8 \\ 91.8 \\ 85.2 \\ 80.8 \end{bmatrix}$ Ann, Bob, Carol, Dan, Eric

(C)

						Class averages			
						Test 1	Test 2	Test 3	Test 4
[0.2	0.2	0.2	0.2	0.2]	$M =$	[84.4	81.8	85	87.2]

(9-2)

Chapter 10

Section 10-1

5. $-1, 0, 1, 2$ **7.** $0, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}$ **9.** $4, -8, 16, -32$

11. 6 **13.** $\frac{99}{101}$ **15.** $1 + 2 + 3 + 4 + 5$ **17.** $\frac{1}{10} + \frac{1}{100} + \frac{1}{1{,}000}$

19. $-1 + 1 - 1 + 1$ **21.** $1, -4, 9, -16, 25$

23. 0.3, 0.33, 0.333, 0.3333, 0.33333

25. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}$ **27.** $7, 3, -1, -5, -9$

29. $4, 1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}$ **31.** $1, 2, 5, 12, 29, 70, 169$

33. $-1, 2, 0, 4, 4, 12, 20$ **35.** $a_n = n + 3$ **37.** $a_n = 3n$

39. $a_n = \dfrac{n}{n+1}$ **41.** $a_n = (-1)^{n+1}$ **43.** $a_n = (-2)^n$

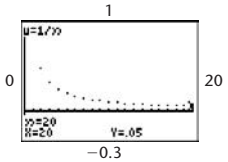
45. $a_n = \dfrac{x^n}{n}$

47. $a_n = 2^n$ and $b_n = 0.5n^2 - 0.5n + 1$ are two possible answers.

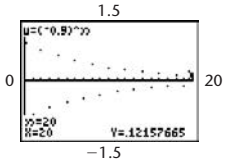
49. $a_n = n^3$ and $b_n = 6n^2 - 11n + 6$ are two possible answers.

51. $a_n = (-1)^n \cdot 3n$ and $a_n = -12n^2 + 45n - 36$ are two possible answers.

53.



55.



57. $\frac{4}{1} - \frac{8}{2} + \frac{16}{3} - \frac{32}{4}$ **59.** $x^2 + \frac{x^3}{2} + \frac{x^4}{3}$

61. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$ **63.** $\sum_{k=1}^{4} k^2$

65. $\sum_{k=1}^{5} \frac{1}{2^k}$ **67.** $\sum_{k=1}^{n} \frac{1}{k^2}$ **69.** $\sum_{k=1}^{n} (-1)^{k+1}k^2$

71. (A) 3, 1.83, 1.46, 1.415 (B) Calculator $\sqrt{2} = 1.4142135\ldots$
(C) $a_1 = 1$; 1, 1.5, 1.417, 1.414

73. The values of c_n are approximately 2.236 (i.e., $\sqrt{5}$) for large values of n.

75. $e^{0.2} = 1.2214000$; $e^{0.2} = 1.2214028$ (calculator—direct evaluation)

79. (A) 5/8 = 0.625 ft; 5/256 ≈ 0.01953 ft (B) 19.98; this is the sum of the heights of all 10 bounces.

81. (A) 40,000; 41,600; 43,264; 44,995; 46,795; 48,666
(B) $a_n = 40{,}000(1.04)^{n-1}$
(C) 265, 319.02; this is the total amount earned in the first 6 years.

Section 10-2

3. Fails at $n = 2$ **5.** Fails at $n = 3$

7. (A) P_1: $2 = 2 \cdot 1^2$; P_2: $2 + 6 = 2 \cdot 2^2$; P_3: $2 + 6 + 10 = 2 \cdot 3^2$
(B) P_k: $2 + 6 + 10 + \cdots + (4k - 2) = 2k^2$
P_{k+1}: $2 + 6 + 10 + \cdots + (4k - 2) + (4k + 2) = 2(k + 1)^2$

9. (A) P_1: $a^5a^1 = a^{5+1}$; P_2: $a^5a^2 = a^5(a^1a) = (a^5a)a = a^6a = a^7 = a^{5+2}$; P_3: $a^5a^3 = a^5(a^2a) = a^5(a^1a)a = [(a^5a)a]a = a^8 = a^{5+3}$
(B) P_k: $a^5a^k = a^{5+k}$; P_{k+1}: $a^5a^{k+1} = a^{5+k+1}$

11. (A) P_1: $9^1 - 1 = 8$ is divisible by 4;
P_2: $9^2 - 1 = 80$ is divisible by 4; P_3: $9^3 - 1 = 728$ is divisible by 4
(B) P_k: $9^k - 1 = 4r$ for some integer r;
P_{k+1}: $9^{k+1} - 1 = 4s$ for some integer s

13. $n = 4$, $p(x) = x^4 + 1$ **15.** $n = 23$

33. P_n: $2 + 4 + 6 + \cdots + 2n = n(n + 1)$

35. $1 + 2 + 3 + \cdots + (n - 1) = \frac{n(n - 1)}{2}, n \geq 2$

36. $2 + 3 + 4 + \cdots + (n - 2) = \frac{n(n - 3)}{2}, n > 3$

41. $3^4 + 4^4 + 5^4 + 6^4 \neq 7^4$

Section 10-3

5. (A) Arithmetic with $d = -5$; $-26, -31$
(B) Geometric with $r = -2$; $-16, 32$ (C) Neither
(D) Geometric with $r = \frac{1}{3}; \frac{1}{54}, \frac{1}{162}$

7. $a_2 = -1$; $a_3 = 3$; $a_4 = 7$ **9.** $a_{15} = 67$; $S_{11} = 242$

11. $S_{21} = 861$ **13.** $a_{15} = -21$

15. $a_2 = 3$; $a_3 = -\frac{3}{2}$; $a_4 = \frac{3}{4}$ **17.** $a_{10} = \frac{1}{243}$ **19.** $S_7 = 3{,}279$

21. $d = 6$; $a_{101} = 603$ **23.** $S_{40} = 200$ **25.** $a_{11} = 2$; $S_{11} = \frac{77}{6}$

27. $a_1 = 1$ **29.** $r = 0.398$ **31.** $S_{10} = -1{,}705$ **33.** $a_2 = 6$; $a_3 = 4$

35. $S_{51} = 4{,}131$ **37.** $S_7 = 547$ **39.** $-1{,}071$

41. $\frac{1{,}023}{1{,}024}$ **43.** 4,446 **47.** $x = 2\sqrt{3}$

49. $a_n = -3 + (n - 1)3$ or $3n - 6$

51. $S_n = n \cdot a_1$ **53.** 66 **55.** 133 **57.** $S_n = \frac{9}{2}$ **59.** No sum

61. $S_n = \frac{8}{5}$ **63.** $\frac{7}{9}$ **65.** $\frac{6}{11}$ **67.** $3\frac{8}{37}$ or $\frac{119}{37}$ **71.** $a_n = (-2)(-3)^{n-1}$

77. $x = -1, y = 2$

79. $S_5 = 31$; $S_{10} = 1{,}023$; $S_{15} = 32{,}767$; $S_{20} = 1{,}048{,}575$. The value of S_n appears to approach infinity as n increases.

81. Firm A: \$501,000; Firm B: \$504,000

83. \$4,000,000 **85.** $P(1 + r)^n$; approximately 12 yr

87. \$700 per year; \$115,500 **89.** 900 **91.** 1,250,000

93. (A) 336 ft (B) 1,936 ft (C) $16t^2$ ft **95.** $A = A_0 2^{2t}$

97. $r = 10^{-0.4} = 0.398$

99. 9.22×10^{16} dollars; 1.845×10^{17} dollars

101. 0.0015 pounds per square inch **103.** 2 **105.** 3,420°

Section 10-4

7. 362,880 **9.** 39,916,800 **11.** 990 **13.** 10 **15.** 35 **17.** 1

21. 60 **23.** 6,497,400 **25.** 10 **27.** 270,725 **29.** (A) Permutation
(B) Combination **31.** $P_{10,3} = 10 \cdot 9 \cdot 8 = 720$

33. $C_{7,3} = 35$ subcommittees; $P_{7,3} = 210$ **35.** $C_{10,2} = 45$

37. No repeats: $6 \cdot 5 \cdot 4 \cdot 3 = 360$; with repeats: $6 \cdot 6 \cdot 6 \cdot 6 = 1{,}296$

39. No repeats: $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 30{,}240$; with repeats:
$10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100{,}000$

41. $C_{13,5} = 1{,}287$

43. $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$ possible license plates; no repeats: $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 = 11{,}232{,}000$

45. $C_{13,5} \cdot C_{13,2} = 100{,}386$ **47.** $5 \cdot 3 \cdot 4 \cdot 2 = 120$

49. $C_{8,3} \cdot C_{10,4} \cdot C_{7,2} = 246{,}960$ **51.** $12 \cdot 11 = 132$

53. There are $C_{4,1} \cdot C_{48,4} = 778{,}320$ hands that contain exactly one king, and $C_{39,5} = 575{,}757$ hands containing no hearts, so the former is more likely.

55. Two people: $5 \cdot 4 = 20$; three people: $5 \cdot 4 \cdot 3 = 60$; four people: $5 \cdot 4 \cdot 3 \cdot 2 = 120$; five people: $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

57. (A) $P_{8,5} = 6{,}720$ (B) $C_{8,5} = 56$ (C) $C_{2,1} \cdot C_{6,4} = 30$

59. (A) $C_{8,2} = 28$ (B) $C_{8,3} = 56$ (C) $C_{8,4} = 70$

61. (B) $r = 0.10$ (C) Each is the product of r consecutive integers, the largest of which is n for $P_{n,r}$ and r for $r!$.

Section 10-5

9. (A) No probability can be negative.
(B) $P(R) + P(G) + P(Y) + P(B) \neq 1$
(C) This is an acceptable probability assignment.

11. $P(R) + P(Y) = .56$ **13.** .1 **15.** .45 **17.** Empirical

19. Theoretical

21. $P(E) = \frac{n(E)}{n(S)} = \frac{1}{720} = .0014$

23. $\frac{C_{26,5}}{C_{52,5}} \approx .025$ **25.** $\frac{C_{16,5}}{C_{52,5}} \approx .0017$

27. Yes; no **29.** $P(E) = \frac{n(E)}{n(S)} = \frac{50}{250} = .2$

31. $P(E) = \frac{n(E)}{n(S)} = \frac{1}{120} \approx .008$ **33.** $\frac{1}{36}$ **35.** $\frac{5}{36}$ **37.** $\frac{1}{6}$

39. $\frac{7}{9}$ **41.** 0 **43.** $\frac{1}{3}$ **45.** $\frac{2}{9}$ **47.** $\frac{2}{3}$

49. $S = \{1, 2, 3, \ldots, 365\}$; $P(e_i) = 1/365$

51. (A) $P(2) \approx .022$, $P(3) \approx .07$, $P(4) \approx .088$, $P(5) \approx .1$, $P(6) \approx .142$, $P(7) \approx .178$, $P(8) \approx .144$, $P(9) \approx .104$, $P(10) \approx .072$, $P(11) \approx .052$, $P(12) \approx .028$

(B) $P(2) = \frac{1}{36}$, $P(3) = \frac{2}{36}$, $P(4) = \frac{3}{36}$, $P(5) = \frac{4}{36}$, $P(6) = \frac{5}{36}$,
$P(7) = \frac{6}{36}$, $P(8) = \frac{5}{36}$, $P(9) = \frac{4}{36}$, $P(10) = \frac{3}{36}$, $P(11) = \frac{2}{36}$,
$P(12) = \frac{1}{36}$

(C) Sum 2 3 4 5 6 7 8 9 10 11 12
Expected
frequency 13.9 27.8 41.7 55.6 69.4 83.3 69.4 55.6 41.7 27.8 13.9

55. $\frac{1}{4}$ **57.** $\frac{1}{4}$ **59.** $\frac{3}{4}$ **61.** $\frac{1}{9}$ **63.** $\frac{1}{3}$ **65.** $\frac{1}{9}$

67. $\frac{4}{9}$ **69.** $\frac{C_{16,5}}{C_{52,5}} \approx .001\,68$ **71.** $\frac{48}{C_{52,5}} \approx .000\,0185$

73. $\frac{4}{C_{52,5}} \approx .000\,0015$ **75.** $\frac{C_{4,2}\,C_{4,3}}{C_{52,5}} \approx .000\,009$

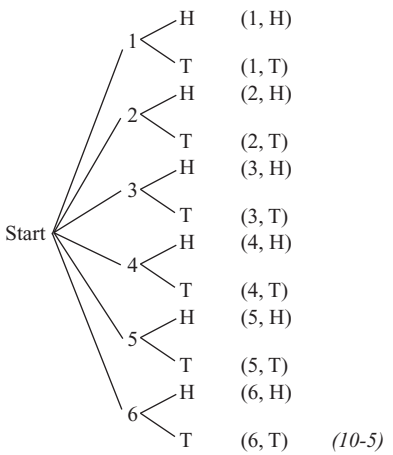
77. (A) 0.15 (B) .222 (C) .169 (D) .958

Section 10-6

5. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
7. $x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$
9. 10 **11.** 6 **13.** 20 **15.** 35 **17.** 84 **19.** 66
21. 2,380 **23.** 230,300 **25.** $m^3 + 3m^2n + 3mn^2 + n^3$
27. $8x^3 - 36x^2y + 54xy^2 - 27y^3$
29. $x^4 - 8x^3 + 24x^2 - 32x + 16$
31. $m^4 + 12m^3n + 54m^2n^2 + 108mn^3 + 81n^4$
33. $32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$
35. $m^6 + 12m^5n + 60m^4n^2 + 160m^3n^3 + 240m^2n^4 + 192mn^5 + 64n^6$
37. $35x^4$ **39.** $-29{,}568x^6$ **41.** $4{,}060{,}938{,}240x^{14}$
43. $15x^8$ **45.** Does not exist **47.** $5{,}005u^9v^6$ **49.** $264m^2n^{10}$
51. $924w^6$ **53.** $-48{,}384x^3y^5$ **55.** $3x^2 + 3xh + h^2$; approaches $3x^2$
57. $5x^4 + 10x^3h + 10x^2h^2 + 5xh^3 + h^4$; approaches $5x^4$
59. 5 **61.** (A) $a_4 = 0.251$ (B) 1 **63.** 1.1046

Review Exercises

1. (A) Geometric (B) Arithmetic (C) Arithmetic (D) Neither
(E) Geometric *(10-1, 10-3)*
2. (A) 5, 7, 9, 11 (B) $a_{10} = 23$ (C) $S_{10} = 140$ *(10-1, 10-3)*
3. (A) 16, 8, 4, 2 (B) $a_{10} = \frac{1}{32}$ (C) $S_{10} = 31\frac{31}{32}$ *(10-1, 10-3)*
4. (A) $-8, -5, -2, 1$ (B) $a_{10} = 19$ (C) $S_{10} = 55$ *(10-1, 10-3)*
5. (A) $-1, 2, -4, 8$ (B) $a_{10} = 512$ (C) $S_{10} = 341$ *(10-1, 10-3)*
6. $S_\infty = 32$ *(10-3)* **7.** 720 *(10-4)*
8. $\frac{22 \cdot 21 \cdot 20 \cdot 19!}{19!} = 9{,}240$ *(10-4)* **9.** 21 *(10-4)*
10. $C_{6,2} = 15$; $P_{6,2} = 30$ *(10-5)*
11. (A) 12 combined outcomes (B) $6 \cdot 2 = 12$ *(10-5)*

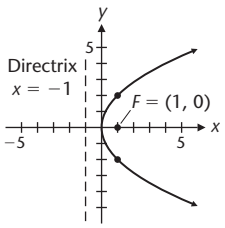


12. $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ *(10-5)* **13.** $P_{6,6} = 6! = 720$ *(10-5)*
14. $\frac{C_{13,5}}{C_{52,5}} \approx .0005$ *(10-5)* **15.** $\frac{1}{P_{15,2}} \approx .005$ *(10-5)*
16. .05 *(10-5)*
17. P_1: $5 = 1^2 + 4 \cdot 1 = 5$; P_2: $5 + 7 = 2^2 + 4 \cdot 2$; P_3: $5 + 7 + 9 = 3^2 + 4 \cdot 3$ *(10-2)*
18. P_1: $2 = 2^{1+1} - 2$; P_2: $2 + 4 = 2^{2+1} - 2$; P_3: $2 + 4 + 8 = 2^{3+1} - 2$ *(10-2)*
19. P_1: $49^1 - 1 = 48$ is divisible by 6; P_2: $49^2 - 1 = 2{,}400$ is divisible by 6; P_3: $49^3 - 1 = 117{,}648$ is divisible by 6 *(10-2)*
20. P_k: $5 + 7 + 9 + \cdots + (2k + 3) = k^2 + 4k$
P_{k+1}: $5 + 7 + 9 + \cdots + (2k + 3) + (2k + 5) = (k + 1)^2 + 4(k + 1)$ *(10-2)*
21. P_k: $2 + 4 + 8 + \cdots + 2^k = 2^{k+1} - 2$
P_{k+1}: $2 + 4 + 8 + \cdots + 2^k + 2^{k+1} = 2^{k+1} - 2$ *(10-2)*
22. P_k: $49^k - 1 = 6r$ for some integer r P_{k+1}: $49^{k+1} - 1 = 6s$ for some integer s *(10-2)*
23. $n = 31$ is a counterexample. *(10-2)*
24. $S_{10} = (-6) + (-4) + (-2) + 0 + 2 + 4 + 6 + 8 + 10 + 12 = 30$ *(10-3)*
25. $S_7 = 8 + 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 15\frac{7}{8}$ *(10-3)*
26. $S_\infty = \frac{81}{5}$ (10-3)
27. $S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{3^k}$; $S_\infty = \frac{1}{4}$ *(10-3)*
28. The probability of an event cannot be negative, but $P(e_2)$ is given as negative. The sum of the probabilities of the simple events must be 1, but it is given as 2.5. The probability of an event cannot be greater than 1, but $P(e_4)$ is given as 2. *(10-5)*
29. $C_{6,3} = 20$ *(10-5)* **30.** $d = 3$, $a_5 = 25$ *(10-3)*
31. 336; 512; 392 *(10-4)*
32. (A) P(2 heads) $\approx .21$; P(1 head) $\approx .48$; P(0 heads) $\approx .31$
(B) $P(E_1) = .25$; $P(E_2) = .5$; $P(E_3) = .25$
(C) 2 heads = 250; 1 head = 500; 0 heads = 250 *(10-5)*
33. (A) $\frac{C_{13,5}}{C_{52,5}}$ (B) $\frac{C_{13,3} \cdot C_{13,2}}{C_{52,5}}$ *(10-5)*
34. $\frac{C_{8,2}}{C_{10,4}} = \frac{2}{15}$ *(10-5)* **35.** (A) $\frac{1}{3}$ (B) $\frac{2}{9}$ *(10-5)* **36.** $\frac{8}{11}$ *(10-3)*
37. (A) $P_{6,3} = 120$ (B) $C_{5,2} = 10$ *(10-4)*
38. 190 *(10-6)* **39.** 1,820 *(10-6)* **40.** 1 *(10-6)*
41. $x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$ *(10-6)*
42. $672x^6$ *(10-6)* **43.** $-1{,}760x^3y^9$ *(10-6)* **47.** 29 *(10-6)*
48. 26 *(10-1)*
49. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$; 6 *(10-4)* **50.** $\frac{49g}{2}$ feet; $\frac{625g}{2}$ feet *(10-3)*
51. 12 *(10-4)*
52. $x^6 + 6ix^5 - 15x^4 - 20ix^3 + 15x^2 + 6ix - 1$ *(10-6)*
53. $1 - \frac{C_{7,3}}{C_{10,3}} = \frac{17}{24}$ *(10-5)*
54. (A) .350 (B) $\frac{3}{8} = .375$ (C) 375 *(10-5)*
60. \$900 *(10-3)* **61.** \$7,200 *(10-3)* **62.** \$895.42; \$1,603.57 *(10-3)*
63. $P_{5,5} = 120$ *(10-4)* **64.** (A) .04 (B) .16 (C) .54 (10-5)
65. $1 - \frac{C_{10,4}}{C_{12,4}} \approx .576$ *(10-5)*

Chapter 11

Section 11-1

7.



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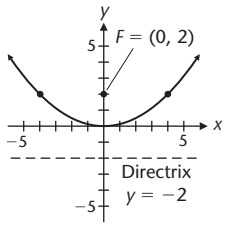

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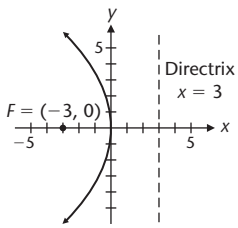

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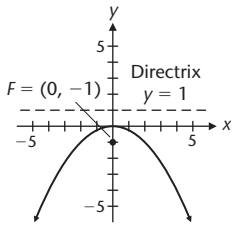

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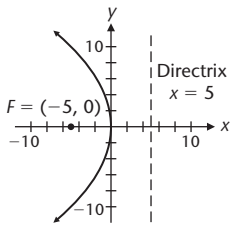

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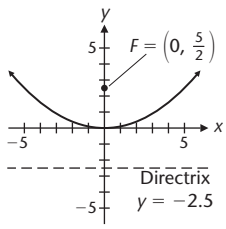
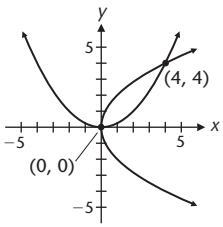


19. (9.75, 0) **21.** (0, −26.25) **23.** (−19.25, 0) **25.** $x^2 = 12y$
27. $x^2 = -28y$ **29.** $y^2 = -24x$ **31.** $y^2 = 8x$ **33.** $x^2 = 8y$
35. $y^2 = -12x$ **37.** $x^2 = -4y$

39.



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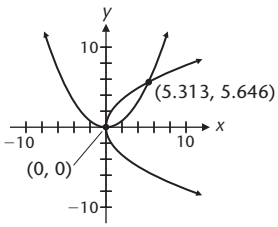


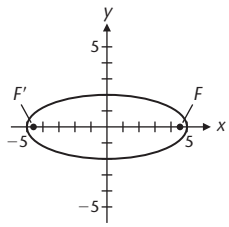
43. (*a*) 2: $x = 0$ and $y = 0$ (*b*) $(0, 0), (4am, 4am^2)$
45. $A = (-2a, a), B = (2a, a)$ **47.** $x^2 - 4x - 12y - 8 = 0$
49. $y^2 + 8y - 8x + 48 = 0$
51. $\sqrt{(x - x)^2 + (y + a)^2} = \sqrt{(x - 0)^2 + (y - a)^2}$

$$(y + a)^2 = x^2 + (y - a)^2$$
$$y^2 + 2ay + a^2 = x^2 + y^2 - 2ay + a^2$$
$$x^2 = 4ay$$

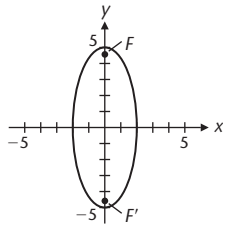
53. $x^2 = -200y$
55. (A) $y = 0.0025x^2 - 100 \le x \le 100$ (B) 25 feet

Section 11-2

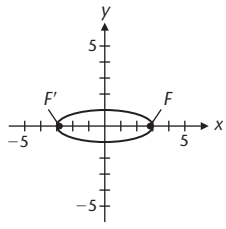
7. Foci: $F' = (-\sqrt{21}, 0)$, $F = (\sqrt{21}, 0)$;
major axis length = 10;
minor axis length = 4



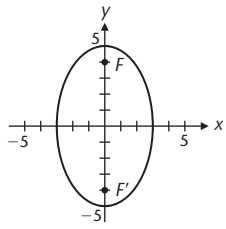
9. Foci: $F' = (0, -\sqrt{21})$, $F = (0, \sqrt{21})$;
major axis length = 10;
minor axis length = 4



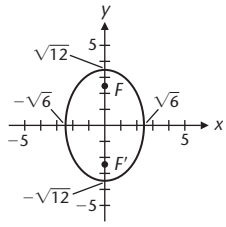
11. Foci: $F' = (-\sqrt{8}, 0)$, $F = (\sqrt{8}, 0)$;
major axis length = 6;
minor axis length = 2



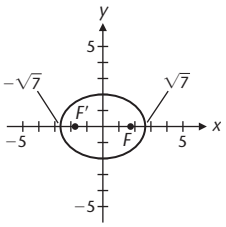
13. (b) **15.** (a)
17. Foci: $F' = (0, -4)$, $F = (0, 4)$;
major axis length = 10;
minor axis length = 6



19. Foci: $F' = (0, -\sqrt{6})$, $F = (0, \sqrt{6})$;
major axis length = $2\sqrt{12} \approx 6.93$;
minor axis length = $2\sqrt{6} \approx 4.90$



21. Foci: $F' = (-\sqrt{3}, 0)$, $F = (\sqrt{3}, 0)$;
major axis length $= 2\sqrt{7} \approx 5.29$;
minor axis length $= 4$



23. $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ **25.** $\dfrac{x^2}{9} + \dfrac{y^2}{36} = 1$

27. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ **29.** $\dfrac{x^2}{64} + \dfrac{y^2}{121} = 1$

31. $\dfrac{x^2}{64} + \dfrac{y^2}{28} = 1$ **33.** $\dfrac{x^2}{100} + \dfrac{y^2}{170} = 1$

35. It does not pass the vertical line test.

37. $\dfrac{x^2}{16} + \dfrac{y^2}{12} = 1$: ellipse

39. All points that lie on the line segment $F'F$.

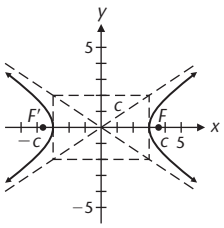
43. $\dfrac{x^2}{400} + \dfrac{y^2}{144} = 1$; 7.94 feet approximately

45. (A) $\dfrac{x^2}{576} + \dfrac{y^2}{15.9} = 1$ (B) 5.13 feet

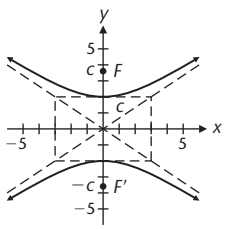
Section 11-3

7. (d) **9.** (c)

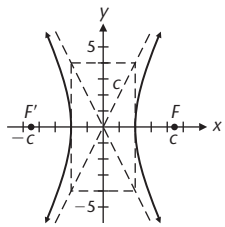
11. Foci: $F' = (-\sqrt{13}, 0)$, $F = (\sqrt{13}, 0)$;
transverse axis length $= 6$;
conjugate axis length $= 4$



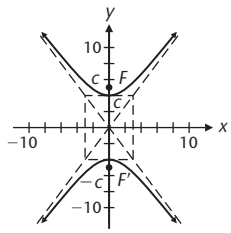
13. Foci: $F' = (0, -\sqrt{13})$, $F = (0, \sqrt{13})$;
transverse axis length $= 4$;
conjugate axis length $= 6$



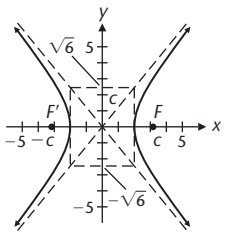
15. Foci: $F' = (-\sqrt{20}, 0)$, $F = (\sqrt{20}, 0)$;
transverse axis length $= 4$;
conjugate axis length $= 8$



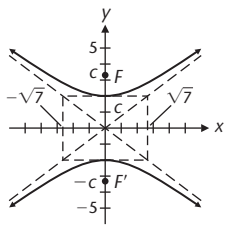
17. Foci: $F' = (0, -5)$, $F = (0, 5)$;
transverse axis length $= 8$;
conjugate axis length $= 6$



19. Foci: $F' = (-\sqrt{10}, 0)$, $F = (\sqrt{10}, 0)$;
transverse axis length $= 4$;
conjugate axis length $= 2\sqrt{6} \approx 4.90$



21. Foci: $F' = (0, -\sqrt{11})$, $F = (0, \sqrt{11})$;
transverse axis length $= 4$;
conjugate axis length $= 2\sqrt{7} \approx 5.29$



23. $\dfrac{x^2}{9} - \dfrac{y^2}{9} = 1$ **25.** $\dfrac{y^2}{16} - \dfrac{x^2}{16} = 1$ **27.** $\dfrac{x^2}{49} - \dfrac{y^2}{25} = 1$

29. $\dfrac{y^2}{144} - \dfrac{x^2}{81} = 1$ **31.** $\dfrac{x^2}{81} - \dfrac{y^2}{40} = 1$ **33.** $\dfrac{y^2}{151} - \dfrac{x^2}{49} = 1$

35. $y = \pm\frac{2}{5}x$ **37.** $y = \pm\frac{1}{2}x$ **39.** $y = \pm 3x$ **41.** $y = \pm\dfrac{\sqrt{3}}{\sqrt{2}}x$

43. (A) Infinitely many; $\frac{x^2}{a^2} - \frac{y^2}{1-a^2} = 1\ (0 < a < 1)$

(B) Infinitely many; $\frac{x^2}{a^2} + \frac{y^2}{a^2-1} = 1\ (a > 1)$ (C) One; $y^2 = 4x$

45. (A) $(2/\sqrt{3}, 1/\sqrt{3}), (-2/\sqrt{3}, -1/\sqrt{3})$
(B) No intersection points
The graphs intersect at $x = \pm 1/(\sqrt{1-m^2})$ and $y = \pm m/(\sqrt{1-m^2})$ for $-1 < m < 1$.

47. (A) No intersection points (B) $(1/\sqrt{5}, 3/\sqrt{5}), (-1/\sqrt{5}, -3/\sqrt{5})$
The graphs intersect at $x = \pm 1/(\sqrt{m^2-4})$ and $y = \pm m/(\sqrt{m^2-4})$ for $m < -2$ or $m > 2$.

51. The empty set 51. $\frac{x^2}{4} - \frac{y^2}{5} = 1$; hyperbola

57. $\frac{y^2}{16} - \frac{x^2}{8} = 1$; 5.38 feet above vertex **59.** $y = \frac{4}{3}\sqrt{x^2 + 30^2}$

Section 11-4

1. (A) $x' = x - 3, y' = y - 5$ (B) $x'^2 + y'^2 = 81$ (C) Circle

3. (A) $x' = x + 7, y' = y - 4$ (B) $\frac{x'^2}{9} + \frac{y'^2}{16} = 1$ (C) Ellipse

5. (A) $x' = x - 4, y' = y + 9$ (B) $y'^2 = 16x'$ (C) Parabola

7. (A) $x' = x + 8, y' = y + 3$ (B) $\frac{x'^2}{12} + \frac{y'^2}{8} = 1$ (C) Hyperbola

9. (A) $\frac{(x-3)^2}{9} - \frac{(y+2)^2}{16} = 1$ (B) Hyperbola

11. (A) $\frac{(x+5)^2}{5} + \frac{(y+7)^2}{6} = 1$ (B) Ellipse

13. (A) $(x+6)^2 = -24(y-4)$ (B) Parabola

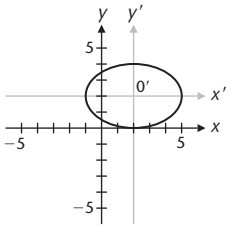
15. $(\sqrt{3}/2, -1/2), (1/2, \sqrt{3}/2), ((\sqrt{3}-1)/2, (-1-\sqrt{3})/2), ((-3\sqrt{3}+4)/2, (3+4\sqrt{3})/2)$

17. $(\sqrt{2}/2, -\sqrt{2}/2), (\sqrt{2}/2, \sqrt{2}/2), (-3\sqrt{2}/2, -\sqrt{2}/2), (-\sqrt{2}, -2\sqrt{2})$

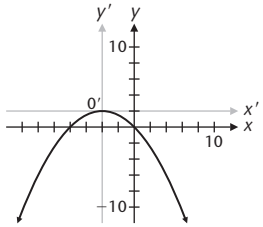
19. y' axis: $y = -\sqrt{3}x$; x' axis: $y = \frac{1}{\sqrt{3}}x$

21. y' axis: $y = -x$; x' axis: $y = x$

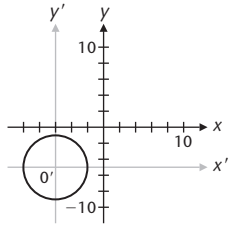
23. $\frac{(x-2)^2}{9} + \frac{(y-2)^2}{4} = 1$; ellipse



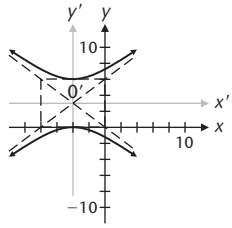
25. $(x+4)^2 = -8(y-2)$; parabola



27. $(x+6)^2 + (y+5)^2 = 16$; circle



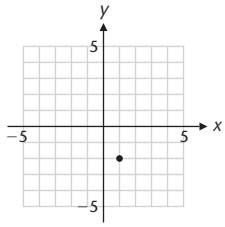
29. $\frac{(y-3)^2}{9} - \frac{(x+4)^2}{16} = 1$; hyperbola



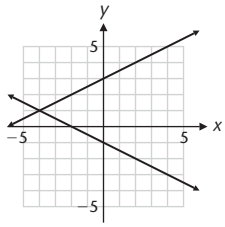
31. $F' = (-\sqrt{5}+2, 2)$ and $F = (\sqrt{5}+2, 2)$

33. $F = (-4, 0)$ **35.** $F' = (-4, -2)$, $F = (-4, 8)$

37. $(x-1)^2 + (y+2)^2 = 0$; the point $(1, -2)$ (a degenerate circle)

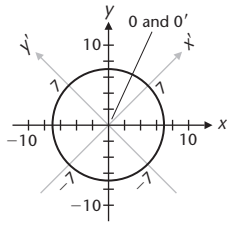


39. $(x+4)^2 - 4(y-1)^2 = 0$; the lines $y = 0.5x + 3$ and $y = -0.5x - 1$, intersecting at $(-4, 1)$ (a degenerate hyperbola)

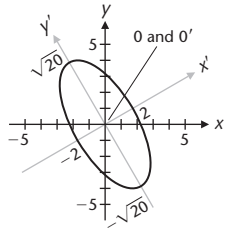


41. $h = \frac{-D}{2A}, k = \frac{D^2 - 4AF}{4AE}$

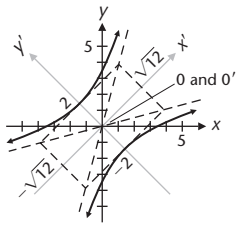
43. $x'^2 + y'^2 = 49$; circle



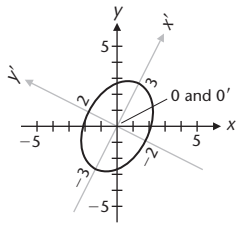
45. $\frac{x'^2}{4} + \frac{y'^2}{20} = 1$; ellipse



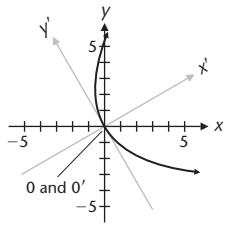
47. $\frac{y'^2}{4} - \frac{x'^2}{12} = 1$; hyperbola $\theta = 45°$



49. $\frac{x'^2}{9} + \frac{y'^2}{4} = 1$; ellipse, $\theta = 63.43°$



51. $y'^2 = 8x'$; parabola $\theta = 30°$



53. $y + 2 = \pm(x - 3)$ **55.** $y + 1 = \pm\frac{5}{2}x$ **57.** $y - 5 = \pm\frac{4}{3}(x + 2)$

59. $y + 4 = \pm\frac{1}{\sqrt{3}}x$ **61.** $x = 0, y = 0$ **63.** $x^2 - 4x + 4y - 16 = 0$

65. $x^2 + 4y^2 + 4x + 24y + 24 = 0$

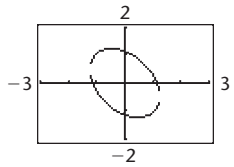
67. $25x^2 + 9y^2 - 200x + 36y + 211 = 0$

69. $4x^2 - y^2 - 16x + 6y + 11 = 0$

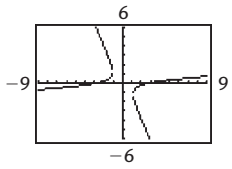
71. $9x^2 + y^2 + 36x - 2y + 28 = 0$

73. $x^2 - 2y^2 - 2x + 8y - 8 = 0$

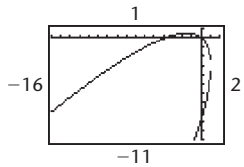
75. Ellipse



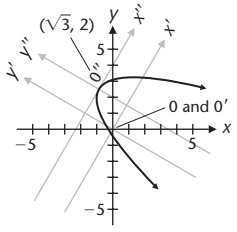
77. Hyperbola



79. Parabola



81. $\theta = 60°$; $x'^2 - 2\sqrt{3}x' + 2y' - 1 = 0$;
translate $0'$ to $(\sqrt{3}, 2)$; $x''^2 = -2y''$; parabola



Section 11-5

3. $(-12, 5), (-12, -5)$ **5.** $(2, 4), (-2, -4)$ **7.** $(5, -5), (-5, 5)$

9. $(4 + 2\sqrt{3}, 1 + \sqrt{3}), (4 - 2\sqrt{3}, 1 - \sqrt{3})$

11. $(2, 4), (2, -4), (-2, 4), (-2, -4)$

13. $(1, 3), (1, -3), (-1, 3), (-1, -3)$

15. $(1 + \sqrt{5}, -1 + \sqrt{5}), (1 - \sqrt{5}, -1 - \sqrt{5})$

17. $(\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2}), (2, 1), (-2, -1)$

19. $(2, 2i), (2, -2i), (-2, 2i), (-2, -2i)$

21. $(2, \sqrt{2}), (2, -\sqrt{2}), (-1, i), (-1, -i)$

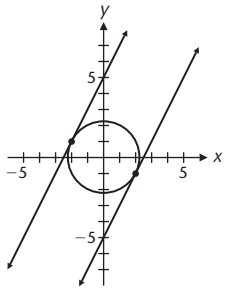
23. $(3, 0), (-3, 0), (\sqrt{5}, 2), (-\sqrt{5}, 2)$

25. $(2, 1), (-2, -1), (i, -2i), (-i, 2i)$

27. $(-1, 4), (3, -4)$ **29.** $(0, 0), (3, 6)$

31. $(1, 4), (4, 1)$ **33.** $(-1, 3), (4, 8)$

35. (A) The lines are tangent to the circle.



(B) $b = 5$, intersection point is $(2, -1)$; $b = -5$, intersection point is $(-2, 1)$

(C) The line $x + 2y = 0$ is perpendicular to all the lines in the family and intersects the circle at the intersection points found in part B. Solving the system $x^2 + y^2 = 5, x + 2y = 0$ would determine the intersection points.

37. $(-5, -\frac{3}{5}), (-\frac{3}{2}, -2)$ **39.** $(0, -1), (-4, -3)$

41. $(2, 2), (-2, -2), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$

43. $(-3, 1), (3, -1), (-i, i), (i, -i)$

45. $(-1.41, -0.82), (-0.13, 1.15), (0.13, -1.15), (1.41, 0.82)$

47. $(-1.66, -0.84), (-0.91, 3.77), (0.91, -3.77), (1.66, 0.84)$

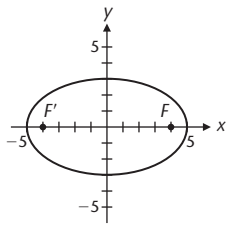
49. $(-2.96, -3.47), (-0.89, -3.76), (1.39, 4.05), (2.46, 4.18)$

51. $\frac{1}{2}(3 - \sqrt{5}), \frac{1}{2}(3 + \sqrt{5})$ **53.** 5 in. and 12 in. **55.** 6 in. by 4.5 in.

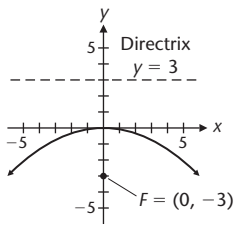
57. 22 ft by 26 ft **59.** Boat *A:* 30 mph; boat *B:* 25 mph

Review Exercises

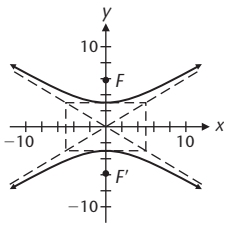
1. Foci: $F' = (-4, 0), F = (4, 0)$; major axis length $= 10$; minor axis length $= 6$ *(11-2)*



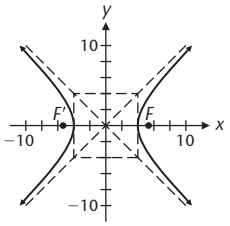
2. *(11-1)*



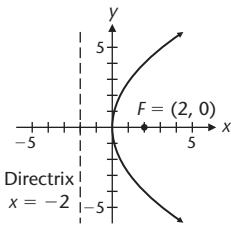
3. Foci: $F' = (0, -\sqrt{34})$, $F = (0, \sqrt{34})$;
transverse axis length = 6;
conjugate axis length = 10 *(11-3)*



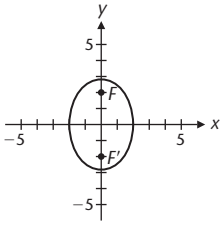
4. Foci: $F' = (-4\sqrt{2}, 0)$, $F = (4\sqrt{2}, 0)$;
transverse axis length = 8;
conjugate axis length = 8 *(11-3)*



5. *(11-1)*



6. Foci: $F' = (0, -2)$, $F = (0, 2)$;
major axis length = $4\sqrt{2}$;
minor axis length = 4 *(11-2)*



7. (A) $\dfrac{(y+2)^2}{25} - \dfrac{(x-4)^2}{4} = 1$ (B) Hyperbola *(11-4)*

8. (A) $(x+5)^2 = -12(y+4)$ (B) Parabola *(11-4)*

9. (A) $\dfrac{(x-6)^2}{9} + \dfrac{(y-4)^2}{16} = 1$ (B) Ellipse *(11-4)*

10. (A) $((3\sqrt{3}+4)/2, (-3+4\sqrt{3})/2)$ (B) $(7\sqrt{2}/2, \sqrt{2}/2)$
(C) $((3+4\sqrt{3})/2, (-3\sqrt{3}+4)/2)$ *(11-4)*

12. (5, −3), (−1, 3) *(11-5)* **13.** (1, −1), (1.4, −0.2) *(11-5)*

14. (1, 3), (1, −3), (−1, 3), (−1, −3) *(11-5)*

16. $\dfrac{x^2}{36} + \dfrac{y^2}{25} = 1$ *(11-2)* **17.** $\dfrac{x^2}{36} + \dfrac{y^2}{100} = 1$ *(11-2)*

18. $\dfrac{y^2}{7} - \dfrac{x^2}{9} = 1$ *(11-3)* **19.** $\dfrac{x^2}{49} - \dfrac{y^2}{64} = 1$ *(11-3)*

20. (−4, 2), (4, −2) *(11-5)* **21.** (−3, 3.2), (3, 3.2) *(11-5)*

22. (1, 3), (1, −3), (−1, 3), (−1, −3) *(11-5)*

23. $(2, \sqrt{2})$, $(2, -\sqrt{2})$, $(-1, i)$, $(-1, -i)$ *(11-5)*

24. (1, −2), (−1, 2), (2, −1), (−2, 1) *(11-5)*

25. $(2, -2)$, $(-2, 2)$, $(\sqrt{2}, \sqrt{2})$, $(-\sqrt{2}, -\sqrt{2})$ *(11-5)*

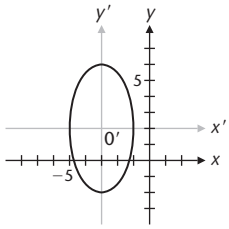
26. $x^2 = -20y$ *(11-1)* **27.** $(-3\sqrt{3}, 0)$, $(3\sqrt{3}, 0)$ *(11-2)*

28. $(0, -\sqrt{5})$, $(0, \sqrt{5})$ *(11-3)* **29.** $x = 4$ *(11-1)*

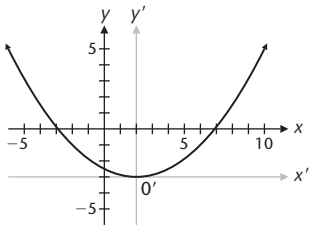
30. (0, 0), (−4, 2) *(11-1)* **31.** $(-4\sqrt{3}, 0)$, $(4\sqrt{3}, 0)$ *(11-2)*

32. $(0, -4\sqrt{5})$, $(0, 4\sqrt{5})$ *(11-3)*

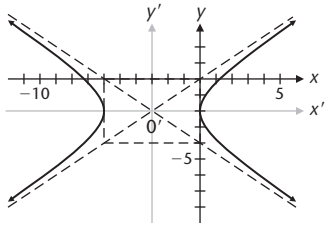
33. $\dfrac{(x+3)^2}{4} + \dfrac{(y-2)^2}{16} = 1$; ellipse *(11-4)*



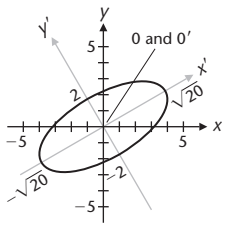
34. $(x-2)^2 = 4(2)(y+3)$; parabola *(11-4)*



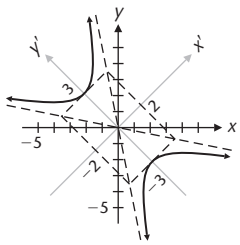
35. $\dfrac{(x+3)^2}{9} - \dfrac{(y+2)^2}{4} = 1$; hyperbola *(11-4)*



36. $\dfrac{x'^2}{20} + \dfrac{y'^2}{4} = 1$; ellipse *(11-4)*



37. $\frac{y'^2}{9} - \frac{x'^2}{4} = 1$; hyperbola; $\theta = 45°$ *(11-4)*



39. $(y-4)^2 = -8(x-4)$ or $y^2 - 8y + 8x - 16 = 0$ *(11-1)*

40. $\frac{x^2}{4} - \frac{y^2}{12} = 1$; hyperbola *(11-3)* **41.** $\frac{x^2}{36} + \frac{y^2}{20} = 1$; ellipse *(11-2)*

42. $F' = (-3, -\sqrt{12} + 2)$ and $F = (-3, \sqrt{12} + 2)$ *(11-4)*

43. F = (2, −1) *(11-4)*

44. $F' = (-\sqrt{13} - 3, -2)$ and $F = (\sqrt{13} - 3, -2)$ *(11-4)*

45. $y = \pm\frac{5}{7}x$ *(11-3)* **46.** $y = \pm 4x$ *(11-3)* **47.** $y = \pm 2x$ *(11-3)*

48. 4 feet *(11-1)* **49.** $\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$ *(11-2)* **50.** 4.72 feet deep *(11-3)*

Cumulative Review

1. (A) Arithmetic (B) Geometric (C) Neither (D) Geometric (E) Arithmetic *(10-3)*

2. (A) 10, 50, 250, 1,250 (B) $a_8 = 781{,}250$ (C) $S_8 = 976{,}560$ *(10-3)*

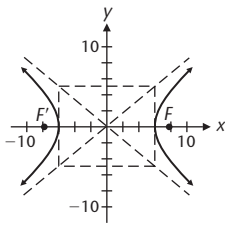
3. (A) 2, 5, 8, 11 (B) $a_8 = 23$ (C) $S_8 = 100$ *(10-3)*

4. (A) 100, 94, 88, 82 (B) $a_8 = 58$ (C) $S_8 = 632$ *(10-3)*

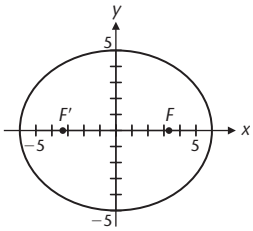
5. (A) 40,320 (B) 992 (C) 84 *(10-4)*

6. (A) 21 (B) 21 (C) 42 *(10-4, 10-5)*

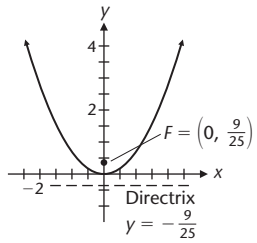
7. Foci: $F' = (-\sqrt{61}, 0)$, $F = (\sqrt{61}, 0)$; transverse axis length = 12; conjugate axis length = 10 *(11-3)*



8. Foci: $F' = (-\sqrt{11}, 0)$, $F = (\sqrt{11}, 0)$; major axis length = 12; minor axis length = 10 *(11-2)*

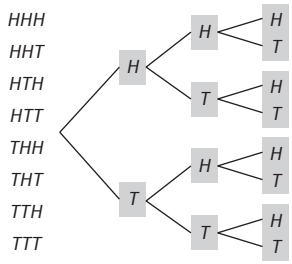


9. *(11-1)*



10. (1, 1) (−1/5, −7/5) *(11-5)* **11.** Ellipse *(11-4)*

12. (A) 8 combined outcomes: (B) $2 \cdot 2 \cdot 2 = 8$ *(10-5)*



13. (A) $4 \cdot 3 \cdot 2 \cdot 1 = 24$ (B) $P_{4,4} = 4! = 24$ *(10-5)*

14. $\frac{C_{13,3}}{C_{52,3}} \approx .0129$ *(10-5)* **15.** $\frac{1}{P_{10.4}} \approx .0002$; $\frac{1}{C_{10.4}} \approx .0048$ *(10-5)*

16. .62 *(10-5)*

17. P_1: $1 = 1(2 \cdot 1 - 1)$; P_2: $1 + 5 = 2(2 \cdot 2 - 1)$; P_3: $1 + 5 + 9 = 3(2 \cdot 3 - 1)$ *(10-2)*

18. P_1: $1^2 + 1 + 2 = 4$ is divisible by 2; P_2: $2^2 + 2 + 2 = 8$ is divisible by 2; P_3: $3^2 + 3 + 2 = 14$ is divisible by 2 *(10-2)*

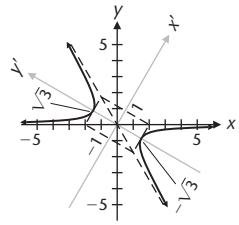
19. P_k: $1 + 5 + 9 + \cdots + (4k-3) = k(2k-1)$; P_{k+1}: $1 + 5 + 9 + \cdots + (4k-3) + (4k+1) = (k+1)(2k+1)$ *(10-2)*

20. P_k: $k^2 + k + 2 = 2r$ for some integer r; P_{k+1}: $(k+1)^2 + (k+1) + 2 = 2s$ for some integer s *(10-2)*

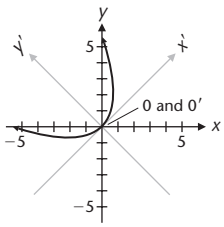
21. $y = -2x^2$ *(11-1)* **22.** $\frac{x^2}{25} + \frac{y^2}{16} = 1$ *(11-2)*

23. $\frac{x^2}{64} - \frac{y^2}{25} = 1$ *(11-3)*

24. $\frac{y'^2}{3} - x'^2 = 1$; hyperbola; $\theta = 60°$ *(11-4)*

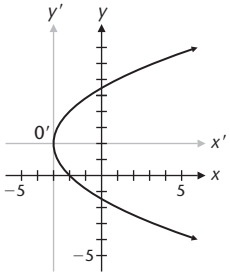


25. $x'^2 = 4y'$; parabola; $\theta = 45°$ *(11-4)*

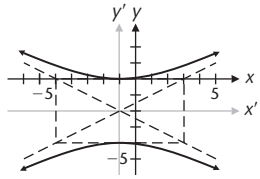


26. $(1, 1), (-1, -1), (\sqrt{3}, \sqrt{3}/3), (-\sqrt{3}, -\sqrt{3}/3)$ *(11-5)*
27. $(0, i), (0, -i), (1, 1), (-1, -1)$ *(11-5)*
28. $(-1.35, 0.28), (-0.87, -1.60), (0.87, 1.60), (1.35, -0.28)$ *(11-4)*
29. $1 + 4 + 27 + 256 + 3{,}125 = 3{,}413$ *(10-1)*
30. $\sum_{k=1}^{6} (-1)^{k+1} \frac{2^k}{(k+1)!}$ *(10-1)*
31. 81 *(10-3)* **32.** 360; 1,296; 750 *(10-4)*
33. $\frac{C_{10,3}}{C_{12,5}} = \frac{5}{33} = .\overline{15}$ *(10-5)* **34.** (A) .365 (B) $\frac{1}{3}$ *(10-5)*
35. $n = 22$ *(10-3)*
36. (A) 6,375,600 (B) 53,130 (C) 53,130 *(10-4, 10-6)*
37. $a^6 + 3a^5b + \frac{15}{4}a^4b^2 + \frac{5}{2}a^3b^3 + \frac{15}{16}a^2b^4 + \frac{3}{16}ab^5 + \frac{1}{64}b^6$ *(10-6)*
38. $153{,}090x^6y^4$; $-3{,}240x^3y^7$ *(10-6)*
41. 61,875 *(10-3)* **42.** $\frac{27}{11}$ *(10-3)* **43.** $a_{27} = 0.236$; 8 terms *(10-6)*

44. $4(x + 3) = (y - 2)^2$; parabola *(11-1, 11-4)*



45. $\frac{(y+2)^2}{4} - \frac{(x+1)^2}{16} = 1$; hyperbola *(11-3, 11-4)*



46. $\frac{(x-2)^2}{9} + \frac{(y+3)^2}{4} = 1$; ellipse *(11-2, 11-4)*
47. 10^9; 3,628,800 zip codes *(10-1)* **49.** $\frac{2}{5}$; $\frac{2}{5}$ *(10-5)*
50. $x^6 - 12ix^5 - 60x^4 + 160ix^3 + 240x^2 - 192ix - 64$ *(10-6)*
51. $x^2 - 12x + 4y + 28 = 0$ *(11-1)* **52.** $\pm 2\sqrt{3}$ *(11-2)*
53. 8 *(11-3)* **54.** $C_{7,3} = 35$ *(10-4)*
57. $x^2 - 8y^2 - 2x - 8y + 17 = 0$; hyperbola *(11-3)*

58. $1 - \frac{C_{8,3}}{C_{12,3}} = \frac{41}{55} = .7\overline{45}$ *(10-5)* **59.** \$6,000,000 *(10-3)*
60. 4 meters by 8 meters *(11-4)* **61.** 4 inches *(11-1)*
62. 32 feet, 14.4 feet *(11-2)* **63.** (A) .13 (B) .17 (C) .32 *(10-5)*

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